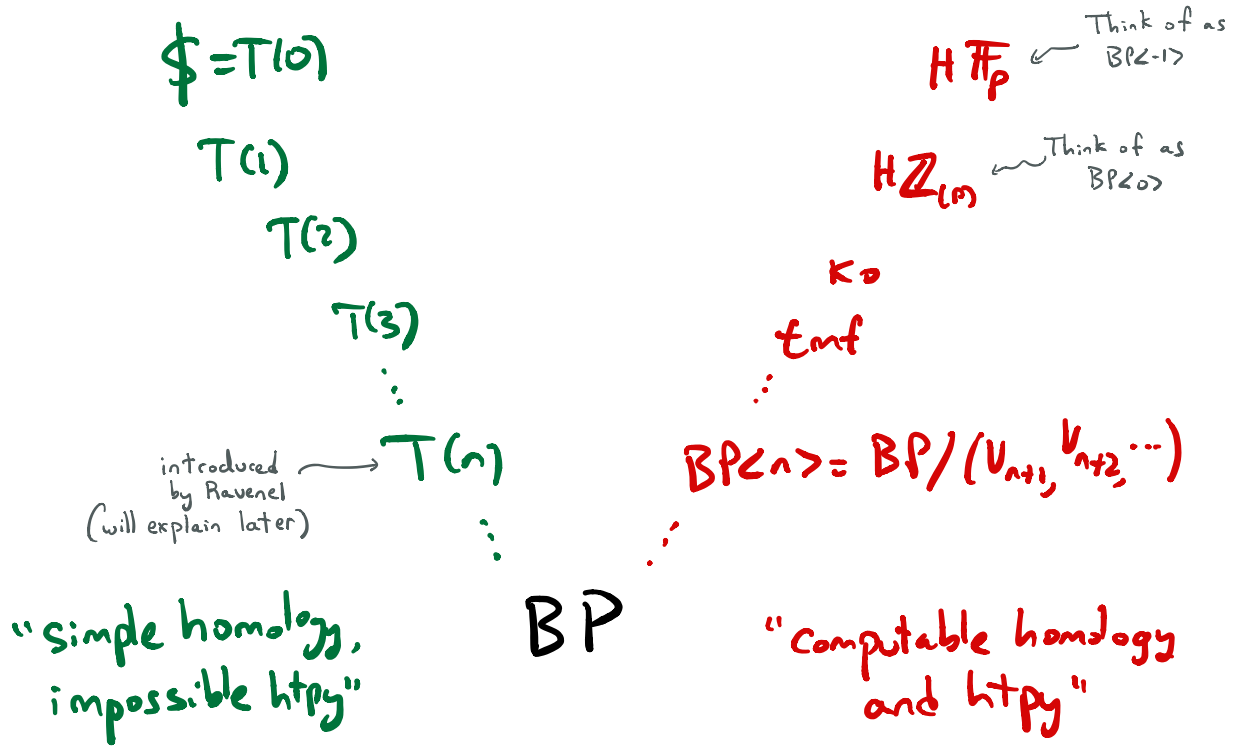


Chromatic analogues of the Hopkins-Mahowald theorem

Broad goal: relate two different "stratifications" of stable htpy thry

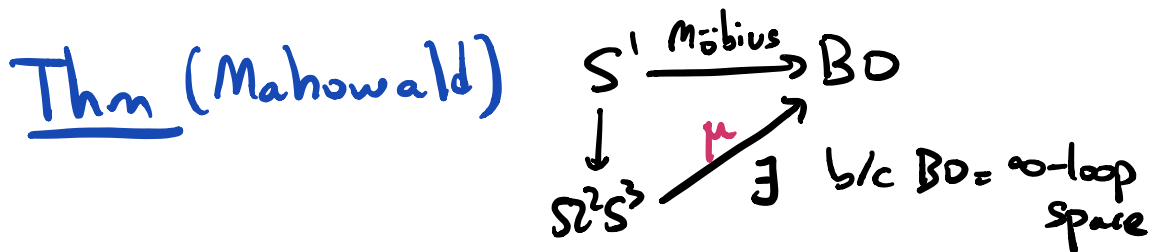
Fix prime p , which we will localize everything at
 Thus $MU_{(p)} = \bigvee \varepsilon^? BP$ with $\pi_* BP = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$
 for some $|v_i| = 2(p^i - 1)$
 (eg, $v_0 = p$)



For now, just observe $\mathbb{Z}_{(p)}[v_1, \dots, v_n] \subseteq \pi_* T(n)$
 \uparrow
 $\pi_* BP\langle n \rangle$

(so $T(n)$ has sth to do w/ height n)

Q: How exactly are the $B\mathbb{P}\langle n \rangle$ and the $T(m)$ related?



Then, the Thom spectrum $(\Omega^2 S^3)^\mu$ has an \mathbb{E}_2 -ring structure, and is equivalent to $HA\mathbb{F}_2$.

Thom spectrum of a v.-bundle only involves the associated spherical fibration

Spherical fibrations are classified by " $BGL(\mathbb{F})$ " and the spherical fibration associated to a v. bdl is given by

$\underbrace{\hspace{10em}}_{\text{an } \infty\text{-loop space}}$

the J-homomorphism $J: B\mathbb{O} \rightarrow BGL, \mathbb{F}$

(Will summarize properties of BGL , later;
 for now, just note:
 if $R = \mathbb{E}_i$ -ring, then BGL, R exists and $\pi_1 BGL, (R) \cong (\pi_0 R)^*$)

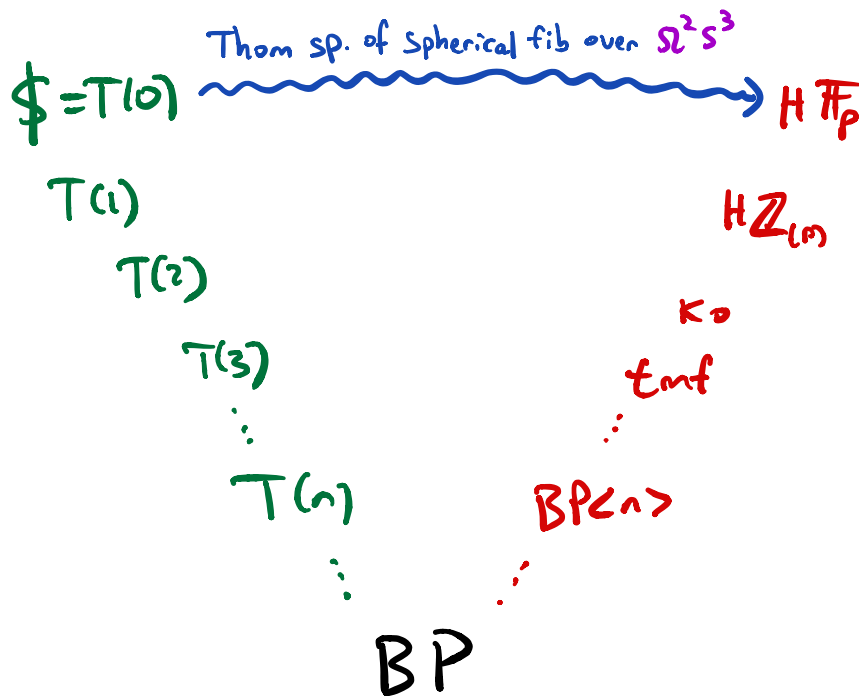
Thm (Hopkins) Consider $1-P \in \pi_1 BGL_1(\mathbb{F}_p)$

This gives:

$$\begin{array}{ccc}
 S^1 & \xrightarrow{1-P} & BGL_1(\mathbb{F}_p) \\
 \downarrow & \nearrow \mu & \\
 \Omega^2 S^3 & &
 \end{array}$$

Then, the Thom spectrum $(\Omega^2 S^3)^\mu$ has an \mathbb{E}_2 -ring structure, and is equivalent to $H\mathbb{F}_p$.

So:



In fact, one can even construct $H\mathbb{Z}(p)$ as the Thom spectrum of a spherical fibration!

Thm (Hopkins-Mahowald)

Recall \exists canonical map $S^3 \xrightarrow{f} K(\mathbb{Z}, 3)$

Define

$$\Omega^2 S^3 \langle 3 \rangle = \text{fib} \left(\Omega^2 S^3 \xrightarrow{\Omega^2 f} \Omega^2 K(\mathbb{Z}, 3) \right)$$

\parallel
 S^1

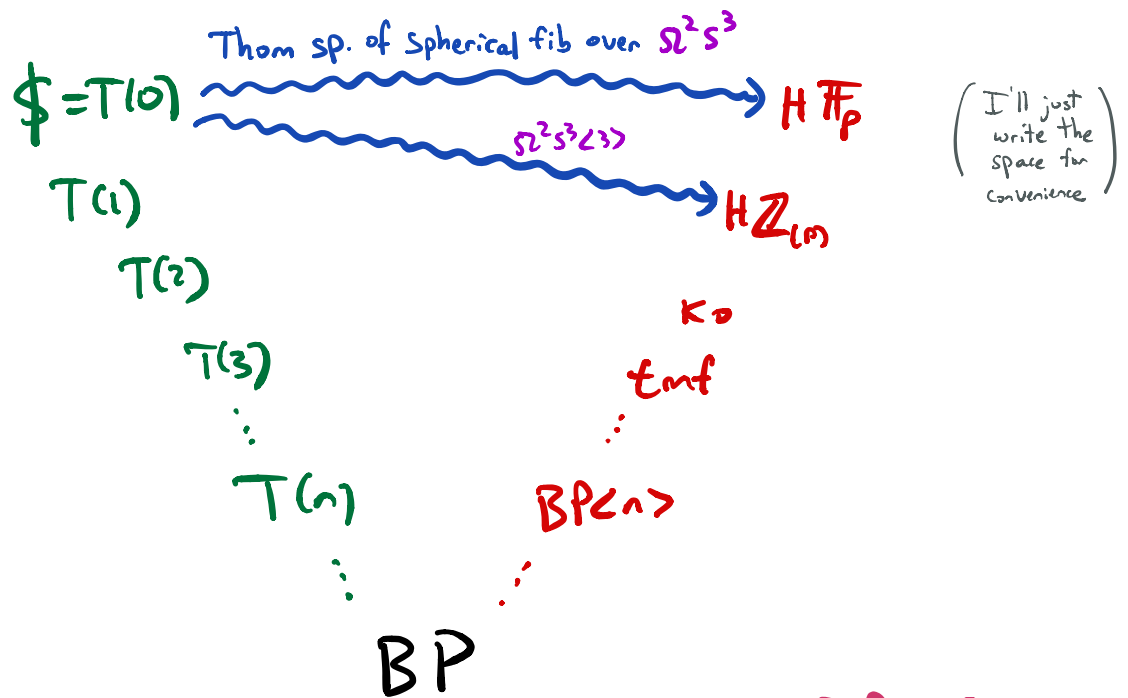
There is a map

$$\Omega^2 S^3 \langle 3 \rangle \rightarrow \Omega^2 S^3 \xrightarrow{\mu} BGL_1(\mathbb{F}_p)$$

which I'll also call μ .

Then: The Thom spectrum $(\Omega^2 S^3 \langle 3 \rangle)^\mu$ is an \mathbb{E}_2 -ring which is equivalent to $H\mathbb{Z}(p)$.

So, now, fill in another arrow in our diagram:



Q: Go from $\$$ to K_0 ? tmf ? $BP\langle n \rangle$?

A: No (Mahowald, Rudyak, Priddy, Chatham, ...)

However, one of the main results in this talk is that one can go from $T(n)$ to $BP\langle n-1 \rangle$ and $BP\langle n \rangle$ (if you assume certain conjectures)

To explain this, need to talk about BGL_1 and $T(n)$.

Facts about BGL_1

- $R = \mathbb{E}_n$ -ring $\Rightarrow BGL_1(R)$ is an \mathbb{E}_{n-1} -space

and

$$\pi_i BGL_1(R) \cong \begin{cases} \pi_{i-1}(R) & i > 1 \\ \pi_0(R)^\times & i = 1 \\ 0 & i \leq 0 \end{cases}$$

- $X = \text{space}$

$$\mu: X \rightarrow BGL_1(R)$$

\leadsto Thom spectrum $X^\mu \in \text{LMod}_R$

- If $F \xrightarrow{i} E \rightarrow X$ is a fiber sequence and you have $\mu: E \rightarrow BGL_1(R)$

then $\exists X \xrightarrow{\nu} BGL_1(F^{\mu \circ i})$ such that

$X^\nu \cong E^\mu$ as R -modules.

Pictorially:

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & \longrightarrow & X \\ & & \downarrow \mu & & \downarrow \nu \\ & & E^\mu & & X^\nu \\ & & \downarrow & \longrightarrow & \downarrow \\ & & BGL_1(R) & \longrightarrow & BGL_1(F^{\mu \circ i}) \end{array}$$

(Warning: need $F^{\mu \circ i}$ to at least be an \mathbb{E}_1 -ring in this formulation. But, can bypass this.)

Let us see an example:

$$\begin{array}{ccc} \Omega^2 S^3 \langle 3 \rangle & \longrightarrow & \Omega^2 S^3 & \longrightarrow & S^1 \\ & & \text{HFF}_p \downarrow \mu & & \downarrow \nu \\ & & BGL_1(\mathbb{Z}_{(p)}) & \longrightarrow & BGL_1(\mathbb{H}\mathbb{Z}_{(p)}) \end{array}$$

The map ν detects $1-p \in \pi_0(\mathbb{H}\mathbb{Z}_{(p)})^x$
 and so $(S^1)^\nu = \mathbb{H}\mathbb{Z}_{(p)} / p \cong \text{HFF}_p$ (as expected)

Facts about $T(n)$

- $\pi_* BPC(n) \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n] \subseteq \pi_* T(n)$
- $T(n)$ is a summand of an \mathbb{F}_2 -ring called $X(p^{n+1}-1)$

- There is a ^{p-torsion} class $\sigma_n \in \pi_{2p^{n+1}-3} T(n)$ such that:

$$\begin{array}{ccc} S^{2p^{n+1}-2} & \xrightarrow{\sigma_n} & BGL, T(n) \\ \downarrow & & \nearrow \exists \\ \Omega S^{2p^{n+1}-1} & & \text{Thom spectrum is } T(n+1). \end{array}$$

elusive, crucial to the nilpotence thm

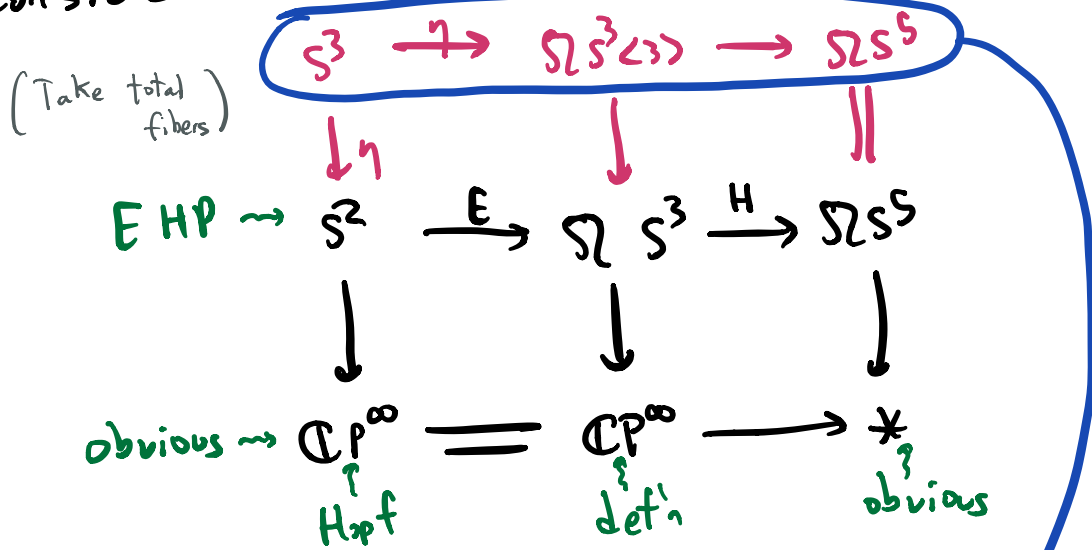
For eg, if $n=0$, then $T(0) = \mathbb{Z}(p)$
 and $\sigma_0 = \alpha_1 \in \pi_{2p-3} \mathbb{Z}(p)$

So $T(1) =$ Thom spectrum of
 $\Omega S^{2p-1} \xrightarrow{\alpha_1} BGL_1 \mathbb{Z}(p)$.

(Eg if $p=2$, then $\alpha_1 = \eta$, and
 $T(1) = \text{Thom}(\Omega S^3 \rightarrow BGL_1 \mathbb{Z}(2))$.)

Let us now use this to prove a baby
 version of the main result of this talk.

Consider the map of fiber sequences:



Toda

baby Cohen-Moore-Neisendorfer

Take loops, get fiber sequence

$$\Omega S^3 \xrightarrow{\eta} \Omega^2 S^3 \langle 3 \rangle \longrightarrow \Omega^2 S^5$$

"TCU" "Hopkins-Mahowald" ?

Map this into $BGL_1(\mathbb{Z}_{(2)})$:

$$\begin{array}{ccccc} \Omega S^3 & \xrightarrow{\eta} & \Omega^2 S^3 \langle 3 \rangle & \longrightarrow & \Omega^2 S^5 \\ & \searrow T(1) & \downarrow H\mathbb{Z}_{(2)} & & \downarrow \mu \\ & & BGL_1(\mathbb{Z}_{(2)}) & \longrightarrow & BGL_1(T(1)) \end{array}$$

Know the Thom sp. is $H\mathbb{Z}_{(2)}$!

So: what is the map μ ?

$$\left[S^3 \longrightarrow \Omega^2 S^5 \xrightarrow{\mu} BGL_1(T(1)) \right]$$

$$\uparrow$$

$$\pi_3 BGL_1(T(1))$$

$$\parallel$$

$$\pi_2 T(1) \cong \mathbb{Z}_{(2)} \cdot v_1$$

The class is just v_1 !

(not immediate; requires proof. Roughly, express v_1 as Toda bracket $\langle 2, \eta, \text{unit} \rangle$.)

We have showed:

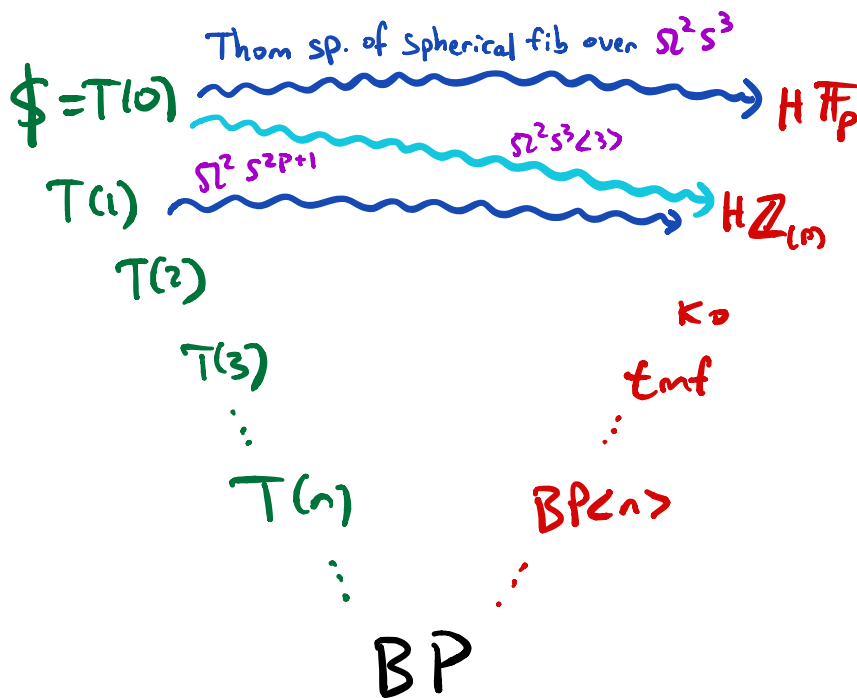
Thm: There is a map

$$\begin{array}{ccc} S^3 & \xrightarrow{\nu_1} & BGL_1(T(1)) \\ \downarrow & \nearrow \mu & \\ \Omega^2 S^5 & & \end{array}$$

Such that $(\Omega^2 S^5)^{\wedge n} \cong H\mathbb{Z}_{(2)}$.

(Same thing works at odd primes:
find $(\Omega^2 S^{2p+1})^{\wedge n} \cong H\mathbb{Z}_{(p)}$)

so:



These Thom Spectra seem to appear in different "flavors", depending on the height shift. Hence the color shift.

Key things needed:

(a) The fiber sequence

$$S^{2p-1} \rightarrow \Omega S^3 \langle 3 \rangle \rightarrow \Omega S^{2p+1}$$

such that the composite

$$S^{2p-1} \xrightarrow{E^2} \Omega^2 S^{2p+1} \xrightarrow{\text{going around}} S^{2p-1} \text{ is degree } P$$

(b) The map $\Omega^2 S^3 \langle 3 \rangle \xrightarrow{\mu} BGL_1 \mathbb{Z}/p$

We'd like to generalize both of these to $T(n)$.
 (Assume $p \geq 2$ now for simplicity)

For (a), we have:

Thm (Cohen-Moore-Neisendorfer)

For any $n \geq 1$, there is a map $\Omega^2 S^{2p^n+1} \xrightarrow{\phi} S^{2p^n-1}$
 such that the composite

$$S^{2p^n-1} \xrightarrow{E^2} \Omega^2 S^{2p^n+1} \xrightarrow{\phi} S^{2p^n-1} \text{ is degree } P$$

The conjecture we need is:

Conj: There is a map $\Omega^2 S^{2p^n+1} \xrightarrow{\phi_n} S^{2p^n-1}$

(Cohen-Moore-Neisendorfer, Gray, Mahowald, ...)

Satisfying the conclusion of CMW's thm such that $\Omega^2(P^{2p^n+1}(p))$

$$\Omega^2(S^{2p^n}/p) \cong \text{fib}(\phi_n) \times \text{other stuff}$$

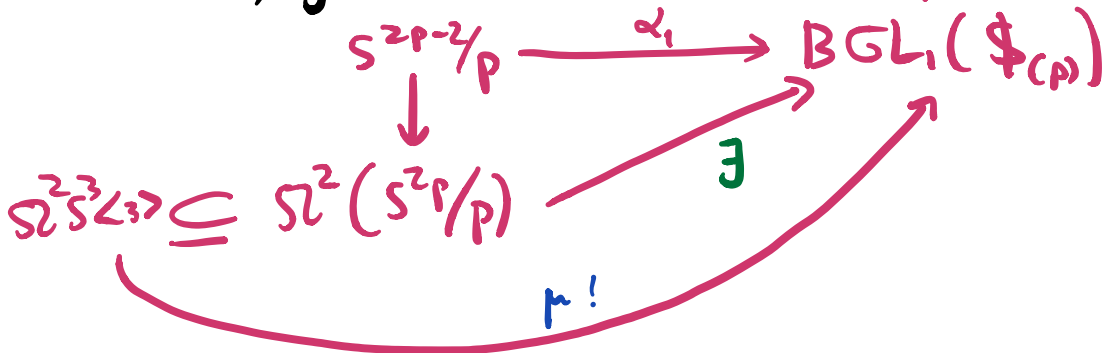
Moore space (in some mildly structured way)

Eg, if $n=1$, then $\text{fib}(\phi_1) = \Omega^2 S^3 \langle 3 \rangle$, so this is asking that

$$\Omega^2(S^{2p}/p) \cong \Omega^2 S^3 \langle 3 \rangle \times \text{other stuff}$$

Can use this to get analogue of (b)!

Indeed, you can then construct μ as



But this won't work for $T(n)$ for general n (issue is that $BGL_1 T(n)$ is not a double loop space)

This is the purpose of the second conj:

Conj: The element $\sigma_n \in \pi_{2p^{n+1}-3} T(n)$ lifts to the \mathbb{E}_3 -center of $T(n)$.

Finally, then:

Thm (D.): Let $n \geq 0$, and assume these two conjectures.

Then there are maps

$$\Omega^2 S^{2p^{n+1}} \xrightarrow[\text{"v}_n"]{\mu} BGL_1(T(n))$$

and

$$\text{fib}(\phi_{n+1}) \xrightarrow[\text{"}\sigma_n\text{"}]{\nu} BGL_1(T(n))$$

Cohen-Moore-Neisendorfer

such that there are equivs of Thom spectra

$$(\Omega^2 S^{2p^{n+1}})^{\mu} \simeq BP\langle n-1 \rangle$$

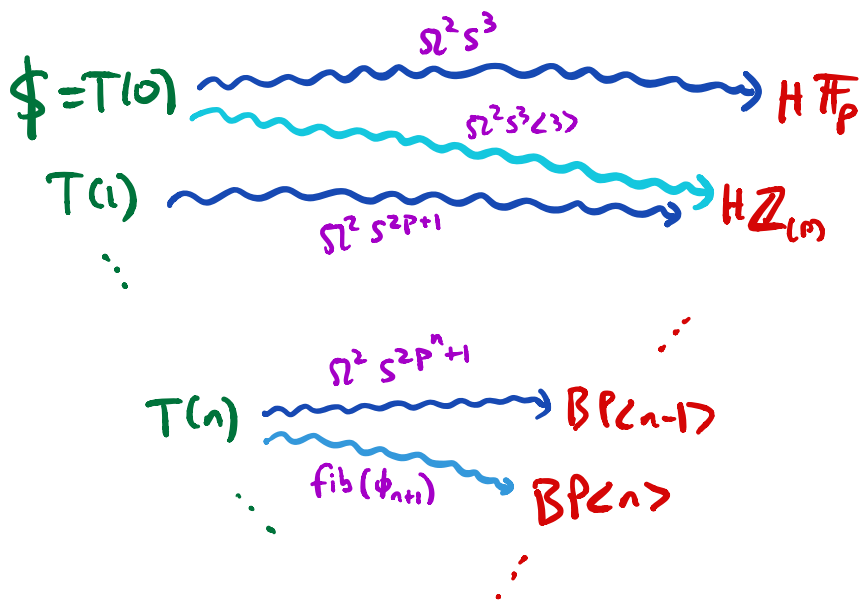
$$\text{fib}(\phi_{n+1})^{\nu} \simeq BP\langle n \rangle$$

- Recovers Hopkins-Mahowald when $n=0$

$$\left(\begin{array}{l} \text{so } T(0) = \mathbb{F}_2 \\ BP\langle 0 \rangle = H\mathbb{Z}\langle p \rangle \\ BP\langle -1 \rangle = H\mathbb{F}_p \end{array} \right)$$

- Can rephrase pf of nilpotence thm via this result.

So:



BP

Epilogue: Other chromatic spectra like k_0 , tmf , $\mathcal{K}(n)$, $\mathcal{K}_2(n)$?

Thm(D.) There's a similar story:
you only need to find appropriate
replacements for the $T(n)$;
no need to change $\text{fib}(\phi_{n+1})$.

Some examples:

- For $k(n)$, need to replace $T(n)$ with the Mahowald-Ravenel-Shick $y(n)$ = Thom sp. of a bdl over $\Omega T_{p-1}(S^2)$
- For ko , need to replace $T(1)$ with A , which is the Thom spectrum $(\Omega S^5)^{\mu}$ of the map

$$\begin{array}{ccc}
 S^4 = \#P^1 & \longrightarrow & BSU \\
 \downarrow & \nearrow \mu & \\
 \Omega S^5 & &
 \end{array}$$

$$\begin{array}{ccc}
 \Omega \mathbb{E}(S^2) & & \Omega \mathbb{E}(S^2) \\
 \parallel & & \parallel \\
 \Omega S^9 & \longrightarrow & N \longrightarrow \Omega S^{13} \\
 H_* N = \mathbb{F}_2[x_8, x_{12}] & & \\
 \begin{matrix} \downarrow \\ \downarrow \\ \downarrow \end{matrix} & & \begin{matrix} \downarrow \\ \downarrow \\ \downarrow \end{matrix} \\
 8 & & 8 \\
 4 & & 4 \\
 2 & & 2
 \end{array}$$

- For tmf , there's also a replacement of $T(2)$, which I call B . It's a little complicated to define, but it is also a Thom spectrum of some cplx bdl over a loop space N .

Corollary (D.) Again assume the conjectures.

Then both $MSpin \xrightarrow{\hat{A}} ko$
and $Mstring \xrightarrow{witten} tmf$

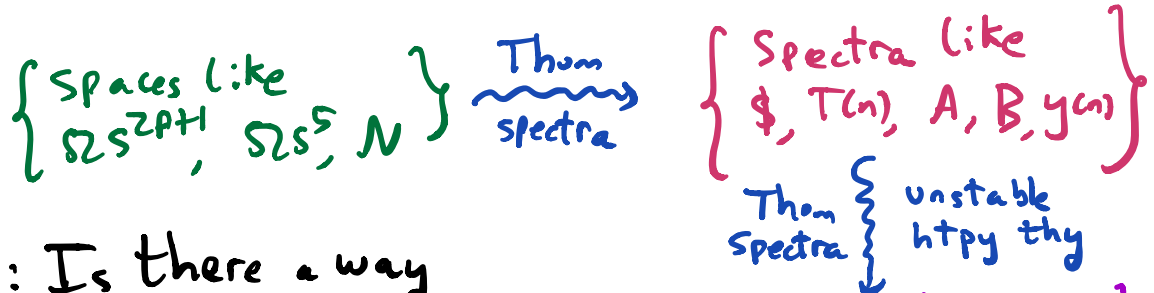
admit splittings!

$$\begin{array}{ccc}
 B \longrightarrow Mstring & & \\
 \searrow \text{unit} & \downarrow & \nearrow \exists \\
 & tmf &
 \end{array}$$

$$\begin{array}{ccc}
 A \longrightarrow \overbrace{MSpin} & & \\
 \searrow \text{unit} & \downarrow \hat{A} & \nearrow \exists \\
 & ko &
 \end{array}$$

$MSpin_{\nu}(m)$
 $\text{index} = \hat{A} \downarrow \nearrow \exists$
 $\text{Dirac} \quad ko_{\nu}(m)$

Moral. Chromatic spectra are built as "iterated Thom spectra"



Q: Is there a way to codify the structures present here? eg, like Mahowald-Rezk did for \rightsquigarrow



Some more questions:

- Prove the conjectures!
- CMN's results had applications to exponents of unstable htpy grs of spheres. Is there a way to use this and the thm above to get bds on nilpotence exponents of $\pi_x(\mathbb{Z}_p)$?
- Another result I showed is that the maps $A \rightarrow ko$ and $B \rightarrow tmf$ are surjective on π_x , but pf is computational. Conceptual explanation?

- relationship between Wood equivalence

$$ku = ko \wedge C\eta \leftarrow T(1) \simeq A \wedge C\eta$$

and

$$S^2 \longrightarrow \Omega S^3 \longrightarrow \Omega S^5$$

$\left\{ \begin{array}{c} \text{"C}\eta\text{"} \\ \text{"T}(1)\text{"} \\ \text{"A"} \end{array} \right.$

\leftarrow 2-local

Generalize to tmf.

- Analogue of this story for the elusive eo_{p-1} ?

$$tmf \wedge \underbrace{DA(1)}_{8\text{-cell cplx}} \simeq B\mathbb{P}\langle 2 \rangle$$

$$B \wedge DA(1) \simeq T(2)$$

\uparrow
analogue of the EHP
sequence involving
the space N

$$\begin{array}{ccc} S^4 & \longrightarrow & BSp \\ \downarrow & \nearrow & \\ \Omega S^5 & \longrightarrow & BSO \end{array}$$

$$fib(\phi_2) \xrightarrow{\gamma} BGL_1(A) \quad \leftarrow A = \text{Thom}(\Omega S^5 \rightarrow BSO)$$

$$\begin{array}{ccc} & \uparrow & \\ \Omega^2 S^{2?+1} & \longrightarrow & S^{2?-1} \end{array}$$

$$fib(\phi_2)^\gamma \simeq ko$$

(as A -modules)

\downarrow
left