THE SMITH FIBER SEQUENCE AND INVERTIBLE FIELD THEORIES

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ABSTRACT. Smith homomorphisms are maps between bordism groups that change both the dimension and the tangential structure. We give a completely general account of Smith homomorphisms, unifying the many examples in the literature. We provide three definitions of Smith homomorphisms, including as maps of Thom spectra, and show they are equivalent. Using this, we identify the cofiber of the spectrum-level Smith map and extend the Smith homomorphism to a long exact sequence of bordism groups, which is a powerful computation tool. We discuss several examples of this long exact sequence, relating them to known constructions such as Wood's and Wall's sequences. Furthermore, taking Anderson duals yields a long exact sequence of invertible field theories, which has a rich physical interpretation. We developed the theory in this paper with applications in mind to symmetry breaking in quantum field theory, which we study in [DDK+24].

CONTENTS

1. INTRODUCTION

Let M be a closed, smooth n-manifold together with a real line bundle $\pi\colon L \to M$. For any section $s\colon M \to L$ of π transverse to the zero section, standard theorems in differential topology imply $N_s \coloneqq s^{-1}(0)$ is a smooth, $(n-1)$-dimensional submanifold of M. We would like to make N_s into an invariant of M and L, but its diffeomorphism type depends on s: consider the trivial line bundle over S^1 with the constant section valued in 1 versus any section intersecting the zero section. However, all choices of N_s are *bordant*: given two sections $s_1, s_2\colon M \to L$, there is a compact n-manifold X whose boundary is diffeomorphic to $N_{s_1} \amalg N_{s_2}$. Thus the image of N_s in

the *bordism group* Ω_{n-1}^{O}, the set of bordism equivalence classes with group structure given by disjoint union, is a well-defined invariant of M and L.

More is true: one can refine the bordism equivalence relation to extend the line bundle $L|_{N_{s_1}} \amalg L|_{N_{s_2}}$ across X, obtaining an invariant valued in the larger group $\Omega_{n-1}^{O}(BO(1))$ of bordism classes of the data of a closed $(n-1)$-manifold and a real line bundle. The value of this invariant also only depends on the bordism class of M and L and is additive in disjoint unions, re-expressing our invariant as a homomorphism of abelian groups

$$(1.1) \qquad \mathrm{sm}_\sigma \colon \Omega_n^{O}(BO(1)) \longrightarrow \Omega_{n-1}^{O}(BO(1)).$$

This map was first studied by Conner-Floyd [CF64, Theorem 26.1], who called it the *Smith homomorphism* after P. A. Smith. Subsequently, many authors studied similarly-defined maps between other bordism groups, focusing on two methods of generalization:[1]

   (1) Generalize from real line bundles to real or complex vector bundles of other ranks.
   (2) Keep track of other topological data on M, and how it is affected by passing to M.

For example, suppose we give M an orientation structure. The submanifold N_s does not inherit an induced orientation, and can be unorientable. However, since $TM|_{N_s} = TN_s \oplus L|_{N_s}$ has an orientation, it follows that $L|_{N_s}$ is the orientation line bundle of N. It follows that $L|_{N_s}$ gives no additional structure at all, and this variant of the Smith homomorphism factors through $\Omega_{n-1}^{O} \subset \Omega_{n-1}^{O}(BO(1))$. That is, we have a map

$$(1.2) \qquad \mathrm{sm}_\sigma \colon \Omega_n^{SO}(BO(1)) \longrightarrow \Omega_{n-1}^{O}.$$

To the best of our knowledge, this was first written down by Komiya [Kom72, §5]. Other examples in the literature show the same phenomenon: if one places a tangential structure on M in the sense of Lashof [Las63], the Smith homomorphism lands in a bordism group whose degree and tangential structure are in general different from those of the domain.

Despite the variety of known examples, the general theory of the Smith homomorphism does not appear in the literature. The first objective of this paper is to tell the general story.

Our other major objective is to apply the Smith homomorphism to quantum physics. Our inspiration for this paper came from [HKT20], where they study the physical process of defect anomaly matching as modelled by Smith homomorphisms. Work of Freed-Hopkins-Teleman [FHT10], Freed-Hopkins [FH21], and Grady [Gra23] classifies various kinds of invertible field theories (IFTs) in terms of Anderson duals[2] to bordism homology theories.[3] Therefore, the Anderson dual of the Smith homomorphism is a map of invertible field theories. As invertible field theories can be understood as anomalies of quantum field theories (see Section 8.1), the Anderson-dualized Smith homomorphism provides an anomaly-matching formula expressing the anomalies of certain QFTs in terms of anomalies of lower-dimensional *defect theories* [HKT20, COSY20].

However, in [HKT20], they noted in section 4.4 that they were missing a mathematical way to compute their homomorphisms of interest. The fiber sequence of spectra we studied to answer this question led to more interesting physics: in forming a fiber sequence, hence a long exact

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[1]Kirby-Taylor [KT90b, Theorem 6.11, Remark 6.15] generalize the Smith homomorphism in a different direction in the setting of *characteristic bordism*; that generalization is out of scope of this paper.

[2]See §8.2.2 and [FH21, §5.3] for more on Anderson duality and how we use it.

[3]There are additional classification theorems for invertible field theories due to Yonekura [Yon19], Rovi-Schoenbauer [RS22], and Kreck-Stolz-Teichner (unpublished). All of these are closely related to [FHT10, FH21] and to each other, but for the purposes of our paper, we need the homotopical approach presented in [FH21].

sequence, we found a tool to easily compute Smith homomorphisms. Furthermore, two new maps in the long exact sequence also have interesting physical interpretations in the context of spontaneous symmetry breaking. In our companion paper [DDK$^+$24], we provide detailed physical interpretations of the entire symmetry breaking long exact sequence (SBLES) and address many physical examples and applications. We provide a summary in Section 8 in this paper.

Next we outline the results of this paper. We make use of standard definitions in bordism theory, which we review in §2. Fix a tangential structure $\xi\colon B \to B\mathrm{O}$, a space X, a virtual vector bundle $V \to X$ of rank r_V, and a vector bundle $W \to X$ of rank r_W.

By an (X, V)-*twisted ξ-structure* on a virtual vector bundle $E \to M$, we mean the data of a map $f\colon M \to X$ and a ξ-structure on $E \oplus f^*(V)$ (Definition 3.1); this is a tangential structure in its own right (Lemma 3.3), which we denote $\xi + (X, V)$. In particular, there is a notion of bordism of (X, V)-twisted ξ-manifolds, and by Corollary 3.4 the corresponding bordism groups are naturally isomorphic to the ξ-bordism groups of the Thom spectrum X^{V-r_V} of the rank-zero virtual bundle $V - r_V \to X$:[4]

$$(1.3) \qquad \Omega_n^{\xi+(X,V)} \xrightarrow{\cong} \Omega_n^\xi(X^{V-r_V}).$$

Our first result is to define a Smith homomorphism sm_W associated to the data of ξ, X, V, and W. This Smith homomorphism will have type signature

$$(1.4) \qquad \mathrm{sm}_W\colon \Omega_n^\xi(X^{V-r_V}) \longrightarrow \Omega_{n-r_W}^\xi(X^{V\oplus W-r_V-r_W}).$$

That is, passing through (1.3), the Smith homomorphism passes from the bordism group of n-dimensional (X, V)-twisted ξ-manifolds to the bordism group of $(n-r_W)$-dimensional $(X, V \oplus W)$-twisted ξ-manifolds. We actually provide three different definitions of sm_W:

(1) First, in Definition 3.7 we define sm_W as the map sending the bordism class of a manifold M with map $f\colon M \to X$ to the bordism class of the zero locus of a section of $f^*W \to M$ transverse to the zero section.

(2) We then define the Smith homomorphism in Definition 3.13 as the map of bordism groups induced by a map of Thom spectra $X^V \to X^{V\oplus W}$, itself induced by the map of total spaces of vector bundles $V \to V \oplus W$ sending $v \mapsto (v, 0)$.

(3) Our third definition, in Definition 4.27, defines sm_W as the cap product homomorphism with the Euler class of V in (twisted) ξ-cobordism, following a construction of Euler classes in twisted generalized cohomology in §4.1.

Theorem (Proposition 3.17 and Corollary 4.39)**.** *The above three definitions are equivalent.*

Each definition has its own advantages: the first and third allow for a direct comparison with preexisting special cases in the literature and the second is an essential ingredient for constructing long exact sequences. Specifically, in Section 5, we reprove the following well-known theorem.

Theorem 5.1. *With X, V, W, and ξ as above, the fiber of the map of spectra $X^V \to X^{V\oplus W}$ in definition 2 is the map $p\colon S(W)^{p^*V} \to X^V$, where $S(W)$ denotes the unit sphere bundle of W and $p\colon S(W) \to X$ is the bundle projection map.*

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[4]This perspective on twisted bordism, and these results, are not new, and we are not sure who originally developed them. We follow the language and perspective of [DDHM23, §10], and note this is not the first approach to this material.

This is not a new result; in [KZ18, Remark 3.14] it is attributed to James. Our contribution is to relate it to the fully general definition of the Smith homomorphism.

From the fiber sequence of spectra in Section 5, we derive two long exact sequences. First, in bordism:

Corollary 5.8. *Let X, V, W, and ξ be as above. There is a long exact sequence of bordism groups:*

(1.5)
$$\cdots \longrightarrow \Omega_n^\xi(S(W)^{p^*V-r_V}) \xrightarrow{p} \Omega_n^\xi(X^{V-r_V}) \xrightarrow{\mathrm{sm}_W} \Omega_{n-r_W}^\xi(X^{V\oplus W-r_V-r_W}) \xrightarrow{\delta} \Omega_{n-1}^\xi(S(W)^{p^*V-r_V}) \longrightarrow \cdots$$

As we discuss in Remark 5.11, this is a generalized Gysin sequence. As a computational tool, it turns out to be remarkably convenient. Different vector bundles W can be combined to calculate bordism groups, often avoiding difficult spectral sequence calculations. For example, we use this idea in Appendix A to address an extension problem; other papers using this or closely related techniques to do computations include [HS13, Deb23, DDHM23, DL23, DYY23].

Next, there are many examples of Smith homomorphisms in the literature, so we devote some time in this paper to explicating the Smith homomorphism for various choices of ξ, X, V, and W. One phenomenon that we address in §6 is that Smith homomorphisms come in "families": if you iterate sm_W with $V = 0, W, 2W, 3W, \ldots$, often the domain and codomain recur with a finite period p because the notions of (X, kW)-twisted and $(X, (k+p)W)$-twisted ξ-structures are equivalent. In §6, we use the "fake vector bundle twists" of [DY23a, §1] to establish the following Smith families.

- Example 6.11: when $\xi = \mathrm{id} \colon BO \to BO$ (unoriented bordism), the period $p = 1$ for all X and W: like in (1.1), the tangential structure does not change.
- Examples 6.12 and 6.13: for $\xi \colon BG \to BO$ for $G = \mathrm{SO}$, Spin^c, or U, $p = 1$ if W is orientable and 2 if W is unorientable.
- Example 6.14: for $\xi \colon B\mathrm{Spin} \to BO$, p can be 1, 2, or 4 depending on the first two Stiefel-Whitney classes of W.

These may be thought of as versions of James periodicity over bases other than $\mathbb{S}$; see Example 6.24. For some other common choices of ξ, including SU-structure, string structure, and stable framing, the period is harder to determine, as we discuss in §6.4. We would be interested in learning tools for computing such periods, and suspect that these periodicities are related to the image of the J-homomorphism; see Remark 6.35.

We also study examples where we fix X, V, and W, but let ξ vary, recovering known Smith families from the literature.

(1) In §7.1, we let $X = \mathbb{RP}^\infty$, $V = k\sigma$, and $W = \sigma$, where $\sigma \to \mathbb{RP}^\infty$ is the tautological line bundle.
   - In Example 7.4, we apply this to unoriented bordism, where it reproduces Conner-Floyd's original Smith homomorphism (1.1) [CF64, Theorem 26.1].
   - For oriented bordism, see Example 7.6, where we get Komiya's [Kom72, §5] 2-periodic family of Smith homomorphisms exchanging $\mathrm{SO} \times \mathbb{Z}/2$- and O-bordism.
   - On spin^c bordism (Example 7.11), this recovers the Smith homomorphisms between $\mathrm{Spin}^c \times \mathbb{Z}/2$ and Pin^c bordism studied in [BG87a, HS13].

- On spin bordism (Example 7.8), this Smith family is 4-periodic, involving bordism for the groups $\mathrm{Spin} \times \mathbb{Z}/2$, Pin^-, $\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/2$ and Pin^+. This family or subsets of it have been discussed in [Pet68, ABP69, Gia73a, Kre84, KT90b, HS13, KTTW15, TY19, HKT20, WWZ20, BR23].
- On string bordism (Example 7.10), this family has period 8 and to our knowledge has not appeared in the literature. It would be interesting to study this family in more detail.

(2) In §7.2, we instead consider the tautological complex line bundle over $\mathbb{CP}^\infty$. For some tangential structures (O, SO, Spin^c, U) this family is 1-periodic (Example 7.22).

- For spin bordism (Example 7.26), this family is 2-periodic, exchanging $\mathrm{Spin} \times \mathrm{U}(1)$ and spin^c bordism, recovering work of Kirby-Taylor [KT90b, Corollary 6.12, Remark 6.14].
- In Example 7.28, we pull L back along $B\mathbb{Z}/n \to B\mathrm{U}(1)$, obtaining a 2-periodic family exchanging bordism for the groups $\mathrm{Spin} \times \mathbb{Z}/n$ and $\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/2n$ also studied in [DDHM23, Appendix E].

These are not the only examples we consider—see §7 for more.

In Section 8, we leverage our theory toward physical applications. Let $\Omega_\xi^*(-)$ denote the generalized cohomology theory which is Anderson dual to ξ-bordism. As mentioned above, $\Omega_\xi^*(X)$ classifies anomalies of X-families of field theories with ξ tangential structure. In particular, when $X = BG$, this classifies field theories with G-symmetry. Using the fiber sequence, we show the following.

Corollary 8.19. *Let X, V, W, and ξ be as above. There is a long exact sequence of invertible field theories:*

(1.6)
$$\cdots \longrightarrow \Omega_\xi^{n-1}(S(W)^{p^*V - r_V}) \xrightarrow{\mathrm{Ind}_W} \Omega_\xi^{n-r_W}(X^{V \oplus W - r_V - r_W}) \xrightarrow{\mathrm{Def}_W} \Omega_\xi^n(X^{V-r_V}) \xrightarrow{\mathrm{Res}_W} \Omega_\xi^n(S(W)^{p^*V - r_V}) \longrightarrow \cdots$$

This is our mathematical model for the SBLES of [DDK$^+$24], and we have labelled the maps as in that paper. The map dual to the Smith map sm_W is the defect anomaly matching map Def_W studied in [HKT20] and [COSY20]. Our framework allows for the study of more physical examples, as well as provides a way of explicitly computing each map and thus extracting more physical information. Moreover, it expands our physical understanding of this form of symmetry breaking to include two other processes: residual anomaly obstructions, which prevent certain symmetry breaking patterns, and index anomaly matching. We study these phenomena in detail in [DDK$^+$24].

Finally, we have two appendices. In Appendix A, we explicate a Smith long exact sequence from Example 7.35, which interchanges pin$^-$ and pin$^+$ bordism, with the third term in the long exact sequence identified with a certain twisted spin bordism of $\mathbb{RP}^1$. In Appendix B, we explain why we use Euler classes in cobordism, rather than in ordinary cohomology: the latter is not compatible with the Smith long exact sequence, and in Theorem B.2 we give an explicit example to this effect. As part of our investigation of this example, we prove a theorem that may be of independent interest.

Theorem B.4. *Let $V \to X$ be a rank-3 vector bundle with spin structure and $\mathcal{S} \to X$ be the spinor bundle of V. If $\eta \in ko^{-1}(\mathrm{pt}) \cong \mathbb{Z}/2$ is the unique nonzero element and $p_1^\mathbb{H} \in ko^4(B\mathrm{Sp}(1))$*

denotes the first symplectic ko-Pontrjagin class (see Proposition B.3), then the ko-cohomology Euler class of V is

$$(1.7) \qquad e^{ko}(V) = \eta \cdot p_1^{\mathbb{H}}(\mathcal{S}) \in ko^3(X).$$

An interesting direction for future work is to investigate what happens in the absence of unitarity. The mathematical backbone of our work generalizes nicely to the nonunitary case: Freed-Hopkins-Teleman [FHT10] classify invertible topological field theories in the absence of a reflection positivity structure using unstable Madsen-Tillmann spectra, and the Smith long exact sequence generalizes to this case (see, e.g., Example 7.57). Anomalies of nonunitary theories are not so well-studied, but some examples appear in [CL21, HTY22], and the fact that the Smith long exact sequence generalizes suggests our methods do too.

From there one could ask: the appearance of Madsen-Tillmann spectra in the classification of invertible TFTs is due to theorems of Galatius-Madsen-Tillmann-Weiss [GMTW09], Nguyen [Ngu17], and Schommer-Pries [SP17] establishing Madsen-Tillmann spectra as classifying spectra for bordism (higher) categories. Can one lift the Smith homomorphism to a morphism of bordism categories? This is a question in pure mathematics whose affirmative answer would suggest a generalization of our methods to noninvertible TFTs, and therefore to the symmetry breaking of noninvertible symmetries of field theories, as studied in, e.g., [LTL+21, ABC+23, CHZ23, DAC23, DY23b].

2. Bordism and Thom spectra

In this section, we review virtual bundles, tangential structures, and their Thom spectra. We also review the Pontrjagin-Thom theorem, which relates the homotopy groups of Thom spectra to bordism groups.

2.1. **Virtual vector bundles and tangential structures.** Everything in this subsection is well-worn mathematics; see [Fre19, FH21, DY23a] and the references therein for additional references for this material.

Definition 2.1. A *virtual vector bundle* $V \to X$ is the data of two vector bundles $V_1, V_2 \to M$, which we think of as "$V_1 - V_2$."

An *isomorphism of virtual vector bundles* f between $V = (V_1, V_2)$ and $W = (W_1, W_2)$ over a common base space X is the data of vector bundles $E_1, E_2 \to X$ and isomorphisms $f_1 \colon V_1 \oplus E_1 \xrightarrow{\cong} W_1 \oplus E_2$ and $f_2 \colon V_2 \oplus E_1 \xrightarrow{\cong} W_2 \oplus E_2$.

The idea behind this definition of isomorphism is that we would like the following pairs of virtual vector bundles to be isomorphic.

(1) (V_1, V_2) and (W_1, W_2) when $V_1 \cong W_1$ and $V_2 \cong W_2$.
(2) (V_1, V_2) and $(V_1 \oplus E, V_2 \oplus E)$: adding and subtracting E should not change the isomorphism type of the vector bundle.

A vector bundle V defines a virtual vector bundle as the pair $(V, 0)$. In the future we will make this assignment implicitly.

Let $B\mathrm{O}$ denote the classifying space of the infinite-dimensional orthogonal group $\mathrm{O} := \varinjlim_n \mathrm{O}_n$.

Lemma 2.2. *The space $\mathbb{Z} \times B\mathrm{O}$ classifies virtual vector bundles: for a space X with the homotopy type of a CW complex, the set $[X, \mathbb{Z} \times B\mathrm{O}]$ is naturally in bijection with the set of isomorphism classes of virtual vector bundles on X.*

Projection $\mathbb{Z} \times B\mathrm{O} \to \mathbb{Z}$ onto the first component defines a numerical invariant of virtual vector bundles; this is the *rank* $\mathrm{rank}(V) := \dim(V_1) - \dim(V_2)$. Thus $B\mathrm{O}$, thought of as $B\mathrm{O} \times \{0\}$, is the classifying space for rank-zero virtual vector bundles.

Definition 2.3. A *(stable) tangential structure* is a map $\xi \colon B \to B\mathrm{O}$, and given ξ, a ξ-*structure* on a (rank-zero virtual) vector bundle $V \to X$ is a homotopy class of a lift of its classifying map $f_V \colon X \to B\mathrm{O}$ as in the diagram

$$(2.4) \qquad\qquad \begin{array}{ccc} & & B \\ & \overset{\widetilde{f_V}}{\nearrow} & \downarrow{\scriptstyle \xi_n} \\ X & \xrightarrow{\ f_V\ } & B\mathrm{O}, \end{array}$$

i.e. a homotopy class of maps $\widetilde{f_V} \colon X \to B$ such that $f_V \simeq \xi \circ \widetilde{f_V}$.

If M is a manifold, a ξ-*structure* on M means a ξ-structure on the virtual vector bundle defined by TM. One also sees *normal ξ-structures* on M, which are ξ-structures on $-TM$, the virtual vector bundle defined by the pair $(0, TM)$.

Example 2.5. For $\xi \colon B\mathrm{SO} \to B\mathrm{O}$, a ξ-structure is equivalent data to an orientation. For $\xi \colon B\mathrm{Spin} \to B\mathrm{O}$, a ξ-structure is equivalent to a spin structure.

If M is a manifold with boundary, the outward unit normal vector field defines a trivialization of the normal bundle to $\partial M \hookrightarrow M$, so $T(\partial M) \oplus \underline{\mathbb{R}} \cong TM|_{\partial M}$, and therefore a ξ-structure on M induces a ξ-structure on ∂M. It is therefore possible to define a notion of bordism of manifolds with ξ-structure, as Lashof [Las63] did; we let Ω_n^ξ denote the set of bordism classes of n-manifolds with ξ-structure, which becomes an abelian group under disjoint union.

Often, one studies groups G with maps $\rho \colon G \to \mathrm{O}$, and lets $\xi := B\rho \colon BG \to B\mathrm{O}$. In this case it is common to denote Ω_*^ξ as Ω_*^G (e.g. $G = \mathrm{O}, \mathrm{SO}, \mathrm{Spin}, \mathrm{Pin}^\pm$, etc.).

Remark 2.6. The category of tangential structures is the slice category $\mathsf{Top}_{/B\mathrm{O}}$, i.e. the objects are spaces with a map to $B\mathrm{O}$, and the morphisms are maps which commute with the maps to $B\mathrm{O}$. Bordism groups are covariantly functorial in this category.

2.2. **Construction of Thom spectra.** First, recall the classical construction of a Thom space: if $V \to X$ is a vector bundle, choose a Euclidean metric on V. Let $D(V)$ be the *disc bundle* of

vectors in V of norm at most 1 and $S(V)$ be the *sphere bundle* of vectors of norm exactly 1; write $\mathrm{Th}(X;V) := D(V)/S(V)$.

Example 2.7. Let $\underline{\mathbb{R}}^n \to X$ be a trivial bundle and let X_+ be the space X with a disjoint basepoint. Then the Thom space is the n-fold suspension $\mathrm{Th}(X;\underline{\mathbb{R}}^n) \simeq \Sigma^n X_+$.

Example 2.8. Let $X = \mathbb{RP}^n$ and let $V = \sigma$ be the tautological line bundle. Then the Thom space is $\mathrm{Th}(\mathbb{RP}^n;\sigma) \simeq \mathbb{RP}^{n+1}$.

Proposition 2.9. *If X is compact, then $\mathrm{Th}(X;V)$ is the one point compactification of the disk bundle $D(V)$.*

Let $V \to X$ be a rank d real vector bundle (*not* merely a virtual vector bundle), and also write $V\colon X \to BO(d)$ for the classifying map. Let Top denote the ∞-category of spaces and Top_* denote the ∞-category of pointed spaces. The action of $\mathrm{O}(d)$ on $\mathbb{R}^d$ induces an action on $S^d = S^{\mathbb{R}^d}$, and this induces a functor from the fundamental ∞-groupoid of $BO(d)$ to Top_*.

Proposition 2.10. *The Thom space $\mathrm{Th}(X;V)$ is naturally homotopy equivalent to the colimit of the X-shaped diagram[5]*

$$(2.11) \qquad\qquad X \xrightarrow{V} BO(d) \longrightarrow \mathsf{Top}_*.$$

We need a similar construction for virtual bundles on X. It has the structure of a *spectrum*.[6] The reader unfamiliar with spectra is encouraged to think of them as similar to topological spaces, in that one can take homotopy, (co)homology, and generalized (co)homology groups of them. See Freed-Hopkins [FH21, §6.1] or Beaudry-Campbell [BC18, §2] for precise definitions and [DDHM23, §10.3] for a longer but still heuristic overview. We write Sp for the category of spectra.

We follow [ABG+14a] in the rest of this section. By a *local system of spectra* over a space X we mean a functor $\mathcal{L}$ from the fundamental ∞-groupoid of X to spectra. We will usually denote this as $\mathcal{L}\colon X \to \mathsf{Sp}$. The fiber of a local system at a point $p \in X$ is obtained by composing $\mathcal{L}$ with the functor $\mathrm{pt} \to X$ given by inclusion at p; a functor out of pt is equivalent to a single spectrum, and we call this the fiber of $\mathcal{L}$ at p.

Definition 2.12 ([DL59, ABG+14a]). A *stable spherical fibration* is a local system of spectra valued in the full sub-∞-category of spectra with objects $\Sigma^n \mathbb{S}$, $n \in \mathbb{Z}$.

Here $\mathbb{S}$ denotes the *sphere spectrum*.

Definition 2.13. Let X be a space and $V \to X$ be a vector bundle of rank r. Let $\mathbb{S}^V \to X$ denote the associated stable spherical fibration, whose fiber at a point $x \in X$ is the suspension spectrum of the one-point compactification of V_x.

Now fix a base space X and a virtual vector bundle $V \to X$, which is equivalently a map $V\colon X \to BO \times \mathbb{Z}$. There is a canonical spectrum called the *Thom spectrum X^V* associated to X, V constructed as follows. There is a functor $J\colon BO \times \mathbb{Z} \to \mathsf{Sp}$, generalizing the map $BO(d) \to \mathsf{Top}_*$ above. It maps into spectra now, instead of spaces, because for a virtual bundle $V_1 - V_2$, we want to assign the sphere $S^{V_1} \wedge S^{-V_2}$, but S^{-V_2} doesn't make sense as a space, since spheres of

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[5]When we say "X-shaped diagram," we mean a functor out of the fundamental ∞-groupoid of X.

[6]Spectra in stable homotopy theory are etymologically unrelated to spectra in algebraic geometry, operator theory, physics, etc.

negative dimension don't exist. However, the sphere spectrum $\mathbb{S}$ can be desuspended, and the Thom spectrum associated to a virtual bundle is defined as follows.

Definition 2.14. Given a virtual bundle $V \colon X \to BO \times \mathbb{Z}$, the Thom spectrum X^V is the colimit (in spectra) of the composite $X \xrightarrow{V} BO \times \mathbb{Z} \xrightarrow{J} \mathsf{Sp}$.

Here's the compatibility between the Thom space and Thom spectrum construction.

Lemma 2.15. *Let $V \colon X \to BO(d)$ be a vector bundle and let $\xi \colon X \to BO(d) \to BO \times \mathbb{Z}$ be the corresponding virtual bundle. Then the Thom spectrum of ξ is the suspension spectrum of the Thom space of V; i.e. $X^\xi \simeq \Sigma^\infty_+ X^V$.*

Here by Σ^∞_+ we mean first taking the disjoint union with a single point, which we take as the basepoint, then taking the suspension spectrum.

Proof. This follows from the fact that $\Sigma^\infty_+ \colon \mathsf{Top} \to \mathsf{Sp}$ preserves colimits. $\qquad\square$

Using Lemma 2.15, one can directly check that the Thom spectrum of the trivial bundle $\underline{\mathbb{R}}^n \to X$ is homotopy equivalent to a suspension of the suspension spectrum $\Sigma^n \Sigma^\infty_+ X$.

Lemma 2.16 ([Ati61a, Lemma 2.3])**.** *Let $V \to X$ and $W \to Y$ be virtual vector bundles. Then the Thom spectrum of $V \boxplus W \to X \times Y$ is homotopy equivalent to $X^V \wedge Y^W$.*

Here $\boxplus$ is the external direct sum, i.e. the direct sum of the pullbacks of V and W across the projection maps $X \times Y \to X$, resp. $X \times Y \to Y$.

One can often combine Lemma 2.16 with the observation that Thom spectra of trivial bundles are suspensions to simplify Thom spectra appearing in examples. For example, $X^{V+\underline{\mathbb{R}}^n}$, often denoted X^{V+n}, is homotopy equivalent to $\Sigma^n X^V$. Since we are working with virtual vector bundles, n may be any integer.

Let us discuss a variant for tangential structures.

Definition 2.17. Let $\xi \colon B \to BO$ be a tangential structure. Then its inverse (as a virtual vector bundle) $-\xi$ is often denoted $\xi^\perp$. Equivalently, $\xi^\perp$ is the composition of ξ with the map $-1 \colon BO \to BO$, which is the inverse map in the E_∞-structure on BO induced by direct sum. Therefore $\xi^\perp$ is also a tangential structure; its Thom spectrum $B^{-\xi}$ is called a *Madsen-Tillmann spectrum* [MT01, MW07] and is often denoted $MT\xi$. If $B \to BO$ is obtained from a family of Lie group homomorphisms $H(n) \to O(n)$ in the (co)limit $n \to \infty$, $MT\xi$ is often written MTH.

Likewise, the Thom spectrum of the pullback of $-V_n \to BO(n)$ across a map $\xi_n \colon B_n \to BO(n)$ is denoted $MT\xi_n$; if $B = BH(n)$ for a Lie group $H(n)$, this is often written $MTH(n)$.

$MT\xi$ has two key properties:

(1) (Pontrjagin-Thom theorem) There is a natural isomorphism $\pi_n(MT\xi) \xrightarrow{\cong} \Omega^\xi_n$.[7]

---

[7]It is most common to define Thom spectra and bordism in terms of the stable normal bundle, rather than the tangent bundle; the resulting spectra are written $M\xi$. The spectra $MT\xi$ and $M\xi$ coincide for the tangential structures O, SO, Spin^c and Spin, but not in general: $MTPin^\pm \simeq MPin^\mp$. By composing with the map $-1 \colon BO \to BO$, one can pass between normal bordism and tangential bordism and therefore pass between our definition and the standard one.

(2) (Thom isomorphism theorem) Let A be a commutative ring. Then there is a natural[8] isomorphism $H^*(B; A_{w_1}) \overset{\cong}{\to} H^*(MT\xi; A)$, where A_{w_1} denotes the pullback by ξ of the orientation local system on $B\mathrm{O}$.

In the Thom isomorphism theorem, the use of twisted cohomology can be avoided by assuming $A = \mathbb{Z}/2$ or by choosing an orientation of the virtual vector bundle classified by the map ξ.

When ξ is the result of applying the classifying space functor to a group homomorphism $G \to \mathrm{O}$, we often write MTG for $MT\xi$.

3. Maps of spectra inducing Smith homomorphisms

This section is the technical heart of the paper—we provide a general definition of the Smith homomorphism, then lift it to a map sm of bordism spectra. The map of spectra has been studied, though its identification with the Smith homomorphism is new; using this, we can write down the cofiber of sm (Theorem 5.1) and therefore obtain Smith long exact sequences of bordism groups and Anderson-dualized bordism groups (Corollaries 5.8 and 8.19).

3.1. (X, V)-twisted tangential structures.
Twisted tangential structures are an important ingredient in the Smith homomorphism—they determine its domain and codomain. We take this subsubsection to define them and point out why they arise in the Smith homomorphism setting.

Throughout this subsubsection, we fix a topological space X, a vector bundle $V \to X$ of rank r, and a tangential structure $\xi \colon B \to B\mathrm{O}$.

Definition 3.1. Let $W \to Y$ be a vector bundle. An (X, V)-*twisted ξ-structure* on W is the data of a map $f \colon Y \to X$ and a ξ-structure on $W \oplus f^*(V)$.

There is a space of (X, V)-twisted ξ-structures on W, and just like for tangential structures, we will think of two such structures as the same if they lie in the same connected component.

Twisted ξ-structures provide a convenient way to describe a more complicated tangential structure in terms of a simpler one.

Example 3.2. Recall that a spin^c structure on an oriented vector bundle $W \to Y$ is the data of a complex line bundle $L \to Y$ and an identification $w_2(L) = w_2(W)$. The data of L is equivalent to a map $Y \to B\mathrm{U}(1)$ such that L is the pullback of the tautological complex line bundle $S \to B\mathrm{U}(1)$. The identification $w_2(L) = w_2(W)$ is equivalent by the Whitney sum formula to $w_2(W \oplus L) = 0$.

Choosing a spin structure on $W \oplus L$ first provides an orientation of $W \oplus L$, which since L is canonically oriented by its complex structure is equivalent to an orientation of W; then it additionally provides an identification $w_2(W \oplus L) = 0$. Therefore the data of a spin^c structure on W is equivalent to the data of L and a spin structure on $W \oplus L$, meaning that a spin^c structure is equivalent to a $(B\mathrm{U}(1), S)$-twisted spin structure.

In a similar way, one can show that if $\sigma \to B\mathbb{Z}/2$ is the tautological real line bundle, pin^- structures are equivalent to $(B\mathbb{Z}/2, \sigma)$-twisted spin structures, pin^+ structures are equivalent to $(B\mathbb{Z}/2, 3\sigma)$-twisted spin structures, and pin^c structures are equivalent to $(B\mathbb{Z}/2, \sigma)$-twisted spin^c structures.

---

[8]Naturality here is for maps of tangential structures as in Remark 2.6; this map typically does not commute with the action of cohomology operations.

It turns out that all of these twisted tangential structures can also be "untwisted" into ordinary tangential structures.

Lemma 3.3 (Shearing). *Let $T \to BO$ denote the tautological rank-zero virtual vector bundle and $\zeta \colon B \times X \to BO$ be classified by the rank-zero virtual vector bundle $\xi^*(T) \boxplus (V - r)$. Then (X, V)-twisted ξ-structures are equivalent to ζ-structures.*

The proof is given in [DDHM23, Lemma 10.18] for $\xi = \mathrm{Spin}$; the general case is completely analogous. Invoking the Pontrjagin-Thom theorem, we then learn:

Corollary 3.4. *There is a notion of bordism of manifolds with (X, V)-twisted ξ-structures, corresponding to the Thom spectrum $MT\xi \wedge X^{V - r_V}$; thus the bordism groups of these manifolds are $\Omega_*^\xi(X^{V - r_V})$.*

Here we use the fact that the Thom spectrum functor sends external direct sums to smash products, which is Lemma 2.16.

Lemma 3.5. *Suppose X is a closed smooth manifold with a ξ-structure and $M \subset X$ is an embedded submanifold such that the image of the mod 2 fundamental class of M in $H_*(X; \mathbb{Z}/2)$ is Poincaré dual to $e(V) \in H^r(X; \mathbb{Z}/2)$. Then M has a canonical (X, V)-twisted ξ-structure.*

Proof. Because the homology class of M is Poincaré dual to the mod 2 Euler class of V, the normal bundle to $M \hookrightarrow X$ is isomorphic to $V|_M$. Choose a Riemannian metric on X; this is a contractible choice, so will not change the connected component of the data we obtain, so as discussed above different choices of metric lead to the same (X, V)-twisted ξ-structure in the end.

Using the Levi-Civita connection induced by the metric, we may split the short exact sequence of vector bundles over M,

$$(3.6) \qquad 0 \longrightarrow TM \longrightarrow TX|_M \longrightarrow \nu \longrightarrow 0,$$

thereby obtaining an isomorphism $TM \oplus V|_M \cong TX|_M$. Since TX has a ξ-structure, this implies $TM \oplus V|_M$ has a chosen ξ-structure, i.e. that we have put a (X, V)-twisted ξ-structure on M. $\square$

3.2. **Smith homomorphisms induced by maps of Thom spectra.** We will now apply the previous discussions of Thom spectra and shearing to understand a class of homomorphisms between bordism groups called *Smith homomorphisms*. These map between bordism groups of manifolds of different dimensions and with different tangential structures.

Fix a tangential structure $\xi \colon B \to BO$ such that its bordism spectrum $MT\xi$ is a ring spectrum (e.g. O, SO, Spin^c, Spin). Fix also a virtual vector bundle $V \to X$ of rank r_V and $W \to X$ a vector bundle of rank r_W.

Definition 3.7. The *Smith homomorphism* associated to ξ, V, and W is the homomorphism

$$(3.8) \qquad \mathrm{sm}_W \colon \Omega_n^\xi(X^{V - r_V}) \longrightarrow \Omega_{n - r_W}^\xi(X^{V \oplus W - r_V - r_W})$$

that sends a closed n-manifold $[M]$ to the bordism class $[N]$, where $N \subset M$ is the submanifold defined as follows: pull back W from X to M and choose a section $s \colon M \to f^*W$ transverse to the zero section. Then, $N := s^{-1}(0)$ is an $(n - r_W)$-dimensional manifold whose mod 2 homology class is Poincaré dual to $e(W)$, hence by Lemma 3.5 has a $(X, V \oplus W)$-twisted ξ-structure, and we define $\mathrm{sm}_W([M]) := [N]$.

Proposition 3.9 ([HKT20] §4.2)**.** *The bordism class $[N] \in \Omega^{\xi}_{n-r_W}(X^{V \oplus W - r_V - r_W})$ is independent of the choice of section.*

Example 3.10. Let $\xi \colon B\mathrm{Spin} \to B\mathrm{O}$, $X = B\mathbb{Z}/2$, $V = 0$, and $W = \sigma \to B\mathbb{Z}/2$, where σ is the tautological line bundle. The corresponding Smith homomorphism is

$$(3.11) \qquad \Omega^{\mathrm{Spin}}_n(B\mathbb{Z}/2) \xrightarrow{\mathrm{sm}_\sigma} \Omega^{\mathrm{Spin}}_{n-1}((B\mathbb{Z}/2)^{\sigma - 1}).$$

After shearing (Lemma 3.3), we recognize this as

$$(3.12) \qquad \Omega^{\mathrm{Spin} \times \mathbb{Z}/2}_n \xrightarrow{\mathrm{sm}_\sigma} \Omega^{\mathrm{Pin}^-}_{n-1}.$$

Letting $V = 0$, σ, 2σ, and 3σ produces the maps in the four-periodic family discussed in Example 7.8.

Later, in Section 7, we thoroughly discuss the history of Smith maps and present many more examples. For the rest of this section, we discuss how Smith homomorphisms are induced by maps of Thom spectra. Let X be a topological space and V be a rank r real vector bundle on X. We abuse notation and also denote the associated classifying map by $V \colon X \to B\mathrm{O}(r)$. The inclusion $0 \hookrightarrow W$ induces a zero section map $X \to X^W$. More generally, we have the following.

Definition 3.13. Let V and W be vector bundles on X. Let $S^V \to S^{V \oplus W}$ be the map of finite-dimensional spheres over X induced by the zero section map on W. The *Smith map* associated to X, V, and W is the map of Thom spaces

$$(3.14) \qquad \mathrm{sm}_W \colon \mathrm{Th}(X;V) \to \mathrm{Th}(X;V \oplus W)$$

formed as the colimit of the map of spheres.

Definition 3.15. In the case that we have a virtual bundle V, the zero section map induces a map of stable spherical fibrations $\mathbb{S}^V \to \mathbb{S}^{V \oplus W} \simeq \mathbb{S}^V \wedge \mathbb{S}^W$ over X. Taking the colimit, we get a map of Thom spectra

$$(3.16) \qquad \mathrm{sm}_W \colon X^V \to X^{V \oplus W}$$

which we also call a *Smith map.*

Proposition 3.17. *The map on ξ-bordism groups induced by the map* (3.16) *of spectra is equal to the Smith homomorphism as defined in Definition 3.7.*

This follows by unpacking the Pontrjagin-Thom isomorphism.

4. Euler classes and Smith homomorphisms

In the next section, we develop an alternate definition of the Smith homomorphism via the Euler class.

4.1. **Euler classes in generalized cohomology.** Fix $\xi \colon X \to B\mathrm{O}$ a tangential structure and $W \colon X \to B\mathrm{O}(r_W)$ a vector bundle on X. We would like to describe the Smith homomorphism on ξ bordism groups as taking a manifold $(M, p \colon M \to X)$ with ξ-structure to a smooth representative of the Poincaré dual of $e(p^*W)$, where $e(p^*W) \in H^{r_W}(M; \mathbb{Z})$ is the Euler class of W. This, however, is *not* true in general, as we show in Appendix B—we need to upgrade what we mean by the Euler class.

We will define an Euler class living in twisted cobordism. More generally, for $\mathcal{R}$ an $\mathbb{E}_1$ ring spectrum, we define a $\mathcal{R}$-valued Euler class in the $\mathcal{R}$-cohomology of X^{-W}. In the case we have an untwisting, given by a $\mathcal{R}$-orientation on W, we will see in Lemma 4.24 that the *untwisted* Euler class is the pullback of the Thom class $U^{\mathcal{R}}(W) \in \mathcal{R}^{r_W}(\mathrm{Th}(X;W))$ along the 0 section $X \to \mathrm{Th}(X;W)$ (e.g. in [Bec70, §13]), so that our definition deserves to be called an Euler class; we also generalize to the twisted setting where there is no Thom class.

Recall the setup of Definition 3.15. Let 0 be the vector bundle over X of rank zero. The zero section gives a map $0 \to W$ of vector bundles over X. Therefore we get a map of stable spherical fibrations

$$(4.1a) \qquad\qquad z \colon \mathbb{S}^0 \longrightarrow \mathbb{S}^W,$$

i.e. a fiberwise map of spectra. Because 0 is the trivial rank-zero vector bundle, $\mathbb{S}^0$ is the constant stable spherical fibration $\mathbb{S}$ with fiber $\mathbb{S}$.

Apply the duality $\mathrm{Map}(-, \mathbb{S})$ fiberwise to obtain another map

$$(4.1b) \qquad\qquad z^\vee \colon \mathbb{S}^{-W} \longrightarrow \mathbb{S}^0.$$

Because the codomain of $z^\vee$ is constant as a functor $X \to \mathsf{Sp}$, there is an induced map of spectra:

$$(4.1c) \qquad\qquad e^{\mathbb{S}}(W) \colon X^{-W} = \mathrm{colim}_X \mathbb{S}^{-W} \to \mathbb{S}.$$

Definition 4.2. The class $e^{\mathbb{S}}(W)$ is called the *stable cohomotopy Euler class* of W. Usually, we will interpret generalized cohomology of $X^{r_W - W}$ as the $(-W)$-twisted cohomology of X, meaning $e^{\mathbb{S}}(W)$ is an element of the degree-r_W $(-W)$-twisted stable cohomotopy of X.

Remark 4.3. This cohomology class of $e^{\mathbb{S}}(W)$ lives in $(\mathbb{S})^0(X^{-W})$. By the Pontrjagin-Thom isomorphism, this is equivalent to the twisted cobordism group $\Omega^0_{\mathrm{fr}}(X, -W)$.

Definition 4.4. Let $\mathcal{R}$ be a $(\mathbb{E}_1)$-ring spectrum, so that there is a unique ring map $1_{\mathcal{R}} \colon \mathbb{S} \to \mathcal{R}$. The *R-cohomology Euler class of W*, denoted $e^{\mathcal{R}}(W)$, is the composition $1_{\mathcal{R}} \circ e^{\mathbb{S}}(W)$. As in the previous definition, we interpret this as an element of the degree-r_W $(-W)$-twisted R-cohomology of X.

Now we see how the Euler class and Smith homomorphism are related:

Proposition 4.5.

    (1) Let 0 be the trivial rank 0 vector bundle on X; then $e^{\mathbb{S}}(0) \colon \Sigma^\infty_+ X \to \mathbb{S}$ is the infinite suspension of the crush map $X \to *$.

    (2) Let W be a vector bundle on X and $\mathrm{sm}_W \colon X^{-W} \to X$ be the Smith map. Then $e^{\mathbb{S}}(W) = (\mathrm{sm}_W)^*(e^{\mathbb{S}}(0))$.

Proof. For part 1: 0 defines the trivial stable spherical fibration on X, which factors through a point. Therefore the Euler class of 0 is the pullback of the Euler class of the trivial bundle over a point.

For part 2: this follows from the fact that $e^{\mathbb{S}}(W) \colon X^{-W} \to \mathbb{S}$ factors through

$$X^{-W} \xrightarrow{\mathrm{sm}_W} X \xrightarrow{e^{\mathbb{S}}(0)} 0. \qquad\qquad\qquad \square$$

We immediately learn that Smith maps pull back Euler classes.

Corollary 4.6. *Given a virtual vector bundle V and a vector bundle W, let sm_W denote the Smith homomorphism $\mathrm{sm}_W \colon X^{-V \oplus -W} \to X^{-V}$. Then*

$$\tag{4.7} \mathrm{sm}_W^*(e^{\mathbb{S}}(V)) = e^{\mathbb{S}}(V \oplus W).$$

We can thus recover the Smith homomorphism from capping with the twisted Euler class.

Proposition 4.8. *For any virtual bundle V on X, the Smith map $X^V \to X^{V \oplus W}$ can be defined as the following composition:*

$$\tag{4.9} X^V \simeq X^{(V \oplus W) \oplus -W} \xrightarrow{\Delta} (X \times X)^{(V \oplus W) \boxplus -W} \simeq X^{V \oplus W} \wedge X^{-W} \xrightarrow{e^{\mathbb{S}}(W)} X^{V \oplus W}.$$

The map $X^{(V \oplus W) \oplus -W} \xrightarrow{\Delta} (X \times X)^{(V \oplus W) \boxplus -W}$ is induced by the diagonal map $\Delta \colon X \to X \times X$.

Proof. The Euler map for the trivial rank 0 vector bundle

$$\tag{4.10} X^0 \simeq \Sigma_+^\infty X \xrightarrow{e^{\mathbb{S}}(0)} \mathbb{S}.$$

is the counit for the $\mathbb{E}_\infty$-coalgebra structure on $\Sigma_+^\infty X$. By Proposition 4.5, the Euler class $e^{\mathbb{S}}(W)$ factors through (4.10) as

$$\tag{4.11} X^{-W} \longrightarrow X^{-W \oplus W} \simeq \Sigma_+^\infty X \xrightarrow{e^{\mathbb{S}}(0)} \mathbb{S}.$$

This implies that (4.9) can be written as

(4.12)

$$X^V \simeq X^{(V \oplus W) \oplus -W} \xrightarrow{\Delta} (X \times X)^{(V \oplus W) \boxplus -W} \longrightarrow (X \times X)^{(V \oplus W) \boxplus 0} \xrightarrow{\simeq} X^{V \oplus W} \wedge \Sigma_+^\infty X \xrightarrow{e^{\mathbb{S}}(W)} X^{V \oplus W}$$

$$\Delta \big\uparrow$$

$$\xrightarrow{\phi} \quad X^{V \oplus W}$$

Since the map $X^{V \oplus W} \to (X \times X)^{(V \oplus W) \boxplus 0} \simeq X^{V \oplus W} \wedge \Sigma_+^\infty X$ comes from the comodule structure of $X^{V \oplus W}$ over $\Sigma_+^\infty X$, the composite $X^{V \oplus W} \to (X \times X)^{(V \oplus W) \boxplus 0} \to X^{V \oplus W}$ is the identity map. Therefore it is sufficient to show that the map ϕ in (4.12) is homotopy equivalent to the spectral Smith map sm_W, and this follows by restricting to the diagonal in the map $(X \times X)^{(V \oplus W) \boxplus -W} \to (X \times X)^{(V \oplus W) \boxplus 0}$ along the top of (4.12), which is induced from $\mathrm{id} \boxplus \mathrm{sm}_W$. $\qquad\square$

We see that the Euler class records all the "Smith" information about W. We will therefore refer to the Smith homomorphism as capping with the Euler class or as the map of Thom spectra interchangeably.

The dual version of Proposition 4.8 also holds.

Proposition 4.13. *Let $\mathcal{R}$ be a ring spectrum. Then the pullback map on $\mathcal{R}$-cohomology $\mathrm{sm}_W^* \colon \mathcal{R}^*(X^{V \oplus W}) \to \mathcal{R}^*(X^V)$ is equal to the cup product with $e^{\mathcal{R}}(W)$.*

Remark 4.14. The long exact sequence of field theories we shall discuss in Section 8 is cohomological in nature: it is given by applying $I_{\mathbb{Z}} MT\xi$-cohomology to sm_W. However, Proposition 4.13 does not apply: the Smith homomorphism there cannot be described as taking the product with an $I_{\mathbb{Z}}\mathcal{R}$-Euler class. This is because if $\mathcal{R}$ is a ring spectrum, $I_{\mathbb{Z}}\mathcal{R}$ usually admits no ring spectrum structure. However, $I_{\mathbb{Z}}\mathcal{R}$ is an $\mathcal{R}$-module, so we do learn from Proposition 4.13 that this Smith homomorphism is the cup product with $e^{\mathcal{R}}(W)$ using the $\mathcal{R}$-module structure. For example, when we study fermionic invertible phases, we will typically choose $\mathcal{R} = MT\mathrm{Spin}$.

Let us review the standard story that "the Euler class is the pullback of the Thom class along the zero section." First we review orientations and Thom classes. For simplicity, we will define them only for vector bundles, though the story generalizes to virtual bundles and much more.

Definition 4.15. Let W be a vector bundle of rank n on X. Fix $\mathcal{R}$ an $\mathbb{E}_1$-algebra in spectra and let Mod_R be the ∞-category of $\mathcal{R}$-module spectra. An $\mathcal{R}$-*orientation* of W is a natural isomorphism ϕ of functors between

$$(4.16) \qquad \mathcal{R}^W \colon X \xrightarrow{W} BO(n) \to \mathsf{Sp} \xrightarrow{-\wedge \mathcal{R}} \mathsf{Mod}_{\mathcal{R}}$$

and the constant functor valued in $\Sigma^n \mathcal{R}$. An $\mathcal{R}$-orientation of a manifold M means an $\mathcal{R}$-orientation of TM.

Remark 4.17. The map $z^\vee$ from (4.1b) is similar to an orientation on $-W$, in the sense of Ando-Blumberg-Gepner-Hopkins-Rezk, except that $z^\vee$ is in general non-invertible and between different suspensions of the sphere spectrum.

An $\mathcal{R}$-orientation ϕ on W induces an equivalence

$$(4.18) \qquad \mathrm{colim}_X \mathcal{R}^W \simeq \Sigma_+^\infty \mathrm{Th}(X; W) \wedge \mathcal{R} \simeq X \wedge \Sigma^n \mathcal{R} \simeq \Sigma^n \Sigma_+^\infty X \wedge \mathcal{R}.$$

Definition 4.19. The composite

$$(4.20) \qquad U \colon \Sigma_+^\infty \mathrm{Th}(X; W) = X^W \to \Sigma_+^\infty \mathrm{Th}(X; W) \wedge \mathcal{R} \simeq \Sigma^n \Sigma_+^\infty X \wedge \mathcal{R} \to \Sigma^n \mathcal{R}$$

is the *Thom class*. Often we think of U through its homotopy class, which lives in $\mathcal{R}^n(\mathrm{Th}(X; W))$.

Given a $\mathcal{R}$-orientation on W, we can also define the (untwisted) Euler class of W. This is a standard definition (e.g. [Bec70, §13]).

Definition 4.21. Given an $\mathcal{R}$-orientation, we have a natural isomorphism of functors $X \to \mathsf{Mod}_R$

$$(4.22) \qquad R^{-W} \simeq \Sigma^{-n}\underline{\mathcal{R}},$$

where $\Sigma^{-n}\underline{\mathcal{R}}$ is the constant functor valued in $\Sigma^{-n}\mathcal{R}$. The composite

$$(4.23) \qquad \Sigma^{-n} X \longrightarrow \Sigma^{-n} X \wedge \mathcal{R} \simeq X^{-W} \wedge \mathcal{R} \xrightarrow{\mathrm{sm}_W} X \wedge \mathcal{R} \to \mathcal{R}$$

is called the (untwisted) *Euler class* of W.

Unlike the twisted Euler class, this untwisted Euler class depends on the $\mathcal{R}$-orientation.

Finally, we can prove that our definition of the Euler class, Definition 4.4, coincides with the more standard Definition 4.21 when they overlap (i.e. when there is an $\mathcal{R}$-orientation chosen on V).

Lemma 4.24. *Suppose W is $\mathcal{R}$-oriented, and let $U \in \mathcal{R}^r(\mathrm{Th}(X; W))$ denote the Thom class. Then $e^{\mathcal{R}}(W) = z_W^* U$, where $z_W \colon X \to \mathrm{Th}(X; W)$ is the inclusion as the zero section.*

Proof. After suspending, the zero section map becomes the Smith map. Therefore it suffices to show that the following diagram commutes.

$$(4.25) \qquad \begin{array}{ccccc} \Sigma_+^\infty X & \xrightarrow{-\wedge \mathcal{R}} & \Sigma_+^\infty X \wedge \mathcal{R} & \xrightarrow{\simeq} & \Sigma^n X^{-W} \wedge \mathcal{R} \\ \downarrow{\scriptstyle \mathrm{sm}_W} & & \downarrow{\scriptstyle \mathrm{sm}_W \wedge \mathrm{id}_{\mathcal{R}}} & & \downarrow{\scriptstyle \mathrm{sm}_W} \\ \Sigma_+^\infty \mathrm{Th}(X; W) \simeq X^W & \xrightarrow{-\wedge \mathcal{R}} & X^W \wedge \mathcal{R} & \xrightarrow{\simeq} & \Sigma^n X \wedge \mathcal{R}. \end{array}$$

Here the equivalences in the right square are the ones induced by the orientation ϕ.

The left-hand square commutes because smashing with $\mathcal{R}$ is a functor. The right-hand square commutes because the following diagram commutes in $\mathsf{Fun}(X, \mathsf{Mod}_R)$:

$$
(4.26) \qquad
\begin{array}{ccc}
\underline{\mathcal{R}} & \xrightarrow{(\phi \wedge \mathcal{R}^{-W})} & \Sigma^n \mathcal{R}^{-W} \\
{\scriptstyle z^\vee \wedge \mathcal{R}} \downarrow & & \downarrow {\scriptstyle z^\vee \wedge \mathcal{R} \wedge \mathcal{R}^W} \\
\mathcal{R}^W & \xrightarrow{\phi} & \Sigma^n \underline{\mathcal{R}},
\end{array}
$$

which follows from naturality. Recall that $z^\vee \colon \underline{\mathcal{R}} \to \mathcal{R}^W$ is the map of spherical fibrations over X that induces the Smith map. $\qquad\square$

4.2. Smith homomorphisms defined via Atiyah-Poincaré dual of the generalized Euler classes.

Now equipped with the theory of Euler classes, we can give another alternate definition of the Smith homomorphism. Fix $\xi \colon B \to BO$, $V \to X$ of rank r_V, and $W \to X$ of rank r_W as in Definition 3.7. Recall that by Corollary 3.4, a class $c \in \Omega_n^\xi(X^{V-r_V})$ can be represented by a closed n-manifold M with an (X, V)-twisted ξ-structure, which includes the data of a map $f \colon M \to X$.

In this subsubsection, we assume that $MT\xi$ is a ring spectrum.

Definition 4.27. The *Smith homomorphism* associated to ξ, V, and W is the homomorphism

$$
(4.28) \qquad\qquad \mathrm{sm}_W \colon \Omega_n^\xi(X^{V-r_V}) \longrightarrow \Omega_{n-r_W}^\xi(X^{V \oplus W - r_V - r_W})
$$

sending the class $[M]$ to the Poincaré dual of the cobordism Euler class $e^{MT\xi}(f^*W)$.

To show this, we first recall Atiyah duality. There is the standard notion of duals in any symmetric monoidal category C [Lin78, DP80, DM82]. Here for C we take the homotopy category of spectra, which is monoidal with respect to the smash product $\wedge$. If A, B have duals $A^\vee, B^\vee$, then a morphism $f \colon A \to B$ induces a dual morphism, which we write as $f^\vee \colon B^\vee \to A^\vee$.

Theorem 4.29 (Atiyah duality [Ati61b, Proposition 3.2 and Theorem 3.3]). *Let M be a compact manifold; then $(M/\partial M)^\vee \simeq M^{-TM}$. If M is closed and $V \to M$ is a virtual vector bundle, then $(M^V)^\vee \simeq M^{-TM-V}$.*

Furthermore, dual spectra provide isomorphisms between homology and cohomology groups: let X be a spectrum with a dual $X^\vee$; then, for any spectrum $\mathcal{R}$, we have a canonical isomorphism

$$
(4.30) \qquad\qquad \mathcal{R}_*(X) \xrightarrow{\cong} \mathcal{R}^{-*}(X^\vee).
$$

We call two classes $\alpha \in \mathcal{R}_*(X)$ and $\beta \in \mathcal{R}^{-*}(X^\vee)$ *Atiyah-Poincaré dual* if $\alpha \mapsto \beta$ under the isomorphism (4.30).

Furthermore, this is functorial: given a map $f \colon X \to Y$ of dualizable spectra, let $f^\vee \colon Y^\vee \to X^\vee$ be the dual map. We have a commutative square:

$$
(4.31) \qquad
\begin{array}{ccc}
\mathcal{R}_*(X) & \xrightarrow{f_*} & \mathcal{R}_*(Y). \\
{\scriptstyle \simeq} \downarrow & & \downarrow {\scriptstyle \simeq} \\
\mathcal{R}^{-*}(X^\vee) & \xrightarrow{(f^\vee)^*} & \mathcal{R}^{-*}(Y^\vee).
\end{array}
$$

Let $\Omega_*^{\mathrm{fr}}(X)$ denote the stably framed bordism of X, i.e. the bordism groups of manifolds with a map to X and a trivialization of the stable tangent bundle (or equivalently, the stable normal

bundle). The Pontrjagin-Thom theorem identifies these bordism groups with the stable homotopy groups of X. We learn a neat fact:

Lemma 4.32. *Let M be a closed compact d-dimensional manifold. Then M defines a canonical class in $\Omega_d^{\mathrm{fr}}(M, -TM) = \mathbb{S}_0(M^{-TM})$. This is the Atiyah-Poincaré dual to the Euler class for the trivial bundle $e^{\mathbb{S}}(0) \in \mathbb{S}^0(M)$.*

Proof. The Euler class is represented by a map $e\colon M_+ \to S^0$ taking $+$ to the basepoint of S^0 and the entirety of M to the other point. On the other hand, consider an embedding $\iota\colon M \to \mathbb{R}^N$ and let ν be the normal bundle. Then $\Sigma_+^\infty \mathrm{Th}(M; \nu) \simeq \Sigma^{-N} M^{-TM}$. By the Pontrjagin-Thom construction, the tautological class $[M] \in \Omega_d^{\mathrm{fr}}(M, -TM)$ comes from the collapse map $S^N = (\mathbb{R}^N)^+ \to \mathrm{Th}(M; \nu)$, where $(-)^+$ denotes the one point compactification.

The result follows from the finite-dimensional description of the evaluation and co-evaluation map of M and M^{-TM} [Ati61a]: we have an evaluation map $S^N \to M_+ \wedge \mathrm{Th}(M; \nu)$, representing $\mathbb{S} \to M \wedge M^{-TM}$. The composite $S^N \to M_+ \wedge \mathrm{Th}(M; \nu) \xrightarrow{e} S^0 \wedge \mathrm{Th}(M; \nu) = \mathrm{Th}(M; \nu)$ is precisely the Pontrjagin-Thom collapse map. $\square$

Now we see how Atiyah duality interacts with Smith homomorphisms on compact manifolds:

Lemma 4.33. *Fix a closed compact manifold M. Given a virtual bundle $V \to M$ and a vector bundle $W \to M$, the Atiyah dual $(\mathrm{sm}_W)^\vee$ of the Smith map*

$$(4.34) \qquad\qquad \mathrm{sm}_W\colon M^V \longrightarrow M^{V \oplus W}$$

is the Smith map associated to $-TM - V - W$:

$$(4.35) \qquad\qquad \mathrm{sm}_W\colon M^{-TM-V-W} \longrightarrow M^{-TM-V}.$$

Proof. Let us do the case $V = 0$; the general case follows in the same way. First we give a space-level description of the Atiyah dual map. Consider the manifold with boundary $D_M(W)$, the disc bundle of W. Its tangent bundle is $T(D_M(W)) \cong TM \oplus W$, where we are implicitly pulling back W to $D_M(W)$. Now consider an embedding $\mu_D\colon D_M(W) \to \mathbb{R}^N$. Then M, sitting as the zero section, also gets an embedding $\mu_M\colon M \to D_M(W) \to \mathbb{R}^N$.

Let ν_D, resp. ν_M be the normal bundle of μ_D, resp. μ_M. As virtual bundles,

$$(4.36a) \qquad\qquad \nu_D \cong \mathbb{R}^N - TM - W$$

$$(4.36b) \qquad\qquad \nu_M \cong \mathbb{R}^N - TM.$$

Note that $\nu_M = \nu_D \oplus W$. Now let $N_D(\mu)$ be a tubular neighborhood of $D_M(W)$ and $N_M(\mu)$ the same for M. Observe that $N_D(\mu)$ and $N_M(\mu)$ are diffeomorphic to μ_D, resp. μ_M.

Using the standard Pontrjagin-Thom collapse argument, the open embedding $i\colon N_M(\mu) \to N_D(\mu)$ induces a map of one-point compactifications $i^+\colon N_D(\mu)^+ \to N_M(\mu)^+$. By Proposition 2.9, we can write this as $\mathrm{Th}(D_M(W); \nu_D) \simeq \mathrm{Th}(M; \nu_D) \to \mathrm{Th}(M; \nu_D)$. Recall that $D_M(W)$ is homotopy equivalent to W.

After passing to spectra, Equation (4.36) gives a map

$$(4.37) \qquad\qquad \Sigma^n M^{-TM-W} \longrightarrow \Sigma^n M^{-TM}.$$

This is the Atiyah dual map of the Smith map.

To see this is the Smith map for $-TM - V - W$ as claimed, notice that the composite $\mathrm{Th}(M; \nu_D) \to \mathrm{Th}(D_M(W); \nu_D) \to \mathrm{Th}(M; \nu_D \oplus W)$ is induced by the inclusion of disk bundles,

a.k.a. the Smith homomorphism on Thom spaces, which suspends to the Smith map on Thom spectra.   □

Lemma 4.38. *Let W be a rank r_W vector bundle on a closed compact d-manifold M, and let $[M] \in \Omega_d^{\mathrm{fr}}(M, -TM)$ be the tautological class. Then $(\mathrm{sm}_W)_*([M]) \in \Omega_d^{\mathrm{fr}}(M, -TM + W) = \mathbb{S}_0(M^{-TM+W})$ is the Atiyah-Poincaré dual of the Euler class $e^{\mathbb{S}}(W) \in \Omega_{\mathrm{fr}}^{r_W - d}(M, -W)$.*

Proof. By Equation (4.31), $\mathrm{sm}_{W*}([M])$ is Atiyah-Poincaré dual to $((\mathrm{sm}_W)^\vee)^*(e^{\mathbb{S}}(0))$, where 0 denotes the zero vector bundle. By Lemma 4.33, $(\mathrm{sm}_W)^\vee$ is still sm_W. By Proposition 4.5, $(\mathrm{sm}_W)^*(e^{\mathbb{S}}(0))$ is $e^{\mathbb{S}}(W)$.   □

Now we can collect our prize: we show that Definitions 3.15 and 4.27 are equivalent definitions of the Smith homomorphism. In other words, the Smith homomorphism as we first defined it is the same as the map taking the Poincaré dual of the Euler class, as it is often described in the literature.

Corollary 4.39. *Let $V \to X$ be a virtual vector bundle and $W \to X$ be a rank r_W vector bundle. Choose a bordism class in $\Omega_d^{\mathrm{fr}}(X, V)$ (i.e. (X, V)-twisted framed bordism) and let M be a closed manifold representative of that class. Let $[N] \in \Omega_d^{\mathrm{fr}}(M, -TM \oplus W)$ be the Atiyah-Poincaré dual of the Euler class $e^{\mathbb{S}}(W|_M)$. Then the image of $[N]$ in $\Omega_d^{\mathrm{fr}}(X, V \oplus W)$ is $\mathrm{sm}_W([M])$.*

Proof. Since M has a (X, V)-twisted framing, the map $M \to X$ Thomifies to a map $f \colon M^{-TM} \to X^V$. The Smith map is functorial, so we get a commutative square:

(4.40)
$$
\begin{array}{ccc}
M^{-TM} & \xrightarrow{\;\;f\;\;} & X^V \\
\downarrow{\scriptstyle \mathrm{sm}_{W|M}} & & \downarrow{\scriptstyle \mathrm{sm}_W} \\
M^{-TM \oplus W} & \xrightarrow{\;\;f\;\;} & X^{V \oplus W}.
\end{array}
$$

Furthermore, $[M] \in \Omega_d^{\mathrm{fr}}(X, V)$ is the f-pushforward of the tautological class in $\Omega_d^{\mathrm{fr}}(M, -TM)$. The result now follows from Lemma 4.38.   □

Remark 4.41. This tells us that given a bordism class represented by M, $\mathrm{sm}_W([M])$ is represented by a manifold N that is Atiyah-Poincaré dual (in the bordism homology theory) to the twisted cobordism Euler class of M.

5. THE SMITH FIBER SEQUENCE

In this section we extend the Smith map into a fiber sequence, which allows us to derive a long exact sequence of bordism groups and, dually, a long exact sequence of field theories.

For any vector bundle $E \to X$ of rank r, let E also denote the classifying map $X \to BO(r)$. Which of these two things we mean by E will be clear from context. In this section, we will write $S_X(E)$ and $D_X(E)$ for the sphere, resp. disc bundles of E, because there will be places where it will help to remember which base space we work over.

The following result is not new; see, e.g. [KZ18, Remark 3.14], where it is attributed to James.

Theorem 5.1. *Let V, W be real vector bundles over X. Then there is a cofiber sequence in pointed spaces:*

(5.2)
$$
S_X(W)^V \to X^V \to X^{W \oplus V}.
$$

Similarly, if V is a virtual bundle, we have a (co)fiber sequence in spectra:

$$(5.3) \qquad S_X(W)^V \to X^V \to X^{V \oplus W}.$$

Proof. We will do the case where V is an actual vector bundle; the virtual bundle case is analogous. Given an r-dimensional vector space W, we have a cofiber sequence in pointed spaces:

$$(5.4) \qquad S(W)_+ \to D(W)_+ \simeq S^0 \to S^W.$$

Now since $\mathrm{Aut}(W) \cong \mathrm{O}(r)$ acts on W, we can upgrade (5.4) to a cofiber sequence of spaces with $\mathrm{O}(r)$-actions; equivalently, (5.4) is a cofiber sequence of functors $B\mathrm{O}(r) \to \mathsf{Top}_*$. Pulling back to X via the map $X \to B\mathrm{O}(r)$ classifying W, we get a cofiber sequence of functors $X \to \mathsf{Top}_*$. Now smash with S^V: we get a cofiber sequence of the form

$$(5.5) \qquad S(W)_+ \wedge S^V \to D(W)_+ \wedge S^V \to S^W \wedge S^V \simeq S^{V \oplus W}.$$

This cofiber sequence is in the category of functors $X \to \mathsf{Top}_*$.

Since taking the colimit over X preserves cofiber sequences, it is sufficient to show that the colimit of (5.5) over X is

$$(5.6) \qquad S_X(W)^V \longrightarrow X^V \longrightarrow X^{V \oplus W}.$$

For the last term $S^{V \oplus W}$ in (5.5), this follows directly from the definition of the Thom spectrum (Definition 2.14).

For $\mathrm{colim}_X(D(W)_+ \wedge S^V)$, note that $D(W)_+ \simeq S^0$, so $D(W)_+ \wedge S^V \simeq S^V$, so Definition 2.14 once again tells us the colimit is X^V. It also follows that the map $X^V \to X^{V \oplus W}$ on colimits is the Smith map.

Lastly, the colimit of $S(W)_+$ over X is the associated sphere bundle $S_X(W)$. It follows that the colimit of $S(W)_+ \wedge S^V$ over X is equivalent to the colimit of (the pullback of) S^V over $S_X(W)$, which is $S_X(W)^V$. $\square$

Remark 5.7. Everything here is functorial, so given a map $Y \to X$, we get maps between cofiber sequences and therefore a map of long exact sequences of homotopy groups.

Corollary 5.8. *Applying π_* to the fiber sequence, we get a long exact sequence of bordism groups:*

$$(5.9) \qquad \cdots \longrightarrow \Omega_k^\xi(S_X(W)^V) \longrightarrow \Omega_k^\xi(X^V) \longrightarrow \Omega_{k-r}^\xi(X^{V+W-r}) \longrightarrow \Omega_{k-1}^\xi(S_X(W)^V) \longrightarrow \cdots$$

Though written as bordism groups of Thom spectra, these are also twisted ξ-bordism groups thanks to Corollary 3.4. We work through an explicit example long exact sequence on the level of manifold generators in Appendix A.

Remark 5.10. Suppose $X = BG$ for a compact Lie group G and that $W \to X$ is the vector bundle associated to an orthogonal representation of G such that G acts transitively on the unit sphere in W. Then the sphere bundle has a particularly simple form: if G_v is the stabilizer subgroup for a point $v \in S(W)$, then the bundle map $S_X(W) \to X$ is homotopy equivalent to the map $BG_v \to BG$ induced by the inclusion $G_v \hookrightarrow G$. Thus, as we use in [DDK+24] and allude to in Section 8, the obstruction for an invertible field theory to be in the image of the Anderson-dualized Smith homomorphism is its restriction from manifolds with G-bundles (and some sort of tangential structure) to manifolds with G_v-bundles (and the corresponding tangential structure).

Remark 5.11 (Smith and Gysin long exact sequences). The reader looking at the type signatures of (5.9) and (8.20) might notice that they resemble Gysin sequences: long exact sequences involving (co)homology groups of the base space and total space of a sphere bundle, especially because one of the maps can be interpreted as a product with an Euler class. And indeed, if one takes ordinary homology or cohomology, the Smith long exact sequence becomes the Gysin long exact sequence, as can be verified by comparing the three maps in the long exact sequence.

Thus, the Smith long exact sequence can be thought of as the generalization of the Gysin long exact sequence to arbitrary vector bundle twists of generalized cohomology theories.

5.1. **Comparison with Conner-Floyd's long exact sequence.** Suppose $\xi_1\colon B_1 \to BO$ and $\xi_2\colon B_2 \to BO$ are tangential structures and $\eta\colon \xi_1 \to \xi_2$ is a map of tangential structures, i.e. a map $B_1 \to B_2$ such that $\xi_2 \circ \eta = \xi_1$. The map η induces a map of Thom spectra $MT\xi_1 \to MT\xi_2$ and hence also maps of bordism groups; we will also denote both of these maps by η.

Conner-Floyd [CF66a, §16] concoct a long exact sequence

$$(5.12) \qquad\qquad \cdots \longrightarrow \Omega_k^{\xi_1} \xrightarrow{\ \eta\ } \Omega_k^{\xi_2} \xrightarrow{\ j\ } \Omega_k^{\xi_2/\xi_1} \xrightarrow{\ \partial\ } \Omega_{k-1}^{\xi_1} \longrightarrow \cdots$$

where $\Omega_k^{\xi_2/\xi_1}$ is the bordism group of k-dimensional ξ_2/ξ_1-manifolds (see below).

Definition 5.13. A ξ_2/ξ_1-*manifold* is a compact manifold M equipped with the data of

(1) a ξ_2-structure $\mathfrak{r}$ on M,
(2) a ξ_1-structure $\mathfrak{r}_\partial$ on ∂M, and
(3) an identification of ξ_2-structures $\eta(\mathfrak{r}_\partial) \overset{\cong}{\to} \mathfrak{r}|_{\partial M}$.

Thus a ξ_2/ξ_1-structure makes precise the notion of a ξ_2-manifold equipped with a ξ_1-structured boundary.

Conner-Floyd (*ibid.*) defined a notion of bordism for ξ_2/ξ_1-manifolds, and showed that the corresponding bordism groups $\Omega_k^{\xi_2/\xi_1}$ fit into the long exact sequence (5.12),[9] where j regards a closed ξ_2-manifold as having an empty boundary and ∂ takes the boundary of a ξ_2/ξ_1-manifold. Like with the Smith long exact sequence, one can verify by hand that (5.12) is exact at each entry.

Where this story meets ours is that Smith long exact sequences are special cases of (5.12). Specifically, given the data (ξ, X, V, W) that we used to construct the Smith long exact sequence in Corollary 5.8, let ξ_1 be $(S(W), V)$-twisted ξ-structure, ξ_2 be (X, V)-twisted ξ-structure, and let η be induced from the bundle map $S(W) \to X$. Then two-thirds of the Conner-Floyd and Smith long exact sequences coincide, so by the five lemma all three must (homotopically, they are both the homotopy groups of equivalent cofiber sequences of spectra):

(5.14)

$$\cdots \longrightarrow \Omega_k^{\xi}(S(W)^V) \longrightarrow \Omega_k^{\xi}(X^V) \longrightarrow \Omega_{k-r}^{\xi}(X^{V+W-r}) \longrightarrow \Omega_{k-1}^{\xi}(S(W)^V) \longrightarrow \cdots$$

$$\Big\| \qquad\qquad \Big\| \qquad\qquad \Big\downarrow{\scriptstyle\varphi} \qquad\qquad \Big\|$$

$$\cdots \longrightarrow \Omega_k^{\xi_1} \xrightarrow{\ \eta\ } \Omega_k^{\xi_2} \xrightarrow{\ j\ } \Omega_k^{\xi_2/\xi_1} \xrightarrow{\ \partial\ } \Omega_k^{\xi_1} \longrightarrow \cdots$$

---

[9]Conner-Floyd do not work in this level of generality, only looking at a few tangential structures; nevertheless, their arguments go through in general. Some other examples with more tangential structures appear in [Sto68, Ale75, Mit75, RST77, Lau00, Bun15, Deb23, TY23a, TY23b, JFY24].

That is, the map φ from $(X, V \oplus W)$-twisted ξ-bordism to ξ_2/ξ_1-bordism (i.e. (X, V)-twisted ξ-manifolds with an $(S(W), V)$-twisted ξ-structure on the boundary) must be an equivalence. But we can do better: we will provide a geometric reason for this equivalence.

- The map φ is: given an $(X, V \oplus W)$-twisted manifold M, with structure map $f \colon M \to X$, $\varphi(M)$ is the disc bundle of f^*W; the boundary $S(f^*W)$ has a canonical reduction of structure across $S(W) \to X$, which is to say the map $f \colon D(f^*W) \to M \to X$, when pulled back to $S(f^*W)$, canonically lifts across $S(W) \to X$. Therefore the output of φ is indeed a ξ_2/ξ_1-manifold.
- Going backwards, given a ξ_2/ξ_1-manifold M, it is possible to construct a bordism of ξ_2/ξ_1-manifolds from M to a tubular neighborhood of a manifold representative of the Poincaré dual of the Euler class of W. The ξ_1-structure on the boundary means this Poincaré dual does not intersect ∂M; now we have replaced M with a disc bundle, so we can directly invert φ by restricting to the zero section.

The reader can then check that j and ∂ coincide with their analogues in the Smith long exact sequence.

6. Periodicity of twists and shearing

The goal of this section is to provide tools for working with twists of tangential structures. We are interested in collections of similar twists over the same base space; this provides an organizing principle for different Smith homomorphisms that we will use many times in the next section.

6.1. **Families of Smith homomorphisms.**

Definition 6.1. Fix a space X, a virtual vector bundle $V \to X$ of rank r_V, a vector bundle $W \to X$ of rank r_W, and a tangential structure ξ. The *family of Smith homomorphisms* associated to this data is the set of Smith homomorphisms

$$(6.2) \qquad \mathrm{sm}_W \colon \Omega_n^\xi(X^{V-r_V+k(W-r_W)}) \longrightarrow \Omega_{n-r_W}^\xi(X^{V-r_V+(k+1)(W-r_W)})$$

for $k \in \mathbb{Z}$, i.e. the Smith homomorphisms from $(X, V \oplus kW)$-twisted ξ-bordism to $(X, V \oplus (k+1)W)$-twisted ξ-bordism.

If there is some $\ell \in \mathbb{Z}$ and an identification of $(X, V \oplus kW)$-twisted ξ-structures with $(X, V \oplus (k+\ell)W)$-twisted tangential structures for all k that commutes with the Smith homomorphisms (6.2), we say this Smith family is *periodic* with period the smallest positive such ℓ.

This definition may seem too specific to be very applicable, but we will soon see many examples of periodic families.

The main new result in this section is Proposition 6.10, providing a way to calculate the periodicity of a family of Smith homomorphisms. We also review the theory of shearing in and around Lemma 6.18, which is a convenient way to split the Thom spectra for a wide class of twisted bordism theories, and is an essential step in identifying the terms in Smith long exact sequences. Our perspective on shearing follows [DY23a, §1], so see there for some more details; see also [FH21, Bea17, Ste22, DDHM23, BLM23] for additional approaches.

Definition 6.3. Let $\xi \colon B \to BO$ be a tangential structure. *Two-out-of-three data* for ξ is the data of:

- for each pair of ξ-structured virtual vector bundles $V, W \to X$, a natural ξ-structure on $V \oplus W$; and
- for each ξ-structured virtual vector bundle $V \to X$, a natural ξ-structure on $-V \to X$.

The reason for this name is that, given this data, a ξ-structure on any two of V, W, and $V \oplus W$ induces a ξ-structure on the third. Unfortunately, this is sometimes called a "two-out-of-three property."

Example 6.4. The tangential structures O, SO, Spin^c, Spin, String, U, SU, and Sp all have two-out-of-three data. $\mathrm{Pin}^\pm$ and Pin^c do not.

If M and N are manifolds, $T(M \times N) \cong p_1^*(TM) \oplus p_2^*(TN)$, where p_1 and p_2 are the projections of $M \times N$ onto M, resp. N, so two-out-of-three data induces a ring structure on Ω_*^ξ given by the direct product of manifolds. More abstractly, this data makes B into a grouplike E_∞-space and ξ into an E_∞-map, where BO has the direct sum E_∞-structure. This implies by work of Lewis [LMSM86, Theorem IX.7.1] (see also [May77, ABG18]) that $MT\xi$ is an E_∞-ring spectrum.

For R an E_∞-ring spectrum, May [May77, §III.2] defines a grouplike E_∞-space $\mathrm{GL}_1(R)$, and Ando-Blumberg-Gepner-Hopkins-Rezk [ABG⁺14a, ABG⁺14b] associate to a map $f \colon X \to B\mathrm{GL}_1(R)$, which we call a *twist* of R, a Thom spectrum $Mf \in \mathsf{Mod}_R$. The f-twisted R-homology groups of X are by definition the homotopy groups of Mf [ABG⁺14a, Definition 2.27]. Homotopy-equivalent twists induce equivalent Thom spectra. All of this generalizes our discussion around Definition 2.14, for which $R = \mathbb{S}$.

Example 6.5 (Vector bundle twists)**.** We have been using (rank-zero virtual) vector bundles to define twists of bordism theories, and these two notions of twist are compatible: rank-zero virtual vector bundles $V \to X$ are classified by maps $f_V \colon X \to BO$, and the J-homomorphism is a map $BO \to B\mathrm{GL}_1(\mathbb{S})$; then, if ξ is a tangential structure with two-out-of-three data, the unit map $e \colon \mathbb{S} \to MT\xi$ induces a map $e \colon B\mathrm{GL}_1(\mathbb{S}) \to B\mathrm{GL}_1(MT\xi)$. The Thom spectrum for (X, V)-twisted ξ-bordism, as we characterized it in Corollary 3.4, is naturally equivalent to the Thom spectrum $M(e \circ J \circ f_V)$ of the map

$$(6.6) \qquad\qquad X \xrightarrow{f_V} BO \xrightarrow{J} B\mathrm{GL}_1(\mathbb{S}) \xrightarrow{e} B\mathrm{GL}_1(MT\xi).$$

This is a combination of theorems of Lewis [LMSM86, Chapter IX] and Ando-Blumberg-Gepner-Hopkins-Rezk (see [ABG⁺14a, Corollary 3.24] and [ABG⁺14b, §1.2]).

Theorem 6.7 (Beardsley [Bea17, Theorem 1])**.** *There is a canonical null-homotopy of the map*

$$(6.8) \qquad\qquad e \circ J \circ \xi \colon B \to B\mathrm{GL}_1(MT\xi),$$

and therefore $e \circ J$ factors through the cofiber BO/B,[10] and in fact through $B\mathrm{GL}_1(\mathbb{S})/B$.

In other words, the homotopy type of the Thom spectrum for (X, V)-twisted ξ-bordism depends only on the image of V in BO/B. And the key slogan is that the orders of elements in $[X, BO/B]$ control the periodicity of families of Smith homomorphisms for twisted ξ-bordism; the group structure on $[X, BO/B]$ uses the fact that BO/B is the cofiber of a map of grouplike E_∞-spaces, hence is also a grouplike E_∞-space, so homotopy classes of maps into BO/B naturally form abelian groups.

---

[10]Beardsley's proof is more abstract, more general, and more powerful than this statement: see [DY23a, Lemma 1.13] for a simpler proof of just this part of Beardsley's theorem.

Definition 6.9 (Bhattacharya-Chatham [BC22, Definition 2.9]). The *$MT\xi$-orientation order* of a virtual vector bundle $V \to X$, written $\Theta(V, MT\xi)$, is the smallest positive integer k such that $V^{\oplus k}$ is $MT\xi$-oriented, or infinity if no such k exists.

Equivalently, $\Theta(V, MT\xi)$ is the order of the classifying map of $e \circ J \circ f_V \colon X \to B\mathrm{GL}_1(MT\xi)$, where $f_V \colon X \to BO$ is the classifying map of V. By Theorem 6.7, $e \circ J \circ f_V$ factors through $[X, BO/B]$, so $\Theta(V, MT\xi)$ divides the exponent of $[X, BO/B]$. We will use this fact below to make quick estimates of orientation orders.

Proposition 6.10. *Let $V \to X$ be a vector bundle. If $\epsilon := \Theta(V, MT\xi)$ is finite, the Smith homomorphism family of (X, kV)-twisted ξ-bordism is ϵ-periodic.*

This bound is not sharp, as we discuss in §6.4.

Proof. The image $\overline{f_V}$ of the classifying map $f_V \colon X \to BO$ in $[X, BO/B]$ satisfies $(k + \epsilon)\overline{f_V} = k\overline{f_V}$. Since the homotopy type of the Thom spectrum for (X, W)-twisted ξ-bordism only depends on $\overline{f_W} \in [X, BO/B]$, this implies that the notions of (X, kV)-twisted ξ-bordism and $(X, (k + \epsilon)V)$-twisted spin bordism coincide, so the Smith family $\{(X, kV) : k \in \mathbb{Z}\}$ is ϵ-periodic. $\square$

6.2. **Examples of periodic Smith families.** Though Proposition 6.10 seems abstract, it lends itself readily to examples.

Example 6.11 (Unoriented bordism families are 1-periodic). Proposition 6.10 implies that when $\xi = \mathrm{id} \colon BO \to BO$, the periodicity of a Smith family of (X, kV)-twisted unoriented bordism divides the exponent of $[X, BO/BO] = 0$. In other words, all Smith families of twisted unoriented bordism are 1-periodic.

We will see some examples of Smith families for unoriented bordism in Examples 7.4, 7.22, and 7.39.

Example 6.12 (Oriented bordism families are 2-periodic). Because BSO is the fiber of $w_1 \colon BO \to K(\mathbb{Z}/2, 1)$, and the Whitney sum formula implies w_1 is a map of E_∞-spaces, the cofiber BO/BSO is equivalent to $K(\mathbb{Z}/2, 1)$ as grouplike E_∞-spaces. Thus for all spaces X, $[X, BO/BSO]$ is annihilated by 2, so all Smith families for twisted oriented bordism are 2-periodic (or 1-periodic).

We will see some examples of Smith families for oriented bordism in Examples 7.6, 7.13, 7.22, 7.35, and 7.39.

Example 6.13 (Complex and spin^c bordism families are 2-periodic). If V is a real vector bundle, then $V \oplus V$ has a canonical complex structure (think of this bundle as $V \oplus iV$), and therefore also a canonical spin^c structure. Therefore for any map $f \colon X \to BO$, $2f$ lifts to BU and to $B\mathrm{Spin}^c$. Therefore the image of the map $[X, BO] \to [X, BO/BU]$ has exponent 2 (and likewise for Spin^c), so by Proposition 6.10 all Smith families of complex and spin^c bordism are at most 2-periodic.

For examples of Smith families for spin^c bordism, see Examples 7.11, 7.22, and 7.39 and Footnote 20.

Example 6.14 (Spin bordism families are 4-periodic). $BO/B\mathrm{Spin}$ is not equivalent to a product of Eilenberg-Mac Lane spaces even as an E_1-space [DY23a, Lemma 1.37], so we cannot reuse the strategy of (6.12). However, there is a cofiber sequence of grouplike E_∞-spaces (heuristically, an extension of abelian ∞-groups) [DY23a, §1.2.3]

$$K(\mathbb{Z}/2, 2) \longrightarrow B\mathrm{Spin}/BO \longrightarrow K(\mathbb{Z}/2, 1), \tag{6.15}$$

inducing a long exact sequence on $[X, –]$. Since $[X, K(\mathbb{Z}/2, 2)]$ and $[X, K(\mathbb{Z}/2, 1)]$ both have exponent at most 2 for any X, exactness implies $[X, BO/B\mathrm{Spin}]$ has exponent at most 4. Thus using Proposition 6.10 we conclude that all twisted spin bordism Smith families are at most 4-periodic; in fact, Example 7.8 has period exactly 4, which implies (6.15) does not split. One could also argue 4-periodicity similarly to Example 6.13.

If we restrict to oriented vector bundles, we can do better, as periodicity is controlled by maps into $BSO/B\mathrm{Spin} \simeq K(\mathbb{Z}/2, 2)$ (the argument is similar to $BO/BSO \simeq K(\mathbb{Z}/2, 1)$ from Example 6.12). Therefore we conclude that twisted spin Smith families using an oriented vector bundle are 2-periodic.

We will discuss several examples of 1-, 2-, and 4-periodic Smith families for spin bordism in Examples 7.8, 7.26, 7.33, 7.35, 7.39, and 7.45.

Remark 6.16. Periodicity for spin bordism also implies periodicity for *ko* and *KO*. Bhattacharya-Chatham [BC22, Main Theorem 1.5] generalize this periodicity of *KO*-orientability to higher real K-theories EO_Γ.

Example 6.17 (Families of twisted string structures). The space $B\mathrm{Spin}$ is 3-connected, with $\pi_4(B\mathrm{Spin}) \cong \mathbb{Z}$; if one kills this homotopy group by taking the 4-connected cover, one gets a space $B\mathrm{String}$, and the corresponding tangential structure is called a *string structure* [ST04, Definition 5.0.3] (see also [Gia71, §1]). The generator of $H^4(B\mathrm{Spin}; \mathbb{Z})$ is not the first Pontrjagin class p_1, but rather is a class λ with $2\lambda = p_1$ [Tho62, Theorem 1.2]. Thus a string structure on a spin vector bundle is equivalent data to a trivialization of λ.

As grouplike E_∞-spaces, $BO/B\mathrm{String}$ is an extension of $BO/B\mathrm{Spin}$ by $B\mathrm{Spin}/B\mathrm{String} \simeq K(\mathbb{Z}, 4)$ (see [DY23a, §1.2.4]); since $BO/B\mathrm{Spin}$ is itself an extension of $K(\mathbb{Z}/2, 1)$ by $K(\mathbb{Z}/2, 2)$, if X is a 3-connected space, $[X, BO/B\mathrm{String}] \cong H^4(X; \mathbb{Z})$. Thus for a general space X, Proposition 6.10 provides no information on Smith families for twisted string bordism: it reports that the period is at most infinity. We will nevertheless prove in Corollary 6.33 that all twisted string Smith families have finite period, though our proof does not provide an effective computation of the period.

In special cases, though, Proposition 6.10 allows us to provide sharper bounds: for example, because $H^*(B\mathbb{Z}/2; \mathbb{Z})$ is 2-torsion in positive degrees and $[B\mathbb{Z}/2, BO/B\mathrm{Spin}]$ has exponent 4, the long exact sequence associated to the cofiber sequence $K(\mathbb{Z}, 4) \to BO/B\mathrm{String} \to BO/B\mathrm{Spin}$ implies $[B\mathbb{Z}/2, BO/B\mathrm{String}]$ has exponent at most 8, implying that all Smith families of $(B\mathbb{Z}/2, V)$-twisted string bordism are at most 8-periodic; an 8-periodic example appears in Example 7.10.

6.3. **Examples of twisted bordism.** In this subsection, we discuss how to use the perspective we have been developing to concretely identify examples of twists of ξ-bordism for the tangential structures SO, Spin^c, and Spin.

Lemma 6.18 (Shearing [ABG$^+$14b, §1.2]). *If a twist $f \colon X \to B\mathrm{GL}_1(M\xi)$ factors through a map $g_V \colon X \to BO$ classifying a rank-zero virtual vector bundle $V \to X$ as in (6.6), then $Mf \simeq MT\xi \wedge X^V$.*

We will use this lemma as follows: first, for the four tangential structures $\xi \colon BG \to BO$ mentioned above, we compute the homotopy type of BO/BG and understand the map $BO \to BO/BG$, to recognize when a map $X \to BO/BG$ comes from a (virtual rank-zero) vector bundle $V \to X$. In that situation, Lemma 6.18 describes the corresponding twisted ξ-bordism groups as $\Omega_*^\xi(X^V)$, so we can use the Smith homomorphism tools we developed in this paper.

Example 6.19 (Twists of oriented bordism). Recall from Example 6.12 that $BO/BSO \simeq K(\mathbb{Z}/2, 1)$; the argument there implies the map $BO \to BO/BSO \xrightarrow{\simeq} K(\mathbb{Z}/2, 1)$ is the first Stiefel-Whitney class. Given a map $a\colon X \to BO/BSO$, the Thom spectrum of the corresponding twist $f_a\colon X \to B\mathrm{GL}_1(MT\mathrm{SO})$ of $MT\mathrm{SO}$ is the bordism spectrum whose homotopy groups are the bordism groups of manifolds M with a map $\phi\colon M \to X$ and a trivialization of $w_1(M) - \phi^*(a)$.[11]

Every class $a \in H^1(X; \mathbb{Z}/2)$ is the first Stiefel-Whitney class of some line bundle $L_a \to X$, so for any twist $f\colon X \to B\mathrm{GL}_1(MT\mathrm{SO})$ described by a map $f_a\colon X \xrightarrow{a} K(\mathbb{Z}/2, 1) \simeq BO/BSO \to B\mathrm{GL}_1(MT\mathrm{SO})$, there is a homotopy equivalence

$$(6.20) \qquad\qquad Mf \xrightarrow{\simeq} MT\mathrm{SO} \wedge X^{L_a - 1}.$$

For example, unoriented bordism is an example of such a twist: every manifold M has a canonical map to $K(\mathbb{Z}/2, 1)$, given by $w_1(M)$, and $w_1(M) - w_1(M)$ has a canonical trivialization. Therefore unoriented bordism is twisted oriented bordism for the twist $K(\mathbb{Z}/2, 1) \xrightarrow{\simeq} BO/BSO$, and Lemma 6.18 implies $MT\mathrm{O} \simeq MT\mathrm{SO} \wedge (K(\mathbb{Z}/2, 1))^{\sigma - 1}$, where $\sigma \to B\mathbb{Z}/2 \simeq K(\mathbb{Z}/2, 1)$ is the tautological line bundle; this is a theorem of Atiyah [Ati61a, Proposition 4.1].

For another example of how to use Lemma 6.18, let $\mathcal{W}$ denote the Thom spectrum for the notion of bordism of manifolds M equipped with a lift of $w_1(M)$ to a class $\alpha \in H^1(M; \mathbb{Z})$. The class α is equivalent to a map $\phi\colon M \to B\mathbb{Z} = S^1$, and $\alpha = \phi^* x$, where $x \in H^1(S^1; \mathbb{Z})$ is the generator; rephrased in this way, the condition that $\alpha \bmod 2 = w_1(M)$ is equivalent to a trivialization of $w_1(M) - \phi^*(x \bmod 2)$. Therefore $\mathcal{W}$-bordism is twisted oriented bordism for $(S^1, x \bmod 2)$, and as $x \bmod 2$ is w_1 of the Möbius bundle $\sigma \to S^1$, we learn from Lemma 6.18 that $\mathcal{W} \simeq MT\mathrm{SO} \wedge (S^1)^{\sigma - 1}$. This is also due to Atiyah [Ati61a, §4].

Example 6.21 (Twists of spin^c bordism). There is an equivalence of spaces, but not E_1-spaces, $BO/B\mathrm{Spin}^c \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3)$ [DY23a, Proposition 1.20, Lemma 1.30], and the map $BO \to BO/B\mathrm{Spin}^c$ is picked out by $(w_1, \beta(w_2))$, where β is the integral Bockstein. The fact that $\beta(w_2)$ is not linear in the direct sum of vector bundles is why this decomposition of $BO/B\mathrm{Spin}^c$ does not respect the E_1-structure.

Given data $a \in H^1(X; \mathbb{Z}/2)$ and $c \in H^3(X; \mathbb{Z})$, if $Mf_{a,c}$ is the Thom spectrum for the corresponding twist

$$(6.22) \qquad f_{a,c}\colon X \xrightarrow{(a,c)} K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \simeq BO/B\mathrm{Spin}^c \longrightarrow B\mathrm{GL}_1(MT\mathrm{Spin}^c),$$

then the homotopy groups of $Mf_{a,c}$ are the bordism groups of manifolds M with maps $\phi\colon M \to X$ and trivializations of $w_1(M) - \phi^*(a)$ and $\beta(w_2(M)) - \phi^*(c)$; the proof is essentially the same as Hebestreit-Joachim's [HJ20, Corollary 3.3.8] (Footnote 11 still applies: what appears is the stable normal bundle, but the characteristic classes are the same). If there is a (rank-zero, virtual) vector bundle $V \to X$ with $w_1(V) = a$ and $\beta(w_2(V)) = c$, then Lemma 6.18 implies $Mf_{a,c} \simeq MT\mathrm{Spin}^c \wedge X^V$ and we can invoke the Smith homomorphism on V.

For example, a pin^c structure on a manifold M is a trivialization of $\beta(w_2(M))$ (i.e. the spin^c condition without the trivialization of w_1). Thus a pin^c structure is equivalent to a twisted spin^c structure where $X = B\mathbb{Z}/2$, a is the generator of $H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$, and $c = 0$: as in Example 6.19, $w_1(M)$ gives us a canonical map to $B\mathbb{Z}/2$, there is a canonical trivialization of

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[11]Strictly speaking, what one trivializes is $w_1(\nu) - \phi^*(a)$, where $\nu \to M$ is the stable normal bundle, but there is a canonical identification of $w_1(M)$ and $w_1(\nu)$. This nuance will matter for spin structures.

$w_1(M) - w_1(M)$, and $c = 0$ means this twisted spin^c condition does not modify $\beta(w_2)$. So this twisted spin^c condition is that $\beta(w_2) = 0$ and w_1 is arbitrary, i.e. a pin^c structure. And if $\sigma \to B\mathbb{Z}/2$ is the tautological line bundle, $w_1(\sigma) = a$ and $\beta(w_2(\sigma)) = 0 = c$, so Lemma 6.18 implies $MT\text{Pin}^c \simeq MT\text{Spin}^c \wedge (B\mathbb{Z}/2)^{\sigma-1}$, reproving a theorem of Bahri-Gilkey [BG87a, BG87b].

Other examples of twists of spin^c bordism which can be realized by vector bundles include the spin-U(2) bordism of Davighi-Lohitsiri [DL20, DL21] and the tangential structure corresponding to Stehouwer's alternate class AI fermionic groups [Ste22, §2.2].

Not every choice of (a, c) can be realized by a vector bundle; for example, $\beta(w_2)$ is always 2-torsion, but c need not be. There are also examples with 2-torsion c, as a consequence of work of Gunawardena-Kahn-Thomas [GKT89, §2].

Example 6.23 (Twists of spin bordism)**.** The most commonly studied examples of twisted ξ-bordism in mathematical physics are twists of spin bordism. The story is closely analogous to Example 6.21, with $K(\mathbb{Z}, 3)$ replaced with $K(\mathbb{Z}/2, 2)$, and the map $BO \to BO/B\text{Spin} \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$ is (w_1, w_2). Given classes $a \in H^1(X; \mathbb{Z}/2)$ and $b \in H^2(X; \mathbb{Z}/2)$, the homotopy groups of the Thom spectrum of the corresponding twist $f_{a,b} \colon X \to B\text{GL}_1(MT\text{Spin})$ are the bordism groups of manifolds M with maps $\phi \colon M \to X$ and trivializations of $w_1(\nu) - \phi^*(a)$ and $w_2(\nu) - \phi^*(b)$ [HJ20, Corollary 3.3.8], where $\nu \to M$ is the stable normal bundle. Now, unlike in Footnote 11, the distinction between TM and ν matters: $w_1(TM) = w_1(\nu)$, but $w_2(TM) + w_1(TM)^2 = w_2(\nu)$, providing a formula for the nontrivial transition from tangential to normal data. If $a = w_1(V)$ and $b = w_2(V)$ for a rank-zero virtual vector bundle $V \to X$, Lemma 6.18 implies $Mf_{a,b} \simeq MT\text{Spin} \wedge X^V$. See [DY23a, §1.2.3] for more information.

Many commonly studied tangential structures arise as vector bundle twists of spin structures.

(1) A pin^- structure is a trivialization of $w_2(M) + w_1(M)^2$, with no condition on w_1. Thus this is equivalent to a trivialization of $w_2(\nu)$. Like in Examples 6.19 and 6.21, we can ask for a map $\phi \colon M \to B\mathbb{Z}/2$ and a trivialization of $w_1(\nu) - \phi^*(a)$, where $a \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ is the generator, and this is no data at all; then we also want to impose $w_2(\nu) = 0$. So pin^- bordism is the Thom spectrum of the twist $f_{a,0} \colon B\mathbb{Z}/2 \to B\text{GL}_1(MT\text{Spin})$. The classes a and 0 are w_1 and w_2 of $\sigma \to B\mathbb{Z}/2$, so we learn that $MT\text{Pin}^- \simeq MT\text{Spin} \wedge (B\mathbb{Z}/2)^{\sigma-1}$, a splitting first written down by Peterson [Pet68, §7].

(2) A pin^+ structure is a trivialization of $w_2(M)$, with no condition on w_1. Switching to the stable normal bundle, we want a trivialization of $w_2(\nu) + w_1(\nu)^2$. Just as for pin^- structures, pick a map $\phi \colon M \to B\mathbb{Z}/2$ and ask for a trivialization of $w_1(\nu) - \phi^*(a)$, which is no data; then we want to trivialize $w_2(\nu) + \phi^*(a^2)$. Thus pin^+ bordism is the Thom spectrum of the twist $f_{a,a^2} \colon B\mathbb{Z}/2 \to B\text{GL}_1(MT\text{Spin})$. The classes a and a^2 are w_1, resp. w_2 of the virtual vector bundle $-\sigma$, so Lemma 6.18 tells us $MT\text{Pin}^+ \simeq MT\text{Spin} \wedge (B\mathbb{Z}/2)^{1-\sigma}$, a result of Stolz [Sto88, §8].[12]

(3) A spin^c structure is data of a trivialization of $w_1(TM)$ and a class $c_1 \in H^2(M; \mathbb{Z})$ such that $c_1 \bmod 2 = w_2(TM)$; in this case there is no difference between $w_2(TM)$ and $w_2(\nu)$. This is a twisted spin structure where $X = B\text{U}(1) = K(\mathbb{Z}, 2)$, $a = 0$, and b is the

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[12]As $[B\mathbb{Z}/2, BO/B\text{Spin}]$ has exponent 4 by Example 6.14, $[1 - \sigma] = [3\sigma - 3]$, so the reader who prefers to avoid virtual vector bundles can write $MT\text{Pin}^+ \simeq MT\text{Spin} \wedge (B\mathbb{Z}/2)^{3\sigma-3}$.

generator of $H^2(K(\mathbb{Z},2);\mathbb{Z}/2) \cong \mathbb{Z}/2$. As 0, resp. b are the first and second Stiefel-Whitney classes of the tautological complex line bundle $L \to B\mathrm{U}(1)$, Lemma 6.18 implies $MT\mathrm{Spin}^c \simeq MT\mathrm{Spin} \wedge (B\mathrm{U}(1))^{L-2}$, which is known due to Bahri-Gilkey [BG87a, BG87b].

(4) A spin-$\mathbb{Z}/2k$ structure on a manifold M is data of a principal $\mathbb{Z}/k$-bundle $P \to M$ together with trivializations of $w_1(M)$ and $w_2(M) - w_2(V_P)$, where V is the standard one-dimensional complex representation of $\mathbb{Z}/k$ as rotations and $V_P \to M$ is the associated complex line bundle to P. Thus, analogous to the spin^c argument above, this structure is a twisted spin structure for $X = B\mathbb{Z}/k$, $a = 0$, and $b = w_2(V)$, and Lemma 6.18 implies $MT(\mathrm{Spin}\text{-}\mathbb{Z}/2k) \simeq MT\mathrm{Spin} \wedge (B\mathbb{Z}/k)^{V-2}$, reproving a theorem of Campbell [Cam17, §7.9].

(5) A spin^h structure is data of a trivialization of $w_1(M)$ and a rank-3 oriented vector bundle $E \to M$ and a trivialization of $w_2(M) - w_2(E)$. Again tangential vs. normal does not matter here, and one can use the same line of reasoning to show that spin^h structures are twisted spin structures for $X = B\mathrm{SO}_3$, $a = 0$, and $b = w_2$. As these are w_1, resp. w_2 of the tautological vector bundle $V \to B\mathrm{SO}_3$, Lemma 6.18 tells us $MT\mathrm{Spin}^h \simeq MT\mathrm{Spin} \wedge (B\mathrm{SO}_3)^{V-3}$, which is due to Freed-Hopkins [FH21, §10].

There are many more examples of vector bundle twists of spin bordism, including the examples in, e.g., [FH21, Guo18, DL20, GOP+20, WW20a, DL21, Ste22, DDHM23]. But one can find twists of spin bordism not described by vector bundle twists, even in physically motivated examples: see [DY22, Theorem 4.2] for an example where $X = B\mathrm{SU}_8/\{\pm 1\}$, with a few more examples given in [DY23a, §3.1]. The Smith-theoretic techniques in our paper do not apply in those situations.

Example 6.24 (James periodicity as Smith periodicity). James periodicity [Jam59] is a classical result in homotopy theory that the homotopy types of the *stunted projective spaces* $\mathbb{RP}^n_k :=$ $\mathbb{RP}^n/\mathbb{RP}^k$ (here $k < n$) are periodic, with periodicity dependent on n and k. There are also results for the analogously defined stunted complex and quaternionic projective spaces $\mathbb{CP}^n_k := \mathbb{CP}^n/\mathbb{CP}^k$ and $\mathbb{HP}^n_k := \mathbb{HP}^n/\mathbb{HP}^k$. These periodicities can be thought of in terms of periodic Smith families for framed bordism—or conversely, the periodicities in the previous several examples can be thought of as generalizations of James periodicity over other ring spectra than $\mathbb{S}$.

Proposition 6.10 is the engine behind our periodicity results; its key idea is that vector bundles inducing equivalent maps to $B\mathrm{GL}_1(R)$ have equivalent R-module Thom spectra. For framed bordism, where $R = \mathbb{S}$, we therefore should look at the image of the homomorphism $[X, B\mathrm{O}] \to [X, B\mathrm{GL}_1(\mathbb{S})]$; following Atiyah [Ati61b, §1], this image is typically denoted $J(X)$. Atiyah (*ibid.*, Lemma 2.5) proves that if $V, W \to X$ have equal images in $J(X)$, then $X^V \simeq X^W$.[13] Therefore we can obtain framed bordism Smith periodicities, or equivalences of Thom spectra, by calculating the groups $J(X)$. Atiyah (*ibid.*, Proposition 1.5) shows that when X is a finite CW complex, $J(X)$ is a finite group, implying the existence of many framed Smith families.

For James periodicity specifically, choose $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Stunted projective spaces are Thom spectra: if $L \to F\mathbb{P}^k$ denotes the tautological (real, complex, or quaternionic) line bundle, there is an equivalence $\Sigma^\infty F\mathbb{P}^n_k \simeq (F\mathbb{P}^k)^{(n-k)L}$ [Ati61b, Proposition 4.3], reducing the proof of James periodicity to the computation of the order of L in $J(F\mathbb{P}^k)$. For example, for $F = \mathbb{R}$ Adams calculates the order of L in $J(\mathbb{RP}^k)$ in [Ada62, Theorem 7.4] and [Ada65a, Example 6.3] to be $2^{\phi(k)}$, where $\phi(k)$ is the number of integers s with $0 < s \le k$ and $s \equiv 0, 1, 2,$ or $4 \bmod 8$. Therefore

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[13]See Held-Sjerve [HS73, Theorem 1.2] for a partial converse to this result.

for all k and n, there is a homotopy equivalence

$$(6.25) \qquad \Sigma^\infty \mathbb{RP}_k^{n+2^{\phi(k)}} \xrightarrow{\simeq} \Sigma^\infty \Sigma^{2^{\phi(k)}} \mathbb{RP}_k^n.$$

(and in fact this is true even before applying Σ^∞ [Mah65]). Additional computations in J-groups of $F\mathbb{P}^k$ are done by Adams-Walker [AW65], Lam [Lam72], Federer-Gitler [FG73, FG77], Sigrist [Sig75], Walker [Wal81], Crabb-Knapp [CK88], Dibağ [Dib99, Dib03], Obiedat [Obi01], and Randal-Williams [RW23, §5.3].

Remark 6.26. There are many other tangential structures ξ that one can study twists of. See [SSS09, Sat10, Sat11a, Sat11b, Sat12, SSS12, Sat15, SW15, SW18, LSW20, SY21, BC22, DY23a] for more examples.

6.4. **The bound in Proposition 6.10 is not sharp.** In this subsection, we discuss ways in which Proposition 6.10 loses information, yielding Smith families with lower-than-expected periodicity. This in particular occurs for twisted string structures.

Proposition 6.10 estimates the periodicity of a Smith family for a tangential structure ξ and vector bundle $V \to X$ in terms of the minimal positive integer k such that $V^{\oplus k}$ has a ξ-structure. As we have seen above in Examples 6.11, 6.12, 6.13, and 6.14, k is finite in many examples of interest, including O-, SO-, Spin^c-, Spin-, and U-structures. However, k is not always finite.

Lemma 6.27.

(1) Let $V \to X$ be a real vector bundle whose rational first Pontrjagin class $p_1(V) \in H^4(X; \mathbb{Q})$ is nonzero. Then for $k \neq 0$, $V^{\oplus k}$ does not admit a string structure.

(2) Let $V \to X$ be a complex vector bundle whose rational first Chern class $c_1(V) \in H^2(X; \mathbb{Q})$ is nonzero. Then for $k \neq 0$, $V^{\oplus k}$ does not admit an SU-structure.

Proof. For part (1), if $E \to M$ is a string vector bundle, then $\lambda(E) = 0$ implies $p_1(E) = 0 \in H^4(M; \mathbb{Z})$ (since $2\lambda = p_1$), which implies the image of $p_1(E)$ in $H^4(M; \mathbb{Q})$ is also 0, so it suffices to show $p_1(V^{\oplus k})$ has nonzero image in $H^q(X; \mathbb{Q})$ for $k \neq 0$. Since $p_1(V)$ is nonzero in this group, it is in particular nontorsion, and the Whitney sum formula implies that in $\mathbb{Q}$-cohomology $p_1(V^{\oplus k}) = kp_1(V)$, so it is also nonzero. Part (2) is analogous, using that the complete obstruction for lifting from a U-structure to an SU-structure is $c_1 \in H^2(B\mathrm{U}; \mathbb{Z})$. $\qquad \square$

There are many bundles satisfying the hypotheses of Lemma 6.27: for example, both are true for the tautological complex line bundle over $B\mathrm{U}(1)$.

Definition 6.28 (Lashof [Las63, §3]). Let $B\mathrm{U}\langle 6\rangle$ denote the 5-connected cover of $B\mathrm{U}$, and let $\xi\langle 6\rangle \colon B\mathrm{U}\langle 6\rangle \to B\mathrm{O}$ be the composition of the covering map $B\mathrm{U}\langle 6\rangle \to B\mathrm{U}$ and the map $B\mathrm{U} \to B\mathrm{O}$ forgetting the complex structure. We will refer to $\xi\langle 6\rangle$-structures as $U\langle 6\rangle$-*structures*.

A $U\langle 6\rangle$-structure induces both an SU-structure and a string structure; the former because the 5-connected covering map always factors through the 3-connected cover, which for $B\mathrm{U}$ is $B\mathrm{SU} \to B\mathrm{U}$, and the latter because the map $B\mathrm{U}\langle 6\rangle \to B\mathrm{O}$ must factor through the 5-connected cover of $B\mathrm{O}$, which is $B\mathrm{String}$.

One can construct two-out-of-three data for $U\langle 6\rangle$-structures using the 5-connected covers of the maps in the two-out-of-three data for $B\mathrm{U}$. Since this data is constructed in this universal way, it is compatible with the two-out-of-three data we have already used for U, SU, String, etc.

Proposition 6.29 (Bauer [Bau03, Lemma 2.1])**.** *Let ξ be a tangential structure with two-out-of-three data admitting a map $\xi\langle 6\rangle \to \xi$ compatible with two-out-of-three data. If $L \to B\mathrm{U}(1)$ denotes the tautological complex line bundle, then there is a homotopy equivalence*

$$(6.30) \qquad\qquad MT\xi \wedge (B\mathrm{U}(1))_+ \xrightarrow{\simeq} MT\xi \wedge (B\mathrm{U}(1))^{24L^* - 48}.$$

Bauer's statement in [Bau03, Lemma 2.1] is only a corollary of this, obtained by base-changing from $MU\langle 6\rangle$ to tmf, but in his proof he proves the version we provide here. One can also pass from L^* to L by pulling back along the complex conjugation map $B\mathrm{U}(1) \to B\mathrm{U}(1)$.

Corollary 6.31. *With ξ as in Proposition 6.29, in particular including string and SU-structures, every Smith family for ξ-structures and a complex line bundle is at most 24-periodic. In particular, Proposition 6.10 is not sharp.*

What went wrong? In this specific case, we didn't use all of the available information in Proposition 6.10. Looking into Bauer's proof, one learns that the map $[B\mathrm{U}(1), B\mathrm{O}/B\mathrm{String}] \to [B\mathrm{U}(1), B\mathrm{GL}_1(MT\mathrm{String})]$ from Theorem 6.7 sends the class of L^*, which is *infinite-order* in $[B\mathrm{U}(1), B\mathrm{O}/B\mathrm{String}]$, to a *finite-order* class in $[B\mathrm{U}(1), B\mathrm{GL}_1(MT\mathrm{String})]$. Specifically, because this twist came from a vector bundle, it also factors through $[B\mathrm{U}(1), B\mathrm{GL}_1(\mathbb{S})/B\mathrm{String}]$ (see Theorem 6.7), and the image of L^* in this group has finite order.

Proposition 6.32. *For any space X homotopy equivalent to a CW complex with finitely many cells in each degree, let $J_*\colon [X, B\mathrm{O}] \to [X, B\mathrm{GL}_1(\mathbb{S})/B\mathrm{String}]$ denote the map induced by the J-homomorphism $B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S})$ followed by taking the cofiber of $B\mathrm{String} \to B\mathrm{O} \xrightarrow{J} B\mathrm{GL}_1(\mathbb{S})$. Then all classes in $\mathrm{Im}(J_*)$ have finite order.*

Corollary 6.33. *Every twisted string structure Smith family over a base space X as in Proposition 6.32 has finite periodicity.*

Proof of Proposition 6.32. Because $B\mathrm{O}$ and $B\mathrm{GL}_1(\mathbb{S})/B\mathrm{String}$ are grouplike E_∞-spaces,[14] one can prove the proposition by lifting the map $B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S})/B\mathrm{String}$ to the equivalent data of a map of spectra $j_{/bstring}\colon bo_0 \to bg\ell_1(\mathbb{S})/bstring$.[15]

The map $j_{/bstring}$ was induced from a map of grouplike E_∞-spaces that factored through $B\mathrm{GL}_1(\mathbb{S})$, and therefore $j_{/bstring}$ factors through $bg\ell_1(\mathbb{S})$. The homotopy groups of this spectrum are torsion, which follows from May's definition [May77, §III.2] of $\mathrm{GL}_1(\mathbb{S})$ and the triviality of the positive-degree rational stable homotopy groups of the sphere [Ser53]. Thus the rationalization $bg\ell_1(\mathbb{S}) \wedge H\mathbb{Q} \simeq 0$. For any class $x \in (bo_0)^0(X)$, if $J_*(x)$ has infinite order, its image in the rationalized $bg\ell_1(\mathbb{S})/bstring$-cohomology of X must be nonzero: because X has finitely many cells in each dimension, its generalized cohomology groups for any finite-type spectrum (including all spectra appearing in this proof) are finitely generated, so infinite-order elements persist through rationalization. Rationally, though, J_* passes through the zero spectrum. $\qquad\square$

Remark 6.34. One way to interpret this phenomenon is that, even though the first Pontrjagin class $p_1\colon B\mathrm{O} \to K(\mathbb{Z}, 4)$ descends to a map $p_1\colon B\mathrm{O}/B\mathrm{String} \to K(\mathbb{Z}, 4)$, this map does not extend to a

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[14]It is not immediately obvious that $B\mathrm{GL}_1(\mathbb{S})/B\mathrm{String}$ is a grouplike E_∞-space, but one can prove it by modifying the argument in [DY23a, Proof of Proposition 1.20].

[15]The name bo_0 instead of bo is because it is traditional to use bo to refer to the spectrum corresponding to the grouplike E_∞-space $\mathbb{Z} \times B\mathrm{O}$, i.e. the spectrum ko.

rationally nontrivial map out of $B\mathrm{GL}_1(\mathbb{S})/B\mathrm{String}$. In the language of [DY23a], "a fake vector bundle with respect to twisted string structures has a first Pontrjagin class, but a fake spherical fibration does not."

Remark 6.35 ($\mathrm{O}\langle n\rangle$-families' periodicity and Bernoulli numbers?). Proposition 6.32 and Corollary 6.33 generalize *mutatis mutandis* to tangential structures further up the Whitehead tower of $B\mathrm{O}$. To wit, given a natural number n, let $\xi\colon B\mathrm{O}\langle n\rangle \to B\mathrm{O}$ be the $(n-1)$-connected covering map. This defines a tangential structure commonly called a $\mathrm{O}\langle n\rangle$-*structure*, and the two-out-of-three data for $B\mathrm{O}$ pull back by the universal property of the $(n-1)$-connected cover to define two-out-of-three data for $\mathrm{O}\langle n\rangle$-structures. (Compare Definition 6.28.) Thus Definition 6.1, Example 6.5, , and Theorem 6.7 define vector bundle twists, non-vector-bundle twists, and Smith families for twisted $\mathrm{O}\langle n\rangle$-structures just like for string structures.

For $n > 4$, the homotopy groups of $B\mathrm{O}/B\mathrm{O}\langle n\rangle$ are not all torsion, which is downstream from the isomorphism $\pi_4(B\mathrm{O}) \cong \mathbb{Z}$ [Ste51, §24, §25]. Therefore, like for $B\mathrm{O}/B\mathrm{String}$, the order of a twist in $[X, B\mathrm{O}/B\mathrm{O}\langle n\rangle]$ is not a particularly good estimate for the value of the corresponding Smith family's period. Indeed, Proposition 6.32 and Corollary 6.33 and their proofs generalize directly from $B\mathrm{String}$ to $B\mathrm{O}\langle n\rangle$, showing all Smith families of twisted $\mathrm{O}\langle n\rangle$-structures have finite order, provided they are over spaces homotopy equivalent to CW complexes with finitely many cells in each dimension. One can also generalize this whole story to the limiting case as $n \to \infty$, which is the tangential structure $E\mathrm{O} \to B\mathrm{O}$, i.e. a stable framing.

Our proof did not give any estimates on the orders of these Smith families, just finiteness. It would be interesting to bound or compute these orders; for example, one could investigate the map of Atiyah-Hirzebruch spectral sequences induced by $j_{/bo\langle n\rangle}$, the generalization of $j_{/bstring}$ in the proof of Proposition 6.32, or generalize Bauer's proof in [Bau03, Lemma 2.1]. We suspect that sharp estimates for periodicity for Smith families of twisted $\mathrm{O}\langle n\rangle$-structures will have formulas involving Bernoulli numbers, because of their appearance in Adams' seminal work computing the image of the J-homomorphism in $\pi_*(\mathbb{S})$ [Ada63, Ada65a, Ada65b, Ada66]. We would be interested in learning whether this is the case.

For $n \leq 16$, twisted $\mathrm{O}\langle n\rangle$-structures as defined here recover familiar twists of familiar tangential structures.

(0) $\mathrm{O}\langle 0\rangle$- and $\mathrm{O}\langle 1\rangle$-structures are canonically equivalent to O-structures, i.e. no data, and so this story is vacuous.

(2) $\mathrm{O}\langle 2\rangle$-structures are equivalent to SO-structures, and this story recovers the twists in Example 6.19.

(3) $\pi_3(B\mathrm{O}) = 0$ [Car36, §IV], and $B\mathrm{O}\langle 3\rangle$ and $B\mathrm{O}\langle 4\rangle$ are equivalent to $B\mathrm{Spin}$. This story recovers the notion of twisted spin structure we discussed in Example 6.23.

(5) $\pi_k(B\mathrm{O})$ vanishes for $k = 5$ [Ste51, Remarks 24.11], $k = 6$ [Eck51, 3.72], and $k = 7$ [BS53, Proposition 19.5], so $B\mathrm{O}\langle 5\rangle$, $B\mathrm{O}\langle 6\rangle$, $B\mathrm{O}\langle 7\rangle$, and $B\mathrm{O}\langle 8\rangle$ all coincide, and are $B\mathrm{String}$. This story recovers the standard story of twists of string bordism that we mentioned in Example 6.17.

(9) Sati-Schreiber-Stasheff [SSS09, Definition 1] call an $\mathrm{O}\langle 9\rangle$-structure a *fivebrane structure*, and in a sequel paper [SSS12, §2.3], they introduce twisted fivebrane structures over a space X given by data of a map $X \to K(\mathbb{Z}, 8)$. It is possible to show that their definition is a special case of ours. Specifically, similarly to the identification

$K(\mathbb{Z}, 4) \simeq B\mathrm{Spin}/B\mathrm{String}$ producing a map from the $K(\mathbb{Z}, 4)$-twists of string bordism to the more general group of $BO/B\mathrm{String}$ twists of string bordism [DY23a, (1.45)], the characteristic class $\frac{1}{6}p_2 \colon B\mathrm{String} \to K(\mathbb{Z}, 8)$ induces an equivalence of grouplike E_∞-spaces $B\mathrm{String}/BO\langle 9\rangle \to K(\mathbb{Z}, 8)$, and this equivalence leads to a map $K(\mathbb{Z}, 8) \simeq B\mathrm{String}/BO\langle 9\rangle \to BO/BO\langle 9\rangle$ carrying Sati-Schreiber-Stasheff's twisted fivebrane structures to a subgroup of the fake vector bundle twists of fivebrane structure.

(10) Sati [Sat15, Definition 2.4] calls $O\langle 10\rangle$-structures "2-orientations." and studies their twists in (*ibid.*, Definition 5.1). Like for twisted fivebrane structures, Sati's twists are classified by maps to $K(\mathbb{Z}/2, 9) \simeq BO\langle 10\rangle/BO\langle 9\rangle$, and map to the twists we considered via the map $BO\langle 10\rangle/BO\langle 9\rangle \to BO/BO\langle 9\rangle$.

(13) As $\pi_{11}(BO) = 0$ [Bot59], $O\langle 11\rangle$- and $O\langle 12\rangle$-structures coincide. Sati refers to this as a "2-spin structure" [Sat15, Definition 2.5], and in (*ibid.*, Definition 5.2) introduces twisted 2-spin structures corresponding to $K(\mathbb{Z}/2, 10) \simeq BO\langle 11\rangle/BO\langle 10\rangle \to BO/BO\langle 10\rangle$.

(14) As $\pi_k(BO) = 0$ for $k = 13$, 14, and 15 [Bot59], $BO\langle 13\rangle = \cdots = BO\langle 16\rangle$. Sati [Sat15, Definition 3.1] names this tangential structure a *ninebrane structure*, and produces twisted ninebrane structures classified by the fractional Pontrjagin class $(1/240)p_3 \colon BO\langle 12\rangle \to K(\mathbb{Z}, 12)$ (*ibid.*, Definition 5.3). As in the previous cases, the map $K(\mathbb{Z}, 12) \simeq BO\langle 13\rangle/BO\langle 11\rangle \to BO/BO\langle 11\rangle$ sends Sati's twists to ours.

Passing to the infinite limit, we obtain an interpretation of maps to BO/EO, i.e. to BO, as classifying vector bundle twists of framed bordism. These twists of framed bordism are studied in [Cru03, FSS24].

7. Examples of Smith fiber sequences

In this section, we implement the discussion from the previous section for some commonly studied vector bundles. We find many previously studied Smith homomorphisms, and also identify a few other well-known cofiber sequences, including Wood's sequences, Wall's sequence, and the cofiber sequences associated to the Hopf maps and to transfer maps, as Smith homomorphisms (Examples 7.13 and 7.16). We include this Pokédex of examples in part to illustrate what kinds of Smith cofiber sequences are out there; in part to make contact with preexisting literature; and in part to illustrate how to put theorems such as Theorem 5.1 into practice to explicitly write down Smith cofiber sequences.

7.1. **Twisting by real line bundles.** Our first family of examples use the tautological line bundle $\sigma \to B\mathbb{Z}/2$; its sphere bundle is the tautological $\mathbb{Z}/2$-bundle $E\mathbb{Z}/2 \to B\mathbb{Z}/2$, whose total space is contractible. Therefore by Theorem 5.1, for any $k \in \mathbb{Z}$, we have a cofiber sequence

$$(7.1) \qquad\qquad \mathbb{S} \longrightarrow (B\mathbb{Z}/2)^{k(\sigma-1)} \xrightarrow{\mathrm{sm}_\sigma} \Sigma(B\mathbb{Z}/2)^{(k+1)(\sigma-1)},$$

where sm_σ is the Smith homomorphism associated to σ. When $k = 0$, this is especially nice: the middle spectrum is $\Sigma^\infty_+ B\mathbb{Z}/2 \simeq \mathbb{S} \vee \Sigma B\mathbb{Z}/2$ and the map $\mathbb{S} \to \mathbb{S} \vee \Sigma B\mathbb{Z}/2$ is the inclusion of the first factor of the wedge sum, leading to a Smith *isomorphism* $\mathrm{sm}_\sigma \colon \Sigma^\infty B\mathbb{Z}/2 \xrightarrow{\simeq} (B\mathbb{Z}/2)^\sigma$. This equivalence is well-known; see Kochman [Koc96, Lemma 2.6.5] for a proof.

Remark 7.2. The Thom spectrum $(B\mathbb{Z}/2)^{k\sigma}$ is often denoted in the homotopy theory literature by $\mathbb{RP}^\infty_k$, so that $(B\mathbb{Z}/2)^{k(\sigma-1)}$ can be identified with its desuspension $\Sigma^{-k}\mathbb{RP}^\infty_k$. One justification

for this notation stems from (7.1): suspending it k times gives a cofiber sequence

$$(7.3) \qquad\qquad \mathbb{S}^k \longrightarrow \mathbb{RP}_k^\infty \longrightarrow \mathbb{RP}_{k+1}^\infty,$$

which exhibits $\mathbb{RP}_{k+1}^\infty$ as the spectrum obtained by crushing the bottom cell of $\mathbb{RP}_k^\infty$.

Example 7.4. Smash (7.1) with $MT\mathrm{O}$. As every virtual bundle has a unique $MT\mathrm{O}$-orientation, this cofiber sequence simplifies to

$$(7.5) \qquad\qquad MT\mathrm{O} \longrightarrow MT\mathrm{O} \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{sm}_\sigma} MT\mathrm{O} \wedge \Sigma(B\mathbb{Z}/2)_+.$$

This was the first Smith homomorphism studied; it was defined and named the Smith homomorphism by Conner-Floyd [CF64, Theorem 26.1]. Thom's celebrated calculation of Ω_*^O implies that $MT\mathrm{O}$ is a sum of shifts of $H\mathbb{Z}/2$; on each of these copies, the Smith map (7.5) is the cap product with the nonzero element of $H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$.

Stong [Sto69, Proposition 5] and Uchida [Uch70] study related examples, where one smashes (7.5) with spaces X; they identify the fiber $MT\mathrm{O} \wedge X$ and show that the long exact sequence of homotopy groups splits. Their papers are among the earliest examples identifying the Smith long exact sequence.[16]

Example 7.6. Smash (7.1) with $MT\mathrm{SO}$. Since σ is not orientable, but 2σ is oriented (see Example 6.12), we obtain a 2-periodic series of codimension-1 Smith homomorphisms between the oriented bordism of $B\mathbb{Z}/2$ and $(B\mathbb{Z}/2, \sigma)$-twisted oriented bordism. The latter can be identified with unoriented bordism: a $(B\mathbb{Z}/2, \sigma)$-twisted orientation on V is data of a line bundle on L and an orientation of $V \oplus L$, which is no data at all: this identifies $L \cong \mathrm{Det}(V)^* \cong \mathrm{Det}(V)$ up to a contractible space of choices, and $V \oplus \mathrm{Det}(V)$ is canonically oriented. So every vector bundle has a canonical $(B\mathbb{Z}/2, \sigma)$-twisted orientation.

Therefore by Theorem 5.1 we obtain a 2-periodic sequence of codimension-1 Smith homomorphisms:

$$(7.7a) \qquad\qquad MT\mathrm{SO} \longrightarrow MT\mathrm{SO} \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{O}$$

$$(7.7b) \qquad\qquad MT\mathrm{SO} \longrightarrow MT\mathrm{O} \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{SO} \wedge (B\mathbb{Z}/2)_+.$$

These maps are obtained by taking smooth representatives of Poincaré duals of w_1 either of the manifold (when the domain is Ω_*^O) or of the principal $\mathbb{Z}/2$-bundle (when the domain is $\Omega_*^\mathrm{SO}(B\mathbb{Z}/2)$). See [DDK$^+$24, §IV.B] for the physical interpretation of the corresponding long exact sequence of Anderson dual groups.

These Smith homomorphisms were first introduced by Komiya [Kom72, §5]; see also Shibata [Shi73, Proposition 2.1]. See Córdova-Ohmori-Shao-Yan [COSY20, Appendix A], Hason-Komargodski-Thorngren [HKT20, §4.4], and Fidkowski-Haah-Hastings [FHH20] for applications of these Smith homomorphisms to physics. The splitting of the $k = 0$ case of (7.1) implies a homotopy equivalence $MT\mathrm{SO} \wedge B\mathbb{Z}/2 \xrightarrow{\cong} \Sigma MT\mathrm{O}$, a theorem of Atiyah [Ati61a, Proposition 4.1].

Example 7.8. Some of the coolest examples of this kind come about by smashing (7.1) with $MT\mathrm{Spin}$. As we discussed in Example 6.14, the periodicity of this family is 1, 2, or 4; a Whitney

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[16]At the time, it was common to think of $\Omega_*^\mathrm{O}(B\mathbb{Z}/2)$ as the bordism groups of manifolds M equipped with a free involution τ, rather than manifolds with a principal $\mathbb{Z}/2$-bundle; Stong and Uchida's results are phrased in that language. To pass between these perspectives, rewrite (M, τ) as the principal $\mathbb{Z}/2$-bundle $M \to M/\tau$; in the other direction, take the deck transformation involution of the total space of a principal $\mathbb{Z}/2$-bundle.

sum formula calculation shows that $k\sigma$ is spin iff k is a multiple of 4, and therefore this Smith family is 4-periodic. The corresponding $(B\mathbb{Z}/2, k\sigma)$-twisted spin bordism groups can be identified with H-bordism for certain Lie groups H, as discussed in Example 6.23; specifically,

(1) a $(B\mathbb{Z}/2, \sigma)$-twisted spin structure is equivalent to a pin$^-$ structure;
(2) a $(B\mathbb{Z}/2, 2\sigma)$-twisted spin structure is equivalent to an H structure, where $H = \mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4$; and
(3) a $(B\mathbb{Z}/2, 3\sigma)$-twisted spin structure is equivalent to a pin$^+$ structure.

Using Theorem 5.1 once again, the 4-periodic sequence of codimension-1 Smith homomorphisms takes the form

$$(7.9a) \qquad MT\mathrm{Spin} \longrightarrow MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{Pin}^-$$

$$(7.9b) \qquad MT\mathrm{Spin} \longrightarrow MT\mathrm{Pin}^- \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4)$$

$$(7.9c) \qquad MT\mathrm{Spin} \longrightarrow MT(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4) \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{Pin}^+$$

$$(7.9d) \qquad MT\mathrm{Spin} \longrightarrow MT\mathrm{Pin}^+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)_+,$$

with each sm_σ obtained by taking a smooth representative of a Poincaré dual of w_1 of the manifold or of a associated principal $\mathbb{Z}/2$-bundle, like in (7.7).

The splitting of the $k = 0$ Smith homomorphism in (7.1) gives us an equivalence $MT\mathrm{Spin} \wedge B\mathbb{Z}/2 \simeq MT\mathrm{Pin}^-$, a theorem of Peterson [Pet68, §7].

This family of Smith homomorphisms has been discussed in the literature before. The piece involving $\mathrm{Spin} \times \mathbb{Z}/2$ and Pin^- was used by Peterson [Pet68, §7] and Anderson-Brown-Peterson [ABP69], who say that it was already "well-known." The long exact sequence corresponding to (7.9c) appears in [Gia73b, Theorem 3.1], where it is attributed to Stong. The Smith homomorphism sm_σ in (7.9d) appears in Kreck [Kre84, §4]. The long exact sequence induced by (7.9b) is used by Botvinnik-Rosenberg [BR23, §2], who also discuss (7.9c) and (7.9d). The composition of two maps in (7.9) in a row to go between pin$^+$ and pin$^-$ bordism appears in Kirby-Taylor [KT90a, Lemma 7]. We work out the corresponding long exact sequences, as well as some physical consequences for all four Smith homomorphisms in [DDK$^+$24, §IV.C].

The full family appears more recently in work of Hambleton-Su [HS13, §4.C], Kapustin-Thorngren-Turzillo-Wang [KTTW15, §8], Tachikawa-Yonekura [TY19, §3.1], Hason-Komargodski-Thorngren [HKT20, §4.4], and Wan-Wang-Zheng [WWZ20, §6.7]. Ekholm [Ekh98] produces the 4-periodic sequence of tangential structures in a different setting but does not discuss the Smith homomorphism.

Example 7.10. As we discussed in Example 6.17, for a general vector bundle $V \to X$, there is no guarantee that kV has a string structure. However, on $B\mathbb{Z}/2$, $k\sigma$ has a string structure iff $k \equiv 0 \bmod 8$, so there is an eight-periodic family of codimension-1 Smith homomorphisms between bordism groups of manifolds with $(B\mathbb{Z}/2, k\sigma)$-twisted string structures for various k.[17]

In Example 7.8, the four twisted spin structures turned out to be equivalent to G-structures for four Lie groups G. An analogous result is true here, but in the world of 2-groups, because the string group is a Lie 2-group [SP11]. One can show that for each $k \in \mathbb{Z}/8$, there is a Lie 2-group

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[17]To prove the claimed fact about string structures on $k\sigma$, first use the Whitney sum formula to show that $w_1(k\sigma)$, $w_2(k\sigma)$, and $w_4(k\sigma)$ all vanish iff $k \equiv 0 \bmod 8$. The reduction mod 2 map $H^4(B\mathbb{Z}/2; \mathbb{Z}) \to H^4(B\mathbb{Z}/2; \mathbb{Z}/2)$ is an isomorphism, so the string obstruction $\lambda(k\sigma)$ vanishes iff its mod 2 reduction does, and $\lambda \bmod 2 = w_4$.

$\mathbb{G}[k]$ and a map $\xi\colon B\mathbb{G}[k] \to BO$ such that $\mathbb{G}[k]$-structures on a smooth manifolds are naturally equivalent to $(B\mathbb{Z}/2, k\sigma)$-twisted string structures. These Lie 2-groups $\mathbb{G}[k]$ are extensions of $\mathrm{Spin} \times \mathbb{Z}/2$, $\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4$, and $\mathrm{Pin}^{\pm}$ by $BU(1)$; such extensions of a compact Lie group G by $BU(1)$ are classified by $H^4(BG; \mathbb{Z})$ [SP11, Wei22], and the $\mathbb{G}[k]$ 2-groups' extension classes are λ of various spin vector bundles over $B\mathrm{Spin} \times B\mathbb{Z}/2$, $B(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4)$, and $B\mathrm{Pin}^{\pm}$. For example, $\mathbb{G}[4] = \mathrm{String} \times_{BU(1)} \mathsf{sLine}$, where sLine is the abelian Lie 2-group of Hermitian super lines.

Example 7.11. If one smashes (7.1) with $MT\mathrm{Spin}^c$, one obtains a very similar story to Example 7.6: twice any vector bundle is complex, hence spin^c, and $(B\mathbb{Z}/2, \sigma)$-twisted spin^c bordism is naturally identified with pin^c bordism, as we discussed in Example 6.21. So taking Poincaré duals of w_1 as in Example 7.6 defines a 2-periodic sequence of codimension-1 Smith homomorphisms

(7.12a) $$MT\mathrm{Spin}^c \longrightarrow MT\mathrm{Spin}^c \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{Pin}^c$$

(7.12b) $$MT\mathrm{Spin}^c \longrightarrow MT\mathrm{Pin}^c \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{Spin}^c \wedge (B\mathbb{Z}/2)_+.$$

To our knowledge, these long exact sequences first appear in Hambleton-Su [HS13, §4.C].

We also obtain an equivalence $MT\mathrm{Spin}^c \wedge B\mathbb{Z}/2 \xrightarrow{\simeq} \Sigma MT\mathrm{Pin}^c$, which was first observed by Bahri-Gilkey [BG87a, §3]. See Shiozaki-Shapourian-Ryu [SSR17, §E.1] and Kobayashi [Kob21, §IV] for applications in condensed-matter physics and [DYY23] for an application of a closely related Smith long exact sequence.

Example 7.13. Pull back (7.1) along the map $B\mathbb{Z} \to B\mathbb{Z}/2$, i.e. $S^1 = \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^\infty$. The sphere bundle of $\sigma \to \mathbb{RP}^1$ is not contractible: it is the double cover $S^1 \to \mathbb{RP}^1$, and its Thom space is $\mathbb{RP}^2$. Therefore we obtain from Theorem 5.1 a cofiber sequence $\Sigma_+^\infty S^1 \to \Sigma^\infty \mathbb{RP}^2 \to \Sigma_+^{1+\infty} \mathbb{RP}^1$, which is a rotated version of the multiplication-by-2 cofiber sequence

(7.14) $$\mathbb{S} \xrightarrow{2} \mathbb{S} \longrightarrow \Sigma^{-1+\infty} \mathbb{RP}^2.$$

The same story applies to the complex, quaternionic, and octonionic Hopf fibrations: their cofibers are the respective projective planes $\Sigma^{-2+\infty} \mathbb{CP}^2$, $\Sigma^{-4+\infty} \mathbb{HP}^2$, and $\Sigma^{-8+\infty} \mathbb{OP}^2$, and in each case the map to the cofiber is a Smith homomorphism for the tautological line bundle over the respective projective line (which is a sphere). In the case of the complex Hopf fibration, after smashing with ko or KO, one obtains the Wood cofiber sequences [Woo63] $\Sigma KO \xrightarrow{\eta} KO \to KU$ and $\Sigma ko \xrightarrow{\eta} ko \to ku$ as rotated versions of Smith cofiber sequences.

Smash (7.14) with $MT\mathrm{SO}$ and you obtain Wall's cofiber sequence [Wal60, Theorem 3]

(7.15) $$MT\mathrm{SO} \xrightarrow{2} MT\mathrm{SO} \longrightarrow \mathcal{W},$$

where $\mathcal{W}$ is the Thom spectrum whose homotopy groups are the bordism groups of manifolds with an integral lift of w_1. This follows from Atiyah's identification of $\mathcal{W} \simeq \Sigma^{-1} MT\mathrm{SO} \wedge \mathbb{RP}^2$ [Ati61a, §4], but it is also easy to directly check that an integral lift of w_1 is equivalent data to a $(\mathbb{RP}^1, \sigma)$-twisted orientation, using that $\mathbb{RP}^1$ is a $B\mathbb{Z}$.

It is also interesting to smash (7.14) with $MT\mathrm{Spin}$; we work out the induced long exact sequence of bordism groups in low degrees in Figure 2, and this long exact sequence also appears in [DYY23].

Example 7.16. Let $\pi\colon E \to B$ be a principal $\mathbb{Z}/2$-bundle and $L := E \times_{\mathbb{Z}/2} \mathbb{R} \to B$ be the associated line bundle. Then we have a Smith homomorphism $\mathrm{sm}_L\colon B^{-L} \to \Sigma_+^\infty B$. The fiber is the Thom spectrum of the pullback of L to its sphere bundle; the sphere bundle is E and $\pi^*(L)$ is

trivial, so Theorem 5.1 gives us a cofiber sequence

$$(7.17) \qquad B^{-L} \xrightarrow{\text{sm}_L} \Sigma_+^\infty B \xrightarrow{\tau} \Sigma_+^\infty E.$$

Lemma 7.18. *The map τ in* (7.17) *is the Becker-Gottlieb transfer* [Rou72, KP72, BG75] *for π.*

Proof. It suffices to work universally with the Smith cofiber sequence $(B\mathbb{Z}/2)^{-\sigma} \to \Sigma_+^\infty B\mathbb{Z}/2 \to \Sigma_+^\infty E\mathbb{Z}/2$, i.e. $(\mathbb{RP}^\infty)^{-\sigma} \to \Sigma_+^\infty \mathbb{RP}^\infty \to \mathbb{S}$, and to show that the latter map is the transfer for $E\mathbb{Z}/2 \to B\mathbb{Z}/2$.

This transfer map admits the following description: consider the map of $\mathbb{Z}/2$-spectra[18] $f\colon \mathbb{S} \to \Sigma^{1-\sigma}(\mathbb{Z}/2)_+$, whose cofiber is $\mathbb{S}^{-\sigma}$. Upon taking homotopy orbits, we obtain a map $f_{h\mathbb{Z}/2}\colon \Sigma_+^\infty \mathbb{RP}^\infty \to \mathbb{S}$, and this is the transfer map.

If G is a finite group and $V \in RO(G)$, there is a natural equivalence of spectra $(\mathbb{S}^V)_{hG} \simeq (BG)^V$.[19] And taking homotopy orbits of G-spectra preserves cofiber sequences, so the fiber of the transfer $f_{h\mathbb{Z}/2}$ is the map $(\mathbb{RP}^\infty)^{-\sigma} \to \Sigma_+^\infty \mathbb{RP}^\infty$ given by the "inclusion" of virtual representations $-\sigma \hookrightarrow 0$, which is the Smith homomorphism we began with. $\square$

In the case that B is a finite CW complex, one can prove Lemma 7.18 more classically by adapting Cusick's calculation [Cus85, Corollary 2.11] identifying the cofibers of transfer maps for double covers.

Remark 7.19. For another example along the lines of (7.17), Morisugi [Mor09, Theorem 1.3] shows that the cofibers of certain Smith homomorphisms over compact Lie groups can be described as Becker-Schultz transfer maps [BS74, §4]. And Uchida [Uch69], motivated by the study of immersions, works out the Smith long exact sequences of a few special cases of Example 7.16, where $E = BO(k) \times BO(k)$ and $B = B(O(1) \ltimes (O(k)^{\times 2}))$, where $O(1)$ acts on $O(k)^{\times 2}$ by swapping the two factors.

Remark 7.20. The ease of modifying the Smith long exact sequence by a vector bundle twist suggests that Example 7.16 could be generalized to some sort of twisted transfer map. The relevant twisted transfer maps have been constructed by Kashiwabara-Zare [KZ18].

7.2. **Twisting by complex line bundles.** Now we consider the analogous family of examples arising from the tautological complex line bundle $L \to B\mathrm{U}(1)$. Its sphere bundle is $E\mathrm{U}(1) \to B\mathrm{U}(1)$, which is contractible, so just like in (7.1), we have for any $k \in \mathbb{Z}$ a cofiber sequence

$$(7.21) \qquad \mathbb{S} \longrightarrow (B\mathrm{U}(1))^{k(L-2)} \xrightarrow{\text{sm}_L} \Sigma^2 (B\mathrm{U}(1))^{(k+1)(L-2)}.$$

Again, when $k = 0$, this sequence splits, yielding another Smith isomorphism $\Sigma^\infty B\mathrm{U}(1) \xrightarrow{\cong} (B\mathrm{U}(1))^L$. This equivalence is well-known, e.g. [Ada74, Example 2.1].

Example 7.22. Let G be one of O, SO, Spin^c, or U; then the tautological line bundle over $B\mathrm{U}(1)$ has a G-structure, and MTG is an E_∞-ring spectrum and we can make sense of G-orientations. The G-orientation on L untwists the Thom spectrum, so smashing (7.21) with MTG has a similar

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[18]This fact, and our argument using it, works for both Borel and genuine $\mathbb{Z}/2$-spectra.

[19]One quick way to prove this uses the Ando-Blumberg-Gepner-Hopkins-Rezk approach to Thom spectra [ABG+14a, ABG+14b]: both $(\mathbb{S}^V)_{hG}$ and $(BG)^V$ are both the colimit of the pt$/G$-shaped diagram whose value on pt is $\mathbb{S}^V$ and whose value on the morphism set G encodes the G-action on $\mathbb{S}^V$ [ABG+14a, Theorem 1.17]. It is also possible to prove this more classically by working with Thom spaces.

effect to Example 7.4: the result is a cofiber sequence

$$(7.23) \qquad MTG \longrightarrow MTG \wedge (B\mathrm{U}(1))_+ \xrightarrow{\ \mathrm{sm}_L\ } \Sigma^2 MTG \wedge (B\mathrm{U}(1))_+.$$

For $G = \mathrm{U}$, this Smith homomorphism was first studied by Conner-Floyd [CF66b, §5].

Lemma 7.24. *For $G = \mathrm{O}$, SO, Spin^c, or U,*

$$(7.25) \qquad MTG \wedge (B\mathrm{U}(1))_+ \simeq \bigvee_{k \geq 0} \Sigma^{2k} MTG.$$

Proof. The zeroth step is splitting off the basepoint: $MTG \wedge (B\mathrm{U}(1))_+ \simeq MTG \vee MTG \wedge (B\mathrm{U}(1))$. As noted above, $\Sigma^\infty B\mathrm{U}(1) \simeq (B\mathrm{U}(1))^L$, and we have a Thom isomorphism $MTG \wedge (B\mathrm{U}(1))^L \simeq MTG \wedge \Sigma^2 (B\mathrm{U}(1))_+$. We are now in the same situation as at the beginning of the proof, but shifted up by 2, and we carry on in a similar way. $\qquad \square$

Example 7.26. Smash (7.21) with $MT\mathrm{Spin}$; the bundle $L \to B\mathrm{U}(1)$ is oriented but not spin, so $2L$ is spin, and therefore we obtain a 2-periodic, codimension-2 family of Smith homomorphisms between the spin bordism of $B\mathrm{U}(1)$ and $(B\mathrm{U}(1), L)$-twisted spin bordism. A $(B\mathrm{U}(1), L)$-twisted spin structure is equivalent data to a spin^c structure, as we discussed in Example 6.23, so this Smith family takes the form

$$(7.27a) \qquad MT\mathrm{Spin} \longrightarrow MT\mathrm{Spin}^c \xrightarrow{\ \mathrm{sm}_L\ } \Sigma^2 MT\mathrm{Spin} \wedge (B\mathrm{U}(1))_+$$

$$(7.27b) \qquad MT\mathrm{Spin} \longrightarrow MT\mathrm{Spin} \wedge (B\mathrm{U}(1))_+ \xrightarrow{\ \mathrm{sm}_L\ } \Sigma^2 MT\mathrm{Spin}^c.$$

The long exact sequence arising from (7.27a) was identified by Kirby-Taylor [KT90b, Corollary 6.12, Remark 6.14]. The splitting of (7.21) when $k = 0$ leads to an equivalence $MT\mathrm{Spin} \wedge B\mathrm{U}(1) \simeq \Sigma^2 MT\mathrm{Spin}^c$, a theorem due to Stong [Sto68, Chapter XI]. We discuss the physical interpretation of (7.27) in [DDK+24, §IV.A].

It would be interesting to study analogues of this example for pin^c or $\mathrm{pin}^{\tilde{c}\pm}$ bordism and applications to invertible phases. Kirby-Taylor [KT90b, Remark 6.15] consider two additional analogues of (7.27a), including a Smith long exact sequence for G-bordism where $G := \mathrm{Spin} \times_{\{\pm 1\}} \mathrm{O}(2)$. Guillou-Marin [GM80] and Stehouwer [Ste22, §4] compute G-bordism groups in low dimensions, and G-bordism also appears in [DDHM22, DDHM23, DYY23]. In addition, Hambleton-Kreck-Teichner [HKT94, §2] study a pin^- and pin^c analogue of Example 7.26.

Example 7.28. Pull back (7.21) along the inclusion $\mathbb{Z}/k \hookrightarrow \mathrm{U}(1)$, giving us Smith homomorphisms $(B\mathbb{Z}/k)^{k(L-2)} \to \Sigma^2 (B\mathbb{Z}/k)^{(k+1)(L-2)}$, where L is the complex line bundle induced by the rotation representation of $\mathbb{Z}/k$ on $\mathbb{C}$. Recall from Theorem 5.1 the fiber sequence

$$(7.29) \qquad S(V_2)^{V_1} \to X^{V_1} \to X^{V_1 \oplus V_2}.$$

For this example, we start with $X = B\mathbb{Z}/n$, $V_2 = i^*L - 2$ (for L as in the previous example and 2 the trivial complex line bundle), and $V_1 = k(i^*L - 2)$. We can compute the sphere bundle $S(i^*L)$ by fitting it into a pullback square:

$$(7.30) \qquad \begin{array}{ccc} S(i^*L) & \longrightarrow & S(L) \simeq * \\ \downarrow{\scriptstyle p} & \lrcorner & \downarrow \\ B\mathbb{Z}/n & \longrightarrow & B\mathrm{U}(1). \end{array}$$

As noted above, $S(L)$ is contractible as it is the total space of the universal fibration. Therefore, the other three corners of the square form a fiber sequence. To compute the fiber of $B\mathbb{Z}/n \to BU(1)$, we notice that applying the classifying space functor to the short exact sequence $\mathbb{Z}/n \hookrightarrow U(1) \xrightarrow{\times n} U(1)$ gives a fibration $B\mathbb{Z}/n \to BU(1) \to BU(1)$. Then, recognizing the map $BU(1) \to BU(1)$ as the classifying map for a principal $U(1)$-bundle over $U(1)$ with total space $B\mathbb{Z}/n$, we conclude that the fiber of the map $B\mathbb{Z}/n \to BU(1)$ is exactly $U(1)$. So, $S(i^*L) \simeq S^1$.

Next, we need to pull back V_1 along the projection $p \colon S(i^*L) \to B\mathbb{Z}/n$. We have that $p^*(k(i^*L - 1)) \cong \bigoplus_k p^*(i^*L)$. Since L is oriented as a real vector bundle, its pullbacks are as well, so p^*i^*L is oriented when considered as a real vector bundle over S^1, and thus it is the trivial 2-plane bundle.

Therefore, we recognize the Thom spectrum $S(i^*L)^{kp^*(i^*L)}$ as

$$S(i^*L)^{kp^*(i^*L)} \simeq (S^1)^k$$
$$\simeq \Sigma^{2k}\mathrm{Th}(S^1;0)$$
$$\simeq \Sigma^{2k}(\Sigma_+^\infty S^1)$$
$$\simeq \Sigma^{2k}(\Sigma^\infty S^1 \oplus \Sigma^\infty S^0)$$
$$\simeq \Sigma^{2k+1}\mathbb{S} \vee \Sigma^{2k}\mathbb{S}.$$

Thus for each $k \geq 0$ we have a Smith cofiber sequence

$$(7.31) \qquad \Sigma^{2k+1}\mathbb{S} \vee \Sigma^{2k}\mathbb{S} \longrightarrow (B\mathbb{Z}/n)^{k \cdot i^*L} \longrightarrow B\mathbb{Z}/n^{(k+1)i^*L}.$$

Finally, we place V_1 in virtual dimension zero by taking $V_1 = k(i^*L - 2)$, to be consistent with the other examples in this section, and obtain the cofiber sequence

$$(7.32) \qquad \Sigma\mathbb{S} \vee \mathbb{S} \longrightarrow (B\mathbb{Z}/n)^{k(i^*L-2)} \xrightarrow{\mathrm{sm}_L} \Sigma^2(B\mathbb{Z}/n)^{(k+1)(i^*L-2)}.$$

Example 7.33. Smash (7.32) with $MT\mathrm{Spin}$. Like in Example 7.26, i^*L is oriented but not spin, and $2i^*L$ is spin, so we obtain a 2-periodic, codimension-2 family of Smith homomorphisms between the spin bordism of $B\mathbb{Z}/n$ and $(B\mathbb{Z}/n, i^*L)$-twisted spin bordism. Campbell [Cam17, §7.9] identifies the latter as bordism for the tangential structure $\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/2n$, explicitly giving us Smith cofiber sequences

$$(7.34a) \qquad \Sigma MT\mathrm{Spin} \vee MT\mathrm{Spin} \longrightarrow MT(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/2k) \xrightarrow{\mathrm{sm}_{i^*L}} \Sigma^2 MT\mathrm{Spin} \wedge (B\mathbb{Z}/k)_+$$

$$(7.34b) \qquad \Sigma MT\mathrm{Spin} \vee MT\mathrm{Spin} \longrightarrow MT\mathrm{Spin} \wedge (B\mathbb{Z}/k)_+ \xrightarrow{\mathrm{sm}_{i^*L}} MT(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/2k).$$

$\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/2k$ bordism appears in the mathematical physics literature in [Bel99, Bla00, BBC17, Cam17, GEM19, Hsi18, Jan18, Li19, GOP+20, DDHM22, Deb21, DL21, DDHM23, HTY22, DYY23]; the case $k = 2$ also appears in [Gia73a, HKT20, TY19, FH20, MV21]. The Smith homomorphisms in (7.34) for $n = 4$ appear in [DDHM23]. We work out the Anderson-dualized long exact sequences corresponding to (7.34) for the $n = 3$ case in [DDK+24, §IV.D], and for the $n = 4$ case in [DDK+24, §IV.E].

Example 7.35. We elaborate on Example 7.33 when $n = 2$. The rotation representation is isomorphic to 2σ, where σ denotes the real sign representation; we will also let σ denote the associated bundle over $B\mathbb{Z}/2$.

Everything in Example 7.33 still works for $n = 2$, but now we have more options: we can start with an odd number of copies of σ. In this case, the fiber of the Smith map is the Thom spectrum of the Möbius bundle $(\sigma - 1) \to U(1)$; one can directly check that the Thom space of σ is $\mathbb{RP}^2$, so the Thom spectrum of $\sigma - 1$ is $\Sigma^{-1+\infty}\mathbb{RP}^2$. Therefore we have a Smith cofiber sequence

$$(7.36) \qquad \Sigma^{-1+\infty}\mathbb{RP}^2 \longrightarrow (B\mathbb{Z}/2)^{(2k-1)(\sigma-1)} \xrightarrow{\mathrm{sm}_{2\sigma}} \Sigma^2(B\mathbb{Z}/2)^{(2k+1)(\sigma-1)}.$$

Out of all the examples we have studied in this section, this is the first one where the pullback of V_2 to the sphere bundle is nontrivial.

As usual, we smash (7.36) with various bordism spectra. The map $\mathrm{sm}_{2\sigma}$ is the composition of two iterations of sm_σ from (7.1), so some of the resulting cofiber sequences look familiar from that perspective. We only discuss a few examples, but plenty more are out there.

- If we smash (7.36) with $MT\mathrm{SO}$, we obtain a cofiber sequence first discussed by Atiyah [Ati61a, (4.3)]:

$$(7.37) \qquad \mathcal{W} \longrightarrow MT\mathrm{O} \xrightarrow{\mathrm{sm}_{2\sigma}} \Sigma^2 MT\mathrm{O},$$

  where $\mathcal{W}$ is Wall's bordism spectrum (see Example 7.13). Here we use the identifications $\Sigma MT\mathrm{O} \simeq MT\mathrm{SO} \wedge B\mathbb{Z}/2$ and $\mathcal{W} \simeq MT\mathrm{SO} \wedge \Sigma^{-1}\mathbb{RP}^2$, both due to Atiyah [Ati61a, §4], that we discussed in Examples 7.6 and 7.13, respectively.
- If we instead smash (7.36) with $MT\mathrm{Spin}$, we obtain a cofiber sequence

$$(7.38) \qquad MT\mathrm{Spin} \wedge \Sigma^{-1}\mathbb{RP}^2 \longrightarrow MT\mathrm{Pin}^\pm \xrightarrow{\mathrm{sm}_{2\sigma}} \Sigma^2 MT\mathrm{Pin}^\mp,$$

  which was first constructed by Kirby-Taylor [KT90a, Lemma 7]. Here we have used the identifications of pin$^+$, resp. pin$^-$ bordism as $(B\mathbb{Z}/2, 3\sigma)$, resp. $(B\mathbb{Z}/2, \sigma)$-twisted spin bordism that we discussed in Example 7.8. In Figure 3, we calculate the long exact sequence on bordism groups corresponding to (7.38) (specifically, the pin$^-$ to pin$^+$ case) in low degrees. See [DDHM23] for an application of a related but different Smith homomorphism in physics.

  The Smith homomorphism in (7.38) is the composition of two of the Smith homomorphisms in the 4-periodic collection of Example 7.8, where we go from pin$^+$ to $\mathrm{Spin} \times \mathbb{Z}/2$ to pin$^-$, or from pin$^-$ to $\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4$ to pin$^+$. The other two compositions, which exchange the spin bordism of $B\mathbb{Z}/2$ with $\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4$ bordism, are (7.34) for $n = 2$.

7.3. **A few more examples.** In this section, we record some examples of Smith cofiber sequences that do not arise from real or complex line bundles.

Example 7.39. Like our previous examples over $B\mathbb{Z}/2$ and $BU(1)$, we can study Smith homomorphisms for the tautological quaternionic line bundle $V \to B\mathrm{SU}(2)$. Once again, the sphere bundle of V is contractible, as it is $E\mathrm{SU}(2) \to B\mathrm{SU}(2)$, so we obtain Smith cofiber sequences like in (7.1) and (7.21):

$$(7.40) \qquad \mathbb{S} \longrightarrow (B\mathrm{SU}(2))^{k(V-4)} \xrightarrow{\mathrm{sm}_V} \Sigma^4(B\mathrm{SU}(2))^{(k+1)(V-4)}.$$

For $k = 0$, this sequence splits, yielding a third Smith isomorphism $\Sigma^\infty B\mathrm{SU}(2) \xrightarrow{\cong} (B\mathrm{SU}(2))^V$. This equivalence is well-known, e.g. [Tam97, §2].

This bundle has a G-structure for G including O, SO, Spin, Spin^c, U, SU, and Sp, and in all of these cases, smashing with MTG produces Smith homomorphisms similar to those in Examples 7.4

and 7.22. The proof of Lemma 7.24 still works in this setting, and for these G we obtain splittings

$$(7.41) \qquad MTG \wedge (B\mathrm{SU}(2))_+ \simeq \bigvee_{k \geq 0} \Sigma^{4k} MTG.$$

When $G = \mathrm{Spin}$, (7.41) is closely related to the splitting in Corollary B.19.

The Smith map (7.40), after smashing with $MTSp$, was studied by Landweber [Lan68, §5].

Example 7.42. As we mentioned in Example 7.10, Lie 2-group extensions of a compact Lie group G by $B\mathrm{U}(1)$ are classified by $H^4(BG; \mathbb{Z})$ [SP11, Wei22]. Let String-SU(2) be the Lie 2-group belonging to the extension

$$(7.43) \qquad 0 \longrightarrow B\mathrm{U}(1) \longrightarrow \text{String-SU}(2) \longrightarrow \mathrm{Spin} \times \mathrm{SU}(2) \longrightarrow 0$$

classified by $\lambda + p_1^{\mathbb{H}} \in H^4(B\mathrm{Spin} \times B\mathrm{SU}(2); \mathbb{Z})$; here $\lambda \in H^4(B\mathrm{Spin}; \mathbb{Z})$ and $p_1^{\mathbb{H}} \in H^4(B\mathrm{SU}(2); \mathbb{Z})$ are the canonical generators of H^4 of a connected, simply connected, simple Lie group. (For $B\mathrm{Spin}$, we may use $B\mathrm{Spin}(n)$ with $n \gg 0$.) Using the usual map $\mathrm{Spin} \to \mathrm{O}$, we obtain a tangential structure $B(\text{String-SU}(2)) \to B\mathrm{O}$; by an argument similar to the one in [DDHM23, §10.4] (see [BDDM24, §3.3.1]), $MT(\text{String-SU}(2)) \simeq MT\mathrm{String} \wedge (B\mathrm{SU}(2))^{L-4}$, where $L \to B\mathrm{SU}(2)$ is the tautological quaternionic line bundle. Thus (7.40) with $k = 0$, smashed with $MT\mathrm{String}$, produces a Smith isomorphism

$$(7.44) \qquad \mathrm{sm}_L \colon \widetilde{\Omega}_*^{\mathrm{String}}(B\mathrm{SU}(2)) \xrightarrow{\cong} \Omega_{*-4}^{\text{String-SU}(2)}.$$

As $L \to B\mathrm{SU}(2)$ is not string, the argument for (7.41) does not apply, and indeed one can show $MT(\text{String-SU}(2))$ does not split in that way. Thus this Smith isomorphism is expressing something nontrivial about string-SU(2) bordism. To our knowledge, (7.44) is the first such nontrivial quaternionic Smith isomorphism known.

String-SU(2) bordism appears in [BDDM24, §3] as an intermediary to other twisted string bordism computations, and Bruner-Rognes [BR21, §1.4, Chapter 8, §12.3, Appendix D] study a closely related object called tmf/ν.

Example 7.45. Consider the Smith homomorphisms coming from the tautological rank-3 vector bundle $V \to B\mathrm{SO}(3)$. Then, like in Example 7.49, the one-point compactification of $\mathfrak{so}(3)/\mathfrak{u}_1$ is isomorphic to $\mathrm{SO}(3)/\mathrm{U}(1) \cong S^2$. Since $\mathfrak{so}(3)/\mathfrak{u}_1 \oplus \mathbb{R}$ is isomorphic to the defining representation V of $\mathrm{SO}(3)$, we obtain a cofiber sequence of spectra

$$(7.46) \qquad (B\mathrm{U}(1))^{k(L-2)} \longrightarrow (B\mathrm{SO}(3))^{k(V-3)} \longrightarrow \Sigma^3 (B\mathrm{SO}(3))^{(k+1)(V-3)}.$$

We are most interested in smashing this sequence with $MT\mathrm{Spin}$.[20] Note that V is not spin, but because V is oriented, $2V$ is spin; therefore we obtain a 2-periodic family of codimension-3 Smith homomorphisms exchanging the spin bordism of $B\mathrm{SO}(3)$ and $(B\mathrm{SO}(3), V)$-twisted spin bordism. Freed-Hopkins [FH21, (10.20)] identify $(B\mathrm{SO}(3), V)$-twisted spin bordism with bordism for the group $G^0 := \mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}(2)$, which is in various sources called spin^h bordism, spin^q bordism,

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[20]It is also interesting to smash (7.46) with $MT\mathrm{Spin}^c$: in this case one obtains a codimension-3, 2-periodic family of Smith homomorphisms exchanging the spin^c bordism of $B\mathrm{SO}(3)$ with "spin-U(2) bordism," i.e. bordism of the group $\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{U}(2) \cong \mathrm{Spin}^c \times_{\{\pm 1\}} \mathrm{SU}(2)$. Davighi-Lohitsiri [DL20, DL21] introduced Spin-U(2) bordism and calculated it in low dimensions; spin-U(2) structures also appear in Seiberg-Witten theory (e.g. [FL02, DW19]) under the name $spin^u$ $structures$.

spin-SU(2) bordism, or G^0 bordism.[21] The fiber we've seen before in Example 7.26: spin^c bordism when k is odd in (7.46), and the spin bordism of $B\mathrm{U}(1)$ when k is even.

In summary, we have two Smith cofiber sequences

$$(7.47a) \qquad\qquad MT\mathrm{Spin}^c \longrightarrow MT\mathrm{Spin}^h \xrightarrow{\mathrm{sm}_V} \Sigma^3 MT\mathrm{Spin} \wedge (B\mathrm{SO}(3))_+$$

$$(7.47b) \qquad MT\mathrm{Spin} \wedge (B\mathrm{U}(1))_+ \longrightarrow MT\mathrm{Spin} \wedge (B\mathrm{SO}(3))_+ \xrightarrow{\mathrm{sm}_V} \Sigma^3 MT\mathrm{Spin}^h.$$

The long exact sequence of bordism groups associated to (7.47a) appears in Theorem B.2 as an example where one must use the cobordism Euler class to calculate the Smith homomorphism: ordinary cohomology Euler classes give the wrong answer. Other works studying anomalies of spin^h QFTs include [FH21, WW19, WWW19, WWZ20, DL20, WW20b, BCD22, DY22, WY22, DYY23].

Remark 7.48. Freed-Hopkins [FH21] also study two unoriented analogues of spin^h structures, called $\mathrm{pin}^{h\pm}$ or $G^\pm$ structures, corresponding to the groups $\mathrm{Pin}^\pm \times_{\{\pm 1\}} \mathrm{SU}(2)$. It would be interesting to work out analogues of the Smith homomorphisms such as the ones in Examples 7.8 and 7.45 for $\mathrm{pin}^{h\pm}$ structures and apply them to symmetry breaking; see [DK24] for some work in that direction. $\mathrm{Pin}^{h\pm}$ manifolds are also studied in [BC18, GPW18, LS19, AM21, DYY23].

Example 7.49. If we pull Example 7.45 back to $B\mathrm{SU}(2)$, we obtain a Smith long exact sequence which makes an appearance both in [DDK⁺24, §IV.F] and in Appendix B.

The tautological quaternionic line bundle over $B\mathrm{SU}(2)$ is *not* isomorphic to the bundle associated to $\mathfrak{su}_2 \oplus \mathbb{R}$, where $\mathfrak{su}_2$ is the adjoint representation of $\mathrm{SU}(2)$. Rather, since $\mathfrak{su}_2 \cong \mathbb{R} \oplus \mathfrak{su}_2/\mathfrak{u}_1$, the map $B\mathrm{U}(1) \to B\mathrm{SU}(2)$ exhibits $B\mathrm{U}(1)$ as the unit sphere bundle in the adjoint representation of $\mathrm{SU}(2)$. It follows that there is a cofiber sequence

$$(7.50) \qquad\qquad B\mathrm{U}(1) \longrightarrow B\mathrm{SU}(2) \xrightarrow{\mathrm{sm}_V} \Sigma^3 (B\mathrm{SU}(2))^{V-3},$$

where $V \to B\mathrm{SU}(2)$ is the vector bundle associated to $\mathfrak{su}(2)$. We claim the first map is induced by the inclusion of a maximal torus into $\mathrm{SU}(2)$. To see that the sphere bundle is $B\mathrm{U}(1)$ as claimed, identify $\mathrm{SU}(2) \to \mathrm{SO}(3)$ with $\mathrm{Spin}(3) \to \mathrm{SO}(3)$ and $U(1) \to \mathrm{SU}(2)$ with $\mathrm{Spin}(2) \to \mathrm{Spin}(3)$; by the third isomorphism theorem, $\mathrm{Spin}(3)/\mathrm{Spin}(2) \cong \mathrm{SO}(3)/\mathrm{SO}(2)$, and in Example 7.45 we identified that quotient with the unit sphere inside $\mathbb{R}^3$. Therefore taking associated bundles, we end up with $B\mathrm{Spin}(2)$ as the fiber in (7.50).

Since $\mathrm{SU}(2)$ is simply connected, $B\mathrm{SU}(2)$ is 2-connected and therefore all of its vector bundles admit spin structures. Thus, when we smash (7.50) with $MT\mathrm{Spin}$, we obtain a cofiber sequence

$$(7.51) \qquad MT\mathrm{Spin} \wedge (B\mathrm{U}(1))_+ \longrightarrow MT\mathrm{Spin} \wedge (B\mathrm{SU}(2))_+ \xrightarrow{\mathrm{sm}_V} \Sigma^3 MT\mathrm{Spin} \wedge (B\mathrm{SU}(2))_+.$$

The Thom spectrum $(B\mathrm{SU}(2))^{\mathfrak{su}(2)}$ is known as James' "quasiprojective space" (see [Jam76]). The Anderson dual of (7.51) appears in a physics application in [DDK⁺24, §IV.F.].

The same Thom isomorphism applies for any $MT\mathrm{Spin}$-oriented ring spectrum, such as $MT\mathrm{SO}$ or ko; if we used ko instead of $MT\mathrm{Spin}$ in (7.51), we would obtain the cofiber sequence in (B.26).

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[21]To the best of our knowledge, spin^h structures were first studied in [BFF78] in the context of quantum gravity; they have also been applied to Seiberg-Witten theory [OT96], index theory, e.g. in [May65, Nag95, Bär99, FH21, Che17], almost quaternionic geometry, e.g. in [Nag95, Bär99, AM21], immersion problems [Bär99, AM21], and the study of invertible field theories [FH21, BC18, WWW19, DY22]. See [Law23] for a review and [BM23, Hu23, Mil23, DK24] for additional related work.

Example 7.52. In [DDK+24, §III.B.1] , we study the SBLES in twisted spin bordism corresponding to the vector bundle $2L \to B\mathrm{U}(1)$, where L denotes the tautological bundle. Since $2L$ is spin, we obtain a one-periodic family of Smith homomorphisms of the form

$$(7.53a) \qquad S(2L) \longrightarrow B\mathrm{U}(1) \xrightarrow{\mathrm{sm}_{2L}} \Sigma^4 (B\mathrm{U}(1))^{2L-4}.$$

The new wrinkle is showing that $S(2L) \to B\mathrm{U}(1)$ is homotopy equivalent to the map $S^2 \to B\mathrm{U}(1)$ given by the inclusion of the 2-skeleton. But this is not so hard: using the long exact sequence in cohomology associated to the cofiber sequence, one learns that if C is the cofiber of sm_{2L}, $\widetilde{H}^*(C;\mathbb{Z})$ vanishes except in degree 3, where it is $\mathbb{Z}$; this characterizes S^3, so the fiber, which is the total space of the sphere bundle, is S^2. Stably this splits as $\mathbb{S} \vee \Sigma^2 \mathbb{S}$, so our cofiber sequence is

$$(7.53b) \qquad \mathbb{S} \vee \Sigma^2 \mathbb{S} \longrightarrow \Sigma_+^\infty B\mathrm{U}(1) \xrightarrow{\mathrm{sm}_{2L}} (B\mathrm{U}(1))^{2L-4}.$$

This cofiber sequence is a complexified version of (7.32). One therefore wonders what happens if we consider it within its family

$$(7.54) \qquad \mathrm{sm}_{2L} \colon (B\mathrm{U}(1))^{kL-2k} \longrightarrow \Sigma^4 (B\mathrm{U}(1))^{(k+1)L-2k-2}.$$

If we smash with $MT\mathrm{Spin}$, this is a 2-periodic family: it only matters whether k is odd or even. For k even we reduce to (7.53b) above; for k odd, we have a very similar cofiber sequence, but the sphere bundle does not split: we obtain for the fiber $(\mathbb{CP}^1)^{\mathcal{O}(-1)-2} \simeq \mathbb{CP}^2$:

$$(7.55) \qquad MT\mathrm{Spin} \wedge \mathbb{CP}^2 \longrightarrow MT\mathrm{Spin}^c \longrightarrow \Sigma^4 MT\mathrm{Spin}^c,$$

using the identification $MT\mathrm{Spin} \wedge (B\mathrm{U}(1))^{L-2} \simeq MT\mathrm{Spin}^c$ from Example 6.23. This is the complex analogue of (7.38).

Remark 7.56. There is a related example where one uses $L \oplus L^* \to B\mathrm{U}(1)$ instead of $2L$; the corresponding long exact sequence in twisted SU-bordism was studied by Conner-Floyd [CF66b, §§6, 14, 17]. When L is odd, the third term in the long exact sequence, corresponding to the sphere bundle, is the bordism of manifolds with c_1-*aspherical structures* or *complex Wall structures*, first introduced by Conner-Floyd [CF66b], and also discussed by Stong [Sto68, Chapter VIII]. Complex Wall bordism plays an important role in the calculation of Ω_*^{SU} via the Adams-Novikov spectral sequence [Nov67, §7], and has also been studied in the context of complex orientations [Buh72, PC23, Che22].

Example 7.57. The unit sphere bundle to the tautological bundle $V_{n+1} \to B\mathrm{O}(n+1)$ is homotopy equivalent to the map $B\mathrm{O}(n) \to B\mathrm{O}(n+1)$. This is because $S^n \cong \mathrm{O}(n+1)/\mathrm{O}(n)$, so the unit sphere bundle can be described by the mixing construction

$$(7.58) \ S^n \times_{\mathrm{O}(n+1)} E\mathrm{O}(n+1) \cong (\mathrm{O}(n+1)/\mathrm{O}(n)) \times_{\mathrm{O}(n+1)} E\mathrm{O}(n+1) \cong E\mathrm{O}(n+1)/\mathrm{O}(n) \cong B\mathrm{O}(n).$$

More generally, if $\xi_{n+1} \colon B_{n+1} \to B\mathrm{O}(n+1)$ is an unstable tangential structure and $\xi_n \colon B_n \to B\mathrm{O}(n)$ is the pullback of ξ_{n+1} by $B\mathrm{O}(n) \to B\mathrm{O}(n+1)$, the sphere bundle of $\xi_{n+1}^* V_{n+1}$ is the pullback of $S(V_{n+1}) = B\mathrm{O}(n)$ by ξ_{n+1}, which is ξ_n. If you then pull $\xi_{n+1}^* V_{n+1}$ back across $B_n \to B_{n+1}$, it splits as $V_n \oplus \mathbb{R}$, so there is a Smith cofiber sequence

$$(7.59) \qquad \Sigma^{-1} B_n^{n-V_n} \longrightarrow B_{n+1}^{n+1-V_{n+1}} \longrightarrow \Sigma_+^\infty B_{n+1}.$$

This cofiber sequence is due to Galatius-Madsen-Tillmann-Weiss [GMTW09, (3.3), §5]. The spectrum $\Sigma^n B_n^{n-\xi_n^* V_n}$ is often denoted $MT\xi_n$.

8. Long exact sequence of invertible field theories

In this section, we turn to physical applications of the Smith fiber sequence (Theorem 5.1). In Section 8.1 we give a brief overview of anomalies and how they relate to invertible field theories. In Section 8.2 we review the mathematical classification of invertible field theories as Anderson dual groups of Thom spectra. In Section 8.3 we show how the Smith fiber sequence gives rise to a long exact sequence of invertible field theories (Corollary 8.19), which we call the symmetry breaking long exact sequence (SBLES). Finally, we give a mathematically-oriented introduction to our companion work [DDK+24] and a table of cross-listed examples (Table 1).

Quantum field theory is not yet completely mathematically formalized, and the applications of the Smith homomorphism in [HKT20, COSY20, DDK+24] take place at a physical level of rigor, not a mathematical one. As such, parts of this section are also only at a physical level of precision.

8.1. **Anomalies and invertible field theories.** Our homotopy-theoretic techniques in this paper apply to the classification of invertible field theories, while the physical objects we ultimately wish to study are (not necessarily invertible) quantum field theories with potentially anomalous symmetries. To apply our techniques, we take the following perspective on anomalies.[22]

Let us first consider anomalies of symmetries. Physically, we begin with a k-dimensional quantum system (such as a quantum field theory) with partition function Z, as well as a symmetry group G, which is usually a compact Lie group. We can couple the theory to a background G-gauge field A. Then, put simply, the G symmetry is *anomalous* if the partition function evaluated on a pair of a closed k-manifold M and a background gauge field with connection A is *not* gauge invariant but rather transforms with a phase. That is, under a gauge tranformation $A \mapsto A^g$, the partition function transforms as

$$(8.1) \qquad Z(M, A^g) = e^{i\alpha(M, A, g)} Z(M, A),$$

where $e^{i\alpha(M, A, g)}$ is a phase factor that cannot be cancelled by local counter-terms.

Mathematically, this means that when we evaluate the theory on a k-dimensional manifold M with a G-principal bundle with connection A, the partition function is a section of a non-trivial line bundle. The non-triviality of the line bundle is the anomaly.

To relate anomalies to invertible field theories, we use the notion of *anomaly inflow*: under mild hypotheses, and in all known cases, there is a local counterterm $e^{i\omega(K, A)}$ defined in one dimension higher, so that if K is a $(k+1)$-dimensional manifold with boundary and $\partial K = M$ (and A extends into K), then

$$(8.2) \qquad e^{i\omega(K, A^g) - i\omega(K, A)} = e^{i\alpha(M, A, g)}.$$

We may interpret $e^{i\omega(K, A)}$ as the partition function of a $k + 1$-dimensional invertible field theory with G symmetry. It is invertible, as stacking with the $e^{-i\omega(K, A)}$ theory gives the trivial theory. Furthermore, Z naturally lives at the boundary of this invertible theory, and together they are gauge invariant by Equation (8.2). Therefore we can interpret the existence of the G-anomaly, which is the failure of $Z(M, A)$ to be gauge invariant, as the statement that Z is a boundary theory of a non-trivial $(k + 1)$-dimensional invertible field theory, which we call the bulk theory. This

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[22]There are many things called "anomalies" in quantum field theory, and we are not claiming our definitions or approach is universal. Instead, our application is to a broad class of anomalies.

perspective, formulated mathematically by Freed-Teleman [FT14], is the link between anomalies and invertible field theories; see also Freed [Fre23].

We also study anomalies of families of field theories parameterized by a topological space X. An anomaly of an X-family of field theories indicates a failure of the partition function to be consistently defined over the space of background X-fields.[23] In this case, the bulk theory is an one-dimension-higher X-family of invertible field theories. More generally, we can ask for X to be a space with G-action, and consider anomalies of G-equivariant X-families field theories. We refer the reader to [DDK$^+$24, §II.A] and the references therein for more detail.

8.2. **Invertible field theories and Thom spectra.** Here we review the mathematical classification of (reflection-positive) invertible theories. First we must relate symmetries and tangential structures.

8.2.1. *Symmetries and tangential structures.* For any tangential structure in the sense of Definition 2.3, there is a notion of topological field theory. Given a field theory whose anomaly we want to investigate, which tangential structure ξ do we want our invertible field theories to carry?[24]

The answer typically depends only on the symmetries of our field theory, not on its field content (the anomaly itself—which invertible field theory we get out of all the invertible field theories on ξ-manifolds—uses more information from the theory). We follow Freed-Hopkins [FH21, §2], who take the stance that since we typically study QFTs in Minkowski signature but invertible field theories are Euclidean, we should Wick-rotate the group of symmetries to define our tangential structure.

Assume the dimension n is at least 2. Let $\mathcal{I}(1, n-1)$ be the isometry group of Minkowski space, and let $\mathcal{I}(1, n-1)^\uparrow \subset \mathcal{I}(1, n-1)$ be the subgroup of isometries that preserve the direction of time. The group of symmetries of our theory is a Lie group $\mathcal{H}(1, n-1)$ with a map $\rho(n)\colon \mathcal{H}(1, n-1) \to \mathcal{I}(1, n-1)^\uparrow$. Let $K := \ker(\rho(n))$; we assume K is compact. Assume that the normal subgroup of translations $\mathbb{R}^{1,n-1} \subset \mathcal{I}(1, n-1)$ lifts to a normal subgroup of $\mathcal{H}(1, n-1)$, and let $H(1, n-1) := \mathcal{H}(1, n-1)/\mathbb{R}^{1,n-1}$. Now:

(1) Let $\mathrm{O}(1, n-1)^\uparrow := \mathrm{O}(1, n-1) \cap \mathcal{I}(1, n-1)$. There is an exact sequence

$$(8.3a) \qquad 0 \longrightarrow K \longrightarrow H(1, n-1) \longrightarrow \mathrm{O}(1, n-1)^\uparrow.$$

(2) This exact sequence can be extended to an exact sequence of complexifications:

$$(8.3b) \qquad 0 \longrightarrow K(\mathbb{C}) \longrightarrow H(n, \mathbb{C}) \longrightarrow \mathrm{O}(n, \mathbb{C}),$$

(3) and then to compact real forms of these complex Lie groups:

$$(8.3c) \qquad 0 \longrightarrow K \longrightarrow H(n) \longrightarrow \mathrm{O}(n).$$

The tangential structure that the anomaly field theory has is $\xi\colon BH(n) \to BO(n)$. Just as it is not a priori clear that the anomaly field theory extends to dimension $n+1$, it is also not necessarily clear that ξ extends to an $(n+1)$-dimensional unstable tangential structure, but Freed-Hopkins [FH21,

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[23]Typically in physics, X carries more structures, such as a smooth structure or Riemannian metric. The anomalies we consider here will not depend on those structures.

[24]For non-topological invertible field theories, there is also the question of enriching the tangential structure to something more geometric, such as including the data of a Riemannian metric or a connection for a principal bundle. At the level of invertible field theories this is accounted for by working with Anderson duals of bordism spectra, so our calculations include these geometric modifications.

Theorem 2.19] prove that it does in nearly every situation one might want, as we discuss below. In this paper, we will always be in the situation that ξ extends to $(n + 1)$-manifolds.

In practice, one can often use an idea of Stehouwer [Ste22] to compute ξ using the formalism of *fermionic groups* (sometimes called *supergroups*).

Definition 8.4 (Benson [Ben88, §7]). A *fermionic group* is a topological group G together with data of:

- a central element squaring to 1, which we call *fermion parity* and denote $-1 \in G$, and
- a group homomorphism $\theta \colon G \to \mathbb{Z}/2$ such that $\theta(-1) = 0$.

We think of θ as defining a $\mathbb{Z}/2$-grading on G, and we refer to elements of G as odd or even. The even elements form a subgroup $G_0 \subset G$, which is itself a fermionic group with θ trivial.

Given two fermionic groups G and H, one can take their *fermionic tensor product* (*ibid.*) $G \otimes H \coloneqq (G \times H)/\langle(-1, -1)\rangle$. This is a fermionic group, with central element $(-1, 1) = (1, -1)$ and grading $\theta((g, h))$ equal to the sum mod 2 of the gradings on g and on h.

Fermionic groups describe symmetries of theories with fermions: -1 acts by fermion parity, which may mix nontrivially with other symmetries in the theory; and θ describes whether elements of G act unitarily or antiunitarily. Given a fermionic group G, Stolz [Sto98, §2.6] defines a tangential structure $\xi_G \colon B \to BO$ as follows: let H be the even subgroup of the fermionic tensor product $\mathrm{Pin}^+ \otimes G$; here, to make Pin^+ into a fermionic group, we use the usual -1, and the grading homomorphism is $\pi_0 \colon \mathrm{Pin}^+ \to \mathrm{O}(1) \cong \mathbb{Z}/2$. Then $B \coloneqq BH$, and the map $\xi \colon B \to BO$ is induced from the usual map $\mathrm{Pin}^+ \to \mathrm{O}$ and the constant map to the identity on the quotient of G_0 by fermion parity. See [Ste22] for several examples of computations of tangential structures from data of the symmetries of a theory.

Tangential structures can encode not only the symmetries of quantum systems, but also the parameter space. For example, S^1-families of fermionic theories with an internal unitary $\mathbb{Z}/2$ symmetry may be described using the tangential structure $B\mathrm{Spin} \times B\mathbb{Z}/2 \times S^1 \to B\mathrm{Spin} \to BO$. There are also variations and twists: if $\mathbb{Z}/2\mathbb{Z}$ acts on S^1, then we replace $B\mathbb{Z}/2\mathbb{Z} \times S^1$ with the homotopy quotient $S^1/(\mathbb{Z}/2\mathbb{Z})$. To encode a time-reversal symmetry T with $T^2 = (-1)^F$, we replace $B\mathrm{Spin} \times B\mathbb{Z}/2$ with $B\mathrm{Pin}^+$. Note that the process of describing the symmetry and parameter space of a field theory can be subtle. See [DDK$^+$24] for more examples of converting the data of symmetries and parameter spaces to tangential structures.

8.2.2. *Invertible field theories and bordism invariants.* At this point in the story we have turned the physics question of determining the possible anomalies of a theory with a given collection of symmetries into the mathematical question of classifying (reflection-positive) invertible field theories for a fixed tangential structure $\xi \colon B \to BO$, with $B = BG$.

In this subsubsection we discuss how this classification question reduces to a well-studied problem in algebraic topology: the computation of groups of bordism invariants. See Freed [Fre19, Lectures 6–9] and Galatius [Gal21] for more detailed reviews of this story.

A field theory $Z \colon \mathsf{Bord}_n^\xi \to \mathsf{C}$ is *invertible* if there is some other theory Z^{-1} such that $Z \otimes Z^{-1}$ is the trivial theory [FM06, Definition 5.7]. This tensor product is evaluated "pointwise," meaning that $(Z \otimes Z^{-1})(M) \coloneqq Z(M) \otimes Z^{-1}(M)$, where M is an object, morphism, etc. in the bordism category; therefore invertibility implies that Z, as a functor, factors through the Picard sub-k-groupoid of units $\mathsf{C}^\times$ inside C, meaning that if X is any object, morphism, or higher morphism in Bord_n^ξ,

$Z(X)$ is invertible: $\otimes$-invertible if X is an object, and composition-invertible if X is a (higher) morphism. If X is invertible, then we must have data of an isomorphism $Z(X^{-1}) \xrightarrow{\cong} Z(X)^{-1}$ because Z is symmetric monoidal; thus, even if X is not invertible, we can heuristically *define* $Z(X^{-1}) \coloneqq Z(X)^{-1}$ as if X^{-1} existed. These definitions are compatible as X varies, in the sense that Z extends to the *Picard k-groupoid completion* $\overline{\mathsf{Bord}}_n^\xi$ of Bord_n^ξ: the Picard k-groupoid defined by formally adding inverses to all objects, morphisms, higher morphisms, etc. of Bord_n^ξ. Thus, an invertible field theory $Z \colon \mathsf{Bord}_n^\xi \to \mathsf{C}$ is equivalent data to a morphism of Picard k-groupoids

$$(8.5) \qquad\qquad Z \colon \overline{\mathsf{Bord}}_n^\xi \longrightarrow \mathsf{C}^\times.$$

So to compute deformation classes of invertible field theories, we should compute the groups of symmetric monoidal functors between these Picard k-groupoids, modulo natural isomorphisms. The homotopy theory of Picard groupoids embeds in the usual stable homotopy category: if D is a Picard groupoid, the geometric realization $|N\mathsf{D}|$ of the nerve of D has an E_∞-structure arising from the monoidal product on D, and the Picard condition implies $|N\mathsf{D}|$ is grouplike. Therefore it is equivalent data to a connective spectrum $|\mathsf{D}|$, which we call the *classifying spectrum* of D. This turns out to be a complete invariant of Picard k-groupoids.

Theorem 8.6 (Stable homotopy hypothesis (Moser-Ozornova-Paoli-Sarazola-Verdugo [MOP$^+$22]))**.** *There is an equivalence of ∞-categories between the ∞-category of Picard k-groupoids and the ∞-category of spectra whose homotopy groups vanish outside of $[0, k]$.*

Remark 8.7. For $k = 1$, the stable homotopy hypothesis was originally a folklore theorem: proofs or sketches appear in [BCC93, HS05, Dri06, Pat12, JO12, GK14]. For $k = 2$, the stable homotopy hypothesis was proven by Gurski-Johnson-Osorno [GJO19].

Therefore we need to compute the group of homotopy classes of maps of spectra $|\overline{\mathsf{Bord}}_n^\xi| \to |\mathsf{C}^\times|$. A reasonable first step would be to identify these two classifying spectra. For the domain, the Picard k-groupoid completion of the bordism category, this is due to Galatius-Madsen-Tillmann-Weiss [GMTW09] and Nguyen [Ngu17] for the bordism $(\infty, 1)$-category and to Schommer-Pries [SP17] for more general (∞, k)-categories.

Theorem 8.8 (Galatius-Madsen-Tillmann-Weiss [GMTW09], Nguyen [Ngu17], Schommer-Pries [SP17])**.** *If Bord_n^ξ denotes the (∞, k)-category of bordisms of ξ_n-structured manifolds in dimensions $n - k, \ldots, n$, then there is a natural equivalence $|\overline{\mathsf{Bord}}_n^\xi| \simeq \Sigma^k MT\xi_n$.*

Here $MT\xi_n$ is a Madsen-Tillmann spectrum as in Definition 2.17.

Freed-Hopkins-Teleman [FHT10] then applied this result to classify invertible field theories in terms of $MT\xi_n$. To do so, we need to determine $|\mathsf{C}^\times|$, which depends on one's choice of C—Freed-Hopkins [FH21, §5.3] argue that the (shifted) *character dual of the sphere spectrum* $\Sigma^n I_{\mathbb{C}^\times}$ is a universal choice, and that a related object called the (shifted) *Anderson dual of the sphere spectrum* $\Sigma^{n+1} I_{\mathbb{Z}}$ should appear when one wants to classify deformation classes of invertible field theories. For applications to anomalies, we are interested in deformation classes, so use $\Sigma^{n+1} I_{\mathbb{Z}}$.

The Anderson dual $I_{\mathbb{Z}}$ is characterized by its universal property that for any spectrum $\mathcal{X}$, there is a short exact sequence [And69, Yos75]

$$(8.9) \quad 0 \longrightarrow \mathrm{Tors}(\mathrm{Hom}(\pi_{n+1}\mathcal{X}, \mathbb{C}^\times)) \xrightarrow{\varphi} [\mathcal{X}, \Sigma^{n+2} I_{\mathbb{Z}}] \xrightarrow{\psi} \mathrm{Hom}(\pi_{n+2}\mathcal{X}, \mathbb{Z}) \longrightarrow 0.$$

We are interested in anomalies of unitary QFTs, hence we expect the anomaly theories to satisfy the Wick-rotated analogue of unitarity: reflection positivity. Freed-Hopkins [FH21, §7.1, §8.1] define reflection positivity for invertible TFTs using $\mathbb{Z}/2$-actions on Bord_n^ξ and C,[25] and prove two key results allowing for a complete classification of reflection positive invertible TFTs following their definition.

Theorem 8.10 (Freed-Hopkins [FH21, Theorem 2.19]). *If $n \geq 3$ and $\xi_n \colon BH(n) \to BO(n)$ is a tangential structure arising from a representation $\rho \colon H(n) \to O(n)$ with $H(n)$ a compact Lie group and $SO(n) \subset \mathrm{Im}(\rho)$, then there is a stable tangential structure $\xi \colon BH \to BO$ such that ξ_n is the pullback of ξ along $BO(n) \to BO$.*

Theorem 8.11 (Freed-Hopkins [FH21, Theorem 5.23], Grady [Gra23]). *Suppose $\xi \colon BH(n) \to BO(n)$ satisfies the hypotheses of Theorem 8.10. The abelian group of deformation classes of n-dimensional, reflection positive invertible field theories on manifolds with ξ-structure is naturally isomorphic to $[MT\xi, \Sigma^{n+1} I_\mathbb{Z}]$.*

So after we require reflection positivity, the classification changes from Madsen-Tillmann bordism to bordism in the usual sense, which is easier to calculate.

Remark 8.12. There are some other approaches to the classification of invertible topological field theories, due to Yonekura [Yon19], Rovi-Schoenbauer [RS22], and Kreck-Stolz-Teichner (unpublished).

Theorem 8.11 has a nice interpretation from the point of view of anomalies. Using the defining property of $I_\mathbb{Z}$, there is a short exact sequence

$$(8.13) \quad 0 \longrightarrow \mathrm{Tors}(\mathrm{Hom}(\Omega_{n+1}^\xi, \mathbb{C}^\times)) \xrightarrow{\;\varphi\;} [MT\xi, \Sigma^{n+2} I_\mathbb{Z}] \xrightarrow{\;\psi\;} \mathrm{Hom}(\Omega_{n+2}^\xi, \mathbb{Z}) \longrightarrow 0,$$

where $\mathrm{Tors}(-)$ denotes the torsion subgroup. The first and third terms in this short exact sequence have anomaly-theoretic interpretations.

- The quotient $\mathrm{Hom}(\Omega_{n+2}^\xi, \mathbb{Z})$ is a free abelian group consisting of characteristic classes of $(n+2)$-dimensional ξ-manifolds; under this identification, the map ψ sends an anomaly field theory to the corresponding anomaly polynomial, which is one degree higher, such as Chern-Simons and Chern-Weil forms. This data is visible to perturbative techniques, and is sometimes called the *local anomaly*.
- The subgroup $\mathrm{Tors}(\mathrm{Hom}(\Omega_{n+1}^\xi, \mathbb{C}^\times))$ is identified with the torsion subgroup of $[MT\xi, \Sigma^{n+2} I_\mathbb{Z}]$; these are the reflection positive invertible field theories which are topological. Such field theories' partition functions are bordism invariants, and the identification of these reflection positive invertible TFTs with $\mathrm{Tors}(\mathrm{Hom}(\Omega_{n+1}^\xi, \mathbb{C}^\times))$ assigns to a reflection positive invertible TFT its partition function. Typically this data is invisible to perturbative methods and is called the *global anomaly*.

Yamashita-Yonekura [YY23] and Yamashita [Yam23] relate the short exact sequence (8.13) to a differential refinement of $\mathrm{Map}(MT\xi, \Sigma^{n+2} I_\mathbb{Z})$.

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[25]The definition of reflection positivity for extended not-necessarily-invertible TFTs is still open: see [JF17, MS23, FHJF+24, Ste24] for work in this direction.

8.3. **Long exact sequence of invertible field theories.** Just as the map of spectra of Section 3.2 induces a Smith homomorphism on bordism, it dually induces a map of invertible field theories. Explicitly, we obtain this by mapping into the Anderson dual and taking homotopy groups. As in Section 3.1, consider a topological space X, a tangential structure $\xi: B \to BO$, a virtual bundle $V \to X$, and a vector bundle $W \to X$ of rank r. As we discuss in the end of Section 8.2.1, the tangential structure encodes both symmetries of the theory as well as the parameter space.

Definition 8.14. The *defect anomaly map*, generalized from [HKT20] section 4.2, is the map

$$(8.15) \qquad \Omega_\xi^{k-r}(X^{V+W-r}) \xrightarrow{\mathrm{Def}_W} \Omega_\xi^k(X^V)$$

of invertible field theories induced by the zero section map.

Physically, we interpret $\Omega_\xi^k(X^V)$ to be classifying anomalies of QFTs in dimension $k-1$ with symmetry according to a (X, V)-twisted ξ-structure. The group mapping in, $\Omega_\xi^{k-r}(X^{V+W-r})$, classifies the anomalies of defect theories in dimension $k - r - 1$. These defect theories are created from the bulk theory, physically, by setting a non-trivial boundary condition on a symmetry-breaking order parameter, which corresponds to a section of the vector bundle W. See [DDK+24, §III.A] for further explanation.

For clarity, we begin a running example.

Example 8.16. Recall the $\mathbb{Z}/2$ family of examples from Example 7.8 and specifically Equation (7.9d):

$$(8.17) \qquad MT\mathrm{Pin}^+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT\mathrm{Spin} \wedge (B\mathbb{Z}/2)_+.$$

Here, we take ξ according to spin bordism and twist with the tautological line bundle $W = \sigma$ over $X = B\mathbb{Z}/2$. We take $V = 3\sigma$ and apply the results of Section 6 to simplify (i.e., we use the that 4σ is spin). The corresponding map on invertible field theories is

$$(8.18) \qquad \Omega_{\mathrm{Spin}\times\mathbb{Z}/2}^{k-1} \xrightarrow{\mathrm{Def}_\sigma} \Omega_{\mathrm{Pin}^+}^k.$$

Physically, we begin with a field theory with a pin$^+$ symmetry; that is, a fermionic theory with an additional time reversal symmetry T that squares to fermion parity: $T^2 = (-1)^F$. We require the physical assumption that there is a $\mathbb{Z}/2$-odd bosonic operator ϕ such that the theory is gapped when the theory is deformed by ϕ. One example is the Majorana mass term for $2 + 1$D Majorana fermions (see [DDK+24, §III.A.1]). We can define the theory on any manifold M with a pin$^+$ structure $P: M \to B\mathrm{Pin}^+$ on its tangent bundle. If we choose generic configuration for the ϕ field, i.e., choose a section of the tautological line bundle $P^*\sigma$, we arrive at an effective theory whose excitations are localized at the zero set of ϕ. We view this as a theory in one dimension lower, which is called a domain wall theory. Note that the bordism class of the zero set of ϕ is precisely the image of $[M]$ under sm_σ in (8.17). The domain wall theory no longer has the symmetry of the bulk theory: instead, it is a fermionic theory with a unitary internal symmetry U squaring to 1; i.e. it has $\mathrm{Spin} \times \mathbb{Z}/2$ tangential structure. We direct the reader to [DDK+24, §III.A] and [HKT20, §3.1] for more details.

In this context, anomaly matching refers to the process of identifying pairs of preimages and images of anomaly classes under the defect anomaly map Def_W. To perform anomaly matching, we must understand not only the two groups classifying the possible bulk and defect theories, but

Smith fiber sequences	SBLES
Example 7.26	[DDK$^+$24, §IV.A]
Example 7.6	[DDK$^+$24, §IV.B]
Example 7.8	[DDK$^+$24, §IV.C]
Example 7.33	[DDK$^+$24, §IV.D & E]
Examples 7.39 and 7.45	[DDK$^+$24, §IV.F]

TABLE 1. Here we present a cross-list of Smith fiber sequences from this paper with the corresponding symmetry breaking long exact sequences (SBLES) that we study in [DDK$^+$24, §IV].

also the kernel and cokernel of the map Def_W. In some cases, one may deduce that information from an understanding of explicit bordism generators, but in general this approach is difficult. To address this question, we derived the map of spectra and identified its fiber, forming the Smith fiber sequence of Section 5. Now, just as for bordism, we may form a long exact sequence.

Corollary 8.19. *Applying $I_{\mathbb{Z}}$ to the cofiber sequence* (5.3), *we obtain the following long exact sequence of Anderson-dualized bordism groups, or in light of Theorem 8.11, groups of invertible field theories:*

(8.20)
$$\cdots \longrightarrow \Omega_\xi^{k-r}(X^{V+W-r}) \xrightarrow{\mathrm{Def}_W} \Omega_\xi^k(X^V) \xrightarrow{\mathrm{Res}_W} \Omega_\xi^k(S_X(W)^V) \xrightarrow{\mathrm{Ind}_W} \Omega_\xi^{k-r+1}(X^{V+W-r}) \longrightarrow \cdots$$

This long exact sequence is our mathematical model for the symmetry-breaking long exact sequence (SBLES) of [DDK$^+$24] (see Table 1). In addition to the defect anomaly map Def_W defined in Definition 8.14, we call Res_W the *residual anomaly map* and Ind_W the *index anomaly map*.

$$\Omega_{\mathrm{Spin}\times\mathbb{Z}/2}^{k-1} \xrightarrow{\mathrm{Def}_\sigma} \Omega_{\mathrm{Pin}^+}^k \xrightarrow{\mathrm{Res}_\sigma} \Omega_{\mathrm{Spin}}^k$$

-1	0	0	$\mathbb{Z}$
0	$\mathbb{Z} \longrightarrow \mathbb{Z}/2$		0
1	0	0	$\mathbb{Z}/2$
2	$(\mathbb{Z}/2)^2 \longrightarrow \mathbb{Z}/2$		$\mathbb{Z}/2$
3	$(\mathbb{Z}/2)^2 \longrightarrow \mathbb{Z}/2$		$\mathbb{Z}$
4	$\mathbb{Z} \oplus \mathbb{Z}/8 \longrightarrow \mathbb{Z}/16$		0

FIGURE 1. Long exact sequence of field theories associated to Equation (7.9d). Observe that all maps in low degrees are determined by exactness. This long exact sequence also appears in [DDK$^+$24, §IV.C]

Example 8.21. In degree $k = 4$, there is a map of invertible field theories

$$(8.22) \qquad \Omega^3_{\text{Spin} \times \mathbb{Z}/2} \cong \mathbb{Z} \oplus \mathbb{Z}/8 \xrightarrow{\text{Def}_\sigma} \mathbb{Z}/16 \cong \Omega^4_{\text{Pin}+}.$$

The $\mathbb{Z}/16$ classifies anomalies of Majorana fermions in $2+1$ dimensions. An associated domain wall theory often has $1 + 1$d chiral fermion modes. To answer the question of what particular chiral fermions can live on the domain wall, we need to identify the map Def_σ in (8.22). To do so, we turn to the long exact sequence of invertible field theories, which we draw out in Figure 1.

By exactness, we deduce that the defect matching map in degree $k = 4$ sends $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}/8$ to $-a + 2b \in \mathbb{Z}/16$. This matches with the physical computation in [HKT20, §3.1], and its physical significance is discussed further there and in [DDK$^+$24, §III.C.3].

As a computational tool, the long exact sequence allows us to determine the defect matching maps with ease. Moreover, the other two maps in the SBLES (8.20) also have physical interpretations. The residual anomaly map classifies the obstruction to gapping a QFT after symmetry breaking, as we explain in [DDK$^+$24, §III.B], while the index anomaly map generalizes the relationship between Berry phases and the ground-state degeneracy in $0 + 1$D systems; see [DDK$^+$24, §III.C].

APPENDIX A. THE LONG EXACT SEQUENCE IN BORDISM

In this appendix, we explicitly describe the Smith long exact sequence of bordism groups and work through an example. As in Section 5, let (X, ξ) be a stable tangential structure, let V be a virtual bundle over X, and let W be a real vector bundle over X of rank r. The corresponding long exact sequence of bordism groups of Corollary 8.19 is

$$(A.1)$$
$$\cdots \to \Omega^\xi_k(S_X(W)^{p^*V}) \xrightarrow{p} \Omega^\xi_k(X^V) \xrightarrow{\text{sm}_W} \Omega^\xi_{k-r}(X^{V+W-r}) \xrightarrow{\delta} \Omega^\xi_{k-1}(S_X(W)^{p^*V}) \to \cdots$$

Here, $p \colon S_X(W) \to X$ is the projection, sm_W is the Smith homomorphism, and δ is the connecting map. In this section, we will be explicit about taking the pullback p^*V of V to $S_X(W)$.

Starting from the left, $\Omega^\xi_k(S_X(W)^{p^*V})$ is the bordism group of k-manifolds M equipped with a map $f \colon M \to S_X(W)$ together with a ξ-structure on $TM \oplus f^*(p^*V)$. Next, $\Omega^\xi_k(X^V)$ is the bordism group of k-manifolds equipped with a map to X with the analogous twisted ξ-structure, and $\Omega^\xi_{k-r}(X^{V+W-r})$ is the bordism group of $(k - r)$-manifolds N equipped with a map g to X with a ξ-structure on $TM \oplus g^*V \oplus g^*W$. Note that the Smith homomorphism lowers the dimension by r and twists the tangential structure condition by W.

Now we describe each map at the level of manifolds.

(1) p: Let M be a closed k-manifold equipped with a map $h \colon M \to S_X(W)$ such that $TM \oplus h^*V$ has a ξ-structure, so that M represents a bordism class in $\Omega^\xi_k(S_X(W)^{p^*V})$. The image of M under p is represented by the same manifold M with an (X, V)-twisted ξ-structure given by the composition with the projection. That is, equip M with the map $M \xrightarrow{h} S_X(W) \xrightarrow{p} X$.

(2) sm_W: Now let M be a closed k-manifold equipped with a map $f \colon M \to X$ such that $TM \oplus f^*V$ has a ξ-structure. Let $s \colon M \to W$ be a generic section, which is transverse to the zero section s_0. Then, the intersection $N := s(M) \pitchfork s_0(M)$ is a $(k - r)$-dimensional manifold. Let g be the composite $g \colon N \hookrightarrow M \xrightarrow{f} X$. Since the normal bundle ν to N satisfies $\nu \cong f^*W|_N = g^*W$, $TM|_N \cong TN \oplus \nu \cong TN \oplus g^*W$, and hence N carries an

$(X, V + W)$-twisted ξ-structure coming from the (X, V)-twisted ξ-structure on M. We have $\mathrm{sm}_W \colon M \mapsto N$.

(3) δ: This is the connecting map in the long exact sequence. Start with a closed $k-r$-manifold N with an $(X, V + W)$-twisted ξ-structure given by, as above, $g \colon N \to X$ and a ξ-structure on $TN \oplus g^*V \oplus g^*W$. Consider the sphere bundle $S_N(g^*W)$ of W restricted to N: it has a map to $S_X(W)$ given by inclusion.

We claim that $S_N(g^*W)$ is the image under δ of N, but it remains to show that $S_N(g^*W)$ has the appropriate tangential structure. This will be a corollary of a general splitting result of tangent bundles of sphere bundles.

Lemma A.2. *For any vector bundle $\pi \colon V \to B$, there is an isomorphism of vector bundles, canonical up to a contractible space of choices,*

$$(A.3) \qquad TS(V) \oplus \mathbb{R} \xrightarrow{\cong} \pi^*(TB) \oplus \pi^*(V).$$

Proof. Choose a metric and connection on V; both of these are contractible choices. For any fiber bundle $\pi \colon E \to B$ of smooth manifolds, the choice of connection splits TE as a direct sum of the horizontal subbundle, which is isomorphic to $\pi^*(TB)$, and the vertical tangent bundle $T_vE = \ker(\pi_*)$, which when pulled back to a fiber is the tangent bundle of that fiber.

Let ν be the normal bundle of $S(V) \hookrightarrow V$. Then there is a canonical isomorphism $T_vS(V) \oplus \nu \cong \pi^*(V)$, which is a parametrized version of the standard isomorphism $TS^n \oplus \nu_{S^n \hookrightarrow \mathbb{R}^{n+1}} \cong \mathbb{R}^{n+1}$. Combining this with the previous paragraph,

$$(A.4) \qquad TS(V) \oplus \nu \cong \pi^*(TB) \oplus T_vS(V) \oplus \nu \cong \pi^*(TB) \oplus \pi^*(V),$$

and the fiberwise outward unit normal vector field trivializes ν. $\qquad \square$

If we now analyze the vertical and horizontal pieces of the tangent bundle to $S_N(g^*W)$, we find that $T(S_N(g^*W)) \oplus \mathbb{R} \cong p^*TN \oplus p^*g^*W$. Then, we can pull back the relationship describing the tangential structure of N to see that $p^*TN \oplus p^*g^*W \oplus p^*g^*V$ over $S_N(g^*W)$ has a ξ-structure. So, $T(S_N(g^*W)) \oplus \mathbb{R} \oplus p^*g^*V$ has a ξ-structure, and thus $S(g^*W)$ has a $(S_X(W), p^*V)$-twisted ξ-structure.

Let us now go through the long exact sequence of bordism groups for the Smith map 7.38. In this case, the Smith homomorphism is a map

$$(A.5) \qquad \mathrm{sm}_{2\sigma} \colon \Omega_k^{\mathrm{Pin}^-} \longrightarrow \Omega_{k-2}^{\mathrm{Pin}^+}$$

between the bordism group of k-dimensional pin$^-$ manifolds to the bordism group of $(k-2)$-dimensional pin$^+$ manifolds, described by sending a pin$^-$ manifold M to any closed submanifold N whose homology class is Poincaré dual to $w_1(M)^2$. Alternatively, in view of Definition 3.7, we could define $\mathrm{sm}_{2\sigma}$ by choosing a section s of the pullback of 2σ to M transverse to the zero section, then letting N be the zero locus of s. Recall from Example 6.23 that a pin$^-$ structure is a trivialization of $w_1(M)^2 + w_2(M)$, while a pin$^+$ structure on M is equivalent to a trivialization of $w_2(M)$. Equivalently, a pin$^-$ manifold M admits a spin structure on $TM \oplus \det(M)$, while a pin$^+$ manifold M admits a spin structure on $TM \oplus 3 \det(M)$. These conditions mean that if N is Poincaré dual to $w_1(M)^2$ inside a pin$^-$ manifold M, then N acquires a pin$^+$ structure.

The third set of groups in this long exact sequence corresponds to the homotopy groups of the fiber, $MT\mathrm{Spin} \wedge \Sigma^{-1}\mathbb{R}P^2$. By Pontrjagin-Thom, these are the groups $\widetilde{\Omega}_{*+1}^{\mathrm{Spin}}(\mathbb{R}P^2)$: bordism

groups of spin manifolds X equipped with maps $f\colon X \to \mathbb{RP}^2$, modulo the subgroup for which f is null-homotopic. Equivalently, we may consider the twisted bordism groups $\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$. Elements of this group are represented by manifolds N with maps $f\colon N \to \mathbb{RP}^1$ such that $TN \oplus f^*\sigma$ is spin.

We next describe the other two maps that appear alongside $\mathrm{sm}_{2\sigma}$ in the bordism long exact sequence and provide several lemmas that help us understand the geometry.

Definition A.6. Define a map $p\colon \Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \to \Omega_*^{\mathrm{Pin}^-}$ by sending $(N, f\colon N \to \mathbb{RP}^1)$ to N.

Lemma A.7. *If N has an $(\mathbb{RP}^1, \sigma)$-twisted spin structure, then N has a canonical pin$^-$ structure (so the map p lands in pin$^-$ bordism as claimed).*

Proof. The orientation of $TN \oplus f^*\sigma$ is equivalent data to an isomorphism $\mathrm{Det}(TN) \overset{\cong}{\to} f^*\sigma$, so we obtain a spin structure on $TN \oplus \mathrm{Det}(TN)$, i.e. a pin$^-$ structure. $\qquad\square$

The third map in the long exact sequence is the connecting map $\delta\colon \Omega_*^{\mathrm{Pin}^+} \to \Omega_{*+1}^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$. The map δ sends a pin$^+$ manifold M to the total space of the sphere bundle $S(2\mathrm{Det}(TM))$.[26] The key to understanding δ is showing that $S(2\mathrm{Det}(TM))$ has a $(\mathbb{RP}^1, \sigma)$-twisted spin structure; in particular, we must cook up a map to $\mathbb{RP}^1$.

Definition A.8. Given a pin$^+$ manifold M, choose a metric on $\mathrm{Det}(TM)$ (a contractible choice); then, given $x \in M$ and $p, q \in \sigma_x$ with $\sqrt{|p^2| + |q^2|} = 1$, so that $(x, p, q) \in S(2\mathrm{Det}(TM))$, the two sections of $\pi^*(2\mathrm{Det}(TM))$

$$
\begin{aligned}
(x, p, q) &\mapsto (p, q) \\
(x, p, q) &\mapsto (-q, p)
\end{aligned}
\tag{A.9}
$$

are everywhere linearly independent, so $\pi^*(2\mathrm{Det}(TM))$ is canonically trivial. This allows us to define a map $\varphi_M\colon S(2\mathrm{Det}(TM)) \to \mathbb{RP}^1$: given $(x, p, q) \in S(2\mathrm{Det}(TM))$ as above, $(p, q) \in (\pi^*(2\mathrm{Det}(TM)))_{(x,p,q)}$, which is canonically identified with $\mathbb{R}^2$, send (p, q) to its image $[p : q] \in \mathbb{RP}^1$ (using that p and q are never both 0).

Definition A.10. Let $\delta\colon \Omega_*^{\mathrm{Pin}^+} \to \Omega_{*+1}^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ be the map sending $M \mapsto (S(2\mathrm{Det}(TM)), \varphi_M)$, where φ_M is defined above in Definition A.8.

If $\sigma \to \mathbb{RP}^1$ is the Möbius bundle, then $\varphi_M^*(\sigma) = \pi^*(\mathrm{Det}(TM))$.

Lemma A.11. *If M is pin$^+$, $(S(2\mathrm{Det}(TM)), \varphi_M)$ has a canonical $(\mathbb{RP}^1, \sigma)$-twisted spin structure, up to a contractible space of choices, so that δ lands in $\Omega_{*+1}^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ as claimed.*

Proof. Plugging in $V = 2\mathrm{Det}(TM)$ to Lemma A.2, we learn

$$
TS(2\mathrm{Det}(TM)) \oplus \underline{\mathbb{R}} \cong \pi^*(TM) \oplus 2\pi^*(\mathrm{Det}(TM)).
\tag{A.12a}
$$

Since $\varphi_M^*(\sigma) \cong \pi^*(\mathrm{Det}(TM))$,

$$
TS(2\mathrm{Det}(TM)) \oplus \varphi_M^*(\sigma) \oplus \underline{\mathbb{R}} \cong \pi^*(TM) \oplus 3\pi^*(\mathrm{Det}(TM)).
\tag{A.12b}
$$

Since M is pin$^+$, the right-hand-side of (A.12b) is spin, so the left-hand side is too; by two-out-of-three, this means $TS(2\mathrm{Det}(TM)) \oplus \varphi_M^*(\sigma)$ is also spin. $\qquad\square$

---

[26]Note that $S(2\mathrm{Det}(TM)) \simeq S(g^*(2\sigma))$.

The maps $\mathrm{sm}_{2\sigma}$, p, and δ assemble into a long exact sequence in bordism, as we will draw in Figure 3. To write out this long exact sequence, we need to know the relevant bordism groups in low dimensions. Giambalvo [Gia73b, §2, §3] computes $\Omega_k^{\mathrm{Pin}^+}$ for $k \leq 12$, more than good enough for us, and gives generating manifolds in all degrees we need except $k = 2, 3$ (though see [KT90a] for a correction); the rest were given by Kirby-Taylor [KT90b, Proposition 3.9, Theorem 5.1]. Anderson-Brown-Peterson [ABP69, Theorem 5.1] computed pin$^-$ bordism groups, with generating manifolds again described by Giambalvo [Gia73b, Theorem 3.4] and Kirby-Taylor [KT90b, Theorem 2.1]. However, the twisted spin bordism of $\mathbb{RP}^1$ is less well-documented, so we calculate it here, using another Smith homomorphism.

Lemma A.13. *There is an abelian group A of order 4 such that*

$$(A.14) \qquad \Omega_k^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \cong \begin{cases} \mathbb{Z}/2, & k = 0, 1, 3, 4 \\ A, & k = 2 \\ 0, & k = 5. \end{cases}$$

Proof. We may start the computation of $\Omega^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ using the observation of Kirby and Taylor [KT90b] that the degree two map

$$(A.15) \qquad \mathbb{S} \xrightarrow{\cdot 2} \mathbb{S} \longrightarrow \Sigma_+^{\infty-1}\mathbb{RP}^2$$

of Example 7.13 induces multiplication by two on spin bordism. Taking the spin bordism long exact sequence of A.15 and inputting the spin bordism of a point, we may deduce the groups $\Omega^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ in low dimensions, up to one ambiguity, as indicated in Figure 2. □
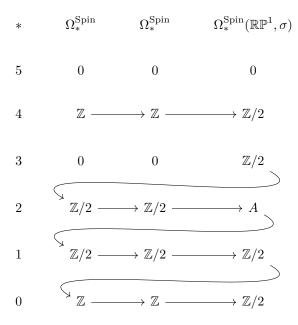

FIGURE 2. Long exact sequence in spin bordism partially determining $\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$

Remark A.16. To address the question as to whether A is isomorphic to $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, one could appeal to geometric arguments or an Adams spectral sequence calculation, but it turns out that the Smith long exact sequence that we will study in Figure 3 provides a cleaner argument that $A \cong \mathbb{Z}/4$.
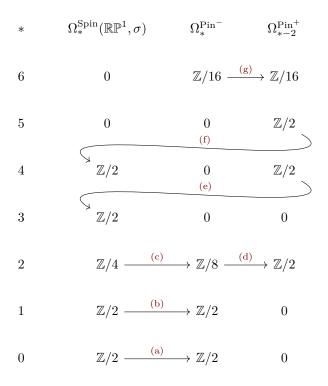
$*$	$\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$	$\Omega_*^{\mathrm{Pin}^-}$	$\Omega_{*-2}^{\mathrm{Pin}^+}$
6	0	$\mathbb{Z}/16 \xrightarrow{\ \text{(g)}\ } \mathbb{Z}/16$	
5	0	0	$\mathbb{Z}/2$
4	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
3	$\mathbb{Z}/2$	0	0
2	$\mathbb{Z}/4 \xrightarrow{\ \text{(c)}\ } \mathbb{Z}/8 \xrightarrow{\ \text{(d)}\ } \mathbb{Z}/2$		
1	$\mathbb{Z}/2 \xrightarrow{\ \text{(b)}\ } \mathbb{Z}/2$		0
0	$\mathbb{Z}/2 \xrightarrow{\ \text{(a)}\ } \mathbb{Z}/2$		0

(f) connects row 5 $\Omega_{*-2}^{\mathrm{Pin}^+}$ to row 4 $\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma)$; (e) connects row 4 $\Omega_{*-2}^{\mathrm{Pin}^+}$ to row 3 $\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma)$.

FIGURE 3. Bordism Long Exact Sequence for $\mathrm{Pin}^- \rightsquigarrow \mathrm{Pin}^+$

We will provide some explicit descriptions of the interesting maps in this sequence using knowledge of the generators of each bordism group, which for pin^+ and pin^- may be found in [KT90b]. For the twisted spin bordism of $\mathbb{RP}^1$, we use what we learned in Lemma A.13.

(a) $* = 0$: The group $\Omega_0^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \cong \mathbb{Z}/2$ is generated by the class of the point equipped with the inclusion i into $\mathbb{RP}^1$. The condition of $T\mathrm{pt} \oplus i^*\sigma$ being spin is satisfied since $i^*\sigma$ is trivial. The map f forgets i, so sends this generator to the point with its pin^- structure, which is a generator of $\Omega_0^{\mathrm{Pin}^-} \cong \mathbb{Z}/2$.

(b) $* = 1$: Consider the circle with spin structure induced from its Lie group framing, denoted S_{nb}^1, equipped with the degree two map $\phi \colon S^1 \to S^1 \simeq \mathbb{RP}^1$. If $x \in H^1(\mathbb{RP}^1; \mathbb{Z}/2)$ is the generator, we have

$$(A.17) \qquad w(TS^1 \oplus \phi^*\sigma) = w(TS^1)\phi^*w(\sigma) = (1)(1 + 2\phi^*(x)) = 1,$$

so (S_{nb}^1, ϕ) has an $(\mathbb{RP}^1, \sigma)$-twisted spin structure. The map p forgets ϕ, so sends the bordism class of (S_{nb}^1, ϕ) to S_{nb}^1, which generates $\Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}/2$ [KT90b, Theorem 2.1].

(c) $* = 2$ (part 1): Exactness of the Smith long exact sequence at $\Omega_2^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \cong A$ implies that A maps injectively to $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$, so $A \cong \mathbb{Z}/4$, and we have resolved the extension problem from Lemma A.13.

The Klein bottle K is an S^1-bundle over $\mathbb{RP}^1$, with the monodromy of the fiber S^1 around the base given by reflection. Therefore $K = S(\sigma \oplus \mathbb{R})$ as S^1-bundles over $\mathbb{RP}^1$. Let $\pi \colon K \to \mathbb{RP}^1$ be the bundle map; then Lemma A.2 defines an isomorphism $TK \oplus \underline{\mathbb{R}} \cong \pi^*(\sigma) \oplus \underline{\mathbb{R}}^2$ (using the Lie group trivialization of $T\mathbb{RP}^1$). The Möbius bundle σ represents the nonzero class in $[\mathbb{RP}^1, BO] = \pi_1(BO) \cong \mathbb{Z}/2$, so 2σ is trivializable,[27] and in particular spin, meaning that (K, π) admits an $(\mathbb{RP}^1, \sigma)$-twisted spin structure (in fact, it admits 4).

That (K, π) generates $\Omega_2^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ depends on which of the four $(\mathbb{RP}^1, \sigma)$-twisted spin structures one chooses. Specifically, each $(\mathbb{RP}^1, \sigma)$-twisted spin structure restricts to a spin structure on the fiber S^1, and we need this to be the spin structure on S^1 induced by the Lie group framing. Two of the four $(\mathbb{RP}^1, \sigma)$-twisted spin structures satisfy this. To then see that either of these two Klein bottles generates, one can play with the Smith long exact sequence from Example 7.13

$$(A.18) \qquad \cdots \longrightarrow \Omega_k^{\mathrm{Spin}} \xrightarrow{\cdot 2} \Omega_k^{\mathrm{Spin}} \longrightarrow \Omega_k^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \xrightarrow{\mathrm{sm}_\sigma} \Omega_{k-1}^{\mathrm{Spin}} \longrightarrow \cdots$$

to see that $\mathrm{sm}_\sigma \colon \Omega_2^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \to \Omega_1^{\mathrm{Spin}}$ is the unique surjective map $\mathbb{Z}/4 \to \mathbb{Z}/2$; the Poincaré dual to $w_1(\sigma)$ is represented by the fiber S^1 in K, which we chose to have the Lie group spin structure, so $\mathrm{sm}_\sigma(K, \pi) = S_{nb}^1$, which generates Ω_1^{Spin}, implying (K, π) generates $\Omega_2^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$.

Now take $f(K, \pi)$, which amounts to forgetting π and finding the pin^- bordism class of K. The Arf-Brown-Kervaire invariant is a complete invariant $\Omega_2^{\mathrm{Pin}^-} \xrightarrow{\cong} \mathbb{Z}/8$ [Bro71, KT90b], so it suffices to compute this invariant on K, as has been explicitly worked out in [Tur20, §II.D]. Our choice of the nonbounding spin structure on the fiber implies that the Arf-Brown-Kervaire map $\Omega_2^{\mathrm{Pin}^-} \xrightarrow{\cong} \mathbb{Z}/8$ sends $[K] \mapsto \pm 2$, so $f \colon \mathbb{Z}/4 \to \mathbb{Z}/8$ sends $1 \mapsto 2$, as required by exactness.

(d) $* = 2$ (part 2): There are two pin^- structures on $\mathbb{RP}^2$, and both are generators of $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ [KT90b, §3]. Pick either of these pin^- structures; the class $w_2(\sigma) \in H^2(\mathbb{RP}^2; \mathbb{Z}/2) \cong \mathbb{Z}/2$ is a generator, and the Smith homomorphism $\Omega_2^{\mathrm{Pin}^-} \to \Omega_0^{\mathrm{Pin}^+}$ maps the input $\mathbb{RP}^2$ to the Poincaré dual of $w_2(2\sigma)$. The class $PD(w_2(2\sigma))$ is $1 \in H_0(\mathbb{RP}^2; \mathbb{Z}/2) \cong \mathbb{Z}/2$ and is represented by a single pin^+ point. The class of the point also corresponds to the zero-dimensional intersection of the zero section and a generic section of 2σ.

(e) $* = 4 \to 3$: $\Omega_2^{\mathrm{Pin}^+} \cong \mathbb{Z}/2$ is generated by the Klein bottle K, where as before we need the nonbounding spin structure on the S^1 fiber of K. The connecting map δ sends K to $S(2\mathrm{Det}(K))$; we saw above in part (c) that $\mathrm{Det}(K) \cong \sigma$ and 2σ is trivialized over K, so $S(2\mathrm{Det}(K)) \cong S^1 \times K$.

Tracking the (twisted) spin structures through this argument, one sees that we obtain the nonbounding spin structure on S^1, so $g(K) = [S_{nb}^1 \times K] \in \Omega_3^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \cong \mathbb{Z}/2$, and $[S_{nb}^1 \times K]$ is indeed the generator.[28]

---

[27]To make this argument carefully, one must know that addition in $[S^1, BO]$ corresponds to direct sum of vector bundles. A priori this is not true—addition in $[S^1, X]$ is built from the pinch map $S^1 \to S^1 \vee S^1$. That this coincides with the group structure on $[S^1, BO]$ arising from direct sum of virtual vector bundles depends on the Eckmann-Hilton argument.

[28]Another way to see this is that because the connecting morphism in the Smith long exact sequence is obtained from a map of spectra by taking homotopy groups, the connecting morphism commutes with the $\pi_*(\mathbb{S})$-actions on $\Omega_*^{\mathrm{Pin}^+}$ and $\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$. The Pontrjagin-Thom theorem identifies this $\pi_*(\mathbb{S})$-action on bordism groups with

(f) $* = 5 \to 4$: $\Omega_3^{\mathrm{Pin}^+} \cong \mathbb{Z}/2$ is generated by $S_{\mathrm{nb}}^1 \times K$ [KT90b, §5], and $\Omega_4^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \cong \mathbb{Z}/2$ is generated by $S_{nb}^1 \times S_{nb}^1 \times K$, with the map to $\mathbb{RP}^1$ induced from the fiber bundle $K \to \mathbb{RP}^1$ from part (c).[29] Thus the story is the same as in (e), crossed with S_{nb}^1.

(g) $* = 6$: The group $\Omega_6^{\mathrm{Pin}^-}$ is generated by $\mathbb{RP}^6$ with either of its two pin$^+$ structures, while $\Omega_4^{\mathrm{Pin}^+}$ is generated by $\mathbb{RP}^4$ with either of its two pin$^-$ structures. Since the normal bundle to $\mathbb{RP}^4$ inside $\mathbb{RP}^6$ is indeed the restriction of 2σ, $\mathbb{RP}^4$ represents the Poincaré dual homology class to $e(2\sigma)$ and is the image of the Smith homomorphism applied to $\mathbb{RP}^6$.

Appendix B. Why we use the cobordism, rather than the cohomology, Euler class

The Smith homomorphism is often defined by taking a Poincaré dual of the $\mathbb{Z}$- or $\mathbb{Z}/2$-cohomology Euler class of a vector bundle $V \to X$, for example in [KTTW15, COSY20, HKT20]. However, in Definition 4.27, we used a different and more abstract definition: the Smith homomorphism for twisted ξ-bordism should use the (possibly twisted) ξ-cobordism Euler class. The purpose of this appendix is to explain that additional effort: we will walk through a concrete, low-dimensional example where the cohomological Euler class does not produce a well-defined Smith homomorphism, and show that the cobordism Euler class does suffice.

Recall that a spin^h structure is a $(B\mathrm{SO}(3), V_3)$-twisted spin structure, where $V_3 \to B\mathrm{SO}(3)$ is the tautological vector bundle. Then, as we discussed in (7.47a), there is a Smith homomorphism

$$(\mathrm{B.1}) \qquad \mathrm{sm}_V \colon \Omega_k^{\mathrm{Spin}^h} \to \Omega_{k-3}^{\mathrm{Spin}}(B\mathrm{SO}(3)).$$

Theorem B.2. *Give S^4 the spin^h structure whose SO_3-bundle is classified by either map $S^4 \to B\mathrm{SO}(3)$ whose homotopy class generates $\pi_4(B\mathrm{SO}(3)) \cong \mathbb{Z}$.*

(1) Exactness forces $\mathrm{sm}_V(S^4)$ to be the bordism class of S_{nb}^1 with constant map to $B\mathrm{SO}(3))$ in $\Omega_1^{\mathrm{Spin}}(B\mathrm{SO}(3))$.

(2) $e(V) \in H^3(S^4; \mathbb{Z}) = 0$, and there is no way to assign every smooth representative of the Poincaré dual of $e(V)$ a spin structure whose bordism class equals that of S_{nb}^1.

(3) The spin cobordism Euler class of V is nonzero, and all smooth representatives of its Poincaré dual have the spin bordism class of S_{nb}^1 and a constant map to $B\mathrm{SO}(3)$.

This is why we use cobordism Euler classes.

We work with $\xi = \mathrm{Spin}$ and its twists throughout this appendix; see Remark B.39 for other tangential structures. Let ko denote the connective real K-theory spectrum; work of Anderson-Brown-Peterson [ABP67] shows that the Atiyah-Bott-Shapiro map $MT\mathrm{Spin} \to ko$ [ABS64] is 7-connected, meaning that as long as we restrict to manifolds of dimension 7 and below, we may replace twisted spin bordism with twisted ko-homology; in particular, we will work with ko-cohomology Euler classes.

Another consequence of the Atiyah-Bott-Shapiro map is that vector bundles with spin structure are oriented for ko-cohomology, meaning that if $V \to X$ is a spin vector bundle, the Euler class

---

taking products with stably framed manifolds; focusing specifically on the nonzero element of $\pi_1(\mathbb{S})$, which is represented by the bordism class of S_{nb}^1. Thus, since $\times S_{nb}^1 \colon \Omega_2^{\mathrm{Pin}^+} \to \Omega_3^{\mathrm{Pin}^+}$ is an isomorphism [KT90b, §5] and the Smith maps $\Omega_{k-2}^{\mathrm{Pin}^+} \to \Omega_k^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ are isomorphisms for $k = 3, 4$ as we saw in the long exact sequence, then $\times S_{nb}^1 \colon \Omega_3^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \to \Omega_4^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ is also an isomorphism.

[29]Another choice of generator is the K3 surface with trivial map to $\mathbb{RP}^1$, as follows from (A.18). The complicated topology of the K3 surface makes this generator harder to work with explicitly.

$e^{ko}(V)$ that a priori lives in $ko^r(X^{V-r})$ in fact can be passed by the Thom isomorphism to $e^{ko}(V) \in ko^r(X)$.

Recall the exceptional isomorphism $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$, and recall that $ko^* \cong \mathbb{Z}[\eta, v, w]/(2\eta, \eta^3, 2v, 4w - v^2)$ with $|\eta| = -1$, $|v| = -4$, and $|w| = -8$.[30]

The following result is stated without proof by Davis-Mahowald [DM79, §2]; see Bruner-Greenlees [BG10, Theorem 5.3.1] for a proof.

Proposition B.3. *There is an isomorphism of ko^*-modules $ko^*(B\mathrm{Sp}(1)) \cong ko^*[[p_1^{\mathbb{H}}]]$ with $|p_1^{\mathbb{H}}| = 4$.*

The class $p_1^{\mathbb{H}}$ is called the *first symplectic ko-Pontrjagin class.* The specific isomorphism in Proposition B.3 can be fixed uniquely by requiring that the image of $p_1^{\mathbb{H}}$ under $ko \to H\mathbb{Z}$ is the usual first symplectic Pontrjagin class, which is positive on the tautological quaternionic line bundle over $\mathbb{HP}^1$.

Given a spin vector bundle $V \to X$, let $\mathcal{S}_V \to X$ be the associated spinor bundle, which is the quaternionic line bundle associated to the accidental isomorphism $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$.

Theorem B.4. *Let $V \to X$ be a rank-3 vector bundle with spin structure. Then $e^{\mathbb{Z}}(V) \in H^3(X; \mathbb{Z})$ and $e^{\mathbb{Z}/2}(V) \in H^3(X; \mathbb{Z}/2)$ both vanish, and*

$$(B.5) \qquad\qquad e^{ko}(V) = \eta p_1^{\mathbb{H}}(\mathcal{S}_V) \in ko^3(X).$$

Remark B.6. Theorem B.4 is new as far as we know. It is a subtle result in that several standard techniques for computing *ko*-Euler classes do not provide any information.

(1) Analogous to the formula for *ku*-Euler classes of complex vector bundles, there is a formula for *ko*-Euler classes of quaternionic vector bundles (see, e.g., Davis-Mahowald [DM79]), but a rank-3 vector bundle cannot be quaternionic.

(2) For non-quaternionic vector bundles, one could compare with Euler classes in *ku*-cohomology or ordinary cohomology, as Davis-Mahowald (*ibid.*, §2) do, but $H^3(B\mathrm{Spin}(3); \mathbb{Z}) = 0$, so comparing with the $\mathbb{Z}$-cohomological Euler class provides no information. Moreover, $ku^*(B\mathrm{Spin}(3))$ is a free ku^*-algebra on generators in even degrees [BG10, Theorem 5.3.1], so $ku^3(B\mathrm{Spin}(3)) = 0$, and therefore we can learn nothing even by comparing to *ku*.

(3) It is more fruitful to compare to *KO*-Euler classes, understood in many cases (see [Cra91, Corollary 3.37(i)] and [FH00, Footnote 13]), but not in rank 3.

(4) The use of the splitting principle to compute Euler classes is stymied by the fact that maximal tori in $\mathrm{Spin}(3)$ can be conjugated into the usual embedding $\mathrm{Spin}(2) \to \mathrm{Spin}(3)$, so the pullback of the Euler class to the maximal torus vanishes, as the pulled-back vector bundle will have a nonvanishing section.

Taking Theorem B.4 for granted now, let us dig into Theorem B.2.

Proof of Theorem B.2 assuming Theorem B.4. Recall from (7.47a) that the Smith homomorphism $\mathrm{sm}_V \colon \Omega_k^{\mathrm{Spin}^h} \to \Omega_{k-3}^{\mathrm{Spin}}(B\mathrm{SO}(3))$ belongs to a long exact sequence whose third term is spin^c bordism:

$$(B.7) \qquad \cdots \to \Omega_2^{\mathrm{Spin}}(B\mathrm{SO}(3)) \to \Omega_4^{\mathrm{Spin}^c} \to \Omega_4^{\mathrm{Spin}^h} \overset{\mathrm{sm}_V}{\to} \Omega_1^{\mathrm{Spin}}(B\mathrm{SO}(3)) \to \Omega_3^{\mathrm{Spin}^c} \to \cdots$$

From Stong [Sto68, Chapter XI] we know $\Omega_3^{\mathrm{Spin}^c} = 0$ and $\Omega_4^{\mathrm{Spin}^c} \cong \mathbb{Z}^2$, from Freed-Hopkins [FH21, Theorem 9.97] we know $\Omega_4^{\mathrm{Spin}^h} \cong \mathbb{Z}^2$, and from Wan-Wang [WW19, §5.5.3] we know $\Omega_1^{\mathrm{Spin}}(B\mathrm{SO}(3)) \cong \mathbb{Z}/2$ and $\Omega_2^{\mathrm{Spin}}(B\mathrm{SO}(3))$ is torsion. Plugging this into (B.7), we see that sm_V is surjective.

---

[30]The negative grading is a feature of generalized cohomology: for any spectrum E, $E^k(\mathrm{pt}) = E_{-k}(\mathrm{pt}) = \pi_{-k}(E)$.

Wan-Wang's argument implies that the map $\Omega_1^{\mathrm{Spin}} \to \Omega_1^{\mathrm{Spin}}(B\mathrm{SO}(3))$ choosing the trivial SO(3)-bundle is an isomorphism, so the generator of $\Omega_1^{\mathrm{Spin}}(B\mathrm{SO}(3))$ is any nonbounding spin 1-manifold with trivial SO(3)-bundle. Hu [Hu23, Appendix A] shows that $\mathbb{CP}^2$ and S^4 generate $\Omega_4^{\mathrm{Spin}^h}$, where $\mathbb{CP}^2$ has spin^h structure induced from its spin^c structure via the standard inclusion $\mathrm{U}(1) \cong \mathrm{SO}(2) \to \mathrm{SO}(3)$, and S^4 has spin^h structure whose principal SO(3)-bundle $V \to S^4$ is induced from the tautological quaternionic line bundle on $\mathbb{HP}^1 \cong S^4$: this has an associated Sp(1)-bundle, and we quotient by $\{\pm 1\}$ to get an SO(3)-bundle. In particular, $\mathbb{CP}^2$ is in the image of $\Omega_4^{\mathrm{Spin}^c} \to \Omega_4^{\mathrm{Spin}^h}$, so because sm_V is surjective, $\mathrm{sm}_V(S^4, V)$ must be S_{nb}^1 with trivial map to $B\mathrm{SO}(3)$, proving the first part of the theorem.

Because $H^3(S^4; \mathbb{Z})$ and $H^3(S^4; \mathbb{Z}/2)$ both vanish, the $\mathbb{Z}$ and $\mathbb{Z}/2$ cohomology Euler classes of V are zero. Therefore any null-homologous 1-manifold in S^4 (i.e. any closed, oriented 1-manifold mapping to S^4) is a smooth representative of the Poincaré dual of $e(V)$. Most of these manifolds, such as the standard $S^1 \subset S^4$, can be given a nonbounding spin structure, but the empty submanifold cannot, even though it is Poincaré dual to $e(V)$. This proves the second part of the theorem.

As discussed above, the Atiyah-Bott-Shapiro map is 7-connected, and therefore for discussing degree-3 spin cobordism of S^4, we may use ko-cohomology without losing information. The Atiyah-Hirzebruch spectral sequence quickly implies

$$(B.8) \qquad ko^*(S^4) \cong ko^*[z]/(z^2), \ |z| = 4.$$

In particular, $ko^3(S^4) \cong \mathbb{Z}/2$, generated by ηz.

The spinor bundle of V is the quaternionic line bundle associated to the identification $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$. Since V came from the identification $S^4 \cong \mathbb{HP}^1$, the spinor bundle of V is the tautological quaternionic line bundle $L_{\mathbb{H}} \to \mathbb{HP}^1$. This is classified by the inclusion $j \colon \mathbb{HP}^1 \to \mathbb{HP}^\infty \simeq B\mathrm{Sp}(1)$ as the 4-skeleton; considering the map of Atiyah-Hirzebruch spectral sequences for ko-cohomology induced by j shows that $p_1^{\mathbb{H}} \in ko^4(B\mathrm{Sp}(1))$ pulls back by j to $z \in ko^4(S^4)$. Thus by Theorem B.4, $e^{ko}(V) = \eta z \neq 0$ in $ko^3(S^4)$.

Because $ko^3(S^4)$ has only one nonzero element, the Poincaré dual of the nonzero element must be the unique nonzero element x of $ko_1(S^4) \cong \mathbb{Z}/2$. Pulling back to spin bordism, the same argument we made for $B\mathrm{SO}(3)$ shows that the smooth representatives of x are precisely the nonbounding spin 1-manifolds with null-bordant map to S^4—and composing with the map $S^4 \to B\mathrm{SO}(3)$ classifying V, we have shown that every smooth representative of the Poincaré dual of $e^{ko}(V)$ (hence also the spin cobordism Euler class) represents the image of (S^4, V) under the Smith homomorphism. $\qquad \square$

The rest of this appendix is devoted to proving Theorem B.4.

Lemma B.9. *Let X be a CW complex with finitely many cells in each dimension, and whose cells are concentrated solely in even degrees. Suppose that the images of the attaching maps of X in ko-homology are never of the form $w^s \eta$ times any other class, where $s > 0$. Then there is an equivalence of ko-module spectra from $ko \wedge X_+$ to a sum of shifts of copies of ko and ku.*

Here ku is a ko-module in the usual way, i.e. through the complexification map $c \colon ko \to ku$. In essence, this is downstream from the way in which $\mathbb{C}$ is an $\mathbb{R}$-module.

Proof. It suffices to prove this when X is a finite-dimensional CW complex, and then take the colimit. Thus we may induct on the dimension of X, as the result is vacuously true when X is 0-dimensional.

If X is n-dimensional (so n is even), with $(n-2)$-skeleton X', then X is the cofiber of the map

$$\text{(B.10a)} \qquad \bigvee_{i=1}^{N} S^{2n-1} \longrightarrow X',$$

classified by $(f_1, \dots, f_N) \in \pi_{2n-1}(X')$, which attaches the n-cells of X. Smash with ko and apply the inductive assumption to deduce that $ko \wedge X$ is the cofiber of a map of ko-modules

$$\text{(B.10b)} \qquad (ko \wedge f_1, \dots, ko \wedge f_N)\colon \bigvee_{i=1}^{N} \Sigma^{2n-1} ko \longrightarrow \bigvee_{i \in \mathcal{I}} \Sigma^{2i} ko \vee \bigvee_{j \in \mathcal{J}} \Sigma^{2j} ku.$$

A map of ko-modules $\Sigma^\ell ko \to M$ is equivalent data to a map of spectra $\Sigma^\ell \mathbb{S} \to M$. Therefore the homotopy class of each $ko \wedge f_i$ is an element of

$$\text{(B.11)} \qquad \bigoplus_{i \in \mathcal{I}} \pi_{2i-(2n-1)} ko \oplus \bigoplus_{j \in \mathcal{J}} \pi_{2j-(2n-1)} ku,$$

and knowledge of these classes for $1 \le i \le N$ suffices to recover $ko \wedge X$ as the cofiber. Moreover, we can compute the cofiber by attaching one sphere at a time, computing the cofiber, and continuing.

The first observation is that $ko \wedge f_i$ is trivial on the $\Sigma^{2j} ku$ summands, because the odd-degree homotopy groups of ku vanish. And on the $\Sigma^{2i} ko$ summands, our only nonzero choices are $w^s \eta$, where $w \in \pi_8(ko)$ is the Bott class. By assumption, $w^s \eta$ does not occur for $s > 0$, so we only need to check the cofibers of 0 and η. The cofiber of $0\colon \Sigma^k ko \to ko$ is $ko \vee \Sigma^{2k+1} ko$, and Wood's theorem implies the cofiber of $\eta\colon \Sigma ko \to ko$ is ku. $\qquad\square$

We will want to know the specific factors in the decomposition promised by Lemma B.9.

Definition B.12. Let $\mathcal{A}$ denote the mod 2 Steenrod algebra and $\mathcal{A}(1) \coloneqq \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle \subset \mathcal{A}$, which acts on the $\mathbb{Z}/2$-cohomology of any space. Since $\mathcal{A}(1)$ is $\mathbb{Z}$-graded ($|\mathrm{Sq}^i| = i$), we consider only $\mathbb{Z}$-graded $\mathcal{A}(1)$-modules. Then, consider the following two $\mathcal{A}(1)$-modules.

(1) $\mathbb{Z}/2$ in degree 0 with trivial $\mathcal{A}(1)$-action.
(2) $C\eta$, which consists of two $\mathbb{Z}/2$ summands in degrees 0 and 2, with a nontrivial Sq^2-action from the former to the latter.

If $k \in \mathbb{Z}$ and M is an $\mathcal{A}(1)$-module, we will let $\Sigma^k M$ (a *suspension* or *shift* of M) denote the same ungraded $\mathcal{A}(1)$-module with the grading of each homogeneous element increased by k. For example, this means that $C\eta \cong \Sigma^{-2} \widetilde{H}^*(\mathbb{CP}^2; \mathbb{Z}/2)$ as $\mathcal{A}(1)$-modules.

Lemma B.13. *With X as in Lemma B.9, there is an $\mathcal{A}(1)$-module isomorphism from $H^*(X; \mathbb{Z}/2)$ to a sum of shifts of $\mathbb{Z}/2$ and $C\eta$.*

Proof. Since X only has cells in even degrees, $H^*(X; \mathbb{Z}/2)$ is concentrated in even degrees, meaning Sq^1 acts trivially on $H^*(X; \mathbb{Z}/2)$. Thus the problem reduces to how Sq^2 can act; the Adem relation $\mathrm{Sq}^2 \mathrm{Sq}^2 = \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1$ means that $\mathrm{Sq}^2 \mathrm{Sq}^2$ acts trivially on $H^*(X; \mathbb{Z}/2)$. Therefore if $\mathrm{Sq}^2(x) \ne 0$ for any $x \in H^k(X; \mathbb{Z}/2)$, then x and $\mathrm{Sq}^2(x)$ generate a $\Sigma^k C\eta \subset H^*(X; \mathbb{Z}/2)$, and this is a direct summand, because x cannot be Sq^1 or Sq^2 of anything. After doing this for all x which Sq^2 acts nontrivially on, the result is a direct sum of shifts of the trivial $\mathcal{A}(1)$-module $\mathbb{Z}/2$. $\qquad\square$

Corollary B.14. *Let X be as in Lemma B.9. If the decomposition of $H^*(X;\mathbb{Z}/2)$ from Lemma B.13 is of the form*

(B.15a) $$H^*(X;\mathbb{Z}/2) \cong \bigoplus_{i\in\mathcal{I}} \Sigma^{m_i}\mathbb{Z}/2 \oplus \bigoplus_{j\in\mathcal{J}} \Sigma^{m_j}C\eta,$$

then there is an equivalence of ko-modules

(B.15b) $$ko \wedge X_+ \simeq \bigvee_{i\in\mathcal{I}} \Sigma^{m_i}ko \vee \bigvee_{j\in\mathcal{J}} \Sigma^{m_j}ku.$$

Proof. By Lemma B.9, we know there are n_k, n_ℓ such that

(B.16) $$ko \wedge X_+ \simeq \bigvee_{k\in\mathcal{K}} \Sigma^{n_k}ko \vee \bigvee_{\ell\in\mathcal{L}} \Sigma^{n_\ell}ku;$$

now we need to match this data to the data coming from cohomology in (B.15a).

Stong [Sto63] showed $H^*(ko;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2$ and Adams [Ada61] showed $H^*(ku;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{E}(1)} \mathbb{Z}/2$, where $\mathcal{E}(1) := \langle \mathrm{Sq}^1, \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1\rangle$; there is an isomorphism $C\eta \cong \mathcal{A}(1) \otimes_{\mathcal{E}(1)} \mathbb{Z}/2$ [BC18, Example 4.5.6], so

(B.17) $$\mathcal{A} \otimes_{\mathcal{A}(1)} C\eta \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \mathcal{A}(1) \otimes_{\mathcal{E}(1)} \mathbb{Z}/2 \cong \mathcal{A} \otimes_{\mathcal{E}(1)} \mathbb{Z}/2 \cong H^*(ku;\mathbb{Z}/2).$$

Therefore (B.16) implies

(B.18a) $$H^*(ko \wedge X_+;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \left(\bigoplus_{k\in\mathcal{K}} \Sigma^{n_k}\mathbb{Z}/2 \oplus \bigoplus_{\ell\in\mathcal{L}} \Sigma^{n_\ell}C\eta \right),$$

and the Künneth formula and Stong's result above imply

(B.18b) $$H^*(ko \wedge X;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} H^*(X;\mathbb{Z}/2).$$

We conclude by plugging (B.15a) into (B.18b) and comparing with (B.18a); a priori information could be lost by tensoring with $\mathcal{A}$, but this tensor product respects direct sums and $\mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2$ and $\mathcal{A} \otimes_{\mathcal{A}(1)} C\eta$ are not isomorphic, so no information is lost. $\square$

Corollary B.19. *As ko-modules,*

(B.20a) $$ko \wedge (B\mathrm{U}(1))_+ \simeq ko \vee \bigvee_{n\geq 0} \Sigma^{4n+2}ku$$

(B.20b) $$ko \wedge (B\mathrm{Sp}(1))_+ \simeq \bigvee_{n\geq 0} \Sigma^{4n}ko.$$

Part (B.20a) is already known, proven by Greenlees-May [GM95, §15]. See (7.41) for a related but different splitting result.

Proof. Once we have shown that $B\mathrm{U}(1)$ and $B\mathrm{Sp}(1)$ satisfy the hypothesis of Lemma B.9, the result follows from Corollary B.14 together with the understanding of $H^*(B\mathrm{U}(1);\mathbb{Z}/2)$ and $H^*(B\mathrm{Sp}(1);\mathbb{Z}/2)$. The latter is a trivial $\mathcal{A}(1)$-module (i.e. a sum of shifts of $\mathbb{Z}/2$) for degree reasons, and the $\mathcal{A}(1)$-module structure on $H^*(B\mathrm{U}(1);\mathbb{Z}/2)$ is computed in [BC18, Example 3.4.2 and Figure 4] to be a direct sum of $\mathbb{Z}/2$ and a $\Sigma^{4n+2}C\eta$ for each $n \geq 0$.

Thus all we have left to do is verify the hypotheses of Lemma B.9. The standard CW decomposition of $B\mathrm{U}(1) \simeq \mathbb{CP}^\infty$ has a k-cell in every nonnegative even degree k, attached to the $(k-2)$-cell (for $k > 0$) by the map $\eta \in \pi_1(\mathbb{S}) \cong \mathbb{Z}/2$, which satisfies the hypothesis, as it maps to

the class we call η in ko_1. For $B\mathrm{Sp}(1) \simeq \mathbb{HP}^\infty$, the standard CW decomposition has a k-cell in each degree $k \equiv 0 \bmod 4$, attached to the $(k-4)$-cell (again $k > 0$) by the map $\nu \in \pi_3(\mathbb{S})$. The image of ν in $ko_3 \cong 0$ vanishes for degree reasons, and so the hypothesis of Lemma B.9 is met. $\square$

Remark B.21. Analogues of Corollary B.19 for the periodic theory *KO* and its generalizations to the *higher real K-theories EO_Γ* are known: see Bousfield [Bou90], Meier [Mei17, Theorem 2.8], Chatham [Cha20, Theorems 5.13 and 5.14], Bhattacharya-Chatham [BC22, Main Theorem 1.7], and Chatham-Hu-Opie [CHO24, Example 2.10].

Definition B.22. Recall the complexification map $c\colon ko \to ku$. The cofiber of c is a map $R\colon ku \to \Sigma^2 ko$, denoted *realification.*

As $ku \not\simeq ko \vee \Sigma^2 ko$, the third map in the cofiber sequence begun by c and R must be nontrivial in

(B.23) $$\pi_0 \mathrm{Map}_{ko}(\Sigma ko, ko) \cong \mathrm{Map}_{\mathbb{S}}(\Sigma \mathbb{S}, ko) \cong \pi_1 ko \cong \mathbb{Z}/2,$$

so must be the unique nontrivial class, namely the Hopf map $\eta\colon \Sigma ko \to ko$. That is, we have found the *Wood cofiber sequence*

(B.24) $$ko \xrightarrow{c} ku \xrightarrow{R} \Sigma^2 ko \xrightarrow{\eta} \Sigma ko \longrightarrow \cdots$$

which we identified as a Smith cofiber sequence in Example 7.13.

Recall from Example 7.57 that the unit sphere bundle inside the tautological rank-3 vector bundle $V_3 \to B\mathrm{Spin}(3)$ is homotopy equivalent to the map $B\mathrm{Spin}(2) \to B\mathrm{Spin}(3)$, which can be identified via accidental isomorphisms to the map $B\mathrm{U}(1) \to B\mathrm{Sp}(1)$ given by the inclusion of a maximal torus. Choose for concreteness the standard maximal torus, given by the map $\mathrm{U}(1) \to \mathrm{SU}(2) \cong \mathrm{Sp}(1)$ defined by

(B.25) $$i\colon z \mapsto \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

Thus there is a Smith cofiber sequence

(B.26) $$ko \wedge (B\mathrm{U}(1))_+ \xrightarrow{i_*} ko \wedge (B\mathrm{Sp}(1))_+ \xrightarrow{\smallfrown e^{ko}(V)} ko \wedge \Sigma^3 (B\mathrm{Sp}(1))^{V_3 - 3},$$

which is the cofiber sequence in Example 7.49 smashed with ko.[31] This sequence is also studied, and placed in context, in Example 7.49.

Since $V_3 \to B\mathrm{Sp}(1)$ is spin, the Thom isomorphism identifies the third term in this sequence with $\Sigma^3 ko \wedge (B\mathrm{Sp}(1))_+$.

Proposition B.27. *The identifications in Corollary B.19 may be chosen to produce the following identifications of ko-module homomorphisms.*

   (1) The map $i_*\colon ko \wedge (B\mathrm{U}(1))_+ \to ko \wedge (B\mathrm{Sp}(1))_+$ is the direct sum of the maps

(B.28a) $$\Sigma^{4n+2} R\colon \Sigma^{4n+2} ku \longrightarrow \Sigma^{4n} ko,$$

   together with the identity $ko \to ko$ on the basepoint.

---

[31]In [DDK+24, §IV.F], we computed the Anderson dual long exact sequence in low degrees.

(2) *The fiber of i_*, which is a map $y\colon ko \wedge \Sigma^2(B\mathrm{Sp}(1))_+ \to ko \wedge (B\mathrm{U}(1))_+$, is the direct sum of the maps*

(B.28b) $$\Sigma^{4n+2}c\colon \Sigma^{4n+2}ko \longrightarrow \Sigma^{4n+2}ku.$$

(3) *The map $\frown e^{ko}(V)\colon ko \wedge (B\mathrm{Sp}(1))_+ \to \Sigma^3 ko \wedge (B\mathrm{Sp}(1))_+$ is the direct sum of the maps*

(B.28c) $$\Sigma^{4n-1}\eta\colon \Sigma^{4n}ko \longrightarrow \Sigma^{4n-1}ko,$$

together with the zero map on the copy of ko in degree 0.

Proof. Using the Wood cofiber sequence (B.24), any one of these three results implies the other two; we will prove (2).

Restricted to $\Sigma^{4k+2}ko$, y is a map

(B.29) $$y|_{\Sigma^{4k+2}ko}\colon \Sigma^{4k+2}ko \longrightarrow ko \vee \bigvee_{\ell \geq 0} \Sigma^{4\ell+2}ku.$$

We will show that it is possible to choose the equivalences in (B.19) to make y "diagonal", i.e. after composing to the projection onto each summand of (B.29) *except* $\Sigma^{4k+2}ku$, $y|_{\Sigma^{4k+2}ko}$ is trivial. We know that the "diagonal terms," i.e. the maps obtained by restricting y to $\Sigma^{4k+2}ko$ and then projecting to the $\Sigma^{4k+2}ku$ summand in the codomain, must be $\pm c$, because this is the only choice compatible with base change along $ko \to H\mathbb{Z}$ inducing maps on $\mathbb{Z}$ cohomology which are isomorphisms in those degrees: this is because

(B.30) $$\pi_0\mathrm{Map}_{ko}(\Sigma^{4k+2}ko, \Sigma^{4k+2}ku) \cong \pi_0\mathrm{Map}_{\mathbb{S}}(\mathbb{S}, ku) \cong \pi_0 ku \cong \mathbb{Z}$$

and c is a generator; thus we must obtain either c or $-c$ on the equal-degree summand.

The map out of $\Sigma^{4k+2}ko$ is trivial when projected to the ko in degree 0, because we need that $\Sigma^0 ko$ summand to map to the degree-0 ko summand in the cofiber $ko \wedge (B\mathrm{Sp}(1))_+$, because that map arose from a basepoint-preserving map of spaces. In the rest of the proof, we will address the $\Sigma^{4\ell+2}ku$ summands.

A map of ko-modules $\Sigma^m ko \to \Sigma^n ku$ is equivalent data to a map of spectra $\Sigma^m \mathbb{S} \to \Sigma^n ku$, which is classified by $\pi_n(ku)$. Since ku is connective, all "off-diagonal terms" vanish unless $4k+2 \geq 4\ell+2$; therefore for our $\Sigma^{4k+2}ko$ summand we may restrict to the map

(B.31) $$y\colon \Sigma^2 ko \vee \cdots \vee \Sigma^{4k+2}ko \longrightarrow \Sigma^2 ku \vee \cdots \vee \Sigma^{4k+2}ku.$$

We may therefore describe y as a $(k+1) \times (k+1)$ matrix. Connectivity of ku implies this matrix is upper triangular.

We saw in (B.30) that if $m \geq \ell$, then $\pi_0\mathrm{Map}_{ko}(\Sigma^{4m+2}ko, \Sigma^{4\ell+2}ku) \cong \pi_{2(m-\ell)}ku \cong \mathbb{Z}$; tracing through the identifications there, we learn that this $\mathbb{Z}$ of maps is the set of scalar multiples of the map $b^{2(m-\ell)}c$, where $b\colon \Sigma^2 ku \to ku$ is the connective version of the Bott periodicity map. Therefore there are integers λ_{ij} for $1 \leq i < j \leq k+1$ such that the map (B.31) is given by the following upper triangular matrix:

(B.32)
$$\begin{bmatrix} \pm c & \lambda_{12}b^2 c & \lambda_{13}b^4 c & \cdots & \lambda_{1(k+1)}b^{2k}c \\ & \pm c & \lambda_{23}b^2 c & \cdots & \lambda_{2(k+1)}b^{2k-2}c \\ & & \ddots & \ddots & \vdots \\ & & & \pm c & \lambda_{k(k+1)}b^2 c \\ & & & & \pm c \end{bmatrix}.$$

This matrix can clearly be row-reduced over ko_* to $c \cdot \mathrm{Id}$, and the requisite row operations correspond to automorphisms of $ko \vee \cdots \vee \Sigma^{4k+2} ko$. The row operations are compatible with adding on more summands by increasing k, so we may conclude.   □

Lemma B.33 (Bruner-Greenlees (see [BPR23, Theorem 3.8])). *Recall* $ko^*(B\mathrm{Sp}(1)) \cong ko^*[[p_1^{\mathbb{H}}]]$ *from Proposition B.3. There is an isomorphism* $\varphi \colon ko_*(B\mathrm{Sp}(1)) \xrightarrow{\cong} ko_*[x]$, *where* $|x| = 4$,[32] *such that the* $ko^*(B\mathrm{Sp}(1))$-*module structure on* $ko_*(B\mathrm{Sp}(1))$ *is the one uniquely specified by*

$$(\text{B.34}) \qquad\qquad p_1^{\mathbb{H}} \frown x^k = x^{k-1}.$$

Finally, we can calculate the ko-Euler class!

Proof of Theorem B.4. It suffices to work universally with the tautological bundle $V_3 \to B\mathrm{Spin}(3)$; the spinor bundle is the tautological quaternionic line bundle associated to $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$, and so $p_1^{\mathbb{H}}(\mathcal{S}_{V_3})$ is the class we called $p_1^{\mathbb{H}} \in ko^4(B\mathrm{Sp}(1))$ in Proposition B.3.

By Proposition B.27,

$$(\text{B.35}) \qquad\qquad e^{ko}(V_3) \frown x^k = \eta x^{k-1},$$

where we define $x^{-1} = 0$ for convenience.[33] A general element of $ko^3(B\mathrm{Sp}(1))$ is of the form

$$(\text{B.36}) \qquad\qquad \sum_{k \geq 0} \eta (p_1^{\mathbb{H}})^k w^{k-1}.$$

We know how η and w^{k-1} act on $ko_*(B\mathrm{Sp}(1))$ because the ko-theory cap product is linear over ko^*. We know how $p_1^{\mathbb{H}}$ acts on $ko_*(B\mathrm{Sp}(1))$ thanks to Lemma B.33. Using these, we can see that the only class of the form (B.36) whose cap product matches that of $e^{ko}(V_3)$ in (B.35) is $\eta p_1^{\mathbb{H}}$.

Finally, we have to check that $e^{\mathbb{Z}}(V_3)$ and $e^{\mathbb{Z}/2}(V_3)$ both vanish. $B\mathrm{Sp}(1)$ is 3-connected, so $H^3(B\mathrm{Sp}(1); \mathbb{Z})$ and $H^3(B\mathrm{Sp}(1); \mathbb{Z}/2)$ both vanish.   □

Remark B.37 (Euler classes of low-rank spin vector bundles). For $2 \leq n \leq 6$, $\mathrm{Spin}(n)$ participates in an accidental isomorphism[34] with a Lie group satisfying Lemma B.9, and one can run a similar argument to compute ko-Euler classes of other low-rank vector bundles.

  (1) If L is a real line bundle with spin structure, $e^{ko}(L) = 0$, because e^{ko} pulls back from the twisted Euler class over $B\mathrm{SO}(1) = *$. The image of this fact in KO-theory is due to Crabb [Cra91, Corollary 3.37(i)].
  (2) If V_2 has rank 2, one can use the accidental isomorphism $\mathrm{Spin}(2) \cong \mathrm{U}(1)$ and the fact that the map $c \colon ko^*(B\mathrm{U}(1)) \to ku^*(B\mathrm{U}(1))$ is injective [BG10, §5.2] to show that $e^{ko}(V_2)$ is determined by $e^{ku}(V)$, hence also by $e^K(V)$, the image in periodic K-theory. In particular, V_2 acquires the structure of a complex line bundle, and there is a formula for the K-theory Euler classes of complex vector bundles, e.g. in Bott [Bot69, (7.2)].
  (4) If $V_4 \to X$ has rank 4, its spinor bundle factors as $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, where the two factors $\mathcal{S}^\pm$ are quaternionic line bundles associated to the two factors of $\phi \colon \mathrm{Spin}(4) \xrightarrow{\cong} \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. There is a choice of ϕ such that

$$(\text{B.38}) \qquad\qquad e^{ko}(V_4) = p_1^{\mathbb{H}}(\mathcal{S}^+) - p_1^{\mathbb{H}}(\mathcal{S}^-) \in ko^4(X).$$

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[32]Here we use polynomial notation only for conciseness; we have not defined any ring structure on $ko_*(B\mathrm{Sp}(1))$.
[33]As we have not been careful about explicit choices of isomorphisms, there could be a sign factor in the choice of x^k, but since $2\eta = 0$, the possible sign error goes away.
[34]"There are no mistakes, just happy little accidental isomorphisms."

(5) There is an accidental isomorphism $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$, and $ko^*(B\mathrm{Sp}(2)) \cong ko^*[[p_1^{\mathbb{H}}, p_2^{\mathbb{H}}]]$ with $|p_1^{\mathbb{H}}| = 4$ and $|p_2^{\mathbb{H}}| = 8$ (see [DM79, §2] or [BG10, Theorem 5.3.5]). Therefore $ko^5(B\mathrm{Sp}(2)) \cong 0$, so for any rank-5 spin vector bundle V_5, $e^{ko}(V_5) = 0$. The image of this fact in KO-theory is due to Crabb [Cra91, Corollary 3.37(i)].

Remark B.39. We saw above that for twisted spin bordism, the ko-theoretic Euler class suffices. For other tangential structures, one may need more or less information.

- Unoriented bordism decomposes as a sum of shifts of mod 2 homology, and this splitting is compatible with the Smith homomorphism. Therefore in this setting, one can use the $\mathbb{Z}/2$-cohomology Euler class.
- Wall [Wal60] showed that MTSO, localized at 2, splits as a sum of shifts of $H\mathbb{Z}$ and $H\mathbb{Z}/2$. Therefore when one studies Smith homomorphisms for twisted oriented bordism, the $\mathbb{Z}$-cohomology Euler class will be accurate up to odd-primary torsion. On odd-primary torsion, oriented and spin bordism coincide, so in that setting one can use ko-Euler classes for twisted oriented bordism.
- Analogously to spin and ko, one can use ku-theory Euler classes for twisted spin^c bordism Smith homomorphisms.

References

[ABC⁺23]   Andrea Antinucci, Francesco Benini, Christian Copetti, Giovanni Galati, and Giovanni Rizi. Anomalies of non-invertible self-duality symmetries: fractionalization and gauging, 2023. https://arxiv.org/abs/2308.11707. 6

[ABG⁺14a] Matthew Ando, Andrew J. Blumberg, David Gepner, Michael J. Hopkins, and Charles Rezk. An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology. *J. Topol.*, 7(3):869–893, 2014. https://arxiv.org/abs/1403.4325. 8, 22, 35

[ABG⁺14b] Matthew Ando, Andrew J. Blumberg, David Gepner, Michael J. Hopkins, and Charles Rezk. Units of ring spectra, orientations and Thom spectra via rigid infinite loop space theory. *J. Topol.*, 7(4):1077–1117, 2014. https://arxiv.org/abs/1403.4320. 22, 24, 35

[ABG18]   Matthew Ando, Andrew J. Blumberg, and David Gepner. Parametrized spectra, multiplicative Thom spectra and the twisted Umkehr map. *Geom. Topol.*, 22(7):3761–3825, 2018. https://arxiv.org/abs/1112.2203. 22

[ABP67]   D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson. The structure of the Spin cobordism ring. *Ann. of Math. (2)*, 86:271–298, 1967. 55

[ABP69]   D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson. Pin cobordism and related topics. *Comment. Math. Helv.*, 44:462–468, 1969. 5, 33, 52

[ABS64]   M.F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology*, 3(Supplement 1):3–38, 1964. 55

[Ada61]   J. F. Adams. On Chern characters and the structure of the unitary group. *Proc. Cambridge Philos. Soc.*, 57:189–199, 1961. 59

[Ada62]   J. F. Adams. Vector fields on spheres. *Ann. of Math. (2)*, 75:603–632, 1962. 27

[Ada63]   J. F. Adams. On the groups $J(X)$. I. *Topology*, 2:181–195, 1963. 30

[Ada65a]   J. F. Adams. On the groups $J(X)$. II. *Topology*, 3:137–171, 1965. 27, 30

[Ada65b]   J. F. Adams. On the groups $J(X)$. III. *Topology*, 3:193–222, 1965. 30

[Ada66]   J. F. Adams. On the groups $J(X)$. IV. *Topology*, 5:21–71, 1966. 30

[Ada74]   J. F. Adams. *Stable homotopy and generalised homology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, Ill.-London, 1974. 35

[Ale75]   J. C. Alexander. On (U, Sp) bordism. *Amer. J. Math.*, 97(3):617–625, 1975. 20

[AM21]   Michael Albanese and Aleksandar Milivojević. Spin^h and further generalisations of spin. *Journal of Geometry and Physics*, 164:104174, 2021. https://arxiv.org/abs/2008.04934. 40

[And69]   D.W. Anderson. Universal coefficient theorems for K-theory, 1969. `https://faculty.tcu.edu/gfriedman/notes/Anderson-UCT.pdf`. 45

[Ati61a]   M. F. Atiyah. Bordism and cobordism. *Proc. Cambridge Philos. Soc.*, 57:200–208, 1961. 9, 17, 25, 32, 34, 38

[Ati61b]   M. F. Atiyah. Thom complexes. *Proc. London Math. Soc. (3)*, 11:291–310, 1961. 16, 27

[AW65]   J. F. Adams and G. Walker. On complex Stiefel manifolds. *Proc. Cambridge Philos. Soc.*, 61:81–103, 1965. 28

[Bär99]   Christian Bär. Elliptic symbols. *Math. Nachr.*, 201:7–35, 1999. 40

[Bau03]   Tilman Bauer. Elliptic cohomology and projective spaces — a computation, 2003. `https://people.kth.se/~tilmanb/papers/cpinfty.pdf`. 29, 30

[BBC17]   Anna Beliakova, Christian Blanchet, and Eva Contreras. Spin modular categories. *Quantum Topol.*, 8(3):459–504, 2017. `https://arxiv.org/abs/1411.4232`. 37

[BC18]   Agnès Beaudry and Jonathan A. Campbell. A guide for computing stable homotopy groups. In *Topology and quantum theory in interaction*, volume 718 of *Contemp. Math.*, pages 89–136. Amer. Math. Soc., Providence, RI, 2018. `https://arxiv.org/abs/1801.07530`. 8, 40, 59

[BC22]   P. Bhattacharya and H. Chatham. On the EO-orientability of vector bundles. *J. Topol.*, 15(4):2017–2044, 2022. `https://arxiv.org/abs/2003.03795`. 23, 24, 28, 60

[BCC93]   M. Bullejos, P. Carrasco, and A. M. Cegarra. Cohomology with coefficients in symmetric cat-groups. An extension of Eilenberg-MacLane's classification theorem. *Mathematical Proceedings of the Cambridge Philosophical Society*, 114(1):163–189, 1993. 45

[BCD22]   T. Daniel Brennan, Clay Córdova, and Thomas T. Dumitrescu. Line defect quantum numbers & anomalies, 2022. `https://arxiv.org/abs/2206.15401`. 40

[BDDM24]   Ivano Basile, Arun Debray, Matilda Delgado, and Miguel Montero. Global anomalies & bordism of non-supersymmetric strings. *J. High Energy Phys.*, (2):Paper No. 92, 71, 2024. `https://arxiv.org/abs/2310.06895`. 39

[Bea17]   Jonathan Beardsley. Relative Thom spectra via operadic Kan extensions. *Algebr. Geom. Topol.*, 17(2):1151–1162, 2017. `https://arxiv.org/abs/1601.04123`. 21, 22

[Bec70]   James C. Becker. Extensions of cohomology theories. *Illinois J. Math.*, 14:551–584, 1970. 13, 15

[Bel99]   Anna Beliakova. Refined invariants and TQFTs from Homfly skein theory. *J. Knot Theory Ramifications*, 8(5):569–587, 1999. `https://arxiv.org/abs/math/9806090`. 37

[Ben88]   Dave Benson. Spin modules for symmetric groups. *J. London Math. Soc. (2)*, 38(2):250–262, 1988. 44

[BFF78]   Allen Back, Peter G.O. Freund, and Michael Forger. New gravitational instantons and universal spin structures. *Physics Letters B*, 77(2):181–184, 1978. 40

[BG75]   J. C. Becker and D. H. Gottlieb. The transfer map and fiber bundles. *Topology*, 14:1–12, 1975. 35

[BG87a]   Anthony Bahri and Peter Gilkey. The eta invariant, Pin^c bordism, and equivariant $Spin^c$ bordism for cyclic 2-groups. *Pacific J. Math.*, 128(1):1–24, 1987. 4, 26, 27, 34

[BG87b]   Anthony Bahri and Peter Gilkey. Pin^c cobordism and equivariant $Spin^c$ cobordism of cyclic 2-groups. *Proceedings of the American Mathematical Society*, 99(2):380–382, 1987. 26, 27

[BG10]   Robert R. Bruner and J. P. C. Greenlees. *Connective real K-theory of finite groups*, volume 169 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010. 56, 62, 63

[Bla00]   Christian Blanchet. Hecke algebras, modular categories and 3-manifolds quantum invariants. *Topology*, 39(1):193–223, 2000. `https://arxiv.org/abs/math/9803114`. 37

[BLM23]   Jonathan Beardsley, Kiran Luecke, and Jack Morava. Brauer-Wall groups and truncated Picard spectra of K-theory, 2023. `https://arxiv.org/abs/2306.10112`. 21

[BM23]   Jonathan Buchanan and Stephen McKean. KSp-characteristic classes determine $Spin^h$ cobordism, 2023. `https://arxiv.org/abs/2312.08209`. 40

[Bot59]   Raoul Bott. The stable homotopy of the classical groups. *Ann. of Math. (2)*, 70:313–337, 1959. 31

[Bot69]   Raoul Bott. *Lectures on $K(X)$*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York-Amsterdam, 1969. 62

[Bou90]   A. K. Bousfield. A classification of K-local spectra. *J. Pure Appl. Algebra*, 66(2):121–163, 1990. 60

[BPR23]   Boris Botvinnik, Paolo Piazza, and Jonathan Rosenberg. Positive scalar curvature on simply connected spin pseudomanifolds. *J. Topol. Anal.*, 15(2):413–443, 2023. https://arxiv.org/abs/1908.04420. 62

[BR21]   Robert R. Bruner and John Rognes. *The Adams spectral sequence for topological modular forms*, volume 253 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2021. 39

[BR23]   Boris Botvinnik and Jonathan Rosenberg. Positive scalar curvature on Pin$^\pm$- and spin^c-manifolds and manifolds with singularities. In *Perspectives in scalar curvature. Vol. 2*, pages 51–81. World Sci. Publ., Hackensack, NJ, [2023] ©2023. https://arxiv.org/abs/2103.00617. 5, 33

[Bro71]   Edgar H. Brown, Jr. The Kervaire invariant of a manifold. In *Algebraic topology (Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970)*, pages 65–71. Amer. Math. Soc., Providence, R.I., 1971. 54

[BS53]   A. Borel and J.-P. Serre. Groupes de Lie et puissances réduites de Steenrod. *Amer. J. Math.*, 75:409–448, 1953. 30

[BS74]   J. C. Becker and R. E. Schultz. Equivariant function spaces and stable homotopy theory. I. *Comment. Math. Helv.*, 49:1–34, 1974. 35

[Buh72]   V. M. Buhštaber. Projectors in unitary cobordisms that are related to SU-theory. *Uspehi Mat. Nauk*, 27(6(168)):231–232, 1972. 41

[Bun15]   Ulrich Bunke. The universal η-invariant for manifolds with boundary. *Q. J. Math.*, 66(2):473–506, 2015. https://arxiv.org/abs/1403.2030. 20

[Cam17]   Jonathan A. Campbell. Homotopy theoretic classification of symmetry protected phases, 2017. https://arxiv.org/abs/1708.04264. 27, 37

[Car36]   Élie Cartan. La topologie des espaces représentatifs des groupes de Lie. *L'Enseignement Mathématique*, 35(1–2):177–200, 1936. 30

[CF64]   P. E. Conner and E. E. Floyd. *Differentiable periodic maps*. Ergebnisse der Mathematik und ihrer Grenzgebiete, (N.F.), Band 33. Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1964. 2, 4, 32

[CF66a]   P. E. Conner and E. E. Floyd. *The relation of cobordism to K-theories*. Lecture Notes in Mathematics, No. 28. Springer-Verlag, Berlin-New York, 1966. 20

[CF66b]   P. E. Conner and E. E. Floyd. Torsion in SU-bordism. *Mem. Amer. Math. Soc.*, 60:74, 1966. 36, 41

[Cha20]   Hood Chatham. *An Orientation Map for Height p -1 Real E Theory*. PhD thesis, Massachusetts Institute of Technology, 2020. https://arxiv.org/abs/1908.11496. 60

[Che17]   Xuan Chen. *Bundles of Irreducible Clifford Modules and the Existence of Spin Structures*. ProQuest LLC, Ann Arbor, MI, 2017. Thesis (Ph.D.)–State University of New York at Stony Brook. 40

[Che22]   Georgy Chernykh. Landweber exactness of the formal group law in c_1-spherical bordism. 2022. https://arxiv.org/abs/2212.04552. 41

[CHO24]   Hood Chatham, Yang Hu, and Morgan Opie. Enumerating stably trivial vector bundles with higher real K-theory, 2024. https://arxiv.org/abs/2403.04733. 60

[CHZ23]   Clay Córdova, Po-Shen Hsin, and Carolyn Zhang. Anomalies of non-invertible symmetries in (3+1)d, 2023. https://arxiv.org/abs/2308.11706. 6

[CK88]   M. C. Crabb and K. Knapp. James numbers. *Math. Ann.*, 282(3):395–422, 1988. 28

[CL21]   Chi-Ming Chang and Ying-Hsuan Lin. On exotic consistent anomalies in $(1 + 1)d$: a ghost story. *SciPost Phys.*, 10(5):Paper No. 119, 28, 2021. https://arxiv.org/abs/2009.07273. 6

[COSY20]   Clay Córdova, Kantaro Ohmori, Shu-Heng Shao, and Fei Yan. Decorated $\mathbb{Z}_2$ symmetry defects and their time-reversal anomalies. *Physical Review D*, 102(4), aug 2020. https://arxiv.org/abs/1910.14046. 2, 5, 32, 42, 55

[Cra91]   M. C. Crabb. On the $KO_{\mathbf{Z}/2}$-Euler class. I. *Proc. Roy. Soc. Edinburgh Sect. A*, 117(1-2):115–137, 1991. 56, 62, 63

[Cru03]   James Cruickshank. Twisted homotopy theory and the geometric equivariant 1-stem. *Topology Appl.*, 129(3):251–271, 2003. 31

[Cus85]   Larry W. Cusick. The cofibre of the transfer map. *Proc. Amer. Math. Soc.*, 93(3):561–566, 1985. 35

[DAC23]   Jeremias Aguilera Damia, Riccardo Argurio, and Soumyadeep Chaudhuri. When the moduli space is an orbifold: Spontaneous breaking of continuous non-invertible symmetries, 2023. https://arxiv.org/abs/2309.06491. 6

[DDHM22]   Arun Debray, Markus Dierigl, Jonathan J. Heckman, and Miguel Montero. The anomaly that was not meant IIB. *Fortschr. Phys.*, 70(1):Paper No. 2100168, 31, 2022. https://arxiv.org/abs/2107.14227. 36, 37

[DDHM23]   Arun Debray, Markus Dierigl, Jonathan J. Heckman, and Miguel Montero. The chronicles of IIBordia: Dualities, bordisms, and the Swampland, 2023. https://arxiv.org/abs/2302.00007. 3, 4, 5, 8, 11, 21, 27, 36, 37, 38, 39

[DDK+24]   Arun Debray, Sanath Devalapurkar, Cameron Krulewski, Yu Leon Liu, Natalia Pacheco-Tallaj, and Ryan Thorngren. A long exact sequence in symmetry breaking: order parameter constraints, defect anomaly-matching, and higher Berry phases, 2024. https://arxiv.org/abs/2309.16749. 1, 3, 5, 19, 32, 33, 36, 37, 40, 41, 42, 43, 44, 47, 48, 49, 60

[Deb21]   Arun Debray. Invertible phases for mixed spatial symmetries and the fermionic crystalline equivalence principle, 2021. https://arxiv.org/abs/2102.02941. 37

[Deb23]   Arun Debray. Bordism for the 2-group symmetries of the heterotic and CHL strings. 2023. https://arxiv.org/abs/2304.14764. 4, 20

[Dib99]   İ. Dibağ. *J*-approximation of complex projective spaces by lens spaces. *Pacific J. Math.*, 191(2):223–242, 1999. 28

[Dib03]   İ. Dibağ. Determination of the *J*-groups of complex projective and lens spaces. *K-Theory*, 29(1):27–74, 2003. 28

[DK24]   Arun Debray and Cameron Krulewski. Smith homomorphisms and Spin^h structures. 2024. To appear. 40

[DL59]   Albrecht Dold and Richard Lashof. Principal quasi-fibrations and fibre homotopy equivalence of bundles. *Illinois J. Math.*, 3:285–305, 1959. 8

[DL20]   Joe Davighi and Nakarin Lohitsiri. Anomaly interplay in U(2) gauge theories. *J. High Energy Phys.*, 202(5):098, 20, 2020. https://arxiv.org/abs/2001.07731. 26, 27, 39, 40

[DL21]   Joe Davighi and Nakarin Lohitsiri. The algebra of anomaly interplay. *SciPost Phys.*, 10(3):Paper No. 074, 41, 2021. https://arxiv.org/abs/2011.10102. 26, 27, 37, 39

[DL23]   Joe Davighi and Nakarin Lohitsiri. Toric 2-group anomalies via cobordism. *J. High Energy Phys.*, 2023(7):Paper No. 19, 52, 2023. https://arxiv.org/abs/2302.12853. With an appendix by Arun Debray. 4

[DM79]   Donald M. Davis and Mark Mahowald. The Euler class for connective *k*o-theory and an application to immersions of quaternionic projective space. *Indiana Univ. Math. J.*, 28(6):1025–1034, 1979. 56, 63

[DM82]   P. Deligne and J. S. Milne. *Tannakian Categories*, pages 101–228. Springer Berlin Heidelberg, Berlin, Heidelberg, 1982. https://www.jmilne.org/math/xnotes/tc.html. 16

[DP80]   Albrecht Dold and Dieter Puppe. Duality, trace, and transfer. In *Proceedings of the International Conference on Geometric Topology (Warsaw, 1978)*, pages 81–102. PWN, Warsaw, 1980. 16

[Dri06]   Vladimir Drinfeld. Infinite-dimensional vector bundles in algebraic geometry. In Pavel Etingof, Vladimir Retakh, and I. M. Singer, editors, *The Unity of Mathematics: In Honor of the Ninetieth Birthday of I.M. Gelfand*, pages 263–304. Birkhäuser Boston, Boston, MA, 2006. https://arxiv.org/abs/math/0309155. 45

[DW19]   Aleksander Doan and Thomas Walpuski. On counting associative submanifolds and Seiberg-Witten monopoles. *Pure Appl. Math. Q.*, 15(4):1047–1133, 2019. https://arxiv.org/abs/1712.08383. 39

[DY22]   Arun Debray and Matthew Yu. What bordism-theoretic anomaly cancellation can do for U, 2022. https://arxiv.org/abs/2210.04911. 27, 40

[DY23a]   Arun Debray and Matthew Yu. Adams spectral sequences for non-vector-bundle Thom spectra, 2023. https://arxiv.org/abs/2305.01678. 4, 6, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31

[DY23b]   Thibault D. Décoppet and Matthew Yu. Gauging noninvertible defects: a 2-categorical perspective. *Lett. Math. Phys.*, 113(2):Paper No. 36, 42, 2023. https://arxiv.org/abs/2211.08436. 6

[DYY23]   Arun Debray, Weicheng Ye, and Matthew Yu. Bosonization and anomaly indicators of (2+1)-d fermionic topological orders. 2023. https://arxiv.org/abs/2312.13341. 4, 34, 36, 37, 40

[Eck51]   Beno Eckmann. Espaces fibrés et homotopie. In *Colloque de topologie (espaces fibrés), Bruxelles, 1950*, pages 83–99. Georges Thone, Liège, 1951. 30

[Ekh98]   Tobias Ekholm. Immersions in the metastable range and spin structures on surfaces. *Math. Scand.*, 83(1):5–41, 1998. 33

[FG73]   S. Feder and S. Gitler. Stable homotopy types of stunted complex projective spaces. *Proc. Cambridge Philos. Soc.*, 73:431–438, 1973. 28

[FG77]   S. Feder and S. Gitler. The classification of stunted projective spaces by stable homotopy type. *Trans. Amer. Math. Soc.*, 225:59–81, 1977. 28

[FH00]   Daniel S. Freed and Michael Hopkins. On Ramond-Ramond fields and K-theory. *J. High Energy Phys.*, 2000(5):Paper 44, 14, 2000. https://arxiv.org/abs/hep-th/0002027. 56

[FH20]   Daniel S. Freed and Michael J. Hopkins. Invertible phases of matter with spatial symmetry. *Adv. Theor. Math. Phys.*, 24(7):1773–1788, 2020. https://arxiv.org/abs/1901.06419. 37

[FH21]   Daniel S Freed and Michael J Hopkins. Reflection positivity and invertible topological phases. *Geometry & Topology*, 25(3):1165–1330, 2021. https://arxiv.org/abs/1604.06527. 2, 6, 8, 21, 27, 39, 40, 43, 44, 45, 46, 56

[FHH20]   Lukasz Fidkowski, Jeongwan Haah, and Matthew B. Hastings. Exactly solvable model for a $4 + 1$D beyond-cohomology symmetry-protected topological phase. *Phys. Rev. B*, 101:155124, Apr 2020. https://arxiv.org/abs/1912.05565. 32

[FHJF⁺24] Giovanni Ferrer, Brett Hungar, Theo Johnson-Freyd, Cameron Krulewski, Lukas Müller, Nivedita, David Penneys, David Reutter, Claudia Scheimbauer, Luuk Stehouwer, and Chetan Vuppulury. Dagger n-categories, 2024. https://arxiv.org/abs/2403.01651. 46

[FHT10]   Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Consistent orientation of moduli spaces. In *The many facets of geometry*, pages 395–419. Oxford Univ. Press, Oxford, 2010. https://arxiv.org/abs/0711.1909. 2, 6, 45

[FL02]   Paul M. N. Feehan and Thomas G. Leness. SO(3) monopoles, level-one Seiberg-Witten moduli spaces, and Witten's conjecture in low degrees. In *Proceedings of the 1999 Georgia Topology Conference (Athens, GA)*, volume 124, pages 221–326, 2002. https://arxiv.org/abs/math/0106238. 39

[FM06]   Daniel S. Freed and Gregory W. Moore. Setting the quantum integrand of M-theory. *Comm. Math. Phys.*, 263(1):89–132, 2006. https://arxiv.org/abs/hep-th/0409135. 44

[Fre19]   Daniel S. Freed. *Lectures on field theory and topology*, volume 133 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, RI, 2019. Published for the Conference Board of the Mathematical Sciences. 6, 44

[Fre23]   Daniel S. Freed. What is an anomaly? 2023. https://arxiv.org/abs/2307.08147. 43

[FSS24]   Domenico Fiorenza, Hisham Sati, and Urs Schreiber. *The character map in non-abelian cohomology—twisted, differential, and generalized.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, [2024] ©2024. https://arxiv.org/abs/2009.11909. 31

[FT14]   Daniel S. Freed and Constantin Teleman. Relative quantum field theory. *Comm. Math. Phys.*, 326(2):459–476, 2014. https://arxiv.org/abs/1212.1692. 43

[Gal21]   Søren Galatius. Lecture on invertible field theories. In *Quantum field theory and manifold invariants*, volume 28 of *IAS/Park City Math. Ser.*, pages 347–402. Amer. Math. Soc., Providence, RI, [2021] ©2021. https://arxiv.org/abs/1912.08706. 44

[GEM19]   Iñaki García-Etxebarria and Miguel Montero. Dai-Freed anomalies in particle physics. *J. High Energy Phys.*, 2019(8):003, 77, 2019. https://arxiv.org/abs/1808.00009. 37

[Gia71]   V. Giambalvo. On $\langle 8 \rangle$-cobordism. *Illinois J. Math.*, 15:533–541, 1971. 24

[Gia73a]   V. Giambalvo. Cobordism of line bundles with a relation. *Illinois J. Math.*, 17:442–449, 1973. 5, 37

[Gia73b]   V. Giambalvo. Pin and Pin' cobordism. *Proc. Amer. Math. Soc.*, 39:395–401, 1973. 33, 52

[GJO19]   Nick Gurski, Niles Johnson, and Angélica M. Osorno. The 2-dimensional stable homotopy hypothesis. *J. Pure Appl. Algebra*, 223(10):4348–4383, 2019. https://arxiv.org/abs/1712.07218. 45

[GK14]   Nora Ganter and Mikhail Kapranov. Symmetric and exterior powers of categories. *Transformation Groups*, 19(1):57–103, Mar 2014. https://arxiv.org/abs/1110.4753. 45

[GKT89]   J. Gunarwardena, B. Kahn, and C. Thomas. Stiefel-Whitney classes of real representations of finite groups. *J. Algebra*, 126(2):327–347, 1989. 26

[GM80]   Lucien Guillou and Alexis Marin. Une extension d'un théorème de Rohlin sur la signature. In *Seminar on Real Algebraic Geometry (Paris, 1977/1978 and Paris, 1978/1979)*, volume 9 of *Publ. Math. Univ. Paris VII*, pages 69–80. Univ. Paris VII, Paris, 1980. 36

[GM95]   J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. *Mem. Amer. Math. Soc.*, 113(543):viii+178, 1995. 59

[GMTW09]   Søren Galatius, Ib Madsen, Ulrike Tillmann, and Michael Weiss. The homotopy type of the cobordism category. *Acta Mathematica*, 202(2):195–239, 2009. https://arxiv.org/abs/math/0605249. 6, 41, 45

[GOP+20]   Meng Guo, Kantaro Ohmori, Pavel Putrov, Zheyan Wan, and Juven Wang. Fermionic finite-group gauge theories and interacting symmetric/crystalline orders via cobordisms. *Comm. Math. Phys.*, 376(2):1073–1154, 2020. https://arxiv.org/abs/1812.11959. 27, 37

[GPW18]   Meng Guo, Pavel Putrov, and Juven Wang. Time reversal, $SU(N)$ Yang-Mills and cobordisms: interacting topological superconductors/insulators and quantum spin liquids in $3+1D$. *Ann. Physics*, 394:244–293, 2018. https://arxiv.org/abs/1711.11587. 40

[Gra23]   Daniel Grady. Deformation classes of invertible field theories and the Freed–Hopkins conjecture, 2023. https://arxiv.org/abs/2310.15866. 2, 46

[Guo18]   Meng Guo. *Some Calculations of Cobordism Groups and Their Applications in Physics*. PhD thesis, Harvard University, 2018. https://dash.harvard.edu/bitstream/handle/1/40050106/GUO-DISSERTATION-2018.pdf?sequence=4. 27

[HJ20]   Fabian Hebestreit and Michael Joachim. Twisted spin cobordism and positive scalar curvature. *J. Topol.*, 13(1):1–58, 2020. https://arxiv.org/abs/1311.3164. 25, 26

[HKT94]   Ian Hambleton, Matthias Kreck, and Peter Teichner. Nonorientable 4-manifolds with fundamental group of order 2. *Trans. Amer. Math. Soc.*, 344(2):649–665, 1994. 36

[HKT20]   Itamar Hason, Zohar Komargodski, and Ryan Thorngren. Anomaly matching in the symmetry broken phase: Domain walls, CPT, and the Smith isomorphism. *SciPost Physics*, 8(4), Apr 2020. https://arxiv.org/abs/1910.14039. 2, 5, 12, 32, 33, 37, 42, 47, 49, 55

[HS73]   R. P. Held and D. Sjerve. On the stable homotopy type of Thom complexes. *Canadian J. Math.*, 25:1285–1294, 1973. 27

[HS05]   M.J. Hopkins and I.M. Singer. Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452, 07 2005. https://arxiv.org/abs/math/0211216. 45

[HS13]   Ian Hambleton and Yang Su. On certain 5-manifolds with fundamental group of order 2. *Q. J. Math.*, 64(1):149–175, 2013. https://arxiv.org/abs/0903.5244. 4, 5, 33, 34

[Hsi18]   Chang-Tse Hsieh. Discrete gauge anomalies revisited, 2018. https://arxiv.org/abs/1808.02881. 37

[HTY22]   Chang-Tse Hsieh, Yuji Tachikawa, and Kazuya Yonekura. Anomaly inflow and p-form gauge theories. *Comm. Math. Phys.*, 391(2):495–608, 2022. https://arxiv.org/abs/2003.11550. 6, 37

[Hu23]   Jiahao Hu. Invariants of real vector bundles, 2023. https://arxiv.org/abs/2310.05061. 40, 57

[Jam59]   I. M. James. Spaces associated with Stiefel manifolds. *Proc. London Math. Soc. (3)*, 9:115–140, 1959. 27

[Jam76]   I. M. James. *The topology of Stiefel manifolds*. London Mathematical Society Lecture Note Series, No. 24. Cambridge University Press, Cambridge-New York-Melbourne, 1976. 40

[Jan18]   Bas Janssens. Generalised spin structures in general relativity. *Ann. Henri Poincaré*, 19(5):1587–1610, 2018. 37

[JF17]   Theo Johnson-Freyd. Spin, statistics, orientations, unitarity. *Algebr. Geom. Topol.*, 17(2):917–956, 2017. https://arxiv.org/abs/1507.06297. 46

[JFY24]   Theo Johnson-Freyd and Mayuko Yamashita. On the 576-fold periodicity of the spectrum SQFT: The proof of the lower bound via the Anderson duality pairing, 2024. https://arxiv.org/abs/2404.06333. 20

[JO12]   Niles Johnson and Angélica M. Osorno. Modeling stable one-types. *Theory and Applications of Categories*, 26(20):520–537, 2012. http://www.tac.mta.ca/tac/volumes/26/20/26-20.pdf. 45

[Kob21]   Ryohei Kobayashi. Anomaly constraint on chiral central charge of $(2+1)$d topological order. *Phys. Rev. Res.*, 3:023107, May 2021. https://arxiv.org/abs/2101.01018. 34

[Koc96]   S. O. Kochman. *Bordism, stable homotopy and Adams spectral sequences*, volume 7 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996. 31

[Kom72]   Katsuhiro Komiya. Oriented bordism and involutions. *Osaka Math. J.*, 9:165–181, 1972. 2, 4, 32

[KP72]   Daniel S. Kahn and Stewart B. Priddy. Applications of the transfer to stable homotopy theory. *Bull. Amer. Math. Soc.*, 78:981–987, 1972. 35

[Kre84]   M. Kreck. Some closed 4-manifolds with exotic differentiable structure. In *Algebraic topology, Aarhus 1982 (Aarhus, 1982)*, volume 1051 of *Lecture Notes in Math.*, pages 246–262. Springer, Berlin, 1984. 5, 33

[KT90a]   R. C. Kirby and L. R. Taylor. A calculation of Pin$^+$ bordism groups. *Commentarii Mathematici Helvetici*, 65(1):434–447, Dec 1990. 33, 38, 52

[KT90b]   R. C. Kirby and L. R. Taylor. Pin structures on low-dimensional manifolds. In *Geometry of low-dimensional manifolds, 2 (Durham, 1989)*, volume 151 of *London Math. Soc. Lecture Note Ser.*, pages 177–242. Cambridge Univ. Press, Cambridge, 1990. 2, 5, 36, 52, 53, 54, 55

[KTTW15]   Anton Kapustin, Ryan Thorngren, Alex Turzillo, and Zitao Wang. Fermionic symmetry protected topological phases and cobordisms. *J. High Energy Phys.*, 2015(12):052, front matter+20pp, 2015. https://arxiv.org/abs/1406.7329. 5, 33, 55

[KZ18]   Takuji Kashiwabara and Hadi Zare. Splitting Madsen-Tillmann spectra I. Twisted transfer maps. *Bull. Belg. Math. Soc. Simon Stevin*, 25(2):263–304, 2018. https://arxiv.org/abs/1407.7201. 4, 18, 35

[Lam72]   Kee Yuen Lam. Fiber homotopic trivial bundles over complex projective spaces. *Proc. Amer. Math. Soc.*, 33:211–212, 1972. 28

[Lan68]   Peter S. Landweber. On the symplectic bordism groups of the spaces $\mathrm{Sp}(n)$, $\mathrm{HP}(n)$, and $\mathrm{BSp}(n)$. *Michigan Math. J.*, 15:145–153, 1968. 39

[Las63]   R. Lashof. Poincaré duality and cobordism. *Trans. Amer. Math. Soc.*, 109:257–277, 1963. 2, 7, 28

[Lau00]   Gerd Laures. On cobordism of manifolds with corners. *Trans. Amer. Math. Soc.*, 352(12):5667–5688, 2000. 20

[Law23]   H. Blaine Lawson, Jr. Spin^h manifolds. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 19:Paper No. 012, 7, 2023. https://arxiv.org/abs/2301.09683. 40

[Li19]   Huaiyu Li. Discussions on Dai-Freed anomalies. Master's thesis, Uppsala University, 2019. https://www.diva-portal.org/smash/get/diva2:1366363/FULLTEXT01.pdf. 37

[Lin78]   Harald Lindner. Adjunctions in monoidal categories. *Manuscripta Math.*, 26(1-2):123–139, 1978. 16

[LMSM86]   L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. http://www.math.uchicago.edu/~may/BOOKS/equi.pdf. 22

[LS19]   C. I. Lazaroiu and C. S. Shahbazi. Real pinor bundles and real Lipschitz structures. *Asian J. Math.*, 23(5):749–836, 2019. https://arxiv.org/abs/1606.07894. 40

[LSW20]   John A. Lind, Hisham Sati, and Craig Westerland. Twisted iterated algebraic K-theory and topological T-duality for sphere bundles. *Ann. K-Theory*, 5(1):1–42, 2020. https://arxiv.org/abs/1601.06285. 28

[LTL$^+$21] Tsuf Lichtman, Ryan Thorngren, Netanel H. Lindner, Ady Stern, and Erez Berg. Bulk anyons as edge symmetries: Boundary phase diagrams of topologically ordered states. *Phys. Rev. B*, 104:075141, Aug 2021. https://arxiv.org/abs/2003.04328. 6

[Mah65]   Mark Mahowald. A short proof of the James periodicity of $\pi_{k+p}(V_{k+m,m})$. *Proc. Amer. Math. Soc.*, 16:512, 1965. 28

[May65]   Karl Heinz Mayer. Elliptische Differentialoperatoren und Ganzzahligkeitssätze für charakteristische Zahlen. *Topology*, 4:295–313, 1965. 40

[May77]   J. Peter May. E_∞ *ring spaces and E_∞ ring spectra*. Lecture Notes in Mathematics, Vol. 577. Springer-Verlag, Berlin-New York, 1977. With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave. 22, 29

[Mei17]   Lennart Meier. Relatively free *TMF*-modules, 2017. https://webspace.science.uu.nl/~meier007/RelativelyFree4.pdf. 60

[Mil23]   Keith Mills. The structure of the Spin^h bordism spectrum, 2023. https://arxiv.org/abs/2306.17709. 40

[Mit75]   G. E. Mitchell. Bordism of manifolds with oriented boundaries. *Proc. Amer. Math. Soc.*, 47:208–214, 1975. 20

[MOP+22] Lyne Moser, Viktoriya Ozornova, Simona Paoli, Maru Sarazola, and Paula Verdugo. Stable homotopy hypothesis in the Tamsamani model. *Topology Appl.*, 316:Paper No. 108106, 40, 2022. https://arxiv.org/abs/2001.05577. 45

[Mor09]   Kaoru Morisugi. Graph constructions and transfer maps. *Bull. Fac. Ed. Wakayama Univ. Natur. Sci.*, 59(59):1–9, 2009. 35

[MS23]   Lukas Müller and Luuk Stehouwer. Reflection structures and spin statistics in low dimensions, 2023. https://arxiv.org/abs/2301.06664. 46

[MT01]   Ib Madsen and Ulrike Tillmann. The stable mapping class group and $Q(\mathbb{CP}^\infty_+)$. *Invent. Math.*, 145(3):509–544, 2001. 9

[MV21]   Miguel Montero and Cumrun Vafa. Cobordism conjecture, anomalies, and the string lamppost principle. *J. High Energy Phys.*, 2021(1):Paper No. 063, 46, 2021. https://arxiv.org/abs/2008.11729. 37

[MW07]   Ib Madsen and Michael Weiss. The stable moduli space of Riemann surfaces: Mumford's conjecture. *Ann. of Math. (2)*, 165(3):843–941, 2007. https://arxiv.org/abs/math/0212321. 9

[Nag95]   Masayoshi Nagase. Spin^q structures. *J. Math. Soc. Japan*, 47(1):93–119, 1995. 40

[Ngu17]   Hoang Kim Nguyen. On the infinite loop space structure of the cobordism category. *Algebraic & Geometric Topology*, 17(2):1021–1040, 2017. https://arxiv.org/abs/1505.03490. 6, 45

[Nov67]   S. P. Novikov. Methods of algebraic topology from the point of view of cobordism theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 31:855–951, 1967. 41

[Obi01]   Mohammad Obiedat. On J-orders of elements of $KO(\mathbf{C}\mathbf{P}^m)$. *J. Math. Soc. Japan*, 53(4):919–932, 2001. https://arxiv.org/abs/math/9906027. 28

[OT96]   Christian Okonek and Andrei Teleman. Quaternionic monopoles. *Comm. Math. Phys.*, 180(2):363–388, 1996. https://arxiv.org/abs/alg-geom/9505029. 40

[Pat12]   Deepam Patel. De Rham $\mathcal{E}$-factors. *Inventiones mathematicae*, 190(2):299–355, Nov 2012. http://www.math.purdue.edu/~patel471/deRhamep.pdf. 45

[PC23]   T. E. Panov and G. Chernykh. SU-linear operations in complex cobordism and the c_1-spherical bordism theory. *Izv. Ross. Akad. Nauk Ser. Mat.*, 87(4):133–165, 2023. https://arxiv.org/abs/2106.11876. 41

[Pet68]   F. P. Peterson. *Lectures on Cobordism Theory*. Lectures in Mathematics. Kinokuniya Book Store Co., Ltd., 1968. 5, 26, 33

[Rou72]   Fred William Roush. *Transfer in generalized cohomology theories*. ProQuest LLC, Ann Arbor, MI, 1972. Thesis (Ph.D.)–Princeton University. 35

[RS22]   Carmen Rovi and Matthew Schoenbauer. Relating cut and paste invariants and TQFTS. *Q. J. Math.*, 73(2):579–607, 2022. https://arxiv.org/abs/1803.02939. 2, 46

[RST77]   Nigel Ray, Robert Switzer, and Larry Taylor. Normal structures and bordism theory, with applications to $M\mathrm{Sp}_*$. *Mem. Amer. Math. Soc.*, 12(193):ix+66, 1977. 20

[RW23]   Oscar Randal-Williams. Monodromy and mapping class groups of 3-dimensional hypersurfaces, 2023. https://arxiv.org/abs/2308.06397. 28

[Sat10]   Hisham Sati. Geometric and topological structures related to M-branes. In *Superstrings, geometry, topology, and C*-algebras*, volume 81 of *Proc. Sympos. Pure Math.*, pages 181–236. Amer. Math. Soc., Providence, RI, 2010. https://arxiv.org/abs/1001.5020. 28

[Sat11a] Hisham Sati. Geometric and topological structures related to M-branes II: Twisted string and string^c structures. *J. Aust. Math. Soc.*, 90(1):93–108, 2011. https://arxiv.org/abs/1007.5419. 28

[Sat11b] Hisham Sati. Twisted topological structures related to M-branes. *Int. J. Geom. Methods Mod. Phys.*, 8(5):1097–1116, 2011. https://arxiv.org/abs/1008.1755. 28

[Sat12]   Hisham Sati. Twisted topological structures related to M-branes II: twisted Wu and Wu^c structures. *Int. J. Geom. Methods Mod. Phys.*, 9(7):1250056, 21, 2012. https://arxiv.org/abs/1109.4461. 28

[Sat15]   Hisham Sati. Ninebrane structures. *Int. J. Geom. Methods Mod. Phys.*, 12(4):1550041, 24, 2015. https://arxiv.org/abs/1405.7686. 28, 31

[Ser53]   Jean-Pierre Serre. Groupes d'homotopie et classes de groupes abéliens. *Ann. of Math. (2)*, 58:258–294, 1953. 29

[Shi73]   Katsuyuki Shibata. Oriented and weakly complex bordism algebra of free periodic maps. *Trans. Amer. Math. Soc.*, 177:199–220, 1973. 32

[Sig75] François Sigrist. Deux propriétés des groupes $J(CP^{2n})$. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)*, 59(5):413–415 (1976), 1975. 28

[SP11] Christopher J. Schommer-Pries. Central extensions of smooth 2-groups and a finite-dimensional string 2-group. *Geom. Topol.*, 15(2):609–676, 2011. https://arxiv.org/abs/0911.2483. 33, 34, 39

[SP17] Christopher Schommer-Pries. Invertible topological field theories, 2017. https://arxiv.org/abs/1712.08029. 6, 45

[SSR17] Ken Shiozaki, Hassan Shapourian, and Shinsei Ryu. Many-body topological invariants in fermionic symmetry-protected topological phases: Cases of point group symmetries. *Phys. Rev. B*, 95:205139, May 2017. https://arxiv.org/abs/1609.05970. 34

[SSS09] Hisham Sati, Urs Schreiber, and Jim Stasheff. Fivebrane structures. *Rev. Math. Phys.*, 21(10):1197–1240, 2009. https://arxiv.org/abs/0805.0564. 28, 30

[SSS12] Hisham Sati, Urs Schreiber, and Jim Stasheff. Twisted differential string and fivebrane structures. *Comm. Math. Phys.*, 315(1):169–213, 2012. https://arxiv.org/abs/0910.4001. 28, 30

[ST04] Stephan Stolz and Peter Teichner. What is an elliptic object? In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 247–343. Cambridge Univ. Press, Cambridge, 2004. https://math.berkeley.edu/~teichner/Papers/Oxford.pdf. 24

[Ste51] Norman Steenrod. *The Topology of Fibre Bundles*. Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton, NJ, 1951. 30

[Ste22] Luuk Stehouwer. Interacting SPT phases are not Morita invariant. *Lett. Math. Phys.*, 112(3):Paper No. 64, 25, 2022. https://arxiv.org/abs/2110.07408. 21, 26, 27, 36, 44

[Ste24] Luuk Stehouwer. The categorical spin-statistics theorem, 2024. https://arxiv.org/abs/2403.02282. 46

[Sto63] Robert E. Stong. Determination of $H^*(\mathrm{BO}(k, \cdots, \infty), Z_2)$ and $H^*(\mathrm{BU}(k, \cdots, \infty), Z_2)$. *Trans. Amer. Math. Soc.*, 107:526–544, 1963. 59

[Sto68] Robert E. Stong. *Notes on cobordism theory.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968. Mathematical notes. 20, 36, 41, 56

[Sto69] R. E. Stong. Bordism and involutions. *Ann. of Math. (2)*, 90:47–74, 1969. 32

[Sto88] Stephan Stolz. Exotic structures on 4-manifolds detected by spectral invariants. *Invent. Math.*, 94(1):147–162, 1988. 26

[Sto98] Stephan Stolz. Concordance classes of positive scalar curvature metrics, 1998. https://www3.nd.edu/~stolz/preprint.html. 44

[SW15] Hisham Sati and Craig Westerland. Twisted Morava K-theory and E-theory. *J. Topol.*, 8(4):887–916, 2015. https://arxiv.org/abs/1109.3867. 28

[SW18] Hisham Sati and Matthew Wheeler. Variations of rational higher tangential structures. *J. Geom. Phys.*, 130:229–248, 2018. https://arxiv.org/abs/1612.06983. 28

[SY21] Hisham Sati and Aliaksandra Yarosh. Twisted Morava K-theory and connective covers of Lie groups. *Algebr. Geom. Topol.*, 21(5):2223–2255, 2021. https://arxiv.org/abs/1711.05389. 28

[Tam97] Hirotaka Tamanoi. Multiplicative indecomposable splittings of $M\mathrm{Sp}_{[2]}$. *Math. Z.*, 225(4):577–610, 1997. 38

[Tho62] Emery Thomas. On the cohomology groups of the classifying space for the stable spinor groups. *Bol. Soc. Mat. Mexicana (2)*, 7:57–69, 1962. 24

[Tur20] Alex Turzillo. Diagrammatic state sums for 2D pin-minus TQFTs. *J. High Energy Phys.*, 2020(3):019, 26, 2020. https://arxiv.org/abs/1811.12654. 54

[TY19] Yuji Tachikawa and Kazuya Yonekura. Why are fractional charges of orientifolds compatible with Dirac quantization? *SciPost Phys.*, 7:58, 2019. https://arxiv.org/abs/1805.02772. 5, 33, 37

[TY23a] Yuji Tachikawa and Mayuko Yamashita. Anderson self-duality of topological modular forms, its differential-geometric manifestations, and vertex operator algebras, 2023. https://arxiv.org/abs/2305.06196. 20

[TY23b] Yuji Tachikawa and Mayuko Yamashita. Topological modular forms and the absence of all heterotic global anomalies. *Comm. Math. Phys.*, 402(2):1585–1620, 2023. https://arxiv.org/abs/2108.13542. 20

[Uch69]   Fuichi Uchida. Exact sequences involving cobordism groups of immersions. *Osaka Math. J.*, 6:397–408, 1969. 35

[Uch70]   Fuichi Uchida. The structure of the cobordism groups $B(n, k)$ of bundles over manifolds with involution. *Osaka Math. J.*, 7:193–202, 1970. 32

[Wal60]   C. T. C. Wall. Determination of the cobordism ring. *Ann. of Math. (2)*, 72:292–311, 1960. 34, 63

[Wal81]   Grant Walker. Estimates for the complex and quaternionic James numbers. *Quart. J. Math. Oxford Ser. (2)*, 32(128):467–489, 1981. 28

[Wei22]   Christoph Weis. The Drinfel'd centres of String 2-groups, 2022. https://arxiv.org/abs/2202.01271. 34, 39

[Woo63]   R.M.W. Wood. K-theory and the complex projective plane. 1963. 34

[WW19]   Zheyan Wan and Juven Wang. Higher anomalies, higher symmetries, and cobordisms I: classification of higher-symmetry-protected topological states and their boundary fermionic/bosonic anomalies via a generalized cobordism theory. *Ann. Math. Sci. Appl.*, 4(2):107–311, 2019. https://arxiv.org/abs/1812.11967. 40, 56

[WW20a]   Zheyan Wan and Juven Wang. Beyond standard models and grand unifications: anomalies, topological terms, and dynamical constraints via cobordisms. *J. High Energy Phys.*, 2020(7):062, 90, 2020. https://arxiv.org/abs/1910.14668. 27

[WW20b]   Juven Wang and Xiao-Gang Wen. Nonperturbative definition of the standard models. *Phys. Rev. Res.*, 2:023356, Jun 2020. https://arxiv.org/abs/1809.11171. 40

[WWW19]   Juven Wang, Xiao-Gang Wen, and Edward Witten. A new SU(2) anomaly. *J. Math. Phys.*, 60(5):052301, 23, 2019. https://arxiv.org/abs/1810.00844. 40

[WWZ20]   Zheyan Wan, Juven Wang, and Yunqin Zheng. Higher anomalies, higher symmetries, and cobordisms II: Lorentz symmetry extension and enriched bosonic/fermionic quantum gauge theory. *Ann. Math. Sci. Appl.*, 5(2):171–257, 2020. https://arxiv.org/abs/1912.13504. 5, 33, 40

[WY22]   Juven Wang and Yi-Zhuang You. Gauge enhanced quantum criticality beyond the standard model. *Phys. Rev. D*, 106:025013, Jul 2022. https://arxiv.org/abs/2106.16248. 40

[Yam23]   Mayuko Yamashita. Differential models for the Anderson dual to bordism theories and invertible QFT's, II. *J. Gökova Geom. Topol. GGT*, 16:65–97, 2023. https://arxiv.org/abs/2110.14828. 46

[Yon19]   Kazuya Yonekura. On the cobordism classification of symmetry protected topological phases. *Comm. Math. Phys.*, 368(3):1121–1173, 2019. https://arxiv.org/abs/1803.10796. 2, 46

[Yos75]   Zen-ichi Yosimura. Universal coefficient sequences for cohomology theories of CW-spectra. *Osaka J. Math.*, 12(2):305–323, 1975. 45

[YY23]   Mayuko Yamashita and Kazuya Yonekura. Differential models for the Anderson dual to bordism theories and invertible QFT's, I. *J. Gökova Geom. Topol. GGT*, 16:1–64, 2023. https://arxiv.org/abs/2106.09270. 46

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