Examples of Disk Algebras

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Abstract. We produce refinements of the known multiplicative structures on the Brown-Peterson spectrum BP, its truncated variants $\mathrm{BP}\langle n\rangle$, Ravenel's spectra $X(n)$, and evenly graded polynomial rings over the sphere spectrum. Consequently, topological Hochschild homology relative to these rings inherits a circle action.

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1. Introduction

If R is a ring spectrum, then the algebraic K-theory of R is often understood by means of its trace maps to $\mathrm{TC}^-(R) = \mathrm{THH}(R)^{hS^1}$, $\mathrm{TP}(R) = \mathrm{THH}(R)^{tS^1}$, and $\mathrm{TC}(R)$. To compute any of these invariants, it has proven extremely fruitful to approximate the absolute Hochschild homology $\mathrm{THH}(R)$ by Hochschild homology relative to some other base, i.e. perform descent along a map

$$\mathrm{THH}(R) \to \mathrm{THH}(R/A).$$

For example, this is one of the main ideas behind the definition of prismatic cohomology of ring spectra given in [**HRW22**], and is featured in the foundational [**BMS19**, §11]. Works such as [**AKN22, LW22, KN19, Lee22, HW22**] showcase both the computational and theoretical effectiveness of the technique.

To enact the above strategy, one needs A to admit enough structure that $\mathrm{THH}(R/A)$ exists as an S^1-equivariant A-module. The action of S^1 on $\mathbb{R}^2$ by rotation defines an S^1-action on the operad $\mathbb{E}_2$, and hence an S^1-action on the category $\mathsf{Alg}_{\mathbb{E}_2}$ of $\mathbb{E}_2$-algebras. The structure necessary on A to define an S^1-A-module structure on $\mathrm{THH}(R/A)$ is that of a homotopy fixed point for this action.[1] The category $\mathsf{Alg}_{\mathbb{E}_2}^{hS^1}$ goes by many names, such as the category of framed $\mathbb{E}_2$-algebras, $\mathbb{E}_2 \rtimes S^1$-algebras, or $\mathbb{E}_{\mathrm{BU}(1)}$-algebras. Following [**AF15**], we will call these $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-*algebras*; more generally, there is a notion of Disk_n^B-algebra which we review below.

Here, we prove that several familiar and fundamental ring spectra admit extra structure of this form:

Theorem 1.1 (Corollary 3.7, Corollary 3.12, Corollary 4.2, Theorem 5.2)**.** *We have:*

(1) At any prime p, BP admits the structure of a $\mathsf{Disk}_4^{\mathrm{BU}(2)}$-$\mathrm{MU}$-algebra.
(2) At any prime p and for each integer $n \geq 0$, there is a form of $\mathrm{BP}\langle n \rangle$ which is a $\mathsf{Disk}_3^{\mathrm{BU}(1)}$-$\mathrm{MU}$-algebra.
(3) For each integer $n \geq 0$, the Ravenel spectrum $X(n)$ admits the structure of a $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-algebra.
(4) For any integer n, the spherical polynomial algebra $\mathbb{S}[x_{2n}]$ on a degree $2n$ class admits the structure of a $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-algebra.

Remark 1.2. There has been a long history of work equipping the above ring spectra with highly structured multiplications. For example, Basterra–Mandell [**BM13**] proved that BP admits a unique $\mathbb{E}_4$-algebra structure, and in [**HW22**] the second and fifth authors show that there are $\mathbb{E}_3$-MU-algebra forms of $\mathrm{BP}\langle n \rangle$. The above theorem strengthens these results, and can be seen as part of the general effort to equip ring spectra with the maximum possible amount of structure.

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2. Review of Disk Algebras

2.1. Definitions.

We recall the algebraic setup from [**AF15**].

Definition 2.1. Let $B \in \mathsf{Spaces}_{/\mathrm{BTop}(n)}$. Then Disk_n^B [**AF15**, Definition 2.9] is the symmetric monoidal (∞-)category of n-manifolds homeomorphic to finite disjoint unions of n-dimensional Euclidean spaces equipped with a lift of the classifer of their tangent microbundle to B. The symmetric monoidal structure is given by disjoint union. The category of Disk_n^B-algebras in a symmetric monoidal category $\mathcal{C}$ is defined as

$$\mathsf{Alg}_{\mathsf{Disk}_n^B}(\mathcal{C}) := \mathsf{Fun}^{\otimes}(\mathsf{Disk}_n^B, \mathcal{C}).$$

Remark 2.2. The symmetric monoidal category Disk_n^B is the symmetric monoidal envelope of the ∞-operad $\mathbb{E}_B$ of [**Lur17**, 5.4.2.10]. Combining [**Lur17**, 2.2.4.9 and

[1] This appears to be folklore, but we give a short proof in Corollary 2.9.

2.3.3.4], we learn that the map $B \to \mathrm{BTop}(n)$ produces a local system of categories of $\mathbb{E}_n$-algebras and that there is an equivalence:

$$\mathsf{Alg}_{\mathsf{Disk}_n^B}(\mathcal{C}) \simeq \lim_B \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C}).$$

2.2. Disk algebras in spaces. Given any pointed space X, the functor of compactly-supported maps

$$\mathrm{Map}_c(-,X) : \mathsf{Disk}_n \to \mathsf{Spaces}$$

is symmetric monoidal for the structure of disjoint union on the source and cartesian product on the target. Observe that, upon restriction to the full subcategory spanned by $\mathbb{R}^n$, we obtain the local system

$$\mathrm{BTop}(n) \to \mathsf{Spaces}$$

associated to the action of $\mathrm{Top}(n)$ on $\mathrm{Map}_c(\mathbb{R}^n, X) = \Omega^n X$. We will denote this local system by $\Omega^{\lambda_n} X$.

Proposition 2.3. *The above construction refines to an adjunction*

$$\mathsf{Alg}_{\mathsf{Disk}_n^B}(\mathsf{Spaces}) \underset{\Omega^{\lambda_n}}{\overset{\mathrm{B}^{\lambda_n}}{\rightleftarrows}} \mathsf{Psh}(B)_*.$$

This restricts to an equivalence between group-like algebras and pointwise n-connective presheaves.

Proof. In the discussion above we produced a functor

$$\mathsf{Spaces}_* \to \mathsf{Alg}_{\mathsf{Disk}_n^B}(\mathsf{Spaces}) = \lim_B \mathsf{Alg}_{\mathbb{E}_n}(\mathsf{Spaces}).$$

given by $X \mapsto \Omega^{\lambda_n} X$. This is the same data as a map

$$\mathsf{Spaces}_* \to \mathsf{Alg}_{\mathbb{E}_n}(\mathsf{Spaces})$$

of presheaves on B, where the source is regarded as a constant presheaf. Taking global sections then produces the desired functor Ω^{λ_n}. The existence of a left adjoint and the restricted equivalence is a formal consequence of the known statement applied pointwise on B. $\square$

We will also need the following computation.

Lemma 2.4. *Suppose $X = \Omega^\infty M$ is an infinite-loop space given the structure of a Disk_n^B-algebra by restriction. Then $\mathrm{B}^{\lambda_n} X = \Omega^\infty \Sigma^{\lambda_n} M$.*

Proof. As in the previous proposition, observe that the construction $Y \mapsto \Omega^{\lambda_n} Y$ refines to a functor (which we temporarily give alternative notation)

$$\Pi^{\lambda_n} : \mathsf{Fun}(B, \mathsf{Sp}) \to \mathsf{Alg}_{\mathsf{Disk}_n^B}(\mathsf{Sp}^\times)$$

that intertwines Ω^∞. Here we have decorated Sp with $\times$ to indicate that we are using the cartesian monoidal structure. Since Sp is stable, this coincides with the cocartesian monoidal structure and thus by [**Lur17**, 2.4.3.8] the forgetful functor

$$\mathsf{Alg}_{\mathsf{Disk}_n^B}(\mathsf{Sp}^\times) \to \mathsf{Fun}(B, \mathsf{Sp})$$

is an equivalence. By design, the composite of Π^{λ_n} with this forgetful functor is Ω^{λ_n}. In other words: the two potentially different (additive) Disk_n^B-algebra structures on the spectrum $\Omega^{\lambda_n} Y$ must coincide for any local system of spectra on B.

It then follows that $\Omega^{\lambda_n}\Omega^\infty\Sigma^{\lambda_n} M \simeq X$ as Disk_n^B-algebras, which proves the result. $\square$

Warning 2.5. If X is an $\mathbb{E}_\infty$-space then, when regarded as a Disk_n^B-algebra, the underlying presheaf on B is constant. However, the presheaf $\mathrm{B}^{\lambda_n} X$ need not be constant. For example, if $X = \mathbb{Z}$ and $B = \mathrm{BO}(1)$, then $\mathrm{B}^{\lambda_1}\mathbb{Z} = S^{\lambda_1}$ is the one-point compactification of the sign representation. This is not (Borel) equivariantly trivial, as seen, for example, from its integral homology.

2.3. Factorization homology. Recall from [**AF15**] that Mfld_n^B denotes the symmetric monoidal (∞-)category of B-framed manifolds, which contains Disk_n^B as a full subcategory.

Definition 2.6. Let A be a Disk_n^B-algebra in a presentably symmetric monoidal ∞-category $\mathcal{C}$. We define the factorization homology functor

$$\int_{(-)} A : \mathsf{Mfld}_n^B \to \mathcal{C}$$

by left Kan extension along the inclusion $\mathsf{Disk}_n^B \hookrightarrow \mathsf{Mfld}_n^B$.

This functor gives a generalization of Hochschild homology by the following theorem.

Theorem 2.7 (Ayala-Francis, Lurie)**.** *If A is a Disk_1^B-algebra in $\mathcal{C}$, then there is a canonical, S^1-equivariant equivalence*

$$\int_{S^1} A \simeq \mathrm{HH}(A),$$

where the latter is defined via the cyclic bar construction in $\mathcal{C}$.

Encoding factorization homology as a functor on B-framed manifolds now allows us to equip relative Hochschild homology with a circle action.

Corollary 2.8. *If A is a $\mathsf{Disk}_{n+2}^{\mathrm{BU}(1)}$-algebra in $\mathcal{C}$ (so that A admits an $S^1 = \mathrm{U}(1)$-action), then the augmentation $\mathrm{HH}(A) \to A$ refines to an S^1-equivariant map of $\mathbb{E}_n$-algebras.*

Proof. Apply functoriality to the inclusion $(\mathbb{R}^2 - \{0\}) \times \mathbb{R}^n \to \mathbb{R}^{n+2}$. Here the $\mathrm{U}(1)$-framing is determined by the maps

$$\mathbb{R}^{n+2} \to * \to \mathrm{BU}(1) \to \mathrm{BTop}(n+2)$$

where the first lift is the standard trivialization, and $\mathrm{U}(1)$ is given the standard action on the first two coordinates. $\square$

Corollary 2.9. *If A is a $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-algebra and $\mathcal{D}$ is an A-linear category, then $\mathrm{THH}(\mathcal{D}/A)$ has a canonical S^1-action.*

Proof. By the previous corollary, base change along $\mathrm{THH}(A) \to A$ takes S^1-equivariant $\mathrm{THH}(A)$-modules to S^1-equivariant A-modules. Whence the claim for

$$\mathrm{THH}(\mathcal{D}/A) = \mathrm{THH}(\mathcal{D}) \otimes_{\mathrm{THH}(A)} A. \qquad \square$$

Remark 2.10. The non-S^1-equivariant analogue of Corollary 2.8 is proved as [**KN18**, Lemma 4.6]. The statement of Corollary 2.8 for $n = 2$ was also mentioned in [**Yua21**, Remark 3.4], where it was attributed to Asaf Horev.

3. Thom spectra

In this section we explain how to equip Thom spectra with Disk-algebra structures in certain situations.

3.1. Disk algebras in spaces and orientability. We now observe that orientability allows us to automatically upgrade some $\mathbb{E}_n$-algebras to Disk-algebras.

Proposition 3.1. *Suppose $X = \Omega^\infty E$ and that there is a chosen equivalence $\Sigma^{\lambda_n} E \simeq \Sigma^n E$ of local systems on B. Then every group-like $\mathbb{E}_n$-algebra Y equipped with a $\mathbb{E}_n$-algebra map $Y \to X$ has a canonical refinement to a Disk_n^B-algebra over X.*

Proof. By assumption, there is a map of spaces

$$\mathrm{B}^n Y \to \mathrm{B}^n X.$$

Using our assumption that $\Sigma^{\lambda_n} E \simeq \Sigma^n E$, we obtain equivalences:

$$\mathrm{B}^n X \simeq \Omega^\infty \Sigma^n E \simeq \Omega^\infty \Sigma^{\lambda_n} E \simeq \mathrm{B}^{\lambda_n} X.$$

Thus we get a map of $\mathsf{Disk}_{2n}^{\mathrm{BU}(n)}$-algebras

$$\Omega^{\lambda_n} \mathrm{B}^n Y \to X$$

which refines the original map. □

Warning 3.2. The action of ΩB on Y constructed above may be nontrivial.

Corollary 3.3. *Let Y be a group-like $\mathbb{E}_{2n}$-algebra over $\mathrm{BU} \times \mathbb{Z}$. Then Y has a canonical refinement to a $\mathsf{Disk}_{2n}^{\mathrm{BU}(n)}$-algebra over $\mathrm{BU} \times \mathbb{Z}$.*

Proof. As ku is a module over MU, a choice of Thom class for the bundle λ_{2n} over $\mathrm{BU}(n)$ gives the desired $\mathrm{U}(n)$-equivariant equivalence

$$\Sigma^{\lambda_{2n}} \mathrm{ku} = \Sigma^{\lambda_{2n}} \mathrm{MU} \otimes_{\mathrm{MU}} \mathrm{ku} \simeq \Sigma^{2n} \mathrm{MU} \otimes_{\mathrm{MU}} \mathrm{ku} \simeq \Sigma^{2n} \mathrm{ku}. \qquad \square$$

The same argument using the Atiyah-Bott-Shapiro orientation gives the following.

Corollary 3.4. *Let Y be a group-like $\mathbb{E}_n$-algebra over $\mathrm{BO} \times \mathbb{Z}$. Then Y has a canonical refinement to a $\mathsf{Disk}_n^{\mathrm{BSpin}(n)}$-algebra over $\mathrm{BO} \times \mathbb{Z}$.*

3.2. Main Result. Let $\mathcal{C}$ be presentably symmetric monoidal and denote by $\mathrm{Pic}(\mathcal{C})$ the $\mathbb{E}_\infty$-groupoid of $\otimes$-invertible objects in $\mathcal{C}$. Recall that, for any map $X \to \mathrm{Pic}(\mathcal{C})$, there is a unique extension to a colimit-preserving functor

$$\mathrm{Th}_{\mathcal{C}} : \mathsf{Spaces}_{/X} \to \mathcal{C};$$

if the map $X \to \mathrm{Pic}(\mathcal{C})$ is one of $\mathbb{E}_\infty$-spaces, this functor is lax symmetric monoidal. See, e.g. [**HL20**, Proposition 3.1.3] and [**CCRY22**, Section 7.1]. We will be mainly concerned with the cases $\mathcal{C} = \mathsf{Sp}$ and $\mathcal{C} = \mathsf{Sp}_{(p)}$. For any ∞-operad $\mathcal{O}$, we then get an induced functor

$$\mathrm{Th}_{\mathcal{C}} : \mathsf{Alg}_{\mathcal{O}}(\mathsf{Spaces}_{/X}) \simeq \mathsf{Alg}_{\mathcal{O}}(\mathsf{Spaces})_{/X} \to \mathsf{Alg}_{\mathcal{O}}(\mathcal{C}).$$

Applying the results from the previous subsection, we immediately deduce the following.

Theorem 3.5. *Let $\mathcal{C}$ be presentably symmetric monoidal. Suppose $X = \Omega^\infty E$, there is a chosen trivialization $\Sigma^{\lambda_n} E \simeq \Sigma^n E$ of local systems on B, and and we are given an $\mathbb{E}_\infty$-algebra map $X \to \mathrm{Pic}(\mathcal{C})$. Suppose $\xi : Y \to X$ is a map of group-like $\mathbb{E}_n$-algebras. Then $\mathrm{Th}_{\mathcal{C}}(\xi)$ admits a canonical Disk_n^B-algebra structure.*

Corollary 3.6. *Suppose $\xi : Y \to \mathrm{BU} \times \mathbb{Z}$ is a map of group-like $\mathbb{E}_{2n}$-algebras. Then the Thom spectrum $\mathrm{Th}(\xi)$ admits a canonical $\mathsf{Disk}_{2n}^{\mathrm{BU}(n)}$-algebra structure.*

Corollary 3.7. *Let $X(n)$ be the Ravenel spectrum from* [**Rav84**, Section 3]*. Then $X(n)$ admits the structure of an $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-algebra.*

Proof. By definition, $X(n)$ is the Thom spectrum of the double loop map $\Omega^2\mathrm{BSU}(n) \to \Omega^2\mathrm{BSU} \simeq \mathrm{BU}$. □

Remark 3.8. Recall that there is a truncated form of the Quillen idempotent ϵ_m on $X(p^m)_{(p)}$ (see [**Hop84**, Proposition 1.3.7]). We will write $T(m)$ to denote the resulting summand of $X(p^m)_{(p)}$, so that $T(m)$ approximates BP in the same way as $X(m)$ approximates MU. At $p = 2$, $T(1)$ admits the structure of an $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-algebra. Indeed, in this case, $T(1) = X(2)$, so the result follows from Corollary 3.7. At $p = 2$, it is also known that $T(2)$ admits the structure of an $\mathbb{E}_2$-ring. Using Corollary 3.6, one can show that $T(2)$ in fact admits the structure of an $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-algebra: indeed, by [**Dev22**, Remark 3.1.9], it is the Thom spectrum of the double loop map $\mu : \Omega\mathrm{Sp}(2) \to \mathrm{BU}$ obtained from taking double loops of the composite

$$\mathrm{BSp}(2) \to \mathrm{BSU}(4) \to \mathrm{BSU} \simeq \mathrm{B}^3\mathrm{U}.$$

Corollary 3.6 can also be used to study polynomial rings over the sphere spectrum. Recall the following construction, e.g., from [**HW22**, Construction 4.1.1] (see also [**Lur15**, Section 3.4]).

Construction 3.9. Fix an integer $n \in \mathbb{Z}$, and let $\mathbb{Z}^{\mathrm{ds}}$ denote the constant simplicial set associated to the set of integers. Then, the free graded $\mathbb{E}_1$-ring $\mathbb{S}[x_{2n,1}]$ on a class in degree $2n$ and weight 1 admits the structure of a graded $\mathbb{E}_2$-ring. This can be viewed as an $\mathbb{E}_2$-monoidal functor $\iota_n : \mathbb{Z}^{\mathrm{ds}} \to \mathsf{Sp}$ sending $1 \mapsto S^{2n}$; this functor factors through the inclusion $\mathsf{Pic}(\mathsf{Sp}) \to \mathsf{Sp}$. Let us write $\mathbb{S}[x_{2n}]$ to denote the underlying $\mathbb{E}_2$-ring in Sp.

Using the $\mathbb{E}_2$-monoidal functor ι_n, one can define a spectral analogue of the "shearing" functor on graded spectra. The following is an adaptation of [**Rak20**, Proposition 3.3.4]. Let $\mathcal{C}$ be a stable presentably symmetric monoidal ∞-category, and let $\mathcal{C}^{\mathrm{gr}} = \mathsf{Fun}(\mathbb{Z}^{\mathrm{ds}}, \mathcal{C})$ denote the ∞-category of graded objects in $\mathcal{C}$. The composite

$$\mathbb{Z}^{\mathrm{ds}} \times \mathcal{C}^{\mathrm{gr}} \xrightarrow{\iota_n \times \mathrm{ev}} \mathsf{Pic}(\mathsf{Sp}) \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C} \tag{1}$$

is a lax $\mathbb{E}_2$-monoidal functor. Using the universal property of Day convolution, this in turn defines a lax $\mathbb{E}_2$-monoidal functor $\mathrm{sh}_n : \mathcal{C}^{\mathrm{gr}} \to \mathcal{C}^{\mathrm{gr}}$ which acts on a graded object by $M(\bullet) \mapsto M(\bullet)[2n\bullet]$. It is easy to see that this functor is in fact $\mathbb{E}_2$-monoidal and defines an equivalence $\mathrm{sh}_n : \mathcal{C}^{\mathrm{gr}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{gr}}$. In fact, $\mathrm{sh}_n \simeq \mathrm{sh}_1^{\circ n}$.

Proposition 3.10. *The shearing equivalence $\mathrm{sh}_n : \mathcal{C}^{\mathrm{gr}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{gr}}$ admits the structure of a $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-monoidal functor.*

Proof. It suffices to show that the composite eq. (1) admits the structure of a $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-monoidal functor. The map $\otimes : \mathsf{Pic}(\mathsf{Sp}) \times \mathcal{C}^{\mathrm{gr}} \to \mathcal{C}$ is evidently symmetric monoidal, so it in turn suffices to show that ι_n admits the structure of a $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-monoidal functor. But ι_n can be factored as the composite

$$\mathbb{Z}^{\mathrm{ds}} \xrightarrow{\cdot n} \mathbb{Z}^{\mathrm{ds}} \to \mathrm{BU} \times \mathbb{Z}^{\mathrm{ds}} \to \mathsf{Pic}(\mathsf{Sp}),$$

where the second map is the inclusion of the factor in the product. The inclusion $\mathbb{Z}^{\mathrm{ds}} \to \mathrm{BU} \times \mathbb{Z}^{\mathrm{ds}}$ is one of group-like $\mathbb{E}_2$-algebras (for instance, it can be obtained via Bott periodicity by taking double loops of the map $\mathrm{BU}(1) \to \mathrm{BU}$), so the claim follows from the discussion in section 3.1. $\square$

Remark 3.11. The functor sh_1 does *not* admit an $\mathbb{E}_3$-monoidal structure. Otherwise, $\mathbb{S}[x_2]$ would admit the structure of an $\mathbb{E}_3$-algebra in Sp. To see that this is impossible, observe that if $\mathbb{S}[x_2]$ did admit the structure of an $\mathbb{E}_3$-algebra, the class $x_2^2 : S^4 \to \mathbb{S}[x_2]$ would factor as

$$\begin{tikzcd} & S^4 \arrow[d] \arrow[dr] & \\ \Sigma^2\mathbb{R}P_2^4 \arrow[r, "\sim"] & \mathrm{Conf}_2(\mathbb{R}^3)_+ \otimes_{\Sigma_2} (S^2)^{\otimes 2} \arrow[r, dashed] & \mathbb{S}[x_2]. \end{tikzcd}$$

Composing with the projection $\mathbb{S}[x_2] \to S^4$, this would show that the bottom cell of $\Sigma^2\mathbb{R}P_2^4$ is unattached; but this is false, since the 4- and 6-cells of $\Sigma^2\mathbb{R}P_2^4$ are connected by η.

It is easier to show that the inclusion $\mathbb{Z}^{\mathrm{ds}} \to \mathrm{BU} \times \mathbb{Z}^{\mathrm{ds}}$ is not a map of $\mathbb{E}_3$-algebras. Otherwise, taking the 3-fold bar construction would show that there is a map $K(\mathbb{Z},3) \to \mathrm{SU}$ which is an isomorphism on $\mathrm{H}^3(-;\mathbb{Z})$. This is impossible: for instance, the resulting composite

$$K(\mathbb{Z},3) \to \mathrm{SU} \to K(\mathbb{Z},3)$$

would be nonzero on $\mathrm{H}^6(-;\mathbb{Z})$; but $\mathrm{H}^6(\mathrm{SU};\mathbb{Z}) = 0$.

Corollary 3.12. *Let $j \in \mathbb{Z}$. Then, $\mathbb{S}[x_{2j,1}]$ admits the structure of a $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-algebra in $\mathsf{Sp}^{\mathrm{gr}}$.*

Proof. Let $\mathbb{S}[x_{0,1}] = \Sigma^\infty_+\mathbb{N}$ denote the free $\mathbb{E}_1$-algebra in graded spectra on a class in weight 1 and degree zero; this in fact admits the structure of an $\mathbb{E}_\infty$-ring in $\mathsf{Sp}^{\mathrm{gr}}$, and $\mathbb{S}[x_{2j,1}] \simeq \mathrm{sh}_j\mathbb{S}[x_{0,1}]$. Since sh_j is $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-monoidal by Proposition 3.10, this implies that $\mathbb{S}[x_{2j,1}]$ admits the structure of a $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-algebra in $\mathsf{Sp}^{\mathrm{gr}}$. $\square$

Remark 3.13. There is an evident generalization of Corollary 3.12 to multi-graded $\mathsf{Disk}_2^{\mathrm{BU}(1)}$-algebras in several variables.

4. Retracts of Complex Bordism

The following result allows us to equip BP with a Disk-algebra structure.

Theorem 4.1. *Every $\mathbb{E}_4$-algebra map $\mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$ refines to a $\mathsf{Disk}_4^{\mathrm{BU}(2)}$-algebra map.*

Proof. We would like to show that the map

$$\mathrm{Map}_{\mathsf{Disk}_2^{\mathrm{BU}(2)}}(\mathrm{MU}_{(p)}, \mathrm{MU}_{(p)}) \to \mathrm{Map}_{\mathbb{E}_4}(\mathrm{MU}_{(p)}, \mathrm{MU}_{(p)})$$

is surjective on path components. Under the identification $\mathsf{Alg}_{\mathsf{Disk}_2^{\mathrm{BU}(2)}} \simeq \mathsf{Alg}_{\mathbb{E}_2}^{h\mathrm{U}(2)}$, we may identify this map with the inclusion of fixed points

$$\mathrm{Map}_{\mathbb{E}_4}(\mathrm{MU}_{(p)}, \mathrm{MU}_{(p)})^{h\mathrm{U}(2)} \to \mathrm{Map}_{\mathbb{E}_4}(\mathrm{MU}_{(p)}, \mathrm{MU}_{(p)})$$

for some $\mathrm{U}(2)$-action on the source. To show that this is surjective on path components, it will suffice to prove that the homotopy fixed point spectral sequence for the source collapses. For this, it further suffices to prove that $\pi_*\mathrm{Map}_{\mathbb{E}_4}(\mathrm{MU}_{(p)}, \mathrm{MU}_{(p)})$ is concentrated in even degrees. Using the Thom isomorphism of [**AB19**, Corollary 3.18], we have:

$$\begin{aligned}
\mathrm{Map}_{\mathbb{E}_4}(\mathrm{MU}_{(p)}, \mathrm{MU}_{(p)}) &\simeq \mathrm{Map}_{\mathbb{E}_4}(\mathrm{MU}, \mathrm{MU}_{(p)}) \\
&\simeq \mathrm{Map}_{\mathbb{E}_4}(\Sigma^\infty_+\mathrm{BU}, \mathrm{MU}_{(p)}) \\
&\simeq \mathrm{Map}_*(\mathrm{BU}\langle 6\rangle, \mathrm{B}^4\mathrm{GL}_1\mathrm{MU}_{(p)}) \\
&\simeq \mathrm{Map}(\Sigma^{-4}\Sigma^\infty_+\mathrm{BU}\langle 6\rangle, \mathrm{gl}_1\mathrm{MU}_{(p)}).
\end{aligned}$$

Since $\mathrm{BU}\langle 6\rangle$ has an even cell decomposition, and the homotopy of $\mathrm{gl}_1\mathrm{MU}_{(p)}$ is concentrated in even degrees, the Atiyah-Hirzebruch spectral sequence collapses and the answer is concentrated in even degrees. This completes the proof. □

Corollary 4.2. BP *admits the structure of a* $\mathsf{Disk}_4^{\mathrm{BU}(2)}$*-algebra under* MU*.*

Proof. Apply the previous theorem to the $\mathbb{E}_4$-algebra idempotent produced by Basterra-Mandell in [**BM13**]. □

Warning 4.3. Unlike the Disk-algebra structures produced on Thom spectra, the refinements of the self-maps of MU, and hence of the algebra structure on BP, are highly non-canonical.

5. Truncated Brown-Peterson Spectra

In [**HW22**], the second and fifth authors produced $\mathbb{E}_3$-MU-algebra forms of $\mathrm{BP}\langle n\rangle$. In this section, we explain how to modify the argument in loc. cit. to produce $\mathsf{Disk}_3^{\mathrm{BU}(1)}$-MU-algebra structures.

5.1. Review of obstruction theory. If $\mathcal{O}$ is an operad, then the deformation theory of an $\mathcal{O}$-algebra A is governed by the cotangent complex, which is an operadic module over A. In the case of interest, it follows from [**Lur17**, 7.3.4.13] that the cotangent complex of a Disk_n^B-algebra A lies in

$$\mathsf{Mod}_A^{\mathsf{Disk}_n^B}(\mathcal{C}) := \lim_B \mathsf{Mod}_A^{\mathbb{E}_n}(\mathcal{C}).$$

The category of $\mathbb{E}_n$-A-modules is equivalent to the category of modules over the enveloping algebra

$$\mathcal{U}^{(n)}(A) := \int_{\mathbb{R}^n - \{0\}} A,$$

so an alternative perspective on Disk_n^B-A-modules is via the equivalence

$$\mathsf{Mod}_A^{\mathsf{Disk}_n^B}(\mathcal{C}) \simeq \mathsf{Mod}_{\mathcal{U}^{(n)}(A)}(\mathsf{Fun}(B, \mathcal{C})).$$

In these terms, the cotangent complex and enveloping algebra are related to one another using the following theorem:

Theorem 5.1 (Lurie, Francis). *If A is a Disk_n^B-algebra, then there is a fiber sequence*

$$\mathcal{U}^{(n)}(A) \to A \to \Sigma^{\lambda_n}\mathbb{L}_A$$

of Disk_n^B-A-modules.

Proof. The proof given in [**Fra13**, Theorem 2.26] applies verbatim for general B. $\square$

5.2. Main result.

Theorem 5.2. *There are forms of* $\mathrm{BP}\langle n\rangle$ *which are* $\mathsf{Disk}_3^{\mathrm{BU}(1)}$-MU-*algebras, and for which the maps*

$$\mathrm{BP}\langle n\rangle \to \mathrm{BP}\langle n-1\rangle$$

are maps of $\mathsf{Disk}_3^{\mathrm{BU}(1)}$-MU-*algebras.*

Proof. The proof in [**HW22**, Theorem 2.0.6] goes through *mutatis mutandis* using the description of the cotangent complex above, except that we replace the use of [**HW22**, Theorem 2.5.5] with the following refinement: for any virtual complex representation V of $\mathrm{U}(1)$, the spectrum

$$\left(\Sigma^V \mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\mathrm{BP}\langle n\rangle, \mathrm{BP}\langle n\rangle)\right)^{h\mathrm{U}(1)}$$

has homotopy groups concentrated in even degrees; moreover, the map

$$\pi_*\left(\Sigma^V \mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\mathrm{BP}\langle n\rangle, \mathrm{BP}\langle n\rangle)\right)^{h\mathrm{U}(1)} \to \pi_*\mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\mathrm{BP}\langle n\rangle, \mathrm{BP}\langle n\rangle)$$

is surjective.

In fact, this statement is an immediate consequence of Theorem 2.5.5. in *loc. cit.*: since the homotopy groups of $\mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\mathrm{BP}\langle n\rangle, \mathrm{BP}\langle n\rangle)$ (and hence of any even suspension) are concentrated in even degrees, the homotopy fixed point spectral sequence for $\mathrm{U}(1)$ collapses and is again concentrated in even degrees. $\square$

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