ANOMALIES AND INVERTIBLE FIELD THEORIES

In this (half of the) talk, I will try to describe how anomalies can be understood in terms of invertible field theories. We will begin by giving an intuitive idea of what an anomaly is supposed to be; more substantive/interesting examples will be given by Ricky in the second half of this talk. Suppose we have some classical field theory with fields $\phi \in \mathcal{A}$ and action functional $S(\phi)$, so that the path integral of the associated quantum field theory is given by $Z = \int_{\mathcal{A}} e^{iS(\phi)} \mathcal{D}\phi$. If we are given some symmetry (i.e., a group G acting on $\mathcal{A}$) which leaves the action functional invariant, it is natural to ask whether this symmetry leaves Z invariant. Clearly, the term $e^{iS(\phi)}$ remains invariant; but there is no reason for the measure $\mathcal{D}\phi$ to be invariant under the action of G. In general, the action of G will introduce a Jacobian factor into the path integral, and this *is* the anomaly. (This is known as the "Fujikawa method" to understand anomalies.) A classical example of an anomaly already arises in quantum mechanics, where a G-symmetry is usually a *projective* representation of G on the Hilbert space of the theory; the obstruction to extending this to an actual representation of G is a class in $\mathrm{H}^2(G; \mathbf{C}^\times)$.

Let us try to mold this picture to fit into more mathematical language. Suppose we have an n-dimensional quantum field theory Z (not extended, for now), i.e., a symmetric monoidal functor to $\mathrm{Vect}_{\mathbf{C}}$ from the category $\mathrm{Bord}_{[n-1,n]}(\mathcal{F})$ of $(n-1)$-manifolds and n-dimensional cobordisms between them, all equipped with fields $\mathcal{F}$. If M is an n- or $(n-1)$-dimensional manifold, we will let $\mathcal{F}(M)$ denote the space of fields on M. Suppose M is a closed n-dimensional manifold. Then the partition function is supposed to be a map $Z : \mathcal{F}(M) \to \mathbf{C}$. Suppose the quantum field theory Z has a symmetry G; in order for the partition function to respect this symmetry, it must descend to a map $Z' : \mathcal{F}(M)/G \to \mathbf{C}$. In general, there is no reason for this to be possible. To understand the obstruction, let us view $Z : \mathcal{F}(M) \to \mathbf{C}$ as a section of the trivial complex line bundle $\epsilon_{\mathcal{F}(M)}$ over $\mathcal{F}(M)$. If the G-symmetry is "anomalous", then we will find that $Z(g \cdot \Phi) = P(g, \Phi) Z(\Phi)$ for some coefficient $P(g, \Phi)$ which depends on g and Φ. This says that Z can be understood as the section of the line bundle $\mathcal{L}_M := (\mathbf{C} \times \mathcal{F}(M))/G$ over $\mathcal{F}(M)/G$, where G acts on $\mathbf{C} \times \mathcal{F}(M)$ by the formula

$$g : (\lambda, \Phi) \mapsto (P(g, \Phi)\lambda, g \cdot \Phi).$$

Therefore, the G-symmetry being anomalous is equivalent to the failure of the line bundle $\mathcal{L}_M$ to be trivial over $\mathcal{F}(M)/G$, i.e., the non-vanishing of $c_1(\mathcal{L}_M) \in \mathrm{H}^2(\mathcal{F}(M)/G; \mathbf{Z})$.

Similarly, suppose N is a closed $(n-1)$-dimensional manifold. Then Z can be viewed as a function from $\mathcal{F}(N)$ to finite-dimensional vector spaces (the categorification of $\mathbf{C}$). Running a similar argument as above, one finds that there is an obstruction to descending Z from $\mathcal{F}(N)$ to $\mathcal{F}(N)/G$, and it is given by the failure of an *invertible gerbe* $\mathcal{G}_N$ to be trivializable. One can think of this obstruction as a class in $\mathrm{H}^3(\mathcal{F}(N)/G; \mathbf{Z})$.

Despite the fact that for a closed n-dimensional manifold M, the map $Z : \mathcal{F}(M) \to \mathbf{C}$ may not descend to a map $Z' : \mathcal{F}(M)/G \to \mathbf{C}$, we see that there is a canonically-defined *line* bundle $\mathcal{L}_M$ over $\mathcal{F}'(M) := \mathcal{F}(M)/G$. We might therefore wish to consider a functor $\alpha : \mathrm{Bord}_{[n-1,n]}(\mathcal{F}') \to \mathrm{LinCat}_{\mathbf{C}}$ which assigns to a closed n-manifold M the line bundle $\mathcal{L}_M$ over $\mathcal{F}'(M)$, and to a closed $(n-1)$-manifold N the invertible gerbe $\mathcal{G}_N$ over $\mathcal{F}'(N) := \mathcal{F}(N)/G$. Since α assigns a vector space (really, vector bundle) to an n-manifold, α is begging to be viewed as an $(n+1)$-dimensional field theory, extended to dimension $n-1$.

Sometimes, α can indeed be viewed as an $(n+1)$-dimensional field theory (but not always; see Footnote 101 in Freed's lectures). We will just assume that this is possible, and view α

as a functor $\mathrm{Bord}_{[n-1,n+1]}(\mathcal{F}') \to \mathrm{Mod}_{\mathrm{Mod}_{\mathbf{C}}}$, where $\mathrm{Mod}_{\mathrm{Mod}_{\mathbf{C}}}$ is a target $(\infty, 2)$-category whose objects are $\mathbf{C}$-linear categories, whose morphisms are $\mathbf{C}$-vector spaces, and whose 2-morphisms are complex matrices. (One might need to replace $\mathrm{Mod}_{\mathrm{Mod}_{\mathbf{C}}}$ with a super-analogue for physical applications.) Moreover, notice that since $\mathcal{L}_M$ and $\mathcal{G}_N$ are invertible, the functor α can be viewed as an *invertible* field theory.

To recover $Z : \mathrm{Bord}_{[n-1,n]}(\mathcal{F}) \to \mathrm{Vect}_{\mathbf{C}}$ from $\alpha : \mathrm{Bord}_{[n-1,n+1]}(\mathcal{F}') \to \mathcal{C}$, recall our observation that for a closed n-manifold M, the partition function can be viewed as a section of $\mathcal{L}_M$, i.e., as a bundle map $\epsilon_M \to \mathcal{L}_M$ from the trivial line bundle over $\mathcal{F}'(M)$ to $\mathcal{L}_M$. Similarly, for a closed $(n-1)$-manifold N, we can view Z on $\mathcal{F}'(N)$ as a "section" of $\mathcal{G}_N$, i.e., as a gerve map $\epsilon_N \to \mathcal{G}_N$ from the trivial gerbe over $\mathcal{F}'(N)$ to $\mathcal{G}_N$. Motivated by this, we make the following definition. Let $\mathcal{D}$ be a symmetric monoidal ∞-category. Define the "trivial" invertible field theory $\mathbf{1} : \mathrm{Bord}_{[n-1,n+1]}(\mathcal{F}') \to \mathcal{D}$ to be the "tensor unit" field theory, i.e., the one which assigns to every closed $(n-1)$-manifold the tensor unit in $\mathcal{C}$, and to every closed n-manifold the tensor unit in $\mathrm{End}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}})$. Then, Z can be viewed as a natural transformation $\mathbf{1} \to \tau_{\leq n}\alpha$, where $\tau_{\leq n}\alpha$ is the restriction of α to $(n-1)$- and n-dimensional manifolds. In this generalized setup, α is called the *anomaly theory*, and the anomaly is trivializable if it is equipped with an isomorphism $\mathbf{1} \xrightarrow{\sim} \alpha$ of (invertible) $(n+1)$-dimensional field theories.

However, in some sense this model does not capture all the richness of anomalies, since we have not gone fully extended. Let us rectify this: suppose $Z : \mathrm{Bord}_n(\mathcal{F}) \to \mathcal{C}$ is an n-dimensional extended QFT. Then an anomaly theory for Z is an $(n+1)$-dimensional invertible extended QFT $\alpha : \mathrm{Bord}_{n+1}(\mathcal{F}') \to \mathcal{C}'$ such that Z is a natural transformation $\mathbf{1} \to \tau_{\leq n}\alpha$, where $\mathcal{C}'$ is a symmetric monoidal $(\infty, n+1)$-category such that $\mathcal{C} = \mathrm{End}_{\mathcal{C}'}(\mathbf{1}_{\mathcal{C}'}) = \Omega\mathcal{C}'$.

Given this observation, let us first try to understand invertible TQFTs $\alpha : \mathrm{Bord}_{n+1}(\mathcal{F}') \to \mathrm{Mod}_{\mathbf{C}}^{(n+1)}$, where $\mathrm{Mod}_{\mathbf{C}}^{(n+1)}$ is the $(\infty, n+1)$-category defined inductively by $\mathrm{Mod}_{\mathbf{C}}^{(1)} := \mathrm{Mod}_{\mathbf{C}}$ and $\mathrm{Mod}_{\mathbf{C}}^{(n+1)} := \mathrm{Mod}_{\mathrm{Mod}_{\mathbf{C}}^{(n)}}$. Let $|\mathrm{Bord}_{n+1}(\mathcal{F}')|$ be the geometric realization of $\mathrm{Bord}_{n+1}(\mathcal{F}')$, so that it is an infinite loop space; let us denote the associated connective spectrum by $\mathrm{MT}(\mathcal{F}')$ (for "Madsen-Tillman"). In a previous talk, we saw that such invertible TQFTs were classified by maps of infinite loop spaces from $|\mathrm{Bord}_{n+1}(\mathcal{F}')|$ to the Picard groupoid of $\mathrm{Mod}_{\mathbf{C}}^{(n+1)}$. But this Picard space is $K(\mathbf{C}^\times, n+1) = \Omega^\infty\Sigma^{n+1}\mathrm{H}\mathbf{C}^\times$, so we see that invertible TQFTs $\alpha : \mathrm{Bord}_{n+1}(\mathcal{F}') \to \mathrm{Mod}_{\mathbf{C}}^{(n+1)}$ are classified by elements of

$$\mathrm{Map}_{\text{inf. loop}}(|\mathrm{Bord}_{n+1}(\mathcal{F}')|, K(\mathbf{C}^\times, n+1)) \simeq \mathrm{Map}_{\mathrm{Sp}}(\mathrm{MT}(\mathcal{F}'), \Sigma^{n+1}\mathrm{H}\mathbf{C}^\times),$$

whose π_0 is $\mathrm{H}^{n+1}(\mathrm{MT}(\mathcal{F}'); \mathbf{C}^\times)$. If $\mathrm{MT}(\mathcal{F}')$ is the Thom spectrum of a map $\mathrm{B}\mathcal{F}' \to \mathrm{BO} \times \mathbf{Z}$ from some space $\mathrm{B}\mathcal{F}'$ (behaving like the moduli space of tangential structures), then the Thom isomorphism gives $\mathrm{H}^{n+1}(\mathrm{MT}(\mathcal{F}'); \mathbf{C}^\times) \simeq \mathrm{H}^{n+1}(\mathrm{B}\mathcal{F}'; \mathbf{C}^\times)$. Initially, this was the group that was assumed to classify deformation classes of anomalies of n-dimensional QFTs; for example, if $\mathrm{B}\mathcal{F}'$ is the classifying space of some group $\mathcal{G}$, then this is the group cohomology $\mathrm{H}^{n+1}(\mathrm{B}\mathcal{G}; \mathbf{C}^\times)$.

But as we have seen, the target category of n-dimensional QFTs should not be $\mathrm{Mod}_{\mathbf{C}}^{(n)}$, but rather something more subtle $\mathcal{C}_{\mathrm{subtle}}$. This subtler object $\mathcal{C}_{\mathrm{subtle}}$ has Picard groupoid given by $\Omega^\infty\Sigma^{n+1}I_{\mathbf{C}^\times}$, where $I_{\mathbf{C}^\times}$ is the Brown-Comenetz dualizing spectrum. Following the above analysis, one posits that deformation classes of anomalies of n-dimensional QFTs are classified by $I_{\mathbf{C}^\times}^{n+1}(\mathrm{MT}(\mathcal{F}'))$. This is closely related to $\mathrm{H}^{n+1}(\mathrm{MT}(\mathcal{F}'); \mathbf{C}^\times)$: the connective cover of $I_{\mathbf{C}^\times}$ is $\mathrm{H}\mathbf{C}^\times$, so we obtain a canonical map $\mathrm{H}^{n+1}(\mathrm{MT}(\mathcal{F}'); \mathbf{C}^\times) \to I_{\mathbf{C}^\times}^{n+1}(\mathrm{MT}(\mathcal{F}'))$.

Let us end this talk by returning to the question of trivializing an anomaly. Suppose that α is the anomaly theory of some n-dimensional QFT. If α is trivializable, then there is an isomorphism $\mathbf{1} \xrightarrow{\sim} \alpha$. However, there could be many choices of such isomorphisms; the space of such isomorphisms is precisely π_1 of the space of invertible field theories $\mathrm{Bord}_{n+1}(\mathcal{F}') \to \mathcal{C}_{\mathrm{subtle}}$, based at α. In other words, it is $\pi_1(\mathrm{Map}_{\mathrm{Sp}}(\mathrm{MT}(\mathcal{F}'), \Sigma^{n+1}I_{\mathbf{C}^\times}), \alpha)$. If $\mathrm{Map}_{\mathrm{Sp}}(\mathrm{MT}(\mathcal{F}'), \Sigma^{n+1}I_{\mathbf{C}^\times})$

is connected (as a simplifying assumption), this group is $I^n_{\mathbf{C}^\times}(\mathrm{MT}(\mathcal{F}'))$. Moreover, this group acts on the space of n-dimensional QFTs with an anomaly trivialization (by changing the trivialization). I would like to understand this better, if anyone has concrete examples they could share.

References

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