Chromatic analogues of the Hopkins-Mahowald theorem

Broad goal: relate two different "stratifications" of stable htpy thy

Fix prime p, which we will localize everything at

Thus $MU_{(p)} = \bigvee \Sigma^{?} BP$ with $\pi_* BP = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ for some $|v_i| = 2(p^i - 1)$
(eg, $v_0 = p$)

$\mathbb{S} = T(0)$

$T(1)$

$T(2)$

$T(3)$

$\vdots$

introduced by Ravenel (will explain later) $\longrightarrow T(n)$

$\vdots$

$H\mathbb{F}_p \leftarrow$ Think of as $BP\langle -1 \rangle$

$H\mathbb{Z}_{(p)} \leftarrow$ Think of as $BP\langle 0 \rangle$

ko

tmf

$\vdots$

$BP\langle n \rangle = BP/(v_{n+1}, v_{n+2}, \ldots)$

$\vdots$

BP

"simple homology, impossible htpy"

"computable homology and htpy"

For now, just observe $\mathbb{Z}_{(p)}[v_1, \ldots, v_n] \subseteq \pi_* T(n)$
$\qquad \underset{\pi_* BP\langle n \rangle}{\shortparallel}$

(so $T(n)$ has s$^{\text{th}}$ to do w/ height n)

Q: How exactly are the $BP\langle n \rangle$ and the $T(n)$ related?

Thm (Mahowald)
$$\begin{tikzcd} S^1 \arrow[r, "\text{Möbius}"] \arrow[d] & BO \\ \Omega^2 S^3 \arrow[ur, "\mu"] & \end{tikzcd}$$
$\exists$ b/c BO = ∞-loop space

Then, the Thom spectrum $(\Omega^2 S^3)^\mu$ has an $\mathbb{E}_2$-ring structure, and is equivalent to $H\mathbb{F}_2$.

Thom spectrum of a v.bundle only involves the associated spherical fibration

Spherical fibrations are classified by "$\underbrace{BGL_1(\mathbb{S})}_{\text{an } \infty\text{-loop space}}$" and the spherical fibration associated to a v.bdl is given by the J-homomorphism $J: BO \to BGL_1\mathbb{S}$

(Will summarize properties of BGL_1 later; for now, just note: if R = $\mathbb{E}_1$-ring, then $BGL_1 R$ exists and $\pi_1 BGL_1(R) = (\pi_0 R)^\times$)

Thm (Hopkins) Consider $1-p \in \pi_1 BGL_1(\mathbb{S}_{(p)}) \cong \mathbb{Z}_{(p)}^\times$

This gives:

$$\begin{tikzcd} S^1 \arrow[r, "1-p"] \arrow[d] & BGL_1(\mathbb{S}_{(p)}) \\ \Omega^2 S^3 \arrow[ur, "\mu"] & \end{tikzcd}$$

Then, the Thom spectrum $(\Omega^2 S^3)^\mu$ has an $\mathbb{E}_2$-ring structure, and is equivalent to $H\mathbb{F}_p$.

So:

$$\begin{tikzcd}[column sep=small] \mathbb{S}=T(0) \arrow[rrr, squiggly, "\text{Thom sp. of spherical fib over } \Omega^2 S^3"] & & & H\mathbb{F}_p \\ T(1) & & & H\mathbb{Z}_{(p)} \\ T(2) & & & ko \\ T(3) & & & tmf \\ \vdots & & & \vdots \\ T(n) & & & BP\langle n\rangle \\ \vdots & & & \vdots \\ & BP & & \end{tikzcd}$$

In fact, one can even construct $H\mathbb{Z}_{(p)}$ as the Thom spectrum of a spherical fibration!

Thm (Hopkins-Mahowald)

Recall $\exists$ canonical map $S^3 \xrightarrow{f} K(\mathbb{Z},3)$

Define

$$\Omega^2 S^3\langle 3\rangle = \mathrm{fib}\left(\Omega^2 S^3 \xrightarrow[\Omega^2 f]{} \underset{\overset{\|}{S^1}}{\Omega^2 K(\mathbb{Z},3)}\right)$$

There is a map

$$\Omega^2 S^3\langle 3\rangle \longrightarrow \Omega^2 S^3 \xrightarrow{\mu} BGL_1(\mathbb{S}_{(p)})$$

which I'll also call μ.

Then: The Thom spectrum $(\Omega^2 S^3\langle 3\rangle)^{\mu}$ is an $\mathbb{E}_2$-ring which is equivalent to $H\mathbb{Z}_{(p)}$.

So, now, fill in another arrow in our diagram:

$$\mathbb{S} = T(0) \xrightarrow{\text{Thom sp. of spherical fib over } \Omega^2 S^3} H\mathbb{F}_p$$

$$T(0) \xrightarrow{\Omega^2 S^3\langle 3\rangle} H\mathbb{Z}_{(p)}$$

(I'll just write the space for convenience)

$$\begin{array}{ccc} T(1) & & H\mathbb{Z}_{(p)} \\ T(2) & & ko \\ T(3) & & tmf \\ \vdots & & \vdots \\ T(n) & & BP\langle n\rangle \\ \vdots & & \vdots \\ & BP & \end{array}$$

Q: Go from $\mathbb{S}$ to ko? tmf? $BP\langle n\rangle$?

A: No (Mahowald, Rudyak, Priddy, Chatham, ...)

However, one of the main results in this talk is that one can go from $T(n)$ to $BP\langle n-1\rangle$ and $BP\langle n\rangle$ (if you assume certain conjectures)

To explain this, need to talk about BGL_1 and $T(n)$.

Facts about BGL_1

- $R = \mathbb{E}_n$-ring $\Rightarrow BGL_1(R)$ is an $\mathbb{E}_{n-1}$-space and
$$\pi_i BGL_1(R) \cong \begin{cases} \pi_{i-1}(R) & i > 1 \\ \pi_0(R)^\times & i = 1 \\ 0 & i \leq 0 \end{cases}$$

- $X =$ space
$\mu: X \to BGL_1(R)$
$\rightsquigarrow$ Thom spectrum $X^\mu \in LMod_R$

- If $F \xrightarrow{i} E \to X$ is a fiber sequence and you have $\mu: E \to BGL_1(R)$ then $\exists\ X \xrightarrow{\nu} BGL_1(F^{\mu \circ i})$ such that
$X^\nu \simeq E^\mu$ as R-modules.

Pictorially:

$$\begin{tikzcd} F \arrow[r, "i"] & E \arrow[r] \arrow[d, "\mu", "E^\mu"'] & X \arrow[d, "\nu", "X^\nu"'] \\ & BGL_1(R) \arrow[r] & BGL_1(F^{\mu \circ i}) \end{tikzcd}$$

(Warning: need $F^{\mu \circ i}$ to at least be an $\mathbb{E}_1$-ring in this formulation. But, can bypass this.)

Let us see an example:

$$\begin{tikzcd}
\Omega^2 S^3\langle 3\rangle \arrow[r] & \Omega^2 S^3 \arrow[rr] \arrow[d, "\mu", "H\mathbb{F}_p"'] & & S^1 \arrow[d, "v"] \\
& BGL_1(\$_{(p)}) \arrow[rr] & & BGL_1(H\mathbb{Z}_{(p)})
\end{tikzcd}$$

The map v detects $1-p \in \pi_0(H\mathbb{Z}_{(p)})^\times$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$
and so $(S^1)^v \simeq H\mathbb{Z}_{(p)}/p$ $\qquad\qquad\qquad\; \mathbb{Z}_{(p)}^\times$

$\simeq H\mathbb{F}_p$ (as expected)

Facts about $T(n)$

- $\pi_* BP\langle n\rangle \cong \mathbb{Z}_{(p)}[v_1, \cdots, v_n] \subseteq \pi_* T(n)$
- $T(n)$ is a summand of an $\mathbb{E}_2$-ring called $X(p^{n+1}-1)$
- There is a p-torsion class $\sigma_n \in \pi_{2p^{n+1}-3} T(n)$ such that:

elusive, crucial to the nilpotence thm

$$\begin{tikzcd}
S^{2p^{n+1}-2} \arrow[r, "\sigma_n"] \arrow[d] & BGL_1 T(n) \\
\Omega S^{2p^{n+1}-1} \arrow[ur, dashed, "\exists"] &
\end{tikzcd}$$

Thom spectrum is $T(n+1)$.

For eg, if $n=0$, then $T(0) = \mathbb{S}_{(p)}$

and $\sigma_0 = \alpha_1 \in \pi_{2p-3}\mathbb{S}_{(p)}$

So $T(1)$ = Thom spectrum of

$$\Omega S^{2p-1} \xrightarrow[\alpha_1]{} BGL_1 \mathbb{S}_{(p)}.$$

(Eg if $p=2$, then $\alpha_1 = \eta$, and

$T(1) = \mathrm{Thom}(\Omega S^3 \longrightarrow BGL_1 \mathbb{S}_{(p)})$.)

Let us now use this to prove a baby version of the main result of this talk.

Consider the map of fiber sequences:

(Take total fibers)

$$\begin{tikzcd}
 & S^3 \arrow[r, "\eta"] \arrow[d, "\eta"] & \Omega S^3\langle 3\rangle \arrow[r] \arrow[d] & \Omega S^5 \arrow[d, equal] \\
\text{EHP} \rightsquigarrow & S^2 \arrow[r, "E"] \arrow[d] & \Omega S^3 \arrow[r, "H"] \arrow[d] & \Omega S^5 \arrow[d] \\
\text{obvious} \rightsquigarrow & \mathbb{C}P^\infty \arrow[r, equal] & \mathbb{C}P^\infty \arrow[r] & *
\end{tikzcd}$$

Hopf def'n obvious

Toda

baby Cohen-Moore-Neisendorfer

Take loops, get fiber sequence

$$\underbrace{\Omega S^3}_{\text{"}T(1)\text{"}} \xrightarrow{\eta} \underbrace{\Omega^2 S^3\langle 3\rangle}_{\text{"Hopkins-Mahowald"}} \longrightarrow \underbrace{\Omega^2 S^5}_{?}$$

Map this into $BGL_1(\mathbb{S}_{(2)})$:

$$\begin{tikzcd}
\Omega S^3 \arrow[r, "\eta"] \arrow[dr, "T(1)"'] & \Omega^2 S^3\langle 3\rangle \arrow[r] \arrow[d, "H\mathbb{Z}_{(2)}"] & \Omega^2 S^5 \arrow[d, "\mu"'] \\
 & BGL_1(\mathbb{S}_{(2)}) \arrow[r] & BGL_1(T(1))
\end{tikzcd}$$

Know the Thom sp. is $H\mathbb{Z}_{(2)}$!

So: what is the map μ?

$$\left[S^3 \longrightarrow \Omega^2 S^5 \xrightarrow{\mu} BGL_1(T(1)) \right]$$

$$\rotatebox{90}{$\in$}$$

$$\pi_3 BGL_1(T(1))$$

$$\rotatebox{90}{$=$}$$

$$\pi_2 T(1) \cong \mathbb{Z}_{(2)} \cdot v_1$$

$\boxed{\text{The class is just } v_1\text{!}}$ (not immediate; requires proof. Roughly, express v_1 as Toda bracket $\langle 2, \eta, \text{unit}\rangle$.)

We have showed:

Thm: There is a map

$$
\begin{tikzcd}
S^3 \arrow[r, "v_1"] \arrow[d] & BGL_1(T(1)) \\
\Omega^2 S^5 \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

such that $(\Omega^2 S^5)^\mu \simeq H\mathbb{Z}_{(2)}$.

(Same thing works at odd primes: find $(\Omega^2 S^{2p+1})^\mu \simeq H\mathbb{Z}_{(p)}$)

so:

$$
\begin{tikzcd}
\mathbb{S}=T(0) \arrow[rrr, squiggly, "\text{Thom sp. of spherical fib over } \Omega^2 S^3"] \arrow[rrrd, squiggly, "\Omega^2 S^3\langle 3\rangle"] & & & H\mathbb{F}_p \\
T(1) \arrow[rrr, squiggly, "\Omega^2 S^{2p+1}"] & & & H\mathbb{Z}_{(p)} \\
T(2) & & & ko \\
T(3) & & & tmf \\
\vdots & & & \vdots \\
T(n) & & & BP\langle n\rangle \\
\vdots & & & \vdots \\
& BP & &
\end{tikzcd}
$$

These Thom spectra seem to appear in different "flavors", depending on the height shift. Hence the color shift.

Key things needed:

(a) The fiber sequence

$$S^{2p-1} \to \Omega S^3\langle 3\rangle \to \Omega S^{2p+1}$$

such that the composite

$$S^{2p-1} \xrightarrow[E^2]{} \Omega^2 S^{2p+1} \xrightarrow[\text{going around}]{} S^{2p-1} \quad \text{is degree } p$$

(b) The map $\Omega^2 S^3\langle 3\rangle \xrightarrow[\mu]{} BGL_1 \mathbb{S}_{(p)}$

We'd like to generalize both of these to $T(n)$.

(Assume $p>2$ now for simplicity)

For (a), we have:

Thm (Cohen-Moore-Neisendorfer)

For any $n \geq 1$, there is a map $\Omega^2 S^{2p^n+1} \xrightarrow[\phi]{} S^{2p^n-1}$ such that the composite

$$S^{2p^n-1} \xrightarrow[E^2]{} \Omega^2 S^{2p^n+1} \xrightarrow[\phi]{} S^{2p^n-1} \quad \text{is degree } p$$

The conjecture we need is:

Conj: There is a map $\Omega^2 S^{2p^n+1} \xrightarrow{\phi_n} S^{2p^n-1}$ satisfying the conclusion of CMN's thm such that

(Cohen-Moore-Neisendorfer, Gray, Mahowald, ...)

$$\Omega^2(S^{2p^n}/p) \simeq \mathrm{fib}(\phi_n) \times \text{other stuff}$$

$\simeq \Omega^2(P^{2p^n+1}(p))$

Moore space

(in some mildly structured way)

Eg, if $n=1$, then $\mathrm{fib}(\phi_1) = \Omega^2 S^3\langle 3\rangle$,
so this is asking that

$$\Omega^2(S^{2p}/p) \simeq \Omega^2 S^3\langle 3\rangle \times \text{other stuff}$$

Can use this to get analogue of (b)!

Indeed, you can then construct μ as

$$\begin{tikzcd}
 & S^{2p-2}/p \arrow[r, "\alpha_1"] \arrow[d] & BGL_1(\mathbb{S}_{(p)}) \\
\Omega^2 S^3\langle 3\rangle \arrow[r, phantom, "\subseteq"] \arrow[urr, bend right, "\mu!"'] & \Omega^2(S^{2p}/p) \arrow[ur, "\exists"'] &
\end{tikzcd}$$

But this won't work for $T(n)$ for general n
(issue is that $BGL_1 T(n)$ is not a double loop space)

This is the purpose of the second conj:

Conj: The element $\sigma_n \in \pi_{2p^{n+1}-3} T(n)$ lifts to the $\mathbb{E}_3$-center of $T(n)$.

Finally, then:

Thm (D.): Let $n \geq 0$, and assume these two conjectures.

Then there are maps

$$\Omega^2 S^{2p^n+1} \xrightarrow[\text{"}v_n\text{"}]{\mu} BGL_1(T(n))$$

and

$$\mathrm{fib}(\phi_{n+1}) \xrightarrow[\text{"}\sigma_n\text{"}]{\nu} BGL_1(T(n))$$

Cohen-Moore-Neisendorfer

such that there are equivs of Thom spectra

$$(\Omega^2 S^{2p^n+1})^{\mu} \simeq BP\langle n-1\rangle$$

$$\mathrm{fib}(\phi_{n+1})^{\nu} \simeq BP\langle n\rangle$$

- Recovers Hopkins-Mahowald when $n=0$
  (so $T(0) = \mathbb{S}$, $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$, $BP\langle -1\rangle = H\mathbb{F}_p$)
- Can rephrase pf of nilpotence thm via this result.

<u>So</u>:

$$\begin{tikzcd}
\mathbb{S}=T(0) \arrow[r, rightsquigarrow, "\Omega^2S^3"] \arrow[dr, rightsquigarrow, "\Omega^2S^3\langle 3\rangle"] & H\mathbb{F}_p \\
T(1) \arrow[r, rightsquigarrow, "\Omega^2S^{2p+1}"'] & H\mathbb{Z}_{(p)} \\
\ddots & \iddots \\
T(n) \arrow[r, rightsquigarrow, "\Omega^2S^{2p^n+1}"] \arrow[dr, rightsquigarrow, "\mathrm{fib}(\phi_{n+1})"'] & BP\langle n-1\rangle \\
\ddots & BP\langle n\rangle \\
& \iddots \\
& BP
\end{tikzcd}$$

<u>Epilogue</u>: other chromatic spectra like ko, tmf, $k(n)$, $k_{\mathbb{Z}}(n)$?

> <u>Thm (D.)</u> There's a similar story: you only need to find appropriate replacements for the $T(n)$; no need to change $\mathrm{fib}(\phi_{n+1})$.

Some examples:

- For $k(n)$, need to replace $T(n)$ with the Mahowald–Ravenel–Shick $y(n)$ = Thom sp. of a bdl over $\Omega J_{p^n-1}(S^2)$
- For ko, need to replace $T(1)$ with A, which is the Thom spectrum $(\Omega S^5)^\mu$ of the map

$$\begin{tikzcd} S^4 = \mathbb{H}P^1 \arrow[r] \arrow[d] & BSU \\ \Omega S^5 \arrow[ur, "\mu"'] & \end{tikzcd}$$

$$\begin{tikzcd} \Omega\Sigma(S^8) \arrow[d, equal] & & \Omega\Sigma(S^{12}) \arrow[d, equal] \\ \Omega S^9 \arrow[r] & N \arrow[r] & \Omega S^{13} \end{tikzcd}$$

$H_* N = \mathbb{F}_2[x_8, x_{12}]$, $x_8 \rightsquigarrow \xi_1^8$, $x_{12} \rightsquigarrow \xi_2^4$

- For tmf, there's also a replacement of $T(2)$, which I call B. It's a little complicated to define, but it is also a Thom spectrum of some cplx bdl over a loop space N.

Corollary (D.) Again assume the conjectures. Then both $MSpin \xrightarrow{\hat{A}} ko$ and $Mstring \xrightarrow{\text{Witten}} tmf$ admit splittings!

$$\begin{tikzcd} B \arrow[r] \arrow[dr, "\text{unit}"'] & Mstring \arrow[d, shift left] \\ & tmf \arrow[u, shift left, "\exists"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[dr, "\text{unit}"'] & MSpin \arrow[d, shift right, "\hat{A}"] \\ & ko \arrow[u, shift right, "\exists"] \end{tikzcd}$$

$$\begin{tikzcd} MSpin_*(M) \arrow[d, shift right, "\text{index}_{\text{Dirac}} = \hat{A}"'] \\ ko_*(M) \arrow[u, shift right, "\exists"'] \end{tikzcd}$$

Moral. Chromatic spectra are built as "iterated Thom spectra"

$$\left\{\begin{matrix}\text{Spaces like} \\ \Omega S^{2p+1},\ \Omega S^5,\ N\end{matrix}\right\} \xrightarrow[\text{spectra}]{\text{Thom}} \left\{\begin{matrix}\text{Spectra like} \\ \$,\ T(n),\ A,\ B,\ y(n)\end{matrix}\right\}$$

$$\Big\downarrow \ \text{Thom spectra} \left\{\begin{matrix}\text{unstable} \\ \text{htpy thy}\end{matrix}\right.$$

$$\left\{\begin{matrix}\text{Spectra like} \\ BP\langle n\rangle,\ ko,\ tmf,\ k(n)\end{matrix}\right\}$$

Q: Is there a way to codify the structures present here? eg, like Mahowald-Rezk did for $\rightsquigarrow$

Some more questions:

- Prove the conjectures!
- CMN's results had applications to exponents of unstable htpy gps of spheres. Is there a way to use this and the thm above to get bds on nilpotence exponents of $\pi_*(\$_{(p)})$?
- Another result I showed is that the maps $A \longrightarrow ko$ and $B \longrightarrow tmf$ are surjective on π_*, but pf is computational. Conceptual explanation?

- rel'nship between wood equivalence

$$ku \simeq ko \wedge C\eta \rightsquigarrow T(1) \simeq A \wedge C\eta$$

2-local

and

$$\begin{tikzcd} S^2 \arrow[r] \arrow[d, phantom, "\wr"] & \Omega S^3 \arrow[r] \arrow[d, phantom, "\wr"] & \Omega S^5 \arrow[d, phantom, "\wr"] \\ \text{"}C\eta\text{"} & \text{"}T(1)\text{"} & \text{"}A\text{"} \end{tikzcd}$$

Generalize to tmf.

- Analogue of this story for the elusive eo_{p-1}?

$$tmf \wedge \underbrace{DA(1)}_{\text{8-cell cplx}} \simeq BP\langle 2\rangle$$

$\uparrow$

$$B \wedge DA(1) \simeq T(2)$$

$\uparrow$

analogue of the EHP sequence involving the space N

$$\begin{tikzcd} S^4 \arrow[r] \arrow[d] & BSp \\ \Omega S^5 \arrow[ur] \arrow[r] & BSU \end{tikzcd}$$

$A = \mathrm{Thom}(\Omega S^5 \to BSU)$

$$\mathrm{fib}(\phi_2) \xrightarrow{\nu} BGL_1(A)$$

$\uparrow$

$$\Omega^2 S^{2^?+1} \longrightarrow S^{2^?-1}$$

$$\mathrm{fib}(\phi_2)^\nu \simeq ko$$

(as A-modules)

left