A Long Exact Sequence in Symmetry Breaking:
order parameter constraints, defect anomaly-matching, and higher Berry phase

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We study defects in symmetry breaking phases, such as domain walls, vortices, and hedgehogs. In particular, we focus on the localized gapless excitations which sometimes occur at the cores of these objects. These are topologically protected by an 't Hooft anomaly. We classify different symmetry breaking phases in terms of the anomalies of these defects, and relate them to the anomaly of the broken symmetry by an anomaly-matching formula. We also derive the obstruction to the existence of a symmetry breaking phase with a local defect. We obtain these results using a long exact sequence of groups of invertible field theories, which we call the "symmetry breaking long exact sequence" (SBLES). The mathematical backbone of the SBLES is the Smith homomorphism, a family of maps between twisted bordism groups. Though many examples have been studied, we give the first completely general account of the Smith homomorphism. We lift it to a map of Thom spectra and identify the cofiber, producing a long exact sequence of twisted bordism groups; the SBLES is the Anderson dual of that long exact sequence. Our work develops further the theory of higher Berry phase and its bulk-boundary correspondence, and serves as a new computational tool for classifying symmetry protected topological phases.

CONTENTS

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I. INTRODUCTION

Over the past twenty years or so, there has been a revolution in the way we understand symmetries and anomalies of many-body quantum systems, both in the continuum and on the lattice, spurred by the discovery of topological insulators and other condensed matter systems exhibiting bulk-boundary correspondence, or anomaly in-flow. In this paper, we study phenomena associated with symmetry breaking at the surface of such phases, and in particular the gapless modes localized at domain walls, vortices, hedgehogs, and other defects in the order parameter, of a broad class that we define in this work. In particular, we provide a complete solution of the anomaly matching problem for such surface defects, relating their classification to that of the bulk phase.

For example, the surface of a 3d topological insulator famously supports a single Dirac cone, protected by charge conservation $U(1)$ and time reversal symmetry. When brought into contact with a superconductor (thought of as a $U(1)$ symmetry breaking state), even if the superconductor is a normal s-wave state, an exotic sort of superconductivity occurs at the interface by proximity effect [FK08, SW16], characterized by Majorana zero modes at vortices. Other famous examples of localized gapless modes include chiral modes along domain walls [JR76] and axion strings [CJH85, GW81].

It turns out that in many cases, the existence of localized gapless modes at such defects is guaranteed by anomaly matching, and holds even at strong coupling. An anomaly matching formula of this type was first provided in [HKT20a], although it was noticed that 1. not all anomalies are consistent with local defects in a symmetry breaking phase and 2. even when it exists, the anomaly of the defect is not determined by the anomaly matching formula. Determining the constraints under which defect anomaly matching can be applied and its ambiguities were left as open problems.

In this work, we devise a general theory of defect anomaly matching, in terms of a mathematical object known as a long exact sequence, which captures both the obstruction to the existence of a symmetry breaking phase with a local defect and the classification of such phases. The results are summarized in Fig. 1, with details to be explained later.

The physical input relies on the recent concept of higher Berry phase and its associated bulk-boundary correspondence [HKT20b, CFLS20, KS20, WQB$^+$21] which we also further develop. In particular, we formulate an interacting version of the Callias index theorem [Cal78, BS78] which we believe will have further applications.

As a computational tool, our long exact sequence turns out to be remarkably convenient. Different symmetry breaking patterns can be combined to calculate the classification of anomalies for a given symmetry group and dimension, often avoiding difficult spectral sequence calculations. For example, we use this idea in §A 1 to address an extension problem; other papers using this or closely related techniques to do computations include [HS13, Deb23, DDHM23, DL23, DYY].

In the remainder of this section we review the description and classification of 't Hooft anomalies in terms of invertible field theories, including some more recent perspectives and family anomalies.

Section II contains the description of the symmetry breaking long exact sequence (SBLES) and our physical results. The SBLES consists of three anomaly-matching formulas/maps: (Section II A) the residual family anomaly which persists after explicitly breaking the global symmetry and

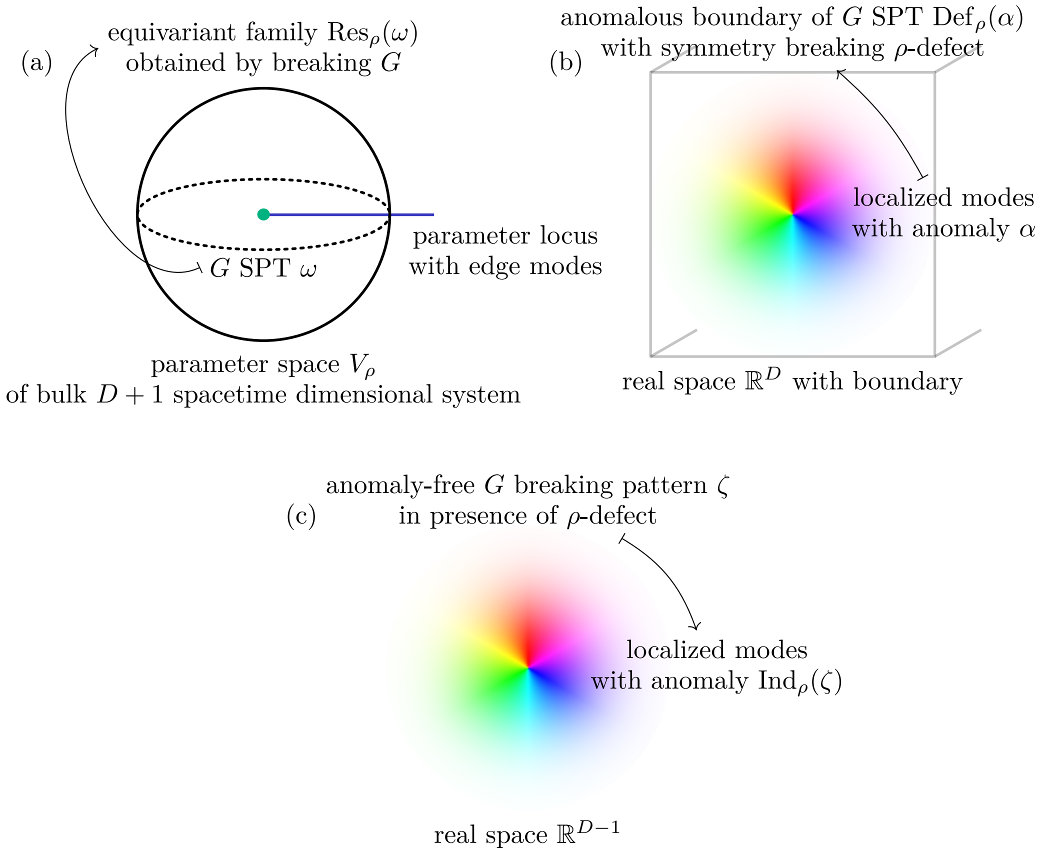

FIG. 1: **The three anomaly-matching maps: (a)** (Section II A) applying a symmetry breaking field transforming in the representation ρ to the G SPT ω produces a G-equivariant invertible family $\mathrm{Res}_\rho(\omega)$ on the unit sphere $S(\rho)$. When this anomaly-free G breaking pattern is topologically nontrivial, there is a parameter locus where the boundary gap closes (a diabolical locus in the sense of [HKT20b]). This locus begins at the origin, where we have G symmetry and protected SPT edge modes, but even though G is broken it extends to infinity. This is the obstruction to a local ρ-defect on the boundary, and we call it the residual family anomaly. **(b)** (Section II B) When an SPT satisfies $\mathrm{Res}_\rho(\omega) = 0$, there is a local ρ-defect on the boundary, a class of defect including domain walls, vortices, hedgehogs, etc, which may host localized modes with anomaly α. The defect anomaly map (aka the Smith homomorphism of [HKT20a]) reconstructs from α the bulk SPT as $\omega = \mathrm{Def}_\rho(\alpha)$. **(c)** (Section II C) The defect anomaly map can reconstruct the boundary anomaly but it cannot generally be inverted to give the anomaly of the defect. Indeed, even in an anomaly-free G equivariant invertible family ζ, we can have a ρ-defect with localized anomalous modes. The index map computes their anomaly as $\alpha = \mathrm{Ind}_\rho(\zeta)$. This gives the ambiguity in the boundary ρ-defect in item (b) and a generalization of the Callias index theorem to interacting systems. In turn, families of the form $\mathrm{Res}_\rho(\omega)$ (as in (a)) are precisely those with trivial index maps, completing the circle (Section II D).

which provides the obstruction to a local defect in the order parameter; (Section II B) the defect anomaly map which reconstructs the bulk anomaly from the anomaly of the local defect, when it exists; and (Section II C) the index map which describes the anomaly of a defect in an invertible family and which determines the ambiguity of the defect anomaly in terms of the classification of topologically distinct symmetry breaking patterns of one lower dimension, thus coming in full circle. We discuss each of these in turn with several examples, before putting them all together in a long exact sequence in Section II D.

Section III contains the mathematical formalism behind our SBLES, in terms of a cofiber sequence of Thom spectra. We begin in §III A by reviewing standard material in homotopy theory relevant for the theory of bordism and Thom spectra, then in §III B go over Freed-Hopkins' work [FH21] establishing a connection between Thom spectra and reflection-positive invertible field theories; this is the bridge between the homotopy-theoretic theorems we prove in §III and the interpretation we give them in §II in terms of invertible field theories.

In §III C, we define the Smith homomorphism, the homotopy-theoretic concept dual to the map Def_ρ in the SBLES. The Smith homomorphism was defined and named by Conner-Floyd [CF64, Theorem 26.1], then generalized by many authors over the years;[1] we are the first to give a completely general account. In §III C 1, we review twisted tangential structures, essential for specifying the domain and codomain of the fully general Smith homomorphism; then, we give three definitions of the Smith homomorphism associated to a tangential structure ξ, a space X, a virtual vector bundle $V \to X$ of rank r_V, and a vector bundle $W \to X$ of rank r_W:

$$\mathrm{sm}_W\colon \Omega_n^\xi(X^{V-r_V}) \longrightarrow \Omega_{n-r_W}^\xi(X^{V\oplus W-r_V-r_W}). \tag{I.1}$$

Here $\Omega_*^\xi(X^E)$, for $E \to X$ a virtual vector bundle, refers to the bordism groups of manifolds with (X, E)-twisted ξ-structures, which we define in Definition III.29.

1. First, in Definition III.35 we define sm_W as the map sending the bordism class of a manifold M with map $f\colon M \to X$ to the bordism class of the zero locus of a section of $f^*W \to M$ transverse to the zero section.
2. We then define the Smith homomorphism in Definition III.42 as the map of bordism groups induced by a map of Thom spectra $X^V \to X^{V\oplus W}$, itself induced by the map of total spaces of vector bundles $V \to V \oplus W$ sending $v \mapsto (v, 0)$.
3. Our third definition, in Definition III.73, defines sm_W as the cap product homomorphism with the Euler class of V in (twisted) ξ-cobordism, following a construction of Euler classes in twisted generalized cohomology in §III C 3.

Theorem (Proposition III.46 and Corollary III.85)**.** *The above three definitions are equivalent.*

Each definition has its own advantages: the first and third allow for a direct comparison with preexisting special cases in the literature; the second is an essential ingredient for the construction of our SBLES, because it allows us to construct the two other maps in the SBLES apart from Def_ρ. Specifically, in §III C 5, we prove the following theorem.

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[1] See §III E for a collection of examples and references to previous literature on the Smith homomorphism.

Theorem III.88. *With X, V, W, and ξ as above, the fiber of the map of spectra $X^V \to X^{V\oplus W}$ in definition 2 is the map $p\colon S(W)^{p^*V} \to X^V$, where $S(W)$ denotes the unit sphere bundle of W and $p\colon S(W) \to X$ is the bundle map.*

This is not a new result, but it allows us to define a long exact sequence of bordism groups including the Smith homomorphism, as well as the Anderson-dual long exact sequence of invertible field theories, which is the mathematical instantiation of the SBLES.

Let $\Omega_\xi^*(-)$ denote the generalized cohomology theory which is Anderson dual to ξ-bordism. By work of Freed-Hopkins [FH21], the group of deformation classes of reflection-positive invertible n-dimensional field theories on manifolds with ξ-structure is isomorphic to $\Omega_\xi^n(\mathrm{pt})$.

Corollaries III.95 and III.97. *Let X, V, W, and ξ be as above. Then there are long exact sequences*

$$\cdots \longrightarrow \Omega_n^\xi(S(W)^{p^*V-r_V}) \xrightarrow{p} \Omega_n^\xi(X^{V-r_V}) \xrightarrow{\mathrm{sm}_W} \Omega_{n-r_W}^\xi(X^{V\oplus W-r_V-r_W}) \longrightarrow \Omega_{n-1}^\xi(S(W)^{p^*V-r_V}) \longrightarrow \cdots \tag{I.2a}$$

$$\cdots \longrightarrow \Omega_\xi^{n-r_W}(X^{V\oplus W-r_V-r_W}) \longrightarrow \Omega_\xi^n(X^{V-r_V}) \xrightarrow{p^*} \Omega_\xi^n(S(W)^{p^*V-r_V}) \longrightarrow \Omega_\xi^{n+1-r_W}(X^{V\oplus W-r_V-r_W}) \longrightarrow \cdots \tag{I.2b}$$

The long exact sequence (I.2b), interpreted as a long exact sequence of groups of reflection-positive invertible field theories, is our mathematical model for the SBLES. Moreover, as we discuss in Remark III.100, these long exact sequences are generalizations of Gysin sequences.

The last two subsections of §III survey many examples of Smith long exact sequences. §III D lays the theoretical framework, including how to identify one tangential structure as a twist of another tangential structure (Examples III.116, III.118, and III.120) and how to compute the periodicity of a family of Smith homomorphisms (Proposition III.108). Finally, in §III E, we focus on many specific examples, including their relationship to SBLESes worked out elsewhere in our paper or to related work in the literature.

We have two appendices. In Appendix A, we explicate a Smith long exact sequence from Example III.157, which involves pin$^-$ and pin$^+$ bordism, with the third term in the long exact sequence identified with certain twisted spin bordism groups of $\mathbb{RP}^1$. In Appendix B, we explain why we use Euler classes in cobordism, rather than in ordinary cohomology: the latter is not compatible with the Smith long exact sequence, and in Theorem B.2 we give an explicit counterexample. As part of our investigation of this counterexample, we prove a theorem that may be of independent interest.

Theorem B.4. *Let $V \to X$ be a rank-3 vector bundle with spin structure and $\mathcal{S} \to X$ be the spinor bundle of V. If $\eta \in ko^{-1}(\mathrm{pt}) \cong \mathbb{Z}/2$ is the unique nonzero element and $p_1^{\mathbb{H}} \in ko^4(B\mathrm{Sp}(1))$ denotes the first symplectic ko-Pontrjagin class (see Proposition B.3), then the ko-cohomology Euler class of V is*

$$e^{ko}(V) = \eta \cdot p_1^{\mathbb{H}}(\mathcal{S}) \in ko^3(X). \tag{I.3}$$

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A. Introduction to anomalies and invertible field theories

1. G-anomalies

An 't Hooft anomaly for a global symmetry G (or just G-anomaly) can be roughly defined as an obstruction to gauging G. This typically appears in some gauge non-invariance when we couple our theory to a background gauge field A. Let us write the partition function on a spacetime X^D with this background as $Z(X^D, A)$. Under a gauge transformation $A \mapsto A^g$, we may have

$$Z(X^D, A^g) = e^{i\alpha(X,A,g)} Z(X^D, A), \tag{I.4}$$

where $e^{i\alpha(X^D,A,g)}$ is some phase factor which signals that $Z(X^D, A)$ is not gauge invariant and there may be an anomaly. More precisely, since $Z(X^D, A)$ is only defined up to local counter-terms, $\alpha(X^D, A, g)$ is only defined up to variations of local counterterms, and if α cannot be cancelled this way, there is a G-anomaly.

Under mild assumptions about $\alpha(X^D, A, g)$ (see Section 5 of [TW21]), and in all known cases, there is a local counterterm $e^{i\omega(Y^{D+1},A)}$ defined in *one greater dimension* so that if $\partial Y^{D+1} = X^D$, then

$$e^{i\omega(Y^{D+1},A^g) - i\omega(Y^{D+1},A)} = e^{i\alpha(X^D,A,g)}. \tag{I.5}$$

This is called anomaly in-flow, since for continuous G it can be interpreted as missing boundary charge flowing into the bulk, and allows us to relate G-anomalies in D-dimensions to local counterterms $e^{i\omega(Y^{D+1},A)}$ in $D + 1$ dimensions. The phase factor $e^{i\omega(Y^{D+1},A)}$ itself is the (phase of the) partition function of a particularly simple type of $D + 1$-dimensional theory known as a G-symmetric invertible field theory. These theories are so named because if we take stacks of such theories (which multiplies their partition functions), each theory has an inverse with which it stacks to the trivial theory.

A famous example is the chiral anomaly in 1+1d. We have a theory of a free Dirac fermion with independently conserved left-movers and right-movers, corresponding to a symmetry group $G = U(1)_L \times U(1)_R$ with generators L and R. If we turn on background gauge fields A_L and A_R each with 2π magnetic flux through spacetime X, then there will be fermion zero modes which must be subtracted from the path integral measure, leading to an imbalance of "axial" $L - R$ charge and a nontrivial gauge variation of $Z_{\text{Dirac}}(X, A_L, A_R)$. This variation is equivalent to the boundary variation of the 2+1d Chern-Simons term [Wit89, Wen95]

$$\omega(Y^3, A) = \frac{1}{4\pi}\int_{Y^3} A_L dA_L - A_R dA_R. \tag{I.6}$$

We can think of the Dirac fermion as living at the boundary of a theory with this partition function. If we make symmetric deformations of the Dirac fermion, such as adding Luttinger interactions, the bulk cannot be affected even at strong coupling, and hence the anomaly does not change, since it is determined by the bulk. This property, known as anomaly matching, makes anomalies very useful for studying phase diagrams of theories and renormalization group flows.

Because of this bulk-boundary correspondence, we can study anomalies by studying the invertible field theory in the bulk. Invertible field theories are rather simple as physical theories, having just a single state in their Hilbert space associated to each closed manifold. However, as mathematical objects they are quite rich, and are expected to form an object called a loop spectrum. This roughly means that a family of invertible field theories in D dimensions parametrized by S^1 (i.e. S^1-family) is equivalent to an invertible field theory in $D-1$ dimensions. The equivalence is via a "Thouless pump", where the $D-1$-dimensional invertible field theory gets "pumped" to the boundary when we go adiabatically around the S^1-family in D dimensions [Kit13, Xio18, GJF19]. The main technical result of our work, the long exact sequence (to be explained in more detail in Section III C 5, specifically Corollary III.97), can be derived from the loop spectrum property. However, for concreteness and ease of calculation, we will demonstrate our physics results using a stronger conjectural description of these theories via cobordism theory, which we presently describe.

The SPT-cobordism conjecture [Kap14, KTTW15, FH21] is that the particular loop spectrum that appears is the so-called Anderson dual of the Thom spectrum, which is related to the cobordism theory of manifolds. We will describe here the basic physics content of this conjecture. First we must define a cobordism. A cobordism between two manifolds M_1 and M_2 is a third manifold N with $\partial N = M_1 \cup M_2$. Note we can define cobordisms for manifolds M_1, M_2 with structures like G gauge fields by asking that the structure extends to the cobordism N. A cobordism invariant is something which is additive under disjoint union, and equal for all manifolds related by cobordism. The second condition can be stated that if $M = \partial N$, all cobordism invariants must be trivial for M, since N gives a cobordism between M and the empty manifold.

The SPT-cobordism conjecture roughly means that $e^{i\omega(Y^{D+1},A)}$ behaves like the holonomy of a $D+1$-form connection integrated over Y^{D+1} [Yam23, Yam21, YY21]. In particular, there are "Chern numbers" associated with this connection, which are integer-valued cobordism invariants of closed $D+2$ manifolds (equipped with a G gauge field and any other relevant structure). We can think of this integer as the winding number of $e^{i\omega(Y^{D+1},A)}$ evaluated along slices of the $D+2$ manifold (compare [HKT20b]). Deformation equivalence classes (meaning continuous deformation within the space of invertible field theories, i.e. π_0 of this space) of invertible field theories are believed to be classified by these invariants.

In practice, this means $e^{i\omega(Y^{D+1},A)}$ can be written as a product of two terms: (1) a Chern-Simons invariant evaluated on Y^{D+1}, which is itself associated with an integer cobordism invariant in $D+2$ dimensions (now two more than the anomalous theory!), e.g.

$$\frac{1}{8\pi^2}\int_{W^4} dA_L dA_L - dA_R dA_R \tag{I.7}$$

is associated with (I.6); and (2) a $U(1)$-valued cobordism invariant in $D+1$-dimensions evaluated on Y, which typically consists of torsion pieces (valued in a finite subgroup of $U(1)$) and theta angles (which are not fixed under deformations). Note that we also equip Y with a metric, so that (1) can also include gravitational Chern-Simons terms.

2. *Family anomalies*

Besides G-anomalies, we are also interested in *family anomalies*, a relatively new concept which has appeared in the study of theories with a parameter space [TE18, KSTZ19, Tho17, CFLS20, KS20, HKT20b, WQB$^+$21]. Suppose we have a theory depending on a parameter space M. We can couple the theory to a background field $\phi(x) \in M$ for these parameters and consider $Z(X^D, \phi)$. It may be that $Z(X^D, \phi)$ cannot be consistently defined over the space of background fields, and instead behaves like a section of a line bundle. This is analogous to how a quantum mechanical system with a nontrivial Berry number cannot have a globally defined ground state.

In practical terms, the family anomaly for a collection of local operators $\mathcal{O}_1, \ldots, \mathcal{O}_n$ is an obstruction to choosing a local Hamiltonian H_0 such that[2]

$$H(c_1, \ldots, c_n) = H_0 + \sum_j c_j \int d^d x \mathcal{O}_j(x) \tag{I.8}$$

has a gapped, nondegenerate ground state for all $\sum_j |c_j|^2 > C$, for some C. This makes family anomalies especially useful for studying phase diagrams.

Family anomalies in D dimensions are associated with boundaries of theories in $D+1$-dimensions with a *higher* Berry number [HKT20b]. We may consider these higher dimensional theories to be invertible field theories for spacetimes equipped with the parameter field ϕ. The boundary partition function $Z(X^D, \phi)$ is then considered a vector in the (1d) state space of this theory (à la relative QFT [FT14]). Considered this way, family anomalies are actually a generalization of G-anomalies, since we may take $M = BG$.

It is interesting to combine family anomalies and G-anomalies, especially when there is *explicit* symmetry breaking. In the simple case with no explicit symmetry breaking, meaning for every value of the parameters M we have G-symmetry, we call this a G-symmetric family. More interesting is the case of a *G-equivariant family*, where G acts nontrivially on M, such that if $m \in M$ is fixed by some subgroup $G_m < G$, the theory at that parameter value has G_m symmetry. Other elements $g \in G$ map states and observables at m to those at $g \cdot m$.

When we have a G-equivariant family and we turn on a background gauge field A, the parameter field ϕ can no longer be a globally defined map to M. Instead, we can think of it as having boundary conditions set by the transition functions of the G gauge bundle $P \to X$ [TE18]. More precisely, we can define the associated M-bundle $P \times_G M := P \times M / G^{\text{diagonal}} \to X$ by the action of G on M, and define ϕ to be a section of this bundle. If it is possible to couple to such a background, we call the family anomaly-free. Note that not all G-equivariant families of invertible field theories are anomaly-free, but the ones characterizing family anomalies always are.

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[2] Note that, just as G-anomalies are only a property of the G action on the microscopic degrees of freedom, not of the dynamics, so too does the family anomaly only depend on the operators we couple to.

G-equivariant family anomalies in D dimensions are thus classified by invertible field theories in $D+1$ dimensions for spacetimes equipped with a G-gauge field A and a parameter field ϕ, which is a section of the associated M-bundle [HKT20b]. These are described in cobordism theory as above, where we ask that ϕ also extends to the cobordism.

We give an example of an equivariant family anomaly occurring at the 0+1d boundary of a 1+1d system. The 1+1d system is constructed beginning with the free Dirac fermion ψ we considered above. We will have a parameter space $M = S^1$ with a 2π-periodic coordinate θ, which parametrizes the mass deformation $\cos\theta\bar{\psi}\psi + i\sin\theta\bar{\psi}\gamma^c\psi$, where $\gamma^c = \gamma^0\gamma^1$. This breaks $G = U(1)_L \times U(1)_R$ to the diagonal "vector" subgroup $U(1)_V$ with generator $V = L + R$. The "axial" subgroup with generator $A = L - R$ is broken down to $\mathbb{Z}/2$, and acts on M as a rotation $\theta \mapsto \theta + 2\alpha$, where α is the angle of the axial rotation. Thus the family is not G-symmetric, but it is G-equivariant since an axial rotation just acts on the parameter θ.

Let us promote the parameter to a background field $\theta = \phi(x,t)$. If we compute the vector current in this model, as a result of the chiral anomaly, we will find a contribution proportional to $\partial_t\phi$, which results in a "Thouless pump": adiabatically taking the parameter around a 2π cycle causes a single $U(1)_V$ charge to be transported across the system [Tho83]. This results in a family anomaly at the boundary, since we cannot define the $U(1)_V$ charge there, consistently over the parameter space. As a result of this anomaly, given any $U(1)_V$-symmetric boundary condition, there will be some value of θ where the boundary gap closes, with two states of different $U(1)_V$ charge crossing in energy [HKT20b]. This generalizes the famous Jackiw-Rebbi domain wall zero mode [JR76].

We can derive the bulk topological term associated with this family anomaly

$$\frac{1}{2\pi}\int_{Y^2} \phi dA_V. \tag{I.9}$$

Indeed, by varying A_V, we find the contribution to the vector current $\partial_t\phi$ which characterizes the Thouless pump. It also defines the (1+1)-dimensional invertible field theory which characterizes the boundary family anomaly. Broadly speaking, the boundary family anomaly in explicit symmetry breaking situations like this one can be derived directly from the bulk anomaly. We will spend much of the paper explaining how this works in general, and also return to this and related examples of massive free fermions.

3. *Twisted tangent structures*

We will also need certain tangent structures on our spacetime manifolds, which are required to consistently define the microscopic degrees of freedom of the theory. For example, we may need a metric and an orientation to define basic kinetic terms, and in this paper we will always ask for these structures. In fermionic theories, we will further ask for a Spin structure, which is needed in the UV to define consistent boundary conditions for fermions. The anomaly typically depends on these choices, and we will need the invertible field theory in one more dimension to be equipped with these data as well.

The presence of the background G gauge field can "twist" these structures. For example, if $G = U(1)$ and we have a spin-charge relation, with all (fermionic) bosonic operators having (half) integer charge, respectively, then fermions can be defined using a Spin^c structure [SW16]. This is slightly weaker than a Spin structure, but requires some compatibility between the background gauge field A and the tangent bundle of spacetime. In particular, we have

$$\oint_\Sigma \frac{1}{2\pi} dA + \frac{1}{2} w_2(TX) \in \mathbb{Z} \tag{I.10}$$

for all closed surfaces Σ, where $w_2(TX)$ is the 2nd Stiefel-Whitney class of the tangent bundle TX, i.e. the obstruction to choosing a Spin structure on TX. We can think of the spin-charge relation in general as defining a central extension of G by fermion parity $\mathbb{Z}/2^F$

$$\mathbb{Z}/2^F \to G_F \to G, \tag{I.11}$$

where G_F has linear (as opposed to projective) representations on fermionic operators. We can classify such central extensions by a class in $H^2(G, \mathbb{Z}/2)$.

The other sort of twist which is important occurs with spacetime-orientation-reversing symmetries. For example, suppose $G = \mathbb{Z}/2$ acts as a time reversal symmetry. If $\gamma \subset X$ is a closed loop in spacetime around which the background gauge field A has nontrivial holonomy

$$\int_\gamma A = 1 \mod 2, \tag{I.12}$$

then it will be impossible to choose a consistent orientation of X around this loop, since we reverse the direction of time as we go around it[3] [Kap14]. Thus we are forced to consider non-orientable spacetimes. Likewise, to define our theory on such manifolds, we *must* have nontrivial holonomy for A along orientation reversing loops such as γ. We can phrase this compatibility condition between A and the tangent bundle as follows:

$$\int_\gamma A = \int_\gamma w_1(TX) \mod 2, \tag{I.13}$$

for all closed loops γ, where $w_1(TX)$ is the 1st Stiefel-Whitney class of TX. We can classify the spacetime-orientation-reversing elements of G as a homomorphism $G \to \mathbb{Z}/2$, or equivalently a class in $H^1(G, \mathbb{Z}/2)$.

A convenient way to encode both these data is to say that we have an orientation, Spin structure, etc. not on the tangent bundle TX of spacetime, but on the direct sum $TX \oplus A^*\eta$, where $A^*\eta$ is a vector bundle associated to the G gauge bundle by some $\mathbb{R}$-linear G representation η. The 1st and 2nd Stiefel-Whitney classes of η, considered as a vector bundle over the classifying space BG, define the twist classes $w_1(\eta) \in H^1(G, \mathbb{Z}/2)$ and $w_2(\eta) \in H^2(G, \mathbb{Z}/2)$ we considered above. Physically, we can think of η as the representation of fermion bilinears in the theory [KTTW15], although our classification will only depend on the classes $w_1(\eta)$ and $w_2(\eta)$.

For example, suppose we study $G = \mathbb{Z}/2$ global symmetry. $\mathbb{Z}/2$ has a single nontrivial irrep, the sign representation σ. Let us take $\eta = n\sigma$, meaning a sum of n copies of the sign representation. We find a four-fold periodic structure

- $n = 0 \mod 4$: ordinary $\mathbb{Z}/2$ symmetry U with $U^2 = 1$, corresponding to a separate Spin structure on TX and a $\mathbb{Z}/2$ gauge field.

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[3] We are working in a Euclidean picture, so there is no special time coordinate.

- $n = 1 \mod 4$: spacetime-orientation-reversing $\mathbb{Z}/2$ symmetry T with $T^2 = 1$, corresponding to a Pin$^-$ structure on TX (see [KT90b] for an introduction to these structures).

- $n = 2 \mod 4$: ordinary $\mathbb{Z}/2$ symmetry U with $U^2 = (-1)^F$, corresponding to a Spin^c structure on TX where the structure group of the determinant line is reduced from $U(1)$ to $\mathbb{Z}/2$.

- $n = 3 \mod 4$: spacetime-orientation-reversing $\mathbb{Z}/2$ symmetry T with $T^2 = (-1)^F$, corresponding to a Pin$^+$ structure on TX

This periodic structure is reflected in the repeated reduction of symmetry to the $\mathbb{Z}/2$ domain wall [HKT20a].

Twisted tangential structures are discussed again in Section III B 1, and the 4-periodic example is discussed again in Example III.131.

4. The group of invertible field theories

With all the data in hand, we are finally ready to define our object of interest:

Definition 1. Let G be a group acting on a space M (the parameter space), s a tangent structure (usually an orientation aka SO structure in the case of bosonic theories or a Spin structure in the case of fermionic theories), η a representation of G. We define $\Omega^n_{G,s,\eta}(M)$ to be the abelian group of deformation classes of invertible field theories defined for n-dimensional spacetimes X equipped with a G-gauge field A, an s-structure on $TX \oplus A^*\eta$, a section ϕ of the M-bundle over X associated with the gauge bundle of A, and a metric.

Note that the group structure on invertible field theories corresponds to "stacking" of physical systems. That is, if we have two D-dimensional systems each with G symmetry and parameter space M depending on the same sort of tangent structure, then we can combine the two systems, initially decoupled, which will have $G \times G$ symmetry, a parameter space $M \times M$, and two tangent structures of the same kind. We want to preserve the diagonal $G < G \times G$, tune the parameters in tandem over the diagonal parameter space $M \hookrightarrow M \times M$, and couple to the same tangent structure in each "layer". Then we will have produced a third system in the same symmetry/parameter space/tangent structure class. We can do the same for the invertible field theories which determines the anomalies of each theory, and by definition the anomaly of the third system will be the sum of those two in the group structure thereof.

II. THE SYMMETRY BREAKING LONG EXACT SEQUENCE

In this section, we will outline our main result, summarized in Fig. 1, which is that three important maps (two of them new) in the theory of anomaly matching fit together into a "long exact sequence". Associated to a symmetry group G and a real (orthogonal) representation ρ of G describing the explicit or spontaneous symmetry breaking of G, the symmetry breaking long exact sequence (SBLES) is

$$\cdots \to \Omega^{D}_{G,s,\eta}(S(\rho)) \xrightarrow{\mathrm{Ind}_\rho} \Omega^{D+1-k}_{G,s,\eta+\rho} \xrightarrow{\mathrm{Def}_\rho} \Omega^{D+1}_{G,s,\eta} \xrightarrow{\mathrm{Res}_\rho} \Omega^{D+1}_{G,s,\eta}(S(\rho)) \to \cdots$$

By anomaly in-flow, we can look at this long exact sequence either from the $D+1$-dimensional point of view of the invertible field theories, or from the D-dimensional point of view of the anomalous theories. From the latter point of view, the players are

- $\Omega^{D+1}_{G,s,\eta}$: Anomalies of G-symmetric theories in D spacetime dimensions, of type s (bosonic or fermionic), and twist η.
- $\Omega^{D+1}_{G,s,\eta}(S(\rho))$: Anomalies of G-equivariant families of theories, parametrized by the unit sphere $S(\rho) \cong S^{k-1}$ in the representation ρ (which has dimension k).

See Section I A for a review of these. Meanwhile, the maps are

- Res_ρ: Measures the residual family anomaly of the D-dimensional theory after breaking the symmetry by an operator transforming in the representation ρ. (Section II A)
- Def_ρ: Describes the reconstruction of the bulk anomaly from the anomaly on a certain defect associated with this symmetry breaking, such as a domain wall. (Section II B)
- Ind_ρ: Encodes a generalized index theorem which associates an anomalous defect to a certain winding configuration in the space of symmetry-broken states. (Section II C)

Each map has the property that its image is the kernel of the map following it. This is what makes it a "long exact sequence." For example, those anomalies which have no residual family anomaly, and so live in the kernel of Res_ρ, are precisely those which can be associated with a special defect, whose anomaly recovers the original anomaly by the map Def_ρ. We show some long subsequences of the whole structure in Section II D.

In this section, we will give physical definitions of each of these maps, arguments for the exactness of the sequence, and many examples of dynamical consequences of these maps. In later sections, we will give more mathematically precise definitions and longer examples.

A. Residual family anomalies

If we have a theory with a G-symmetry and a (t' Hooft) anomaly, there is no G-symmetric deformation of the theory to a nondegenerate, gapped phase. However, in the absence of gravitational anomalies, we can always nondegenerately gap the theory by breaking the symmetry, so long as we reduce to an anomaly-free subgroup $H < G$ (possibly trivial).

A more refined question is, if we have a family of symmetry breaking parameters transforming in a representation ρ of G, when can we nondegenerately gap the theory for all large-enough values of the symmetry breaking parameters?

Definition 2. A theory is **ρ-(nondegenerately)-gappable** if there exists an operator transforming in the representation ρ, such that for all large enough perturbations by this operator (referred to as the symmetry breaking field), the theory has a (nondegenerate) gapped ground state. Equivalently, the ground state for all large enough symmetry breaking fields is **uniformly**

(nondegenerately) gapped, meaning there is a uniform lower bound on the energy gap about the ground states (and further the ground state is unique). For this paper, "nondegenerately" will always be implied. This condition is equivalent to the existence of a local "ρ-defect", defined in Section II B below.

It turns out that depending on the G anomaly and ρ, a theory may not be ρ-gappable. The simplest such obstruction occurs when some unbroken symmetry H is still anomalous, but we will derive the general obstruction. We find there are more subtle obstructions, which can exist even when all unbroken symmetries are anomaly free, and which are related to parameter space anomalies and higher Berry phases [CFLS20, HKT20b, KS20, WQB$^+$21, Tho17].

The general obstruction can be derived by anomaly in-flow, as follows. We can start by thinking of our anomalous system as living at the boundary of a $D+1$-dimensional G-SPT, i.e. a G-symmetric invertible theory, (the equivalence class of) which we may use to label the 't Hooft anomaly. Let V_ρ be the real vector space associated to ρ. For each value of the symmetry breaking field $v \in V_\rho$, we can extend the symmetry breaking into the SPT bulk.

This defines a G-equivariant family of $D+1$-dimensional invertible theories over $S(\rho)$, a $k-1$-sphere of large radius $S(\rho) \subset V_\rho$, with our original anomalous theory with symmetry breaking field defining a G-equivariant family of boundary conditions. The deformation class of the bulk defines a (linear) map

$$\mathrm{Res}_\rho : \Omega^{D+1}_{G,s,\eta} \to \Omega^{D+1}_{G,s,\eta}(S(\rho)). \tag{II.1}$$

We call this map the **residual family anomaly**, since it turns out to be the obstruction to ρ-gappability. Indeed, if our D-dimensional theory is ρ-gappable and has anomaly $\omega \in \Omega^{D+1}_{G,s,\eta}$, then we must have $\mathrm{Res}_\rho\omega = 0$, since this would give us a uniformly gapped, G-equivariant family of boundary conditions for the $D+1$-dimensional invertible family with invariant $\mathrm{Res}_\rho\omega$, which is not possible if $\mathrm{Res}_\rho\omega \neq 0$, by the bulk-boundary correspondence for families [HKT20b]. Conversely, $\mathrm{Res}_\rho\omega$ is very likely the only obstruction to ρ-gappability, as we will argue below.

Recall that using the SPT-cobordism conjecture of Section I A, we can describe the anomaly $\omega \in \Omega^{D+1}_{G,s,\eta}$ as a function $\omega(X, A) \in U(1)$ on pairs of a spacetime $D+1$-manifold X and a background gauge field A. Meanwhile $\mathrm{Res}_\rho\omega \in \Omega^{D+1}_{G,s,\eta}(S(\rho))$ can be described as a function $(\mathrm{Res}_\rho\omega)(X, A, \phi)$ on triples (X, A, ϕ) further consisting of a section ϕ as above. We can define this function by evaluating ω on just (X, A), simply discarding ϕ, giving

$$(\mathrm{Res}_\rho\omega)(X, A, \phi) := \omega(X, A) \tag{II.2}$$

This residual family anomaly generalizes the anomaly of the unbroken symmetry. Indeed, consider the theory at some fixed $v \in S(\rho)$. This theory may have a residual anomaly for the unbroken subgroup $G_v < G$, which prevents us from gapping it without breaking G_v. The residual anomaly is thus also an obstruction to ρ-gappability. In fact, for each v, there is a map (pullback along the inclusion of v in $S(\rho)$)

$$\begin{gathered} v^* : \Omega^{D+1}_{G,s,\eta}(S(\rho)) \to \Omega^{D+1}_{G_v,s,\eta} \\ (v^*\omega)(X, A_v) := \omega(X, A_v, \phi = v), \end{gathered} \tag{II.3}$$

such that the image of the G anomaly under $v^* \circ \mathrm{Res}_\rho$ is the residual G_v anomaly. So the residual family anomaly cannot vanish unless the residual anomaly also vanishes for each $v \in S(\rho)$. However, the examples below in Sections II A 1 and II A 2 demonstrate that even if all residual anomalies vanish, the residual family anomaly might still not. In fact, this is the case *even if the symmetry is completely broken.* This is because we have broken the symmetry in a particular way, and we will be able to use how the symmetry relates theories at different parameter values to observe the residual family anomaly.

One situation where the residual G_v anomaly determines the residual family anomaly is when G acts transitively on $S(\rho)$, meaning for each v, v' there is a $g \in G$ such that $g \cdot v = v'$. Indeed, if the residual G_v anomaly at some v vanishes, then there exists a G_v-symmetric nondegenerate gapping of the theory at v. We can then apply G to that trivially gapped theory to get a uniformly gapped G-equivariant family on $S(\rho)$. In this case one can show v^* above is an isomorphism. See Remark III.99.

We note that in spontaneous symmetry breaking, the ground states are naturally labelled by elements of a single G orbit, since degeneracy between distinct G orbits may be lifted by G-symmetric perturbations. In this case, as above, the residual family anomaly is always determined by the anomaly of the unbroken symmetry group.

1. *Example:* $2+1D$ *Majoranas*

Let us give a simple example of a theory with a residual family anomaly, which is nontrivial even though the symmetry is completely broken. We take a single Majorana fermion (2 component real) ψ in 2+1D transforming under time reversal with $T^2 = (-1)^F$. This is known to be anomalous, and is associated with the generator of a $\mathbb{Z}/16$ group of 3+1D SPTs $\Omega^4_{\text{Pin}+} = \mathbb{Z}/16$ [Wit16a]. This and related symmetry breaking patterns are discussed later in Example III.131.

The mass term $m\bar{\psi}\psi$ is T-odd and completely gaps the theory, so for σ the sign representation of $\mathbb{Z}/2$, the theory is σ-gappable. However, if we take $\rho = \sigma \oplus \sigma$, or equivalently the π rotation representation of $\mathbb{Z}/2$, this theory turns out not to be ρ-gappable. This means that for *any* pair of T-odd operators $\mathcal{O}_1$, $\mathcal{O}_2$, and for any r, there exists a θ such that with the symmetry breaking field

$$r \cos\theta \mathcal{O}_1 + r \sin\theta \mathcal{O}_2, \tag{II.4}$$

the theory is not nondegenerately gapped.

As a somewhat trivial example, if we take $\mathcal{O}_1$ and $\mathcal{O}_2$ to both be the (same) T-odd mass term, then we can always balance the coefficients so they cancel and we have the massless Majorana. This gives a phase diagram as in Fig. 2.

Although this phase diagram is pretty trivial, it allows us to compute the residual family anomaly. Indeed, we can observe that going around the circle by an angle of π is equivalent to changing the sign of the mass. Majoranas with opposite mass differ by an invertible phase known as the $p + ip$ superconductor. We can say that the invertible family pumps a $p + ip$ superconductor or its inverse, a $p - ip$ superconductor, to the boundary as it crosses the $m = 0$ values of the angle; see Fig. 2. Observe that nothing is pumped going around the entire circle[4], since the $p + ip$ and $p - ip$ are inverse phases and cancel. However, this family is still nontrivial, which can be seen as follows.

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[4] We will see this is a general feature of families occurring in the image of the gapping obstruction in Section II D.

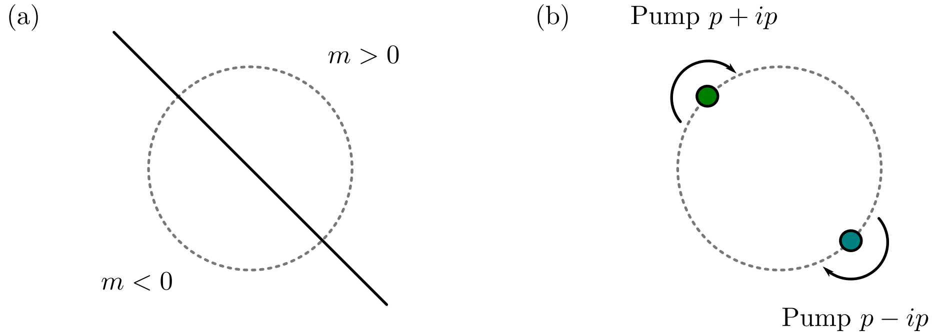

FIG. 2: (a) The phase diagram of a single 2+1D Majorana with redundant mass term. Time reversal acts on this phase diagram by a π rotation. The solid black line is where the Majorana is massless, and the dotted circle represents $S(\rho)$. (b) A representation of the 3+1D invertible family, where upon crossing either the green or blue dot, a $p + ip$ or $p - ip$ superconductor is pumped to the boundary. Observe that there is no total pump in going around the entire circle. However, with time reversal, this family is non-trivial, as can be measured by going half-way around the circle and then applying time reversal to return to the starting point. The number of $p + ip$'s pumped mod 2 this way is an invariant of the equivariant family.

First, one can try to modify the $S^1 = S(\rho)$ family along a short arc by pumping a $p + ip$ and then a $p - ip$ at the beginning and end of said arc. However, such arcs must occur in time reversal symmetric pairs, and by inspection one can show that the number of $p + ip$'s pumped while going around half the circle is an invariant mod 2.

More precisely, in such a family we can go adiabatically half way around the circle, and then return to where we started by applying time reversal, which acts as a π rotation. The invertible phase pumped to the boundary over such a cycle is a sort of equivariant generalization of the Thouless charge pump.

The fact that this family is nontrivial implies that the Majorana is not ρ-gappable for *any* pair of T-odd operators, not just the redundant mass terms. For example we may take $\mathcal{O}_1$ to be the mass term and $\mathcal{O}_2$ to be any other T-odd operator, such as $(\bar{\psi}\psi)^3$.

2. *Example: adjoint QCD*

Let us give a slightly more nontrivial example of a theory with a residual family anomaly, which has some interesting dynamical consequences. We consider $SU(2)$ Yang-Mills theory in 3+1D with Dirac fermions transforming in the complexified adjoint representation (equivalently we have two Majorana fermions transforming in the real adjoint). A recent discussion of this model can be found in [CD18].

There is an ABJ anomaly between the $U(1)_a$ axial symmetry and the $SU(2)$ gauge symmetry, which we can represent by the 6D integer cobordism invariant associated with (see Section I A)

$$8c_1^a c_2^{SU(2)} \in H^6(BU(1)_a \times BSU(2), \mathbb{Z}). \tag{II.5}$$

This means that the classical $U(1)_a$ is broken down to $\mathbb{Z}/8_a$ by $SU(2)$ instantons carrying 8 units of axial charge. Note that for fundamental Diracs the anomaly is $2c_1^a c_2^{SU(2)}$ in this normalization. The relative factor of 4 can be seen by restricting to the maximal torus $U(1) < SU(2)$ for which our complex adjoint Dirac becomes a charge 2, a charge 0, and a charge -2 Dirac, and $2^2 + 0^2 + (-2)^2 = 8$, while for a fundamental we have $1^2 + (-1)^2 = 2$.

This theory has a 1-form $\mathbb{Z}/2$ center symmetry, since the matter fields transform in the adjoint representation. If we gauge this center symmetry, it is equivalent to changing the global structure of the gauge group from $SU(2)$ to $SO(3)$. This allows for $\frac{1}{4}$ instantons that further break $\mathbb{Z}/8_a$ to the $\mathbb{Z}/2^F$ fermion parity subgroup (this is the same factor of 4 as above). This means there is an 't Hooft anomaly we can represent via

$$\omega = \frac{1}{4} A\Pi(B) \in H^5(B\mathbb{Z}/8_a \times B^2\mathbb{Z}/2, U(1)), \tag{II.6}$$

where A is the $\mathbb{Z}/8_a$ gauge field, B is the $B\mathbb{Z}/2$ gauge field, and $\Pi(B) \in H^4(B^2\mathbb{Z}/2, \mathbb{Z}/4)$ is the Pontrjagin square. We will see this anomaly has a residual family anomaly for $\mathbb{Z}/8_a$.

We can consider $\mathbb{Z}/8_a$ chiral symmetry breaking in this theory. A natural order parameter is the charge 2 doublet consisting of the real and chiral mass terms $\bar{\Psi}\Psi$ and $i\bar{\Psi}\gamma^5\Psi$, respectively—these form a basis of V_ρ. Let ϕ be the phase of this order parameter, which parametrizes a $\mathbb{Z}/8_a$-equivariant family on S^1 with $\mathbb{Z}/8_a$ acting as a $\pi/2$ rotation (so the $\mathbb{Z}/2^F$ subgroup acts trivially).

This family is not uniformly gapped over this S^1. We can parametrize it by $\theta/4$, where θ is the 2π-periodic QCD vacuum angle (the factor of 4 is once again the same one). However, for $\theta = \pi$, which corresponds to four different points on this S^1, it is expected that the theory has two degenerate ground states [GKKS17, tH81]. See Fig. 3.

Indeed, we can pass to the class describing the $\mathbb{Z}/8_a$-equivariant S^1-family by replacing A with $A - 4d\phi/2\pi$, where ϕ is 2π periodic and parametrizes the S^1, since a gauge transformation by 1 shifts A by 1 and $d\phi$ by $\pi/2$:

$$\mathrm{Res}_\rho(\omega) = \left(\frac{1}{4}A - \frac{d\phi}{2\pi}\right)\Pi(B) \in H^5_{\mathbb{Z}/8_a}(S^1 \times B^2\mathbb{Z}/2, U(1)). \tag{II.7}$$

Since $\mathbb{Z}/8_a$ acts freely on S^1 through its $\mathbb{Z}/4$ quotient, we can replace S^1 by its quotient, parametrized by the vacuum angle $\theta = 4\phi$, and find

$$\mathrm{Res}_\rho(\omega) = \frac{1}{4}\frac{d\theta}{2\pi}\Pi(B) \in H^5(S^1 \times B\mathbb{Z}/2^F \times B^2\mathbb{Z}/2, U(1)). \tag{II.8}$$

This is a non-trivial order 2 class, and was identified in [CFLS20] as the family anomaly of pure QCD. See also [KSTZ19, GKKS17].

B. The defect anomaly matching condition

Let us assume now there is no residual family anomaly, i.e. $\mathrm{Res}_\rho(\alpha) = 0$ in (II.1), and our system is nondegenerately gapped for all values of the symmetry breaking parameter $v \in S(\rho)$. In this case, we can construct a localized **ρ-defect** as follows. In coordinates, we take the symmetry

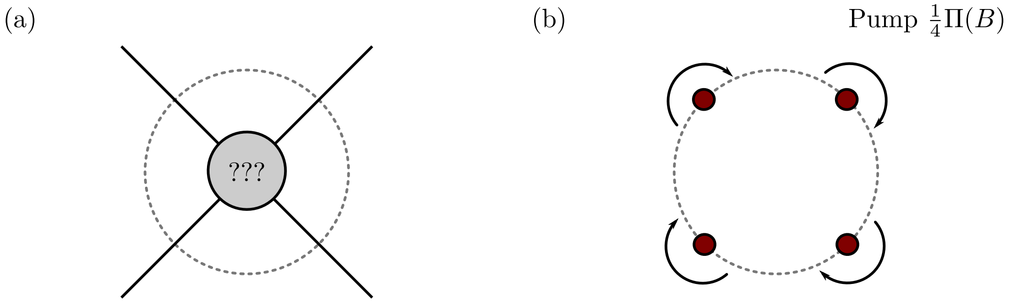

FIG. 3: (a) The phase diagram of 3+1D $SU(2)$ QCD with one adjoint Dirac fermion, deformed by the two mass terms. The $\mathbb{Z}/8_a$ chiral symmetry acts as a $\pi/2$ rotation on this phase diagram. Along the four spokes, we have oblique confinement with a 2-fold degenerate ground state. At the origin, where the fermion is massless, these spokes must merge into a nontrivial point or phase. One consistent proposal is that at the origin we have $SU(2)$ chiral symmetry breaking. The deformation of this state by small masses is analyzed in Section 3.4 of [CD18]. (b) The associated 4+1D invertible family over $S(\rho) \cong S^1$, where upon crossing one of the angles corresponding to oblique confinement, the 3+1D 1-form SPT $\frac{1}{4}\Pi(B)$ is pumped to the boundary. Note as in Fig. 2 there is no total pump around the family, but if we go around by an angle $\pi/2$ and then apply the axial symmetry to return to where we started, we get a well-defined pump invariant.

breaking parameter $v = \phi(x)$ to vary in space, with the form

$$\phi \sim (v_1 x_1 + \cdots + v_k x_k)/\sqrt{x_1^2 + \cdots x_k^2}, \qquad \text{for large } x_1^2 + \cdots x_k^2, \tag{II.9}$$

where $v_1, \ldots, v_k$ is an orthonormal basis of V_ρ, so that ϕ winds once around $S(\rho)$ far away from a defect along $x_1 = \cdots = x_k = 0$, where it must vanish. It is crucial that the system is uniformly gapped on $S(\rho)$ for this defect to define a local $D-k$-dimensional theory.

Following [HKT20a], it is possible to reconstruct the 't Hooft anomaly α by studying the theory on the ρ-defect. Although the symmetry is broken, by combining the G action with Lorentz symmetries (and CPT), we can invent a new symmetry G_ρ (isomorphic to G) which acts on this effective $D-k$-dimensional theory. If the original symmetry was anomalous, there will be localized modes on the ρ-defect which transform nontrivially under G_ρ, and in particular they will have a nontrivial anomaly. We know this is the case because we can actually use this anomaly to reconstruct the bulk anomaly, as follows.

Again we use anomaly in-flow. The key is to realize that the ρ-defect in the anomalous D-dimensional theory can be extended to a ρ-defect in the $D+1$-dimensional G-SPT, so that the core of the ρ-defect in $D+1$-dimensions carries a G_ρ-SPT which controls the anomaly of the ρ-defect in D dimensions. Thus we only need to understand how the $D+1$-dimensional SPT reduces to the ρ-defect.

To this end, suppose we want to compute the partition function of the G-SPT associated with the bulk anomaly on some $D+1$-spacetime X. In the presence of a symmetry breaking field ϕ on X, the G-SPT can be trivialized away from the $Y \subset X$ where $\phi = 0$. For generic smooth ϕ, in the normal bundle of Y we see that this zero set is precisely the bulk ρ-defect. Since the theory is trivialized away from Y, the partition function on X is simply equal to the partition function of the defect anomaly theory on Y.

The map from spacetimes X to zero sets Y can be formalized, once we keep track of all the relevant structures, to define a linear map we call the **defect anomaly map**[5]

$$\mathrm{Def}_\rho : \Omega^{D+1-k}_{G,s,\eta+\rho} \to \Omega^{D+1}_{G,s,\eta}, \tag{II.10}$$

defined by

$$\mathrm{Def}_\rho(\alpha)(X, A) = \alpha(Y, A), \tag{II.11}$$

where Y is a zero set as above. Cobordism invariance implies that this map does not depend on the choice of ϕ or Y. This map encodes the defect anomaly matching, such that if

$$\alpha \in \Omega^{D+1-k}_{G,s,\eta+\rho} \tag{II.12}$$

describes the G anomaly of the ρ-defect, and ω our original anomaly, then we have

$$\mathrm{Def}_\rho(\alpha) = \omega. \tag{II.13}$$

Note that the defect anomaly α determines the bulk anomaly ω by this equation, but not vice versa, and in particular even anomaly-free symmetries can have anomalous ρ-defects, a phenomenon we will explore in Section II C.

The defect anomaly map Def_ρ and the residual family anomaly Res_ρ defined in Section II A fit together in a special way. The kernel of Res_ρ is the image of Def_ρ. This means those anomalies which do not have a residual family anomaly are precisely those which can be reconstructed from the ρ-defect. This gives strong evidence that Res_ρ is the only obstruction to ρ-gappability, since we used this to define the ρ-defect. It also generalizes Theorem 4.2 in [HKT20a] from finite cyclic groups to arbitrary groups and arbitrary representations, answering the question of the cokernel of Def_ρ (ie. the Smith map) which was posed there.

1. *Example: defect anomaly matching for 3+1D Dirac fermion*

Consider a 3+1D Dirac fermion ψ (with four complex components). This has an anomalous chiral symmetry $U(1)_L$ which gives charge 1 to the two left-handed components of ψ and charge 0 to the two right handed ones. There are two Dirac masses $\bar{\psi}\psi$ and $i\bar{\psi}\gamma^5\psi$, which transform together under $U(1)_L$ as a charge 1 doublet ρ. Any combination of the two mass terms completely gaps the fermion, so in this case there is no residual family anomaly and there is a local ρ-defect. This process corresponds to Example III.148.

We can construct the ρ-defect in this theory by choosing a spatially-varying mass profile of the form

$$x_1\bar{\psi}\psi + x_2 i\bar{\psi}\gamma^5\psi. \tag{II.14}$$

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[5] This was called the Smith map in [HKT20a], but we prefer this more descriptive name in this section.

One can solve the Dirac equation for localized modes with this mass profile and find a massless 1+1D Weyl fermion (with one complex component) propagating in the remaining coordinates [CJH85]. We will evaluate Def_ρ for 1+1D theories with this symmetry and show by anomaly-matching that this fermion must have charge 1 under the residual $U(1)_\rho$ symmetry (which could also be concluded by a careful analysis of the localized solutions).

The residual symmetry $U(1)_\rho$ acting on the 1+1D ρ-defect acts as a combination of a $U(1)_L$ rotation and a compensating Spin(2) rotation, where Spin(2) is the rotation in the x_1, x_2 plane, such that the mass profile is invariant under their combination. In particular, a 2π $U(1)_\rho$ rotation is equal to a 2π rotation of this plane, which equals the fermion parity $(-1)^F$. This means we are interested in 1+1D systems with $\mathrm{Spin}^c = (\mathrm{Spin} \times U(1)_\rho)/\mathbb{Z}/2$ structure. A general anomaly for such a theory is given by a Chern-Simons form associated with a 4D integer cobordism invariant (see Section I A)

$$\alpha = k_1 \left(\frac{1}{8}(c_1^\rho)^2 - \frac{1}{24} p_1(TY) \right) + k_2 (c_1^\rho)^2, \tag{II.15}$$

where $k_1, k_2 \in \mathbb{Z}$.

We can compute $\mathrm{Def}_\rho(\alpha)$ in terms of these 4D cobordism invariants. That is, suppose X is a closed 6D Spin manifold with a principal $U(1)_L$ bundle P and a section ϕ of the $\mathbb{C}$ bundle $E_\rho := P \times_{U(1)_L} V_\rho$ associated to the charge 1 representation V_ρ. We take Y to be the analog of the ρ-defect, i.e. it is the zero set of ϕ (we can always perturb ϕ so its zero set is a 4-manifold). A useful fact is that the homology class $[Y] \in H_4(X, \mathbb{Z})$ is Poincaré dual to the first Chern class $c_1^L \in H^2(X, \mathbb{Z})$. This means that for any $\beta \in H^4(X, \mathbb{Z})$,

$$\int_X c_1^L \beta = \int_Y \beta. \tag{II.16}$$

To compute $\mathrm{Def}_\rho(\alpha)$, we want to choose β such that $\beta|_Y = \alpha$, then by definition we will have $\mathrm{Def}_\rho(\alpha) = c_1^L \beta$.

To get the $(c_1^\rho)^2$ terms, we use the fact that the $U(1)_\rho$ bundle over Y is defined by restriction of the $U(1)_L$ bundle, so in particular $c_1^L|_Y = c_1^\rho$. In terms of the defect anomaly map, this means to get $(c_1^\rho)^2$ we should take $\beta = (c_1^L)^2$, so

$$\mathrm{Def}_\rho((c_1^\rho)^2) = (c_1^L)^3. \tag{II.17}$$

The "gravitational" term (involving $p_1(TY)$) is more interesting. If we study the tangent bundle of X restricted to Y we find

$$TX|_Y = TY \oplus NY = TY \oplus E_\rho|_Y, \tag{II.18}$$

where we have identified the normal bundle NY with the restriction of the associated bundle E_ρ, since Y is the zero set of the section ϕ. Using the Whitney sum formula we obtain

$$\begin{aligned} p_1(TX)|_Y &= p_1(TY) + p_1(E_\rho)|_Y \\ &= p_1(TY) + (c_1^L)^2|_Y \\ &= p_1(TY) + (c_1^\rho)^2. \end{aligned} \tag{II.19}$$

So to get

$$\alpha = \frac{1}{8}(c_1^\rho)^2 - \frac{1}{24}p_1(TY) \tag{II.20}$$

we should take

$$\beta = \frac{1}{6}(c_1^L)^2 - \frac{1}{24}p_1(TX), \tag{II.21}$$

hence

$$\mathrm{Def}_\rho(\frac{1}{8}(c_1^\rho)^2 - \frac{1}{24}p_1(TY)) = \frac{1}{6}(c_1^L)^3 - \frac{1}{24}c_1^L p_1(TX). \tag{II.22}$$

This turns out to precisely coincide with the $U(1)_L$ anomaly of the 3+1D Dirac fermion. Thus defect anomaly matching requires $k_1 = 1$, $k_2 = 0$ in (II.15). This is consistent with a 1+1D Weyl fermion with $U(1)_\rho$ charge 1.

The above calculation seems to rely on a choice of β. Actually, it does not, since if $\beta'|_Y = \alpha$, $(\beta - \beta')|_Y = 0$, and so, using Poincaré duality, $c_1^L\beta - c_1^L\beta' = c_1^L(\beta - \beta') = 0$. On the other hand, the existence of such a β is guaranteed by the vanishing of the residual family anomaly, since this guarantees that $\int_X \omega = \int_Y \beta$ for some β.

2. *Example: defect anomaly matching for 3+1D Weyl fermion*

To see the importance of the representation in the above computation, let us consider a closely related example, this time beginning with a left-handed Weyl fermion in $3+1D$. This has a $U(1)_L$ symmetry with the same anomaly as the Dirac in Section II B 1 (since the right-handed Weyl does not contribute anything):

$$\omega = \frac{1}{6}(c_1^L)^3 - \frac{1}{24}c_1^L p_1(TX). \tag{II.23}$$

Above we studied the Dirac mass, which couples the two Weyl components. However, a single Weyl on its own has a Majorana mass that is *charge 2* under $U(1)_L$. Solving the equations of motion for the associated ρ-defect we find a left-handed Majorana-Weyl fermion in 1+1D. This has one real component, so $U(1)_\rho$ must act trivially on it. This situation is modeled by the Smith map in Example III.171.

Let us compute the defect anomaly map in this case and verify that this matches. Note that a 2π rotation in $U(1)_\rho$ is a 4π rotation in V_ρ, which is 1 on the fermion, so there is no Spin^c business here. Anomalies of 1+1D fermions with Spin $\times$ $U(1)_\rho$ symmetry split between a pure gravity and a pure symmetry part, and take the form

$$\alpha = \frac{k_1}{48}p_1(TY) + k_2(c_1^\rho)^2. \tag{II.24}$$

The calculation proceeds as above, although now $[Y] \in H_4(X,\mathbb{Z})$ is Poincaré dual to $2c_1^L \in H^2(X,\mathbb{Z})$, since ρ is a charge 2 representation. Once we compute β such that $\beta|_Y = \alpha$, we will have $\mathrm{Def}_\rho(\alpha) = 2c_1^L\beta$.

Using $c_1^L|_Y = c_1^\rho$, we find

$$\mathrm{Def}_\rho((c_1^\rho)^2) = 2(c_1^L)^3. \tag{II.25}$$

We also have

$$\begin{aligned} p_1(TX)|_Y &= p_1(TY) + p_1(E_\rho)|_Y \\ &= p_1(TY) + 4(c_1^L)^2|_Y \\ &= p_1(TY) + 4(c_1^\rho)^2. \end{aligned} \tag{II.26}$$

Thus we find

$$\mathrm{Def}_\rho(\frac{1}{48}p_1(TY)) = \frac{1}{6}(c_1^L)^3 - \frac{1}{24}c_1^L p_1(TX), \tag{II.27}$$

so the defect anomaly matches correctly with $k_1 = 1$, $k_2 = 0$.

C. The index map and higher Berry phase

Above we described an anomaly matching condition in terms of a map Def_ρ for which the image of the defect anomaly α is the bulk anomaly ω:

$$\mathrm{Def}_\rho(\alpha) = \omega. \tag{II.28}$$

We see the defect anomaly determines the bulk anomaly, but when Def_ρ is not injective, there can be several solutions for α given ω. Thus there is an ambiguity in the defect anomaly. There can even be anomalous defects ($\alpha \neq 0$) in anomalous bulk theories ($\omega = 0$)!

Recall that as long as there is no residual family anomaly, we can perturb things so that for each large enough value of the symmetry-breaking field, we obtain a trivially gapped ground state. This defines a G-equivariant family of invertible field theories over the sphere $S(\rho)$. This family is not typically free of G-anomalies, but it is when $\omega = 0$. In this case, we can couple it to a G gauge field, and classify its topological response by an element

$$\zeta \in \Omega^D_{G,s,\eta}(S(\rho)) \tag{II.29}$$

(cf. Section I A). Given such a family, we can construct the ρ-defect as before, and we want to describe the anomaly.

We can actually construct the anomaly theory of the ρ-defect directly from ζ by compactifying on $S(\rho) \cong S^{k-1}$. The idea is shown in Fig. 4. The compactification defines an element of $\Omega^{D+1-k}_{G,s,\eta+\rho}$, and moreover we get the **index map**

$$\mathrm{Ind}_\rho : \Omega^D_{G,s,\eta}(S(\rho)) \to \Omega^{D-k+1}_{G,s,\eta+\rho}. \tag{II.30}$$

In terms of partition functions, this map is defined as follows. Suppose we have a $D-k+1$-dimensional spacetime Y, equipped with a G connection A and $\eta+\rho$-twisted s-structure ξ. We can define the D-dimensional spacetime W given as the total space of the $S(\rho)$ bundle over W associated to the G gauge bundle. W gets a G connection π^*A by pullback from the projection

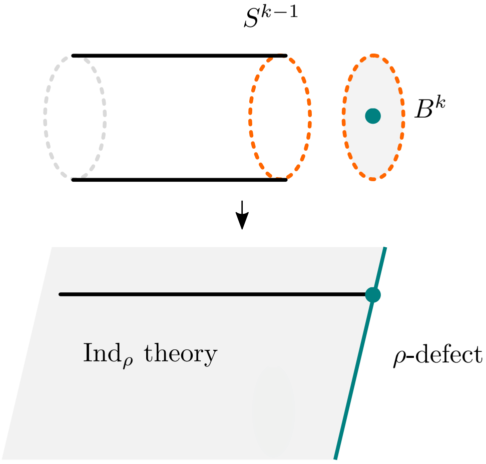

FIG. 4: **Calculating the index map:** the index map Ind_ρ describes the anomaly of a ρ-defect inside an invertible phase via a certain sphere compactification of that phase described in the text. The proof-by-picture of why this works is given here. The ρ-defect is defined on $B^k \times \mathbb{R}^{D-k}$, where B^k is a k-dimensional ball, depicted here as a B^k bundle over $\mathbb{R}^{D-k}$ (blue). Meanwhile we consider the invertible phase defined on $S^{k-1} \times H^{D-k+1}$, where H^{D-k+1} is a $D-k+1$-dimensional half-space, shown as an S^{k-1} bundle over H^{D-k+1} (gray). These have the same boundary (orange), and can be glued together to define a boundary condition of the compactified invertible theory, so long as the order parameter winds around this S^{k-1}. This defines Ind_ρ and thus measures the anomaly of the ρ-defect by anomaly in-flow.

map $\pi : W \to Y$, an η-twisted s structure $\pi^*\xi$, and a canonical section ϕ of the associated $S(\rho)$ bundle over it. Thus, given an element $\zeta \in \Omega^D_{G,s,\eta}(S(\rho))$, we can define

$$\mathrm{Ind}_\rho(\zeta)(Y, A, \xi) = \zeta(W, \pi^*A, \pi^*\xi, \phi). \tag{II.31}$$

We can also consider elements of $\Omega^D_{G,s,\eta}(S(\rho))$ as D-dimensional counterterms which can appear relating different symmetry-breaking patterns of a given theory with the same representation ρ. In particular, we can compare two different G-equivariant $S(\rho)$-families of invertible field theories by stacking one with the orientation reversal of the other. The result is free of G-anomalies and defines an element of $\Omega^D_{G,s,\eta}(S(\rho))$. Thus, the image of Ind_ρ above describes both the ambiguity in the defect anomaly and the kernel of Def_ρ (answering the question of the kernel of the Smith homomorphism in [HKT20a]).

The index map can be thought of as a generalization of the Callias index theorem [Cal78, BS78] which computes the fermion zero modes at the core of a mass defect. Our map gives the G-anomaly of those zero modes (and thus accounts for interactions).

If we define $B(\rho)$ as the ball in V_ρ with boundary $S(\rho)$, the index map is the obstruction to extending the $S(\rho)$ family to a G-equivariant family on $B(\rho)$. In particular, the point $0 \in B(\rho)$ is a G-symmetric invertible field theory, and therefore the kernel of Ind_ρ is the image of Res_ρ! We will explain this further in the next subsection. In terms of bulk-boundary correspondence, the index map is the obstruction to a G-equivariant family admitting a G-symmetry boundary condition which is *independent* of the parameters.

1. *Example: Thouless pump and vortices*

We will consider the relationship between the index map and the Thouless pump. We begin with a 1+1D Dirac fermion (with two complex components) with its anomaly-free $U(1)$ symmetry

$$\psi \mapsto e^{i\theta/2}\psi. \tag{II.32}$$

Suppose we add a $U(1)$-symmetric mass term

$$i((\cos\phi)\bar{\psi}\psi + i(\sin\phi)\bar{\psi}\gamma^c\psi), \tag{II.33}$$

where γ^c is the chirality operator $i\gamma^0\gamma^1$. This defines a $U(1)$-*symmetric* S^1-family of invertible field theories parametrized by ϕ. This family is nontrivial, and can be described by

$$\zeta(W, A, \phi) = \frac{1}{2\pi}\int_W d\phi A, \tag{II.34}$$

where W is the 1+1D spacetime, A is a Spin^c structure, and $\phi : W \to S^1$. As described in Section I A, the physics of this term is we get an A current when adiabatically varying the S^1 parameter, leading to a quantized charge pump (the classic Thouless pump [Tho83]). This situation again corresponds to the setup of Example III.148.

We expect the ρ-defect, which is the operator which creates a vortex in ϕ, to carry a unit A charge which matches the Thouless pump. This will be the result of the index map, which in this case takes

$$\mathrm{Ind}_\rho : \mathbb{Z} \cong \Omega^2_{\mathrm{Spin}^c}(S^1) \to \Omega^1_{\mathrm{Spin}^c} \cong \mathbb{Z}, \tag{II.35}$$

where the latter group can be thought of as the group of A charges. Note that since the image of Ind_ρ is the kernel of Def_ρ, and ρ here is trivial (we have a symmetric family) so $\mathrm{Def}_\rho = 0$, we already know on abstract grounds that this map is surjective, and hence an isomorphism. Let us compute it to check.

To compute the map, we use (II.31). That is, we will associate to ζ in (II.34) a partition function of 0+1D spacetimes Y (which are merely collections of oriented circles) equipped with a Spin^c connection A. We start by forming the associated $S(\rho)$ bundle over Y. Since ρ is trivial, this bundle is simply a product $W = S^1 \times Y$. The canonical section $\phi : W \to W \times S^1$ is the product of diagonal map $S^1 \to S^1 \times S^1$ and the identity map $Y \to Y$. In particular, $d\phi/2\pi$ is the volume form on the S^1 factor. It follows

$$\mathrm{Ind}_\rho(\zeta)(Y, A, \phi) = \zeta(W, \pi^*A, \phi) = \frac{1}{2\pi}\int_W d\phi\pi^*A = \int_{S^1}\frac{d\phi}{2\pi}\int_Y A = \int_Y A, \tag{II.36}$$

which is the generator of $\Omega^1_{\mathrm{Spin}^c}$, as expected.

We study the relationship between projective symmetry and Berry phase via the index map. See also Example III.164.

Let us take $G = SO(3)$ acting on a Hilbert space carrying spin $s/2$, initially with $H = 0$. We can think of this as a $D = 1$ system with anomaly

$$\omega = \frac{1}{2} s w_2 \in H^2(BSO(3), U(1)) \cong \mathbb{Z}/2, \tag{II.37}$$

where w_2 is the generator of $H^2(BSO(3), U(1))$.

We then apply a "magnetic field"

$$H(B) = -\vec{B} \cdot \vec{S} \tag{II.38}$$

to this spin. The parameter $\vec{B} \in \mathbb{R}^3$ transforms in the adjoint representation ρ of $SO(3)$, and so for any nonzero value, $SO(3)$ is broken down to the $SO(2)$ subgroup of rotations around the $\vec{B}$ axis. Furthermore, for any nonzero value, $H(B)$ has a unique ground state. This means that the residual anomaly

$$\mathrm{Res}_\rho \omega = 0, \tag{II.39}$$

and thus we expect ω to be in the image of the defect anomaly map.

The defect anomaly lives in

$$\Omega^{-1}_{SO(3),SO} = H^0(BSO(3), \mathbb{Z}) = \mathbb{Z}, \tag{II.40}$$

and so evidently

$$\mathrm{Def}_\rho : \mathbb{Z} \to \mathbb{Z}/2 \tag{II.41}$$

is reduction mod 2. However, the interpretation of the defect anomaly is not obvious, since it seems to encode an anomaly of a -2-spacetime-dimensional system. The correct interpretation of this $\mathbb{Z}$ (which follows from the definition of Def_ρ) is the Chern number of the Berry bundle over $S(\rho) \cong S^2$ family, which is known to equal the spin s, consistent with the anomaly above.

The index map is

$$\mathrm{Ind}_\rho : \Omega^2_{SO(3),SO}(S(\rho)) \cong \mathbb{Z} \to \mathbb{Z}, \tag{II.42}$$

which by exactness must be multiplication by 2, since its image is the kernel of the quotient $\mathrm{Def}_\rho : \mathbb{Z} \to \mathbb{Z}/2$. We can interpret this map as follows. Suppose the spin $s/2$ is an integer, so we are in the kernel of Def_ρ, meaning there is no anomaly and the Hilbert space carries an honest representation of $SO(3)$.

We can generalize the magnetic field Hamiltonian above, which projects onto a highest weight vector, to one which projects onto a vector of weight l (the magnetic quantum number). For each $l \in \{-s/2, -s/2 + 1, \ldots, s/2\}$, this Hamiltonian transforms in the adjoint of $SO(3)$. We find l is encoded in the $SO(3)$-equivariant S^2 family as the charge of the unbroken $SO(2)$ at any fixed value. This family thus represents $l \in \Omega^2_{SO(3),SO} = \mathbb{Z}$ via the isomorphism

$$\Omega^2_{SO(3),SO} = H^2_{SO(3)}(S^2, \mathbb{Z}) = H^2(BSO(2), \mathbb{Z}), \tag{II.43}$$

where the latter represents the charge of the unbroken $SO(2)$ at a fixed value (see the discussion in Section II A about transitive group actions, just before the examples, and also Remark III.99). Indeed, it is known in this case that the Chern number of the resulting Berry connection is $2l$, which agrees with the index map above.

We note that for representations ρ of dimension greater than 3, since $\Omega^{2-k}_{G,s,\eta} = 0$ for all $k > 3$ and all G, s, η, if there is a projective representation, there is no Berry phase that can match this anomaly by Def_ρ. Since the image of $\mathrm{Def}_\rho = 0$ is the kernel of Res_ρ, the residual family anomaly map is therefore *injective*. In particular, the family is not uniformly gapped over $S(\rho)$.

For example, suppose we take $G = PSU(n)$, with our Hilbert space corresponding to the $SU(n)$ vector representation. This is a projective $PSU(n)$ representation and has anomaly generating the group

$$\omega = \frac{1}{n} u_2 \in H^2(BPSU(n), U(1)) \cong \mathbb{Z}/n. \tag{II.44}$$

The spin-1/2 case above corresponds to $n = 2$, via $PSU(2) = SO(3)$. The analog of the magnetic field Hamiltonian above is

$$H(B) = -\sum_i B^i S_i \tag{II.45}$$

where $S_i \in su(n)$ ranges over a basis of the traceless Hermitian $n \times n$ matrices. As before, $\vec{B}$ transforms in the adjoint representation ρ of $PSU(n)$, which has dimension $n^2 - 1$.

When $n > 2$, the Hamiltonian $H(B)$ does not have a unique gapped ground state for all $B \neq 0$. The issue is that the lowest two (or more, up to $n - 1$) eigenvalues of $H(B)$ may be degenerate, while the other eigenvalues can balance them so $\mathrm{Tr}\, H(B) = 0$, without making $H(B)$ identically zero. We anticipated this based on the long exact sequence, and indeed there is a residual family anomaly, which generates the group

$$\mathrm{Res}_\rho \omega \in H^2_{PSU(n)}(S(\rho), U(1)) \cong \mathbb{Z}/n. \tag{II.46}$$

To see this, we observe that if we take B to be one of the points in $S(\rho)$ with two degenerate lowest energy states, there is an unbroken $PSU(2)$ with $\mathbb{Z}/n$ anomaly $\frac{1}{n}u_2$, which must be given by $B^*\mathrm{Res}_\rho\omega$ (cf. Section II.3 and Eq. (II.3)).

3. *Example: time reversal domain wall for 2+1D Majorana fermions*

Let us analyze an example from [HKT20a] of a situation with ambiguous defect anomaly. We study N_f 2+1D Majorana fermions ψ_j with time reversal

$$T\psi_j = \gamma^0 \psi_j, \tag{II.47}$$

which satisfies $T^2 = (-1)^F$. This has an anomaly $\omega = N_f \omega_4 \in \Omega^4_{\mathrm{Spin}}(B\mathbb{Z}/2, 3\sigma) = \Omega^4_{\mathrm{Pin}^+} \cong \mathbb{Z}/16$, where ω_4 is the generator corresponding to $N_f = 1$ (it can be expressed as an eta invariant of the Dirac operator [Wit16b]). This example is also a member of the 4-periodic family discussed later in Example III.131.

Let us consider $N_f = 2$. Time reversal can be broken by mass terms such as

$$\bar{\psi}_1 \psi_1 \pm \bar{\psi}_2 \psi_2. \tag{II.48}$$

(Each T-odd mass term transforms in the sign representation, which is ρ here.) On the time reversal domain wall there is a unitary $\mathbb{Z}/2$ symmetry U, whose anomaly group is classified by $\Omega^3_{\mathrm{Spin},\mathbb{Z}/2} \cong \mathbb{Z} \oplus \mathbb{Z}/8$, the first part α_3 being purely gravitational and the second part $\alpha_3^{\mathbb{Z}/2}$ involving the internal symmetry U. It turns out that depending on the relative sign, the domain wall has different anomalous modes. If the sign is the same, on the wall we have two 1+1d Majorana modes of the same chirality. However, if we take opposing signs, we get two Majoranas with opposite chirality. These clearly have distinct gravitational anomalies, and it turns out they have distinct U anomalies as well, with U acting trivially in the first case and chirally in the second case.

Although they have different anomalies, both must satisfy the defect anomaly matching condition. Since Def_ρ is linear, we can use the two data points above to compute it, and find, in terms of generators $k_1 \in \mathbb{Z}$, $k_2 \in \mathbb{Z}/8$,

$$\mathrm{Def}_\rho(k_1 \alpha_3 + k_2 \alpha_3^{\mathbb{Z}/2}) = (k_1 - 2k_2)\omega_4, \tag{II.49}$$

where (k_1, k_2) is $(2, 0)$ or $(0, 1)$ in the two domain walls above, and both match the anomaly $2\omega_4$ as expected.

We see that the kernel of Def_ρ is generated by $(2, 1)$, which was noted in [HKT20a]. We can see ambiguity arising from Ind_ρ as follows. We need to start by considering 2+1D $\mathbb{Z}/2^T$-equivariant families of invertible field theories over $S(\rho)$. In this case, $S(\rho) = S^0$ is just two points which get exchanged by T. The generator $\zeta \in \Omega^3_{\mathbb{Z}/2,\mathrm{Spin},3\sigma}(S(\rho)) = \mathbb{Z}$ is defined by taking the generator $\alpha_3 \in \Omega^3_{\mathrm{Spin}} = \mathbb{Z}$ over one of the two points, and its time-reversed partner $-\alpha_3$ over the other point.

To calculate Ind_ρ, we study the interface between these two invertible theories. The result is two fermions of equal chirality (gravitational anomaly $2\alpha_3$), which are swapped by the induced $\mathbb{Z}/2$ symmetry U. This swap has eigenvalues ±1 and we find its anomaly is $\alpha_3^{\mathbb{Z}/2}$. So if ζ is the class of the family above,

$$\mathrm{Ind}_\rho(\zeta) = 2\alpha_3 + \alpha_3^{\mathbb{Z}/2}, \tag{II.50}$$

the image of which is indeed the kernel of Def_ρ we computed above.

This has a physical interpretation in terms of the two mass terms above. If we change the sign of just the $\bar{\psi}_2 \psi_2$ mass term, we can think of this as stacking with either α_3 or $-\alpha_3$, depending whether the sign change is from minus to plus or from plus to minus. This gives the invertible family ζ above.

4. Example: vortices in $p + ip$ superfluid

Now we will discuss the famous Majorana zero modes bound to the vortices of a $p+ip$ superfluid [Vol03], which turn out to have an interesting description in terms of the index map.

We study a single Dirac fermion in 2+1D, carrying charge 1 under $G = U(1)$ symmetry, and undergoing symmetry breaking via a charge 2 complex order parameter coupling to the two Majorana masses. Such a spontaneous symmetry breaking scenario is typically referred to as a $p+ip$ superfluid[6] and again corresponds to Example III.148. The resulting $S(\rho) \cong S^1$ family has a unique gapped ground state for all nonzero values of the order parameter, and the $U(1)$ symmetry is anomaly-free, and it represents a generator of

$$\Omega^3_{\mathrm{Spin},U(1),\rho}(S(\rho)) \cong \mathbb{Z}. \tag{II.51}$$

We want to compute the index map

$$\mathrm{Ind}^{U(1)}_\rho : \Omega^3_{\mathrm{Spin},U(1),\rho}(S(\rho)) \to \Omega^2_{\mathrm{Spin},U(1)} \cong \mathbb{Z}/2. \tag{II.52}$$

It is interesting to consider the map

$$f : \Omega^3_{\mathrm{Spin},U(1),\rho}(S(\rho)) \to \Omega^3_{\mathrm{Spin}}(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \tag{II.53}$$

which forgets the $U(1)$ action, since the index map of the latter, namely

$$\mathrm{Ind}_\rho : \Omega^3_{\mathrm{Spin}}(S^1) \to \Omega^2_{\mathrm{Spin}} \tag{II.54}$$

can be more easily understood. The generators of $\Omega^3_{\mathrm{Spin}}(S^1)$ correspond to the generator of $\Omega^3_{\mathrm{Spin}} \cong \mathbb{Z}$, with trivial parameter dependence, and $\Omega^2_{\mathrm{Spin}} = \mathbb{Z}/2$, via a family which pumps this phase to the boundary as we go around S^1. The index map clearly sends the $\mathbb{Z}$ generator to zero and the $\mathbb{Z}/2$ generator to the generator of $\Omega^2_{\mathrm{Spin}} = \mathbb{Z}/2$.

Because the SBLES is functorial in G, we have a commutative square

$$\begin{tikzcd}
\mathbb{Z} \cong \Omega^3_{\mathrm{Spin},U(1),\rho}(S(\rho)) \arrow[r, "\mathrm{Ind}^{U(1)}_\rho"] \arrow[d, "f"] & \Omega^2_{\mathrm{Spin},U(1)} \cong \mathbb{Z}/2 \arrow[d, "\sim"] \\
\mathbb{Z} \oplus \mathbb{Z}/2 \cong \Omega^3_{\mathrm{Spin}}(S^1) \arrow[r, "\mathrm{Ind}_\rho"] & \Omega^2_{\mathrm{Spin}} \cong \mathbb{Z}/2.
\end{tikzcd}$$

Combined with the information above, we learn $\mathrm{Ind}^{U(1)}_\rho$ must be reduction mod 2. This is reasonable from the physical point of view, since it is known that a vortex in the $p+ip$ superfluid binds an odd number of Majorana zero modes, which carry the gravitational anomaly associated with the generator of Ω^2_{Spin}. We also learn that the map f above sends the generator to the sum of the generators $(1,1) \in \mathbb{Z} \oplus \mathbb{Z}/2$, which is a bit more surprising! We will verify both these facts directly from the definition of these maps.

First we study f. In terms of spacetime manifolds, we want to take a 3-manifold X with spin structure ξ and a map $\phi : X \to S^1$, and construct a Spin^c structure A on X under which ϕ has

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[6] Note that there is a mixed $U(1)$ and time reversal anomaly, and a choice of $U(1)$ symmetric fermion regulator will break time reversal and select either a $p+ip$ or $p-ip$ superfluid.

charge 2, so that A gets Higgs'd to a spin structure. In terms of equations we want

$$\begin{aligned} 2A &= d\phi \\ dA &= \pi w_2(TX) = \pi d\xi, \end{aligned} \tag{II.55}$$

which can be solved by

$$A = \pi\xi + \frac{1}{2}d\phi. \tag{II.56}$$

The two terms here is the essential reason why we get the sum of generators when we compute f. It means when ϕ has an odd winding number around a 1-cycle of X, we twist the spin structure ξ along that cycle, turning it from periodic to antiperiodic or vice versa.

We do the same thing when we compute $\mathrm{Ind}_\rho^{U(1)}$ according to the recipe given at the beginning of this subsection. There, from a spin surface Y we form the manifold $X = Y \times S^1$ with ϕ winding once around the S^1 factor. The spin structure along this S^1 factor becomes twisted. When we evaluate the $\mathbb{Z}$ generator of Ω^3_{Spin} on this spin 3-manifold, we get the Arf invariant of Y and its spin structure, which is the nontrivial element of Ω^2_{Spin}.

D. Completing the circle and long exact sequence examples

By now we have defined our three maps: the residual family anomaly Res_ρ, the defect anomaly Def_ρ, and the index map Ind_ρ. We have seen how they fit together into an exact sequence: the kernel of Res_ρ is the image of Def_ρ and the kernel of Def_ρ is the image of Ind_ρ. In this section we will complete the circle and argue they form a *long* exact sequence, in particular, the kernel of Ind_ρ is the image of Res_ρ *from one lower dimension.*

As we have already mentioned in the previous subsection, the essential reason for this is that the index map is the obstruction to extending the $S(\rho)$ family to a G-equivariant family on $B(\rho)$. In particular, the point $0 \in B(\rho)$ represents a G-symmetric invertible field theory, and the $S(\rho)$ family can be reconstructed by applying Res_ρ to this theory, by definition.

In the rest of this subsection, we will collect a couple longer segments of the SBLES, containing some of the examples of individual maps we have already seen. More such examples can be found in §III E.

1. $U(1)$ *symmetry breaking for fermions*

Let us consider the symmetry breaking long exact sequence for a $U(1)$ symmetry in a fermionic theory and an order parameter transforming in the charge 1 representation ρ. There are two cases to consider, depending on whether we have a spin-charge relation, meaning that fermionic operators have half-integer $U(1)$ charge, or not. In either case the relevant groups of invertible field theories we will need are shown in Table I. To calculate these groups, one applies the universal property of Anderson duality (III.24) to the spin bordism groups, the Spin × U(1) bordism groups, and the spin^c bordism groups, which are known: for spin bordism, see Milnor [Mil63], for $\Omega_*^{\mathrm{Spin}}(B\mathrm{U}(1))$, see Wan-Wang [WW19, §3.1.5], and for spin^c bordism, see Bahri-Gilkey [BG87a].

First we study the case with spin-charge relation, where fermions carry half-charge under $U(1)$

D	Ω^D_{Spin}	$\Omega^D_{U(1),\text{Spin}}$	$\Omega^D_{U(1),\text{Spin},\rho} = \Omega^D_{\text{Spin}^c}$
-1	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0
1	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}$	$\mathbb{Z}$
2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
3	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
4	0	0	0
5	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$
6	0	0	0

TABLE I: Classification of D-spacetime-dimensional fermionic invertible field theories with $\mathbb{Z}/2^F$, $U(1) \times \mathbb{Z}/2^F$, and $U(1)^F$ symmetry, respectively.

and bosons carry integer charge. We consider symmetry breaking by a charge 1 order parameter (charge $2e$ from the point of view of the fermions). We studied such an example in Section II C 4, the $p + ip$ superfluid.

We organize the SBLES into rows associated with this symmetry breaking in each dimension D. The map from the first column to the second is the defect anomaly map Def_ρ, from the second to the third is the residual family anomaly Res_ρ, and the index maps Ind_ρ go from the third column of one row to the first column of the next. We omit arrows for maps that are zero, but the whole long exact sequence is connected.

$$\begin{tikzcd}
D & \Omega^{D-2}_{U(1),\text{Spin}} \arrow[r, "\text{Def}_\eta"] & \Omega^D_{U(1),\text{Spin},\eta} = \Omega^D_{\text{Spin}^c} \arrow[r, "\text{Res}_\eta"] & \Omega^D_{U(1),\text{Spin},\eta}(S(\eta)) = \Omega^D_{\text{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \\
2 & 0 & 0 & \mathbb{Z}/2 \arrow[dll] \\
3 & \mathbb{Z}/2 \oplus \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z} \arrow[dll] \\
4 & \mathbb{Z}/2 & 0 & 0 \\
5 & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2 & 0
\end{tikzcd}$$

The long subsequence beginning in $D = 2$ is

$$\Omega^2_{\text{Spin}} \cong \mathbb{Z}/2 \xrightarrow{\binom{1}{0}} \Omega^1_{U(1),\text{Spin}} \cong \mathbb{Z}/2 \oplus \mathbb{Z} \xrightarrow{\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)} \Omega^3_{\text{Spin}^c} \cong \mathbb{Z}^2 \xrightarrow{(2\ 0)} \Omega^3_{\text{Spin}} \cong \mathbb{Z} \xrightarrow{1} \Omega^2_{\text{Spin}} \cong \mathbb{Z}/2. \tag{II.57}$$

We have discussed the last map in Section II C 4: it corresponds to the Majorana zero mode bound to the vortex of the $p + ip$ superfluid. Let us briefly discuss the computation of the other maps, although they are determined by the exact sequence.

The preceding map $\Omega^3_{\text{Spin}^c} \to \Omega^3_{\text{Spin}}$ measures the residual gravitational anomaly upon breaking the $U(1)$ symmetry. The group $\Omega^3_{\text{Spin}^c}$ represents Chern-Simons terms associated with the four-dimensional invariants

$$k_1 \left(\frac{1}{8} c_1^2 - \frac{1}{24} p_1 \right) + k_2 c_1^2, \tag{II.58}$$

see (II.15). Meanwhile the generator of Ω^3_{Spin} is represented by $-\frac{1}{48} p_1$, so we see the map sends (k_1, k_2) to $2k_1$.

The defect anomaly map $\Omega^1_{U(1),\text{Spin}} \to \Omega^3_{\text{Spin}^c}$ tells us the fermion parity as well as the $U(1)_\rho$ charge of the ρ-defect, i.e. the vortex of the order parameter. A physical model with anomaly $k_1 = 0$ and $k_2 = 1$ is the 1+1D compact boson with $U(1)$ acting only on the left movers. The vortex clearly is parity-even since there are no fermions in the model. However, it carries unit $U(1)_\rho$ charge, as is well-known from the chiral anomaly.

Finally, the index map $\Omega^2_{\text{Spin}} \to \Omega^1_{U(1),\text{Spin}}$ can be understood in terms of the "topological superfluid" in 1+1D. This can be thought of as a $U(1)$-charged Dirac fermion with the $U(1)$ symmetry broken by the two Majorana masses, which form a doublet. This is in the same phase as the Kitaev chain. A vortex operator in this phase, which changes the winding number of the order parameter, also changes the boundary conditions for the fermions, and therefore toggles the fermion parity of the ground state. This is captured by the nonzero index map, landing on the generator of $\Omega^1_{\text{Spin}} \cong \mathbb{Z}/2$ inside $\Omega^1_{U(1),\text{Spin}}$, which gives the "anomaly" of the vortex operator, namely its fermion parity (compare Section II C 1).

Next we collect the SBLES for charge 1 breaking of a $U(1)$ symmetry without spin-charge relation:

$$\begin{tikzcd}
D & \Omega^{D-2}_{U(1),\mathrm{Spin},\eta} = \Omega^{D-2}_{\mathrm{Spin}^c} \arrow[r, "\mathrm{Def}_\eta"] & \Omega^{D}_{U(1),\mathrm{Spin}} \arrow[r, "\mathrm{Res}_\eta"] & \Omega^{D}_{U(1),\mathrm{Spin}}(S(\eta)) = \Omega^{D}_{\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
2 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
3 & \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z} \\
4 & 0 & 0 & 0 \\
5 & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2 & 0
\end{tikzcd}$$

We have studied the map in $D = 5$ in Section II B 1 when we considered breaking of chiral symmetry of a 4+1D Dirac fermion by its Dirac mass terms.

One general observation is that the index map always vanishes. The reason is that in the definition of the index map from Section II C, we produce an S^1 bundle W with spin structure which extends to the disc bundle, since this S^1 always carries anti-periodic spin structure. Moreover, Def_ρ is an isomorphism from $\Omega^{D-2}_{U(1),\mathrm{Spin},\rho}$ to the “reduced” part of $\Omega^{D}_{U(1),\mathrm{Spin}}$, namely those $U(1)$ symmetric invertible phases with no pure gravitational response, in other words which become trivial upon breaking the $U(1)$ symmetry. This the “Smith isomorphism” in [KT90b] (which can be proven following the methods of [HKT20a] and which is discussed further in Example III.144, and more specifically in Example III.148). Meanwhile the pure gravitational part is mapped isomorphically by Res_σ, since by definition we do not need the $\mathbb{Z}/2$ symmetry to detect it, and $\mathbb{Z}/2$ acts transitively on $S(\sigma)$, so the residual family anomaly is determined by the anomaly of the unbroken subgroup, which is just the gravitational part.

2. $\mathbb{Z}/2$ *symmetry breaking for bosons*

Now let us discuss perhaps the simplest example of the SBLES, which describes the breaking of a unitary $\mathbb{Z}/2$ symmetry of a bosonic system by a single real order parameter transforming in the sign representation σ. This corresponds to Example III.129. On the domain wall, this unitary symmetry is transmuted to an anti-unitary symmetry. For reference, the relevant classification groups are shown in Table II, with Ω^D_{SO} denoting D-spacetime-dimensional bosonic invertible field theories, $\Omega^D_{\mathbb{Z}/2,SO}$ denoting those with a unitary $\mathbb{Z}/2$ symmetry, and $\Omega^D_{\mathbb{Z}/2,SO,\sigma}$ denoting those with an anti-unitary $\mathbb{Z}/2$ symmetry. As usual, these groups were obtained by applying Anderson duality (III.24) to oriented bordism, unoriented bordism, and the oriented bordism of $B\mathbb{Z}/2$. See Thom [Tho54, Théorèmes IV.9, IV.13] for oriented and unoriented bordism groups in low degrees. We do not know of an explicit reference for $\Omega^{\mathrm{SO}}_*(B\mathbb{Z}/2)$, but it can be calculated using a result of Wall [Wal60] that implies that the Atiyah-Hirzebruch spectral sequence for oriented bordism collapses for any space whose mod p cohomology vanishes for all odd p.

We collect the SBLES as follows.

D	Ω^D_{SO}	$\Omega^D_{\mathbb{Z}/2,SO,\sigma} = \Omega^D_O$	$\Omega^D_{\mathbb{Z}/2,SO}$
-1	$\mathbb{Z}$	0	$\mathbb{Z}$
0	0	$\mathbb{Z}/2$	0
1	0	0	$\mathbb{Z}/2$
2	0	$\mathbb{Z}/2$	0
3	$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}/2$
4	0	$(\mathbb{Z}/2)^2$	0
5	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$
6	0	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$
7	0	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$

TABLE II: Classification of D-spacetime-dimensional bosonic invertible field theories with no symmetry, time reversal symmetry, and $\mathbb{Z}/2$ symmetry respectively.

$$\Omega^{D-1}_{\mathbb{Z}/2,SO,\sigma} = \Omega^{D-1}_O \overset{\mathrm{Def}_\sigma}{\longrightarrow} \Omega^D_{\mathbb{Z}/2,SO} \overset{\mathrm{Res}_\sigma}{\longrightarrow} \Omega^D_{\mathbb{Z}/2,SO}(S(\sigma)) = \Omega^D_{SO}$$

$$\begin{tikzcd}
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0 \\
2 & 0 & 0 & 0 \\
3 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \\
4 & 0 & 0 & 0 \\
5 & (\mathbb{Z}/2)^2 \arrow[r] & (\mathbb{Z}/2)^3 \arrow[r] & \mathbb{Z}/2 \\
6 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0 \\
7 & (\mathbb{Z}/2)^3 \arrow[r] & (\mathbb{Z}/2)^3 & 0
\end{tikzcd}$$

This has a similar structure to the $U(1) \times \mathbb{Z}/2^F \to \mathbb{Z}/2^F$ breaking we studied above in Section II D 1, splitting into isomorphisms given by Def_σ (the "Smith isomorphism") and Res_σ, with Ind_σ vanishing.

We can also compute the SBLES associated with breaking of a time reversal symmetry by a single real order parameter transforming in the sign representation. This turns out to be more interesting, since we no longer have a Smith isomorphism, and Ind_σ may be nonzero.

$$\Omega^{D-1}_{\mathbb{Z}/2,SO} \xrightarrow{\mathrm{Def}_\sigma} \Omega^{D}_{\mathbb{Z}/2,SO,\sigma} = \Omega^D_O \xrightarrow{\mathrm{Res}_\sigma} \Omega^{D}_{\mathbb{Z}/2,SO,\sigma}(S(\sigma)) = \Omega^D_{SO}$$

$$\begin{tikzcd}
-1 & 0 & 0 & \mathbb{Z} \arrow[dll, bend left] \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 & 0 \\
1 & 0 & 0 & 0 \\
2 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0 \\
3 & 0 & 0 & \mathbb{Z} \arrow[dll, bend left] \\
4 & \mathbb{Z}\oplus\mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2\oplus\mathbb{Z}/2 & 0 \\
5 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
6 & (\mathbb{Z}/2)^3 \arrow[r] & (\mathbb{Z}/2)^3 & 0 \\
7 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0
\end{tikzcd}$$

Consider for example the 3rd to 4th rows. We have the sequence

$$0 \to \Omega^3_{\mathbb{Z}/2,SO,\sigma}(S(\sigma)) \cong \mathbb{Z} \xrightarrow{(2\ 0)} \Omega^3_{\mathbb{Z}/2,SO} \cong \mathbb{Z}\oplus\mathbb{Z}/2 \xrightarrow{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \Omega^4_O \cong \mathbb{Z}/2\oplus\mathbb{Z}/2 \to 0. \tag{II.59}$$

The generator of the first nonzero group is the $S(\sigma) \cong S^0$-family with an E_8 phase [LV12] at one point (the generator of $\Omega^3_{SO} \cong \mathbb{Z}$), and its inverse phase at the other point. To compute the index map, we study a domain wall between the E_8 and its inverse, which with the standard boundary conditions has chiral modes with $c_L = 16$, $c_R = 0$. The induced unitary $\mathbb{Z}/2$ symmetry is anomaly-free, since $k = 0 \bmod 8$ of the modes are charged. This theory represents the anomaly $(2,0) \in \mathbb{Z}\oplus\mathbb{Z}/2 \cong \Omega^3_{\mathbb{Z}/2,SO}$.

The next map sends the E_8 state, representing $(1,0)$ in that group, to the time-reversal symmetric phase described by a gravitational $\theta = \pi$ angle, or $\frac{1}{2}w_2^2$. This encodes the well-known fact that the time reversal domain wall at the boundary of that theory (known as $e_f m_f$ in [WPS14]) hosts $c_L = 8 \bmod 16$ gapless chiral modes. Meanwhile, it sends the Levin-Gu SPT [LG12] associated to $\frac{1}{2}A^3$ and representing $(0,1)$ in $\Omega^3_{\mathbb{Z}/2,SO}$, to the phase associated with $\frac{1}{2}w_1^4$.

3. *$\mathbb{Z}/2$ symmetry breaking for fermions*

Now we turn to the same $\mathbb{Z}/2$ symmetry breaking scenario for fermions. Now four different types of $\mathbb{Z}/2$ symmetry are involved, either unitary with $U^2 = 1$ or $U^2 = (-1)^F$, or time reversing with $T^2 = 1$ or $T^2 = (-1)^F$, as discussed in Section I A. The relevant classifications are collection in Table III, corresponding to low-degree bordism groups that are explicitly calculated in the following references.

- Spin bordism: see Milnor [Mil63].

- Pin^+ bordism: see Giambalvo [Gia73b].

- Pin^- bordism: see Kirby-Taylor [KT90b].

- $\mathrm{Spin}\times\mathbb{Z}/2$ bordism: see García-Etxebarria and Montero [GEM19, (C.18)].[7]

- $\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/4$ bordism: see Giambalvo [Gia73a].

D	Ω^D_{Spin}	$\Omega^D_{\mathbb{Z}/2,\mathrm{Spin}}$	$\Omega^D_{\mathbb{Z}/2,\mathrm{Spin},\sigma}=\Omega^D_{\mathrm{Pin}^-}$	$\Omega^D_{\mathbb{Z}/2,\mathrm{Spin},2\sigma}=\Omega^D_{\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/4}$	$\Omega^D_{\mathbb{Z}/2,\mathrm{Spin},3\sigma}=\Omega^D_{\mathrm{Pin}^+}$
-1	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
0	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
1	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	0
2	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/8$	0	$\mathbb{Z}/2$
3	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/8$	0	$\mathbb{Z}$	$\mathbb{Z}/2$
4	0	0	0	0	$\mathbb{Z}/16$
5	0	0	0	$\mathbb{Z}/16$	0
6	0	0	$\mathbb{Z}/16$	0	0

TABLE III: Fermionic invertible field theories in D spacetime dimensions with symmetry $\mathbb{Z}/2^F$, $\mathbb{Z}/2^U\times\mathbb{Z}/2^F$, $\mathbb{Z}/2^T\times\mathbb{Z}/2^F$, $\mathbb{Z}/4^U$, or $\mathbb{Z}/4^T$, respectively.

There are four different SBLES, concerning symmetry breaking for each of the four types of $\mathbb{Z}/2$ symmetry, as discussed further in Example III.131. We have computed an initial segment of each. First we study $\mathbb{Z}/2^U\times\mathbb{Z}/2^F$ breaking to $\mathbb{Z}/2^F$:

---

[7] This calculation, or more precisely its equivalent analogue in *ko*-homology, was first done by Mahowald-Milgram [MM76], with $ko_*(B\mathbb{Z}/2)$ worked out explicitly by Bruner-Greenlees [BG10, Example 7.3.1], but the cited reference lists spin bordism groups specifically.

$$\begin{tikzcd}
 & \Omega^{D-1}_{\mathrm{Pin}^-} \arrow[r, "\mathrm{Def}_\sigma"] & \Omega^D_{\mathbb{Z}/2,\mathrm{Spin}} \arrow[r, "\mathrm{Res}_\sigma"] & \Omega^D_{\mathbb{Z}/2,\mathrm{Spin}}(S(\sigma)) = \Omega^D_{\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & \mathbb{Z}/2 \arrow[r] & (\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 \\
2 & \mathbb{Z}/2 \arrow[r] & (\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 \\
3 & \mathbb{Z}/8 \arrow[r] & \mathbb{Z}/8 \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \\
4 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 \\
7 & \mathbb{Z}/16 \arrow[r] & \mathbb{Z}/16 \oplus \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2
\end{tikzcd}$$

Next we study $\mathbb{Z}/2^T \times \mathbb{Z}/2^F$ breaking to $\mathbb{Z}/2^F$:

$$\begin{tikzcd}
D & \Omega^{D-1}_{\mathbb{Z}/2,\mathrm{Spin},2\sigma} \arrow[r, "\mathrm{Def}_\sigma"] & \Omega^D_{\mathbb{Z}/2,\mathrm{Spin},\sigma} = \Omega^D_{\mathrm{Pin}^-} \arrow[r, "\mathrm{Res}_\sigma"] & \Omega^D_{\mathbb{Z}/2,\mathrm{Spin},\sigma}(S(\sigma)) = \Omega^D_{\mathrm{Spin}} \\
-1 & 0 & 0 & \mathbb{Z} \arrow[dll, out=-20, in=160] \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 & 0 \\
1 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
2 & \mathbb{Z}/4 \arrow[r] & \mathbb{Z}/8 \arrow[r] & \mathbb{Z}/2 \\
3 & 0 & 0 & \mathbb{Z} \arrow[dll, out=-20, in=160] \\
4 & \mathbb{Z} & 0 & 0 \\
5 & 0 & 0 & 0 \\
6 & \mathbb{Z}/16 \arrow[r] & \mathbb{Z}/16 & 0 \\
7 & 0 & 0 & \mathbb{Z}^2
\end{tikzcd}$$

One generator of $\Omega^2_{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ is represented by a T-odd Majorana zero mode. Upon forgetting the T symmetry, this still has a gravitational anomaly, associated with $\Omega^2_{\mathrm{Spin}} \cong \mathbb{Z}/2$. If we have two T-odd Majoranas $\gamma_{1,2}$, we can write the T-odd pairing $i\gamma_1\gamma_2$ which leads to a unique ground state. Changing the sign of this term toggles the fermion parity of this ground state, so the associated operator has unit charge under the induced unitary symmetry U, since $U^2 = (-1)^F$. This "anomaly" represents a generator of $\Omega^1_{\mathbb{Z}/2,\mathrm{Spin},2\sigma} \cong \mathbb{Z}/4$.

Next we study breaking of a unitary symmetry U with $U^2 = (-1)^F$ down to $\mathbb{Z}/2^F$.

$$\begin{tikzcd}
D & \Omega^{D-1}_{\mathbb{Z}/2,\mathrm{Spin},3\sigma} = \Omega^{D-1}_{\mathrm{Pin}^+} \arrow[r, "\mathrm{Def}_\sigma"] & \Omega^{D}_{\mathbb{Z}/2,\mathrm{Spin},2\sigma} \arrow[r, "\mathrm{Res}_\sigma"] & \Omega^{D}_{\mathbb{Z}/2,\mathrm{Spin},2\sigma}(S(\sigma)) = \Omega^{D}_{\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/4 \arrow[r] & \mathbb{Z}/2 \\
2 & 0 & 0 & \mathbb{Z}/2 \arrow[dll] \\
3 & \mathbb{Z}/2 & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[dll] \\
4 & \mathbb{Z}/2 & 0 & 0 \\
5 & \mathbb{Z}/16 \arrow[r] & \mathbb{Z}/16 & 0 \\
6 & 0 & 0 & 0 \\
7 & 0 & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2
\end{tikzcd}$$

Finally, we have breaking of a time reversal symmetry T with $T^2 = (-1)^F$ down to $\mathbb{Z}/2^F$.

$$\begin{tikzcd}
 & I\mathbb{Z}^{*-1}(\mathit{MTSpin} \times \mathbb{Z}/2) & I\mathbb{Z}^{*}(MT\mathrm{Pin}^+) & I\mathbb{Z}^{*}(\mathit{MTSpin}) \\
-1 & 0 & 0 & \mathbb{Z} \arrow[dll] \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 & 0 \\
1 & 0 & 0 & \mathbb{Z}/2 \arrow[dll] \\
2 & (\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 & \mathbb{Z}/2 \arrow[dll] \\
3 & (\mathbb{Z}/2)^2 \arrow[r] & \mathbb{Z}/2 & \mathbb{Z} \arrow[dll] \\
4 & \mathbb{Z} \oplus \mathbb{Z}/8 \arrow[r] & \mathbb{Z}/16 & 0 \\
5 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 \\
7 & 0 & 0 & \mathbb{Z}^2
\end{tikzcd}$$

The short exact sequence from $D = 3$ to $D = 4$ was analyzed in Section II C 3 in the context of time reversal domain walls of $2+1D$ Majorana fermions.

4. $\mathbb{Z}/3$ *symmetry breaking for fermions*

An interesting case which demonstrates some of the more typical complexity of the SBLES is $\mathbb{Z}/3$ symmetry breaking in fermionic systems via a charge 1 order parameter. Such a symmetry must be unitary and the symmetry group must have the product structure $\mathbb{Z}/3^U \times \mathbb{Z}/2^F$. This situation is described later in Example III.155, with $k = 3$. The relevant classification is shown in Table IV; the new piece of information we need is $\Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/3)$, worked out in degrees 11 and below in [DDHM23, §12.2] using work of Bruner-Greenlees [BG10, Example 7.3.2].

D	Ω^D_{Spin}	$\Omega^D_{\mathbb{Z}/3,\mathrm{Spin}}$
-1	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0
1	$\mathbb{Z}/2$	$\mathbb{Z}/3 \oplus \mathbb{Z}/3$
2	$\mathbb{Z}/2$	$\mathbb{Z}/2$
3	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/3$
4	0	0
5	0	$\mathbb{Z}/9$
6	0	0
7	$\mathbb{Z}^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}/9$
8	0	0
9	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/27$
10	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$
11	$\mathbb{Z}^3$	$\mathbb{Z}^3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/27$

TABLE IV: Classification of invertible field theories with $\mathbb{Z}/2^F$ and $\mathbb{Z}/3 \times \mathbb{Z}/2^F$ symmetry in D spacetime dimensions.

The long exact sequence is the following

$$\begin{tikzcd}
D & \Omega^{D-2}_{\mathbb{Z}/3,\mathrm{Spin}} \arrow[r, "\mathrm{Def}_\rho"] & \Omega^{D}_{\mathbb{Z}/3,\mathrm{Spin}} \arrow[r, "\mathrm{Res}_\rho"] & \Omega^{D}_{\mathbb{Z}/3,\mathrm{Spin}}(S(\rho)) = \Omega^{D}_{\mathrm{Spin}} \oplus \Omega^{D-1}_{\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \\
0 & 0 & 0 & \mathbb{Z} \arrow[dll, "(3)" description] \\
1 & \mathbb{Z} \arrow[r, "\binom{0}{1}"] & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \arrow[r, "(1{,}0)"] & \mathbb{Z}/2 \\
2 & 0 & \mathbb{Z}/2 \arrow[r] & (\mathbb{Z}/2)^2 \arrow[dll, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}" description] \\
3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}"] & \mathbb{Z} \oplus \mathbb{Z}/3 \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"'] & \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[dll, "(0{,}1)" description] \\
4 & \mathbb{Z}/2 & 0 & \mathbb{Z} \arrow[dll, "\begin{pmatrix} -3 \\ 1 \end{pmatrix}" description] \\
5 & \mathbb{Z} \oplus \mathbb{Z}/3 \arrow[r, "(1{,}3)"] & \mathbb{Z}/9 & 0 \\
6 & 0 & 0 & 0 \\
7 & \mathbb{Z}/9 \arrow[r, "(0{,}1)"] & \mathbb{Z}^2 \oplus \mathbb{Z}/9 \arrow[r, "\binom{1}{0}"] & \mathbb{Z}^2 \\
8 & 0 & 0 & \mathbb{Z}/2 \arrow[dll] \\
9 & \mathbb{Z}/2 \oplus \mathbb{Z}/9 \arrow[r] & (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/27 \arrow[r] & (\mathbb{Z}/2)^2 \\
10 & 0 & (\mathbb{Z}/2)^3 \arrow[r] & (\mathbb{Z}/2)^5 \arrow[dll] \\
11 & (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/27 \arrow[r] & \mathbb{Z}^3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/27 \arrow[r] & \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^3
\end{tikzcd} \tag{II.60}$$

Note that because there is no twist, $\Omega^D_{\mathbb{Z}/3,\mathrm{Spin}} = \tilde{\Omega}^D_{\mathbb{Z}/3,\mathrm{Spin}} \oplus \Omega^D_{\mathrm{Spin}}$, where $\tilde{\Omega}^D_{\mathbb{Z}/3,\mathrm{Spin}}$ denotes the subgroup of those phases which become trivial upon breaking $\mathbb{Z}/3$. This subgroup is finite and has no 2-torsion, so Res_ρ is always zero on it, while it maps the Ω^D_{Spin} factor injectively. It follows that the long exact sequence splits into a series of short exact sequences of the form

$$0 \to \Omega^{D-2}_{\mathrm{Spin}} \xrightarrow{\mathrm{Ind}_\rho} \Omega^{D-2}_{\mathbb{Z}/3,\mathrm{Spin}} \xrightarrow{\mathrm{Def}_\rho} \tilde{\Omega}^{D}_{\mathbb{Z}/3,\mathrm{Spin}} \to 0 \tag{II.61}$$

There are four interesting ones:

- $D = 1$: $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/3$
- $D = 2$: $\mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/3 \to \mathbb{Z}/3$.
- $D = 4$: $\mathbb{Z}/2 \to \mathbb{Z}/2$.
- $D = 5$: $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/3 \to \mathbb{Z}/9$.

Let us consider for example $D = 5$. The first $\mathbb{Z} \cong \Omega^3_{\mathrm{Spin}} = \tilde{\Omega}^4_{\mathbb{Z}/3,\mathrm{Spin}}(S(\rho))$ is generated by a 3+1D family which pumps a generator of Ω^3_{Spin} to the boundary over each third of the $S(\rho) \cong S^1$. When we compute the first map, the index map, we look at the vortex where the order parameter windings all the way around $S(\rho)$. This has three 1+1D gapless Majorana modes of the same chirality, with $\mathbb{Z}/3$ acting as a permutation. This can be written as a neutral chiral Majorana and a charge 1 Weyl, so it has anomaly $(3, 1) \in \mathbb{Z} \oplus \mathbb{Z}/3$. The calculation of the next map, the defect anomaly map, follows Section II B 1.

$D = 1$ is also interesting. Since it involves phases in "negative dimension" we need to think in terms of families (compare Section II C 2). The map $\mathrm{Def}_\rho : \mathbb{Z} \to \mathbb{Z}/3$ says that if we have an S^2 family of quantum states, with $\mathbb{Z}/3$ acting as a $2\pi/3$ polar rotation, the difference in the $\mathbb{Z}/3$ charges of the states at the poles equals the Chern number mod 3.

5. $\mathbb{Z}/4$ *symmetry breaking for fermions*

Now we consider symmetry breaking of a unitary symmetry U with $U^4 = (-1)^F$ by a charge 1 order parameter (defining the representation ρ). This corresponds now to Example III.155 with $k = 4$. The relevant classifications are given in Table V; the new bordism groups we need as input are $\Omega^{\mathrm{Spin}}_*(B\mathbb{Z}/4)$ and $\Omega^{\mathrm{Spin}\times_{\mathbb{Z}/2}\mathbb{Z}/8}_*$, which appear explicitly in [DDHM23, §12.1, §13.2] (the former building on a calculation of Bruner-Greenlees [BG10, Example 7.3.3]).

D	Ω^D_{Spin}	$\Omega^D_{\mathbb{Z}/4,\mathrm{Spin}}$	$\Omega^D_{\mathbb{Z}/4,\mathrm{Spin},\rho}$
-1	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0
1	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	$\mathbb{Z}/8$
2	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0
3	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8$	$\mathbb{Z} \oplus \mathbb{Z}/2$
4	0	0	0
5	0	$\mathbb{Z}/4$	$\mathbb{Z}/32 \oplus \mathbb{Z}/2$
6	0	0	0

TABLE V: The classification of $\mathbb{Z}/4$ symmetric invertible field theories in D spacetime dimension. Here ρ is the charge one representation of $\mathbb{Z}/4$, giving a unitary symmetry class with $U^4 = (-1)^F$.

The symmetry breaking long exact sequence is as follows:

$$\begin{tikzcd}[ampersand replacement=\&]
D \& \Omega^{D-2}_{\mathbb{Z}/4,\text{Spin}} \arrow[r, "\text{Def}_\rho"] \& \Omega^{D}_{\mathbb{Z}/4,\text{Spin},\rho} \arrow[r, "\text{Res}_\rho"] \& \Omega^{D}_{\mathbb{Z}/4,\text{Spin},\rho}(S(\rho)) = \Omega^{D}_{\text{Spin}} \oplus \Omega^{D-1}_{\text{Spin}} \\
-1 \& 0 \& \mathbb{Z} \arrow[r, "\cong"] \& \mathbb{Z} \\
0 \& 0 \& 0 \& \mathbb{Z} \arrow[dll, "(4)"] \\
1 \& \mathbb{Z} \arrow[r] \& \mathbb{Z}/8 \arrow[r] \& \mathbb{Z}/2 \\
2 \& 0 \& 0 \& (\mathbb{Z}/2)^2 \arrow[dll, "\begin{pmatrix} 1 \& 0 \\ 0 \& 2 \end{pmatrix}"] \\
3 \& \mathbb{Z}/2 \oplus \mathbb{Z}/4 \arrow[r, "\begin{pmatrix} 0 \& 0 \\ 0 \& 1 \end{pmatrix}"] \& \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[r, "\begin{pmatrix} 2 \& 0 \\ 0 \& 0 \end{pmatrix}"'] \& \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[dll, "\begin{pmatrix} 1 \& 0 \\ 0 \& 1 \end{pmatrix}"] \\
4 \& (\mathbb{Z}/2)^2 \& 0 \& \mathbb{Z} \arrow[dll] \\
5 \& \mathbb{Z} \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2 \arrow[r] \& \mathbb{Z}/32 \oplus \mathbb{Z}/2 \& 0 \\
6 \& 0 \& 0 \& 0
\end{tikzcd} \tag{II.62}$$

Let us study the subsequence from $D = 2$ to $D = 4$ which goes

$$(\mathbb{Z}/2)^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}} \mathbb{Z}/2 \oplus \mathbb{Z}/4 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} (\mathbb{Z}/2)^2. \tag{II.63}$$

The first map is $\text{Ind}_\rho : \Omega^2_{\mathbb{Z}/4,\text{Spin},\rho}(S(\rho)) \to \Omega^1_{\mathbb{Z}/4,\text{Spin}}$. The Ω^2_{Spin} generator is the 1+1D topological superfluid we discussed around (II.57) and gets mapped to the Ω^1_{Spin} generator as we discussed there. The other $\mathbb{Z}/2$ generator pumps four fermionic charges to the boundary when traversing $S(\rho) \cong S^1$. Let $\mathcal{O}_i$, $i = 1, 2, 3, 4$ be the four operators creating these charges, which anticommute. The $\mathbb{Z}/4$ symmetry acts on them by $\mathcal{O}_i \mapsto \mathcal{O}_{i+1}$. The vortex operator of the whole family is the product $\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4$, which we compute is charge 2 under $\mathbb{Z}/4$. This corresponds to $2 \in \mathbb{Z}/4 \cong \tilde{\Omega}^1_{\mathbb{Z}/4,\text{Spin}}$.

The next group is $\Omega^3_{\mathbb{Z}/4,\text{Spin},\rho} \cong \mathbb{Z} \oplus \mathbb{Z}/2$. The $\mathbb{Z}$ generator represents the anomaly of a charge 1/2 (charge 1 under $\mathbb{Z}/8^F$) 1+1D Weyl fermion, while the $\mathbb{Z}/2$ generator represents that of a Dirac fermion with chiral charges $\pm 1/2$ for the left and right handed components. In the second case, if we break the symmetry by adding a Dirac mass (which transforms in the representation ρ) we get a Thouless pump with a unit $\mathbb{Z}/4$-charged vortex operator, matching the defect anomaly map $\mathbb{Z}/4 \to \mathbb{Z}/2$. Res_ρ maps the $\mathbb{Z}$ generator to two times the $\mathbb{Z}$ generator of Ω^3_{Spin}, since a Weyl fermion is two Majorana-Weyl fermions.

Another interesting subsequence goes from $D = 4$ to 5, in particular exactness requires the map

$$\mathrm{Ind}_\rho : \Omega^4_{\mathbb{Z}/4,\mathrm{Spin}}(S(\rho)) \cong \mathbb{Z} \xrightarrow{\begin{pmatrix}4 & 1 & 0\end{pmatrix}^T} \Omega^3_{\mathbb{Z}/4,\mathrm{Spin}} \cong \mathbb{Z} \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2. \tag{II.64}$$

Let us verify this. The generator of the source is a family which pumps the generator of $\Omega^3_{\mathrm{Spin}} \cong \mathbb{Z}$ to the boundary over each quarter of the circle $S(\rho)$. When we form the ρ-defect, we have four copropagating 1+1D chiral Majorana modes, with $\mathbb{Z}/4$ acting as a permutation. This corresponds to a charge 1 and a charge 2 left-handed Weyl. If this was a $U(1)$ symmetry, its chiral anomaly would be $1^2 + 2^2 = 5$, which is indeed coprime to 8, so when $U(1)$ is reduced to $\mathbb{Z}/4$, this is a generator of $\mathbb{Z}/8$.

6. *$SU(2)$ symmetry breaking for fermions*

Now we discuss $SU(2)$ and $SO(3)$ symmetry breaking in fermion systems. There are three cases of interest, $SU(2) \times \mathbb{Z}/2^F$, $SO(3) \times \mathbb{Z}/2^F$, and $SU(2)^F$, where the latter has a spin-charge relation where fermions carry half integer spin and bosons carry integer spin. We will consider symmetry breaking by both spin-1/2 and spin-1 order parameters. The relevant classifications are shown in Table VI. As input, we need Ω^{Spin}_*, as discussed above, and several families of bordism groups that have not yet appeared in this paper.

- $\Omega^{\mathrm{Spin}}_*(B\mathrm{SO}(3))$ is calculated in low degrees by Wan-Wang [WW19, §5.3.3].
- $\Omega^{\mathrm{Spin}}_*(B\mathrm{SU}(2))$ is calculated in low degrees by Lee-Tachikawa [LT21, Appendix B.2].
- $\Omega^{\mathrm{Spin}^h}_*$ is calculated in low degrees by Freed-Hopkins [FH21, Theorem 9.97].

D	Ω^D_{Spin}	$\Omega^D_{SU(2),\mathrm{Spin}}$	$\Omega^D_{SO(3),\mathrm{Spin}}$	$\Omega^D_{SO(3),\mathrm{Spin},\mathbf{1}}$
-1	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0	0
1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	0
3	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
4	0	0	0	0
5	0	$\mathbb{Z}/2$	0	$(\mathbb{Z}/2)^2$
6	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$
7	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$

TABLE VI: Anomaly groups relevant to the $SU(2)$ families of long exact sequences of field theories

First we will consider $SU(2) \times \mathbb{Z}/2^F$ symmetry breaking to $\mathbb{Z}/2^F$ by a complex spin-1/2 order parameter, which is the simplest case (see Example III.161 for $G = \mathrm{Spin}$):

$$\begin{tikzcd}
D & \Omega^{D-4}_{SU(2),\mathrm{Spin}} & \Omega^{D}_{SU(2),\mathrm{Spin}} & \Omega^{D}_{SU(2),\mathrm{Spin}}(S(\rho)) = \Omega^{D}_{\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
2 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
3 & \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z} \\
4 & 0 & 0 & 0 \\
5 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0 \\
6 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & 0 \\
7 & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^2
\end{tikzcd}$$

The generator of $\Omega^5_{SU(2),\mathrm{Spin}} \cong \mathbb{Z}/2$ corresponds to Witten's $SU(2)$ anomaly [Wit82]. For example, we can consider $N_f = 2$ QCD with chiral $SU(2)_L \times SU(2)_R$ symmetry. In the usual chiral symmetry breaking scenario, the order parameters are mass terms and form a complex $SU(2)$ doublet. The defect anomaly map here is capturing the fact that skyrmions in this theory are fermions.

Next we study $SU(2) \times \mathbb{Z}/2^F$ symmetry breaking to $U(1) \times \mathbb{Z}/2^F$ by a real spin-1 order parameter (see Example III.168):

$$\begin{tikzcd}
D & \Omega^{D-3}_{SU(2),\mathrm{Spin}} & \Omega^{D}_{SU(2),\mathrm{Spin}} & \Omega^{D}_{SU(2),\mathrm{Spin}}(S(\rho)) = \Omega^{D}_{U(1),\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & 0 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \oplus \mathbb{Z} \arrow[dll] \\
2 & \mathbb{Z} & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \\
3 & 0 & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2 \arrow[dll] \\
4 & \mathbb{Z}/2 & 0 & 0 \\
5 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 & \mathbb{Z}^2 \arrow[dll] \\
6 & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}/2 & 0 \\
7 & 0 & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^4
\end{tikzcd}$$

The residual family anomaly in $D = 3$ maps the gravitation Chern-Simons term associated with $\Omega^3_{\mathrm{Spin}} \cong \mathbb{Z}$ to itself, while the level 1 $SU(2)$ Chern-Simons term corresponding to the other generator of $\Omega^3_{SU(2),\mathrm{Spin}}$ maps to a level 2 Chern-Simons term for the unbroken $U(1)$ subgroup. If we have a level 1 Chern-Simons term, the ρ-defect acts as a $U(1)$ monopole (this is like an 't Hooft-Polyakov monopole), and is thus fermionic, which is captured by the index map.

Now we study $SO(3)\times\mathbb{Z}/2^F$ symmetry breaking to $U(1)\times\mathbb{Z}/2^F$ by a real spin 1 order parameter (see Example III.164 and specifically (III.166b)):

$$\begin{tikzcd}
D & \Omega^{D-3}_{SO(3),\mathrm{Spin},\mathbf{1}} & \Omega^D_{SO(3),\mathrm{Spin}} & \Omega^D_{SO(3),\mathrm{Spin}}(S(\rho)) \cong \Omega^D_{U(1),\mathrm{Spin}} \\
-1 & 0 & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & 0 & \mathbb{Z}/2 \arrow[r, "{(0,1)}"] & \mathbb{Z}\oplus\mathbb{Z}/2 \arrow[dll, "{(2,0)}" description] \\
2 & \mathbb{Z} \arrow[r, "{(1,0)}"'] & (\mathbb{Z}/2)^{\oplus 2} \arrow[r, "{(0,1)}"] & \mathbb{Z}/2 \\
3 & 0 & \mathbb{Z}^2 \arrow[r, "\cong"] & \mathbb{Z}^2 \\
4 & 0 & 0 & 0 \\
5 & 0 & 0 & \mathbb{Z}^2 \arrow[dll, "{(1,2)}" description] \\
6 & \mathbb{Z}^2 \arrow[r, "{(1,0)}"'] & \mathbb{Z}/2 & 0 \\
7 & 0 & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^4
\end{tikzcd}$$

Finally we study $SU(2)^F$ symmetry breaking to $U(1)^F$ by a real spin 1 order parameter (see Example III.164 and specifically (III.166a)):

$$\begin{tikzcd}
D & \Omega^{D-3}_{SO(3),\mathrm{Spin}} & \Omega^{D}_{SO(3),\mathrm{Spin},\mathbf{1}} & \Omega^{D}_{SO(3),\mathrm{Spin}}(S(\rho)) \cong \Omega^{D}_{\mathrm{Spin}^c} \\
-1 & 0 & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & \mathbb{Z} \arrow[dll, "\cong"] \\
2 & \mathbb{Z} & 0 & 0 \\
3 & 0 & \mathbb{Z}^2 \arrow[r, "{(2,1)}"] & \mathbb{Z}^2 \arrow[dll, "{(1,0)}"] \\
4 & \mathbb{Z}/2 & 0 & 0 \\
5 & (\mathbb{Z}/2)^{\oplus 2} \arrow[r, "\cong"] & (\mathbb{Z}/2)^{\oplus 2} & \mathbb{Z}^2 \arrow[dll, "2"] \\
6 & \mathbb{Z}^2 \arrow[r, "\mathrm{mod}\ 2"'] & (\mathbb{Z}/2)^{\oplus 2} & 0 \\
7 & 0 & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^4
\end{tikzcd}$$

III. MATH

In this section we provide definitions and proofs of the claims in the physics section, as well as discuss mathematics applications for the long exact sequence.

A. Bordism and Thom spectra

The mathematical formalism in this paper is built on the theory of Thom spectra. In this subsection, we introduce virtual bundles, tangential structures, and their Thom spectra. Furthermore, we review the Pontrjagin-Thom theorem, which relates the homotopy groups of Thom spectra to bordism groups. In the next section, §III B, we discuss the relationship between Thom spectra, anomalies, and invertible field theories; this is the bridge between the mathematics and physics in this paper.

1. *Virtual vector bundles and tangential structures*

Everything in this subsection is well-worn mathematics; see [Fre19, FH21, DY23a] and the references therein for additional references for this material.

Definition III.1. A *virtual vector bundle* $V \to X$ is the data of two vector bundles $V_1, V_2 \to M$, which we think of as "$V_1 - V_2$."

An *isomorphism of virtual vector bundles* f between $V = (V_1, V_2)$ and $W = (W_1, W_2)$ over a common base space X is the data of vector bundles $E_1, E_2 \to X$ and isomorphisms $f_1 \colon V_1 \oplus E_1 \xrightarrow{\cong} W_1 \oplus E_2$ and $f_2 \colon V_2 \oplus E_1 \xrightarrow{\cong} W_2 \oplus E_2$.

The idea behind this definition of isomorphism is that we would like the following pairs of virtual vector bundles to be isomorphic.

1. (V_1, V_2) and (W_1, W_2) when $V_1 \cong W_1$ and $V_2 \cong W_2$.

2. (V_1, V_2) and $(V_1 \oplus E, V_2 \oplus E)$: adding and subtracting E should not change the isomorphism type of the vector bundle.

A vector bundle V defines a virtual vector bundle as the pair $(V, 0)$. In the future we will make this assignment implicitly.

Let $B\mathrm{O}$ denote the classifying space of the infinite-dimensional orthogonal group $\mathrm{O} := \varinjlim_n \mathrm{O}_n$.

Lemma III.2. *The space $\mathbb{Z} \times B\mathrm{O}$ classifies virtual vector bundles: for a space X with the homotopy type of a CW complex, the set $[X, \mathbb{Z} \times B\mathrm{O}]$ is naturally in bijection with the set of isomorphism classes of virtual vector bundles on X.*

Projection $\mathbb{Z} \times B\mathrm{O} \to \mathbb{Z}$ onto the first component defines a numerical invariant of virtual vector bundles; this is the *rank* $\mathrm{rank}(V) := \dim(V_1) - \dim(V_2)$. Thus $B\mathrm{O}$, thought of as $B\mathrm{O} \times \{0\}$, is the classifying space for rank-zero virtual vector bundles.

Definition III.3. A *(stable) tangential structure* is a map $\xi\colon B \to B\mathrm{O}$, and given ξ, a *ξ-structure* on a (rank-zero virtual) vector bundle $V \to X$ is a homotopy class of a lift of its classifying map $f_V\colon X \to B\mathrm{O}$ as in the diagram

$$\begin{tikzcd} & B \arrow[d, "\xi_n"] \\ X \arrow[ur, dashed, "\widetilde{f}_V"] \arrow[r, "f_V"'] & B\mathrm{O}, \end{tikzcd} \tag{III.4}$$

i.e. a homotopy class of maps $\widetilde{f}_V\colon X \to B$ such that $f_V \simeq \xi \circ \widetilde{f}_V$.

If M is a manifold, a *ξ-structure* on M means a ξ-structure on the virtual vector bundle defined by TM. One also sees *normal ξ-structures* on M, which are ξ-structures on $-TM$, the virtual vector bundle defined by the pair $(0, TM)$.

Example III.5. For $\xi\colon B\mathrm{SO} \to B\mathrm{O}$, a ξ-structure is equivalent data to an orientation. For $\xi\colon B\mathrm{Spin} \to B\mathrm{O}$, a ξ-structure is equivalent to a spin structure.

If M is a manifold with boundary, the outward unit normal vector field defines a trivialization of the normal bundle to $\partial M \hookrightarrow M$, so $T(\partial M) \oplus \underline{\mathbb{R}} \cong TM|_{\partial M}$, and therefore a ξ-structure on M induces a ξ-structure on ∂M. It is therefore possible to define a notion of bordism of manifolds with ξ-structure, as Lashof [Las63] did; we let Ω_k^ξ denote the set of bordism classes of n-manifolds with ξ-structure, which becomes an abelian group under disjoint union.

Often, one studies groups G with maps $\rho\colon G \to \mathrm{O}$, and lets $\xi \coloneqq B\rho\colon BG \to B\mathrm{O}$. In this case it is common to denote Ω_*^ξ as Ω_*^G (e.g. $G = \mathrm{O}$, SO, Spin, $\mathrm{Pin}^\pm$, etc.).

Remark III.6. The category of tangential structures is the slice category $\mathsf{Top}_{/B\mathrm{O}}$, i.e. the objects are spaces with a map to $B\mathrm{O}$, and the morphisms are maps which commute with the maps to $B\mathrm{O}$. Bordism groups are covariantly functorial in this category.

2. Construction of Thom spectra

First, recall the classical construction of a Thom space: if $V \to X$ is a vector bundle, choose a Euclidean metric on V. Let $D(V)$ be the *disc bundle* of vectors in V of norm at most 1 and $S(V)$ be the *sphere bundle* of vectors of norm exactly 1; write $\mathrm{Th}(X; V) \coloneqq D(V)/S(V)$.

Example III.7. Let $\underline{\mathbb{R}}^n \to X$ be a trivial bundle and let X_+ be the space X with a disjoint basepoint. Then the Thom space is the n-fold suspension $\mathrm{Th}(X; \underline{\mathbb{R}}^n) \simeq \Sigma^n X_+$.

Example III.8. Let $X = \mathbb{RP}^n$ and let $V = \sigma$ be the tautological line bundle. Then the Thom space is $\mathrm{Th}(\mathbb{RP}^n; \sigma) \simeq \mathbb{RP}^{n+1}$.

Proposition III.9. *If X is compact, then $\mathrm{Th}(X; V)$ is the one point compactification of the disk bundle $D(V)$.*

Let $V \to X$ be a rank d real vector bundle (*not* merely a virtual vector bundle!), and also write $V\colon X \to B\mathrm{O}(d)$ for the classifying map. Let Top denote the ∞-category of spaces and Top_* denote the ∞-category of pointed spaces. The action of $\mathrm{O}(d)$ on $\mathbb{R}^d$ induces an action on $S^d = S^{\mathbb{R}^d}$, and this induces a functor from the fundamental ∞-groupoid of $B\mathrm{O}(d)$ to Top_*.

Proposition III.10. *The Thom space* $\mathrm{Th}(X;V)$ *is naturally homotopy equivalent to the colimit of the* X*-shaped diagram*[8]

$$X \xrightarrow{V} B\mathrm{O}(d) \longrightarrow \mathsf{Top}_*. \tag{III.11}$$

We need a similar construction for virtual bundles on X. It has the structure of a *spectrum*.[9] The reader unfamiliar with spectra is encouraged to think of them as similar to topological spaces, in that one can take homotopy, (co)homology, and generalized (co)homology groups of them. See Freed-Hopkins [FH21, §6.1] or Beaudry-Campbell [BC18, §2] for precise definitions and [DDHM23, §10.3] for a longer but still heuristic overview. We write Sp for the category of spectra.

We follow [ABG⁺14a] in the rest of this section. By a *local system of spectra* over a space X we mean a functor $\mathcal{L}$ from the fundamental ∞-groupoid of X to spectra. We will usually denote this as $\mathcal{L}\colon X \to \mathsf{Sp}$. The fiber of a local system at a point $p \in X$ is obtained by composing $\mathcal{L}$ with the functor $\mathrm{pt} \to X$ given by inclusion at p; a functor out of pt is equivalent to a single spectrum, and we call this the fiber of $\mathcal{L}$ at p.

Definition III.12 ([DL59, ABG⁺14a])**.** A *stable spherical fibration* is a local system of spectra valued in the full sub-∞-category of spectra with objects $\Sigma^n\mathbb{S}$, $n \in \mathbb{Z}$.

Here $\mathbb{S}$ denotes the *sphere spectrum*.

Definition III.13. Let X be a space and $V \to X$ be a vector bundle of rank r. Let $\mathbb{S}^V \to X$ denote the associated stable spherical fibration, whose fiber at a point $x \in X$ is the suspension spectrum of the one-point compactification of V_x.

Now fix a base space X and a virtual vector bundle $V \to X$, which is equivalently a map $V\colon X \to BO \times \mathbb{Z}$. There is a canonical spectrum called the *Thom spectrum* X^V associated to X, V constructed as follows. There is a functor $J\colon BO \times \mathbb{Z} \to \mathsf{Sp}$, generalizing the map $BO(d) \to \mathsf{Top}_*$ above. It maps into spectra now, instead of spaces, because for a virtual bundle $V_1 - V_2$, we want to assign the sphere $S^{V_1} \wedge S^{-V_2}$, but S^{-V_2} doesn't make sense as a space, since spheres of negative dimension don't exist. However, the sphere spectrum $\mathbb{S}$ can be desuspended, and the Thom spectrum associated to a virtual bundle is defined as follows.

Definition III.14. Given a virtual bundle $V\colon X \to BO \times \mathbb{Z}$, the Thom spectrum X^V is the colimit (in spectra) of the composite $X \xrightarrow{V} BO \times \mathbb{Z} \xrightarrow{J} \mathsf{Sp}$.

Here's the compatibility between the Thom space and Thom spectrum construction.

Lemma III.15. *Let* $V\colon X \to BO(d)$ *be a vector bundle and let* $\xi\colon X \to BO(d) \to BO \times \mathbb{Z}$ *be the corresponding virtual bundle. Then the Thom spectrum of* ξ *is the suspension spectrum of the Thom space of* V*; i.e.* $X^\xi \simeq \Sigma^\infty_+ X^V$.

Here by Σ^∞_+ we mean first taking the disjoint union with a single point, which we take as the basepoint, then taking the suspension spectrum.

Proof. This follows from the fact that $\Sigma^\infty_+\colon \mathsf{Top} \to \mathsf{Sp}$ preserves colimits. ☐

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[8] When we say "X-shaped diagram," we mean a functor out of the fundamental ∞-groupoid of X.

[9] Spectra in stable homotopy theory are etymologically unrelated to spectra in algebraic geometry, operator theory, physics, etc.

Using Lemma III.15, one can directly check that the Thom spectrum of the trivial bundle $\underline{\mathbb{R}}^n \to X$ is homotopy equivalent to a suspension of the suspension spectrum $\Sigma^n \Sigma^\infty_+ X$.

Lemma III.16 ([Ati61a, Lemma 2.3])**.** *Let $V \to X$ and $W \to Y$ be virtual vector bundles. Then the Thom spectrum of $V \boxplus W \to X \times Y$ is homotopy equivalent to $X^V \wedge Y^W$.*

Here $\boxplus$ is the external direct sum, i.e. the direct sum of the pullbacks of V and W across the projection maps $X \times Y \to X$, resp. $X \times Y \to Y$.

One can often combine Lemma III.16 with the observation that Thom spectra of trivial bundles are suspensions to simplify Thom spectra appearing in examples. For example, $X^{V+\underline{\mathbb{R}}^n}$, often denoted X^{V+n}, is homotopy equivalent to $\Sigma^n X^V$. Since we are working with virtual vector bundles, n may be any integer.

Let us discuss a variant for tangential structures.

Definition III.17. Let $\xi \colon B \to B\mathrm{O}$ be a tangential structure. Then its inverse (as a virtual vector bundle) $-\xi$ is often denoted $\xi^\perp$. Equivalently, $\xi^\perp$ is the composition of ξ with the map $-1 \colon B\mathrm{O} \to B\mathrm{O}$, which is the inverse map in the E_∞-structure on $B\mathrm{O}$ induced by direct sum. Therefore $\xi^\perp$ is also a tangential structure; its Thom spectrum $B^{-\xi}$ is called a *Madsen-Tillmann spectrum* [MT01, MW07] and is often denoted $MT\xi$. If $B \to B\mathrm{O}$ is obtained from a family of Lie group homomorphisms $H(n) \to \mathrm{O}(n)$ in the (co)limit $n \to \infty$, $MT\xi$ is often written MTH.

Likewise, the Thom spectrum of the pullback of $-V_n \to B\mathrm{O}(n)$ across a map $\xi_n \colon B_n \to B\mathrm{O}(n)$ is denoted $MT\xi_n$; if $B = BH(n)$ for a Lie group $H(n)$, this is often written $MTH(n)$.

$MT\xi$ has two key properties:

1. (Pontrjagin-Thom theorem) There is a natural isomorphism $\pi_n(MT\xi) \xrightarrow{\cong} \Omega_n^\xi$.[10]
2. (Thom isomorphism theorem) Let A be a commutative ring. Then there is a natural[11] isomorphism $H^*(B; A_{w_1}) \xrightarrow{\cong} H^*(MT\xi; A)$, where A_{w_1} denotes the pullback by ξ of the orientation local system on $B\mathrm{O}$.

In the Thom isomorphism theorem, the use of twisted cohomology can be avoided by assuming $A = \mathbb{Z}/2$ or by choosing an orientation of the virtual vector bundle classified by the map ξ.

When ξ is the result of applying the classifying space functor to a group homomorphism $G \to \mathrm{O}$, we often write MTG for $MT\xi$.

B. Thom spectra and invertible phases

1. *Symmetries and tangential structures*

Recall from §I A 1 that anomalies are captured by the data of invertible field theories. In this section we ask: what kinds of invertible field theories? For any tangential structure in the sense of

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[10] It is most common to define Thom spectra and bordism in terms of the stable normal bundle, rather than the tangent bundle; the resulting spectra are written $M\xi$. The spectra $MT\xi$ and $M\xi$ coincide for the tangential structures O, SO, Spin^c and Spin, but not in general: $MTPin^\pm \simeq MPin^\mp$. By composing with the map $-1 \colon BO \to BO$, one can pass between normal bordism and tangential bordism and therefore pass between our definition and the standard one.

[11] Naturality here is for maps of tangential structures as in Remark III.6; this map typically does not commute with the action of cohomology operations.

Definition III.3, there is a notion of topological field theory. Given a field theory whose anomaly we want to investigate, which tangential structure ξ do we want our invertible field theories to carry?[12]

The answer typically depends only on the symmetries of our field theory, not on its field content (the anomaly itself—which invertible field theory we get out of all the invertible field theories on ξ-manifolds—uses more information from the theory). We follow Freed-Hopkins [FH21, §2], whose take the stance that since we typically study QFTs in Minkowski signature but invertible field theories are Euclidean, we should Wick-rotate the group of symmetries to define our tangential structure.

Now we describe Freed-Hopkins' procedure. Assume the dimension n is at least 2. Let $\mathcal{I}(1,n-1)$ be the isometry group of Minkowski space, and let $\mathcal{I}(1,n-1)^{\uparrow}\subset\mathcal{I}(1,n-1)$ be the subgroup of isometries that preserve the direction of time. The group of symmetries of our theory is a Lie group $\mathcal{H}(1,n-1)$ with a map $\rho(n)\colon\mathcal{H}(1,n-1)\to\mathcal{I}(1,n-1)^{\uparrow}$. Let $K\coloneqq\ker(\rho(n))$; we assume K is compact. Assume that the normal subgroup of translations $\mathbb{R}^{1,n-1}\subset\mathcal{I}(1,n-1)$ lifts to a normal subgroup of $\mathcal{H}(1,n-1)$, and let $H(1,n-1)\coloneqq\mathcal{H}(1,n-1)/\mathbb{R}^{1,n-1}$. Now:

1. Let $\mathrm{O}(1,n-1)^{\uparrow}\coloneqq\mathrm{O}(1,n-1)\cap\mathcal{I}(1,n-1)$. There is an exact sequence
$$0\longrightarrow K\longrightarrow H(1,n-1)\longrightarrow\mathrm{O}(1,n-1)^{\uparrow}. \tag{III.18a}$$
2. This exact sequence can be extended to an exact sequence of complexifications:
$$0\longrightarrow K(\mathbb{C})\longrightarrow H(n,\mathbb{C})\longrightarrow\mathrm{O}(n,\mathbb{C}), \tag{III.18b}$$
3. and then to compact real forms of these complex Lie groups:
$$0\longrightarrow K\longrightarrow H(n)\longrightarrow\mathrm{O}(n). \tag{III.18c}$$

The tangential structure that the anomaly field theory has is $\xi\colon BH(n)\to B\mathrm{O}(n)$. Just as it is not a priori clear that the anomaly field theory extends to dimension $n+1$, it is also not necessarily clear that ξ extends to an $(n+1)$-dimensional unstable tangential structure, but Freed-Hopkins [FH21, Theorem 2.19] prove that it does in nearly every situation one might want, as we discuss below. In this paper, we will always be in the situation that ξ extends to $(n+1)$-manifolds.

In practice, computing ξ can be simplified using Stehouwer's formalism of *fermionic groups* [Ste22, §2.1].

Definition III.19 (Stehouwer [Ste22, Definition 1])**.** A *fermionic group* is a topological group G together with data of:

- a central element squaring to 1, which we call *fermion parity* and denote $-1\in G$, and
- a group homomorphism $\theta\colon G\to\mathbb{Z}/2$ such that $\theta(-1)=0$.

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[12] For non-topological invertible field theories, there is also the question of enriching the tangential structure to something more geometric. We will not need to worry about this question in this paper.

We think of θ as defining a $\mathbb{Z}/2$-grading on G, and we refer to elements of G as odd or even. The even elements form a subgroup $G_0 \subset G$, which is itself a fermionic group with θ trivial.

Given two fermionic groups G and H, one can take their *fermionic tensor product* (*ibid.*, Definition 5) $G \otimes H \coloneqq (G \times H)/\langle(-1,-1)\rangle$. This is a fermionic group, with central element $(-1,1) = (1,-1)$ and grading $\theta((g,h))$ equal to the sum mod 2 of the gradings on g and on h.

Fermionic groups describe symmetries of theories with fermions: -1 acts by fermion parity, which may mix nontrivially with other symmetries in the theory; and θ describes whether elements of G act unitarily or antiunitarily. Given a fermionic group G, one obtains a tangential structure $\xi_G\colon B \to B\mathrm{O}$ as follows: let H be the even subgroup of the fermionic tensor product $\mathrm{Pin}^+ \otimes G$ (*ibid.*, Definition 7); here, to make Pin^+ into a fermionic group, we use the usual -1, and the grading homomorphism is $\pi_0\colon \mathrm{Pin}^+ \to \mathrm{O}(1) \cong \mathbb{Z}/2$. Then $B \coloneqq BH$, and the map $\xi\colon B \to B\mathrm{O}$ is induced from the usual map $\mathrm{Pin}^+ \to \mathrm{O}$ and the constant map to the identity on the quotient of G_0 by fermion parity.

See [Ste22] for several examples of computations of tangential structures from data of the symmetries of a theory.

2. *From invertible field theories to bordism invariants*

At this point in the story we have turned the physics question of determining the possible anomalies of a theory with a given collection of symmetries into the mathematical question of classifying (reflection-positive) invertible field theories for a fixed tangential structure $\xi\colon B \to B\mathrm{O}$. In this section we discuss how this classification question reduces to a well-studied problem in algebraic topology, the computation of groups of bordism invariants. See Freed [Fre19, Lectures 6–9] and Galatius [Gal21] for more detailed reviews of this story.

Recall from §I A 1 that a field theory $Z\colon \mathsf{Bord}_n^\xi \to \mathsf{C}$ is invertible if there is some other theory Z^{-1} such that $Z \otimes Z^{-1}$ is the trivial theory. This tensor product is evaluated "pointwise," meaning that $(Z \otimes Z^{-1})(M) \coloneqq Z(M) \otimes Z^{-1}(M)$, where M is an object, morphism, etc. in the bordism category; therefore invertibility implies that Z, as a functor, factors through the Picard sub-k-groupoid of units $\mathsf{C}^\times$ inside C, meaning that if X is any object, morphism, or higher morphism in Bord_n^ξ, $Z(X)$ is invertible: $\otimes$-invertible if X is an object, and composition-invertible if X is a (higher) morphism. If X is invertible, then we must have data of an isomorphism $Z(X^{-1}) \xrightarrow{\cong} Z(X)^{-1}$ because Z is symmetric monoidal; thus, even if X is not invertible, we can heuristically *define* $Z(X^{-1}) \coloneqq Z(X)^{-1}$ as if X^{-1} existed. These definitions are compatible as X varies, in the sense that Z extends to the *Picard k-groupoid completion* $\overline{\mathsf{Bord}}_n^\xi$ of Bord_n^ξ: the Picard k-groupoid defined by formally adding inverses to all objects, morphisms, higher morphisms, etc. of Bord_n^ξ. Thus, an invertible field theory $Z\colon \mathsf{Bord}_n^\xi \to \mathsf{C}$ is equivalent data to a morphism of Picard k-groupoids

$$Z\colon \overline{\mathsf{Bord}}_n^\xi \longrightarrow \mathsf{C}^\times. \tag{III.20}$$

So to compute deformation classes of invertible field theories, we should compute the groups of symmetric monoidal functors between these Picard k-groupoids, modulo natural isomorphisms. The homotopy theory of Picard groupoids embeds in the usual stable homotopy category: if D is a Picard groupoid, the geometric realization $|N\mathsf{D}|$ of the nerve of D has an E_∞-structure arising from the monoidal product on D, and the Picard condition implies $|N\mathsf{D}|$ is grouplike. Therefore it is equivalent data to a connective spectrum $|\mathsf{D}|$, which we call the *classifying spectrum* of D. This turns out to be a complete invariant of Picard k-groupoids.

Theorem III.21 (Stable homotopy hypothesis (Moser-Ozornova-Paoli-Sarazola-Verdugo [MOP⁺22]))**.** *There is an equivalence of ∞-categories between the ∞-category of Picard k-groupoids and the ∞-category of spectra whose homotopy groups vanish outside of* $[0, k]$.

Remark III.22. For $k = 1$, the stable homotopy hypothesis was originally a folklore theorem: proofs or sketches appear in [BCC93, HS05, Dri06, Pat12, JO12, GK14]. For $k = 2$, the stable homotopy hypothesis was proven by Gurski-Johnson-Osorno [GJO19].

Therefore we need to compute the group of homotopy classes of maps of spectra $|\overline{\mathsf{Bord}}_n^\xi| \to |\mathsf{C}^\times|$. A reasonable first step would be to identify these two classifying spectra. For the domain, the Picard k-groupoid completion of the bordism category, this is due to Galatius-Madsen-Tillmann-Weiss [GMTW09] and Nguyen [Ngu17] for the bordism $(\infty, 1)$-category and to Schommer-Pries [SP17] for more general (∞, k)-categories.

Theorem III.23 (Galatius-Madsen-Tillmann-Weiss [GMTW09], Nguyen [Ngu17], Schommer–Pries [SP17])**.** *If* Bord_n^ξ *denotes the* (∞, k)*-category of bordisms of* ξ_n*-structured manifolds in dimensions* $n - k, \dots, n$*, then there is a natural equivalence* $|\overline{\mathsf{Bord}}_n^\xi| \simeq \Sigma^k MT\xi_n$.

Here $MT\xi_n$ is a Madsen-Tillmann spectrum as in Definition III.17.

Freed-Hopkins-Teleman [FHT10] then applied this result to classify invertible field theories in terms of $MT\xi_n$. To do so, we need to determine $|\mathsf{C}^\times|$, which depends on one's choice of C—Freed-Hopkins [FH21, §5.3] argue that the (shifted) *character dual of the sphere spectrum* $\Sigma^n I_{\mathbb{C}^\times}$ is a universal choice, and that a related object called the (shifted) *Anderson dual of the sphere spectrum* $\Sigma^{n+1} I_{\mathbb{Z}}$ should appear when one wants to classify deformation classes of invertible field theories. For applications to anomalies, we are interested in deformation classes, so use $\Sigma^{n+1} I_{\mathbb{Z}}$.

The Anderson dual $I_{\mathbb{Z}}$ is characterized by its universal property that for any spectrum $\mathcal{X}$, there is a short exact sequence [And69, Yos75]

$$0 \longrightarrow \mathrm{Tors}(\mathrm{Hom}(\pi_{n+1}\mathcal{X}, \mathbb{C}^\times)) \xrightarrow{\varphi} [\mathcal{X}, \Sigma^{n+2} I_{\mathbb{Z}}] \xrightarrow{\psi} \mathrm{Hom}(\pi_{n+2}\mathcal{X}, \mathbb{Z}) \longrightarrow 0. \qquad \text{(III.24)}$$

We are interested in anomalies of unitary QFTs, hence we expect the anomaly theories to satisfy the Wick-rotated analogue of unitarity: reflection positivity. Freed-Hopkins [FH21, §7.1, §8.1] define reflection positivity for invertible TFTs using $\mathbb{Z}/2$-actions on Bord_n^ξ and C,[13] and prove two key results allowing for a complete classification of reflection positive invertible TFTs following their definition.

Theorem III.25 (Freed-Hopkins [FH21, Theorem 2.19])**.** *If* $n \geq 3$ *and* $\xi_n \colon BH(n) \to B\mathrm{O}(n)$ *is a tangential structure arising from a representation* $\rho \colon H(n) \to \mathrm{O}(n)$ *with* $H(n)$ *a compact Lie group and* $\mathrm{SO}(n) \subset \mathrm{Im}(\rho)$*, then there is a stable tangential structure* $\xi \colon BH \to B\mathrm{O}$ *such that* ξ_n *is the pullback of* ξ *along* $B\mathrm{O}(n) \to B\mathrm{O}$.

Theorem III.26 (Freed-Hopkins [FH21, Theorem 5.23])**.** *Suppose* $\xi \colon BH(n) \to B\mathrm{O}(n)$ *satisfies the hypotheses of Theorem III.25. The abelian group of deformation classes of* n*-dimensional,*

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[13] The definition of reflection positivity for extended not-necessarily-invertible TFTs is still open: see [JF17, MS23] for work in this direction.

reflection positive invertible topological field theories on manifolds with ξ-structure is naturally isomorphic to the torsion subgroup of $[MT\xi, \Sigma^{n+1}I_{\mathbb{Z}}]$.

So after we require reflection positivity, the classification changes from Madsen-Tillmann bordism to bordism in the usual sense, which is easier to calculate.

Freed-Hopkins then conjecture (*ibid.*, Conjecture 8.37) that the entire group $[MT\xi, \Sigma^{n+1}I_{\mathbb{Z}}]$ classifies all n-dimensional reflection positive invertible field theories, topological or not. See Grady-Pavlov [GP21, §5] for some discussion of the nontopological case.

Remark III.27. There are some other approaches to the classification of invertible topological field theories, due to Yonekura [Yon19], Rovi-Schoenbauer [RS22], and Kreck-Stolz-Teichner (unpublished).

Freed-Hopkins' conjecture has a nice interpretation from the point of view of anomalies. Using the defining property of $I_{\mathbb{Z}}$, there is a short exact sequence

$$0 \longrightarrow \mathrm{Tors}(\mathrm{Hom}(\Omega_{n+1}^{\xi}, \mathbb{C}^\times)) \xrightarrow{\varphi} [MT\xi, \Sigma^{n+2}I_{\mathbb{Z}}] \xrightarrow{\psi} \mathrm{Hom}(\Omega_{n+2}^{\xi}, \mathbb{Z}) \longrightarrow 0, \qquad \text{(III.28)}$$

where Tors(–) denotes the torsion subgroup. The first and third terms in this short exact sequence have anomaly-theoretic interpretations.

- The quotient $\mathrm{Hom}(\Omega_{n+2}^{\xi}, \mathbb{Z})$ is a free abelian group consisting of characteristic classes of $(n+2)$-dimensional ξ-manifolds; under this identification, the map ψ sends an anomaly field theory to the corresponding anomaly polynomial, which is one degree higher, such as Chern-Simons and Chern-Weil forms. This data is visible to perturbative techniques, and is sometimes called the *local anomaly*.

- The subgroup $\mathrm{Tors}(\mathrm{Hom}(\Omega_{n+1}^{\xi}, \mathbb{C}^\times))$ is identified with the torsion subgroup of $[MT\xi, \Sigma^{n+2}I_{\mathbb{Z}}]$; these are the reflection positive invertible field theories which are topological. Such field theories' partition functions are bordism invariants, and the identification of these reflection positive invertible TFTs with $\mathrm{Tors}(\mathrm{Hom}(\Omega_{n+1}^{\xi}, \mathbb{C}^\times))$ assigns to a reflection positive invertible TFT its partition function. Typically this data is invisible to perturbative methods and is called the *global anomaly*.

Yamashita-Yonekura [YY21] and Yamashita [Yam21] relate the short exact sequence (III.28) to a differential refinement of $\mathrm{Map}(MT\xi, \Sigma^{n+2}I_{\mathbb{Z}})$.

C. Smith homomorphisms

This section is the technical heart of the math section—we provide a general definition of the Smith homomorphism, then lift it to a map S of bordism spectra. The map of spectra has been studied, though its identification with the Smith homomorphism is new; using this, we can write down the cofiber of S (Theorem III.88) and therefore obtain Smith long exact sequences of bordism groups and Anderson-dualized bordism groups (Corollaries III.95 and III.97). The latter, interpreted by way of Theorem III.26 as long exact sequences of invertible field theories, are the mathematical instantiation of our symmetry breaking long exact sequence in Section II.

1. (X, V)-twisted tangential structures

Twisted tangential structures are an important ingredient in the Smith homomorphism—they determine its domain and codomain. We take this subsubsection to define them and point out why they arise in the Smith homomorphism setting.

Throughout this subsubsection, we fix a topological space X, a vector bundle $V \to X$ of rank r, and a tangential structure $\xi\colon B \to B\mathrm{O}$.

Definition III.29. Let $W \to Y$ be a vector bundle. An (X, V)-*twisted* ξ-*structure* on W is the data of a map $f\colon Y \to X$ and a ξ-structure on $W \oplus f^*(V)$.

There is a space of (X, V)-twisted ξ-structures on W, and just like for tangential structures, we will think of two such structures as the same if they lie in the same connected component.

Twisted ξ-structures provide a convenient way to describe a more complicated tangential structure in terms of a simpler one.

Example III.30. Recall that a spin^c structure on an oriented vector bundle $W \to Y$ is the data of a complex line bundle $L \to Y$ and an identification $w_2(L) = w_2(W)$. The data of L is equivalent to a map $Y \to B\mathrm{U}(1)$ such that L is the pullback of the tautological complex line bundle $S \to B\mathrm{U}(1)$.The identification $w_2(L) = w_2(W)$ is equivalent by the Whitney sum formula to $w_2(W \oplus L) = 0$.

Choosing a spin structure on $W \oplus L$ first provides an orientation of $W \oplus L$, which since L is canonically oriented by its complex structure is equivalent to an orientation of W; then it additionally provides an identification $w_2(W \oplus L) = 0$. Therefore the data of a spin^c structure on W is equivalent to the data of L and a spin structure on $W \oplus L$, meaning that a spin^c structure is equivalent to a $(B\mathrm{U}(1), S)$-twisted spin structure.

In a similar way, one can show that if $\sigma \to B\mathbb{Z}/2$ is the tautological real line bundle, pin$^-$ structures are equivalent to $(B\mathbb{Z}/2, \sigma)$-twisted spin structures, pin$^+$ structures are equivalent to $(B\mathbb{Z}/2, 3\sigma)$-twisted spin structures, and pin^c structures are equivalent to $(B\mathbb{Z}/2, \sigma)$-twisted spin^c structures.

It turns out that all of these twisted tangential structures can also be "untwisted" into ordinary tangential structures.

Lemma III.31 (Shearing)**.** *Let $T \to B\mathrm{O}$ denote the tautological rank-zero virtual vector bundle and $\zeta\colon B \times X \to B\mathrm{O}$ be classified by the rank-zero virtual vector bundle $\xi^*(T) \boxplus (V - r)$. Then (X, V)-twisted ξ-structures are equivalent to ζ-structures.*

The proof is given in [DDHM23, Lemma 10.18] for $\xi = \mathrm{Spin}$; the general case is completely analogous. Invoking the Pontrjagin-Thom theorem, we then learn:

Corollary III.32. *There is a notion of bordism of manifolds with (X, V)-twisted ξ-structures, corresponding to the Thom spectrum $MT\xi \wedge X^{V-r}$; thus the bordism groups of these manifolds are $\Omega_*^\xi(X^{V-r})$.*

Here we use the fact that the Thom spectrum functor sends external direct sums to smash products, which is Lemma III.16.

Lemma III.33. *Suppose X is a closed smooth manifold with a ξ-structure and $M \subset X$ is an embedded submanifold such that the image of the mod 2 fundamental class of M in $H_*(X; \mathbb{Z}/2)$ is Poincaré dual to $e(V) \in H^r(X; \mathbb{Z}/2)$. Then M has a canonical (X, V)-twisted ξ-structure.*

Proof. Because the homology class of M is Poincaré dual to the mod 2 Euler class of V, the normal bundle to $M \hookrightarrow X$ is isomorphic to $V|_M$. Choose a Riemannian metric on X; this is a contractible choice, so will not change the connected component of the data we obtain, so as discussed above different choices of metric lead to the same (X, V)-twisted ξ-structure in the end.

Using the Levi-Civita connection induced by the metric, we may split the short exact sequence of vector bundles over M,

$$0 \longrightarrow TM \longrightarrow TX|_M \longrightarrow \nu \longrightarrow 0, \tag{III.34}$$

thereby obtaining an isomorphism $TM \oplus V|_M \cong TX|_M$. Since TX has a ξ-structure, this implies $TM \oplus V|_M$ has a chosen ξ-structure, i.e. that we have put a (X, V)-twisted ξ-structure on M. ☐

2. *Smith homomorphisms induced by maps of Thom spectra*

We will now apply the previous discussions of Thom spectra and shearing to understand a class of homomorphisms between bordism groups called *Smith homomorphisms*. These map between bordism groups of manifolds of different dimensions and with different tangential structures, and we are study them in this paper since they are Anderson dual to the defect anomaly matching maps Def_ρ of Section II B.

Fix a tangential structure $\xi\colon B \to B\mathrm{O}$ such that its bordism spectrum $MT\xi$ is a ring spectrum (e.g. O, SO, Spin^c, Spin). Fix also a virtual vector bundle $V \to X$ of rank r_V and $W \to X$ a vector bundle of rank r_W.

Definition III.35. The *Smith homomorphism* associated to ξ, V, and W is the homomorphism

$$\mathrm{sm}_W\colon \Omega_n^\xi(X^{V-r_V}) \longrightarrow \Omega_{n-r_W}^\xi(X^{V\oplus W-r_V-r_W}) \tag{III.36}$$

that sends a closed n-manifold $[M]$ to the bordism class $[N]$, where $N \subset M$ is the submanifold defined as follows: pull back W from X to M and choose a section $s\colon M \to f^*W$ transverse to the zero section. Then, $N \coloneqq s^{-1}(0)$ is an $(n - r_W)$-dimensional manifold whose mod 2 homology class is Poincaré dual to $e(W)$, hence by Lemma III.33 has a $(X, V \oplus W)$-twisted ξ-structure, and we define $\mathrm{sm}_W([M]) \coloneqq [N]$.

Proposition III.37 ([HKT20a] §4.2)**.** *The bordism class* $[N] \in \Omega_{n-r_W}^\xi(X^{V\oplus W-r_V-r_W})$ *is independent of the choice of section.*

Remark III.38. The bundle W in the definition above plays the role of the symmetry breaking representation ρ in Section II.

Example III.39. Let $\xi\colon B\mathrm{Spin} \to B\mathrm{O}$, $X = B\mathbb{Z}/2$, $V = 0$, and $W = \sigma \to B\mathbb{Z}/2$, where σ is the tautological line bundle. The corresponding Smith homomorphism is

$$\Omega_n^{\mathrm{Spin}}(B\mathbb{Z}/2) \xrightarrow{\mathrm{sm}_\sigma} \Omega_{n-1}^{\mathrm{Spin}}((B\mathbb{Z}/2)^{\sigma-1}). \tag{III.40}$$

After shearing (Lemma III.31), we recognize this as

$$\Omega_n^{\mathrm{Spin}\times\mathbb{Z}/2} \xrightarrow{\mathrm{sm}_\sigma} \Omega_{n-1}^{\mathrm{Pin}^-}. \tag{III.41}$$

Letting $V = 0$, σ, 2σ, and 3σ produces the maps in the four-periodic family discussed in Example III.131.

Later, in Section III E, we thoroughly discuss the history of Smith maps and present many more examples. For the rest of this section, we discuss how Smith homomorphisms are induced by maps of Thom spectra. Let X be a topological space and V be a rank r real vector bundle on X. We abuse notation and also denote the associated classifying map by $V\colon X \to BO(r)$.The inclusion $0 \hookrightarrow W$ induces a zero section map $X \to X^W$. More generally, we have the following.

Definition III.42. Let V and W be vector bundles on X. Let $S^V \to S^{V\oplus W}$ be the map of finite-dimensional spheres over X induced by the zero section map on W. The *Smith map* associated to X, V, and W is the map of Thom spaces

$$\mathrm{sm}_W\colon \mathrm{Th}(X;V) \to \mathrm{Th}(X;V\oplus W) \tag{III.43}$$

formed as the colimit of the map of spheres.

Definition III.44. In the case that we have a virtual bundle V, the zero section map induces a map of stable spherical fibrations $\mathbb{S}^V \to \mathbb{S}^{V\oplus W} \simeq \mathbb{S}^V \wedge \mathbb{S}^W$ over X. Taking the colimit, we get a map of Thom spectra

$$\mathrm{sm}_W\colon X^V \to X^{V\oplus W} \tag{III.45}$$

which we also call a *Smith map.*

Proposition III.46. *The map on ξ-bordism groups induced by the map* (III.45) *of spectra is equal to the Smith homomorphism as defined in Definition III.35.*

This follows by unpacking the Pontrjagin-Thom isomorphism.

In the next two sections, we develop an alternate definition of the Smith homomorphism via the Euler class.

3. *Euler classes in generalized cohomology*

Fix $\xi\colon X \to BO$ a tangential structure and $W\colon X \to BO(r_W)$ a vector bundle on X. We would like to describe the Smith homomorphism on X bordism groups as taking a manifold $(M, p : M \to X)$ with X tangential structure to a smooth representative of the Poincaré dual of $e(p^*W)$, where $e(p^*W) \in H^{r_W}(M;\mathbb{Z})$ is the Euler class of W. This, however, is *not* true in general, as we show in Appendix B—we need to upgrade what we mean by the Euler class.

We will define an Euler class living in twisted cobordism. More generally, for $\mathcal{R}$ an $\mathbb{E}_1$ ring spectrum, we define a $\mathcal{R}$-valued Euler class in the $\mathcal{R}$-cohomology of X^{-W}. In the case we have an untwisting, given by a $\mathcal{R}$-orientation on W, we will see in Lemma III.70 that the *untwisted* Euler class is the pullback of the Thom class $U^{\mathcal{R}}(W) \in \mathcal{R}^r(\mathrm{Th}(X;W))$ along the 0 section $X \to \mathrm{Th}(X;W)$ (e.g. in [Bec70, §13]), so that our definition deserves to be called an Euler class; we also generalize to the twisted setting where there is no Thom class.

Recall the setup of Definition III.44. Let 0 be the vector bundle over X of rank zero. The zero section gives a map $0 \to W$ of vector bundles over X. Therefore we get a map of stable spherical fibrations

$$z\colon \mathbb{S}^0 \longrightarrow \mathbb{S}^W, \tag{III.47a}$$

i.e. a fiberwise map of spectra. Because 0 is the trivial rank-zero vector bundle, $\mathbb{S}^0$ is the constant stable spherical fibration $\underline{\mathbb{S}}$ with fiber $\mathbb{S}$.

Apply the duality $\mathrm{Map}(–, \mathbb{S})$ fiberwise to obtain another map

$$z^\vee\colon \mathbb{S}^{-W} \longrightarrow \mathbb{S}^0. \tag{III.47b}$$

Because the codomain of $z^\vee$ is constant as a functor $X \to \mathsf{Sp}$, there is an induced map of spectra:

$$e^{\mathbb{S}}(W) : X^{-W} = \mathrm{colim}_X\, \mathbb{S}^{-W} \to \mathbb{S} \tag{III.47c}$$

Definition III.48. The class $e^{\mathbb{S}}(W)$ is called the *stable cohomotopy Euler class* of W. Usually, we will interpret generalized cohomology of X^{r-W} as the $(-W)$-twisted cohomology of X, meaning $e^{\mathbb{S}}(W)$ is an element of the degree-r $(-W)$-twisted stable cohomotopy of X.

Remark III.49. This cohomology class of $e^{\mathbb{S}}(W)$ lives in $(\mathbb{S})^0(X^{-W})$. By the Pontrjagin-Thom isomorphism, this is equivalent to the twisted cobordism group $\Omega^d_{\mathrm{fr}}(X, -W)$.

Definition III.50. Let $\mathcal{R}$ be a $(\mathbb{E}_1)$-ring spectrum, so that there is a unique ring map $1_{\mathcal{R}}\colon \mathbb{S} \to \mathcal{R}$. The *R-cohomology Euler class of* W, denoted $e^{\mathcal{R}}(W)$, is the composition $1_{\mathcal{R}} \circ e^{\mathbb{S}}(W)$. As in the previous definition, we interpret this as an element of the degree-r_W $(-W)$-twisted R-cohomology of X.

Now we see how the Euler class and Smith homomorphism are related:

Proposition III.51.

1. *Let* 0 *be the trivial rank* 0 *vector bundle on* X*; then* $e^{\mathbb{S}}(0)\colon \Sigma^\infty_+ X \to \mathbb{S}$ *is the infinite suspension of the crush map* $X \to *$.
2. *Let* W *be a vector bundle on* X *and* $\mathrm{sm}_W\colon X^{-W} \to X$ *be the Smith map. Then* $e^{\mathbb{S}}(W) = (\mathrm{sm}_W)^* e^{\mathbb{S}}(0)$.

Proof. For part 1: 0 defines the trivial stable spherical fibration on X, which factors through a point. Therefore the Euler class of 0 is the pullback of the Euler class of the trivial bundle over a point.

For part 2: this follows from the fact that $e^{\mathbb{S}}(W)\colon X^{-W} \to \mathbb{S}$ factors through

$$X^{-W} \xrightarrow{\mathrm{sm}_W} X \xrightarrow{e^{\mathbb{S}}(0)} 0. \qquad \square$$

We immediately learn that Smith maps pull back Euler classes.

Corollary III.52. *Given a virtual vector bundle* V *and a vector bundle* W*, let* sm_W *denote the Smith homomorphism* $\mathrm{sm}_W\colon X^{-V\oplus -W} \to X^{-V}$*. Then*

$$\mathrm{sm}_W^*(e^{\mathbb{S}}(V)) = e^{\mathbb{S}}(V \oplus W). \tag{III.53}$$

We can thus recover the Smith homomorphism from capping with the twisted Euler class.

Proposition III.54. *For any virtual bundle V on X, the Smith map $X^V \to X^{V\oplus W}$ can be defined as the following composition:*

$$X^V \simeq X^{(V\oplus W)\oplus -W} \xrightarrow{\Delta} (X\times X)^{(V\oplus W)\boxplus -W} \simeq X^{V\oplus W} \wedge X^{-W} \xrightarrow{e^{\mathbb{S}}(W)} X^{V\oplus W}. \tag{III.55}$$

The map $X^{(V\oplus W)\oplus -W} \xrightarrow{\Delta} (X\times X)^{(V\oplus W)\boxplus -W}$ is induced by the diagonal map $\Delta\colon X \to X\times X$.

Proof. The Euler map for the trivial rank 0 vector bundle

$$X^0 \simeq \Sigma^\infty_+ X \xrightarrow{e^{\mathbb{S}}(0)} \mathbb{S}. \tag{III.56}$$

is the counit for the $\mathbb{E}_\infty$-coalgebra structure on $\Sigma^\infty_+ X$. By Proposition III.51, the Euler class $e^{\mathbb{S}}(W)$ factors through (III.56) as

$$X^{-W} \longrightarrow X^{-W\oplus W} \simeq \Sigma^\infty_+ X \xrightarrow{e^{\mathbb{S}}(0)} \mathbb{S}. \tag{III.57}$$

This implies that (III.55) can be written as

$$\begin{tikzcd}
X^V \simeq X^{(V\oplus W)\oplus -W} \arrow[r, "\Delta"] \arrow[rrd, swap, "\phi"] & (X\times X)^{(V\oplus W)\boxplus -W} \arrow[r] & (X\times X)^{(V\oplus W)\boxplus 0} \arrow[r, "\simeq"] & X^{V\oplus W}\wedge \Sigma^\infty_+ X \arrow[r, "e^{\mathbb{S}}(W)"] & X^{V\oplus W} \\
 & & X^{V\oplus W} \arrow[u, "\Delta"] & &
\end{tikzcd} \tag{III.58}$$

Since the map $X^{V\oplus W} \to (X\times X)^{(V\oplus W)\boxplus 0} \simeq X^{V\oplus W}\wedge \Sigma^\infty_+ X$ comes from the comodule structure of $X^{V\oplus W}$ over $\Sigma^\infty_+ X$, the composite $X^{V\oplus W} \to (X\times X)^{(V\oplus W)\boxplus 0} \to X^{V\oplus W}$ is the identity map. Therefore it is sufficient to show that the map ϕ in (III.58) is homotopy equivalent to the spectral Smith map sm_W, and this follows by restricting to the diagonal in the map $(X\times X)^{(V\oplus W)\boxplus -W} \to (X\times X)^{(V\oplus W)\boxplus 0}$ along the top of (III.58), which is induced from $\mathrm{id}\boxplus \mathrm{sm}_W$. ☐

We see that the Euler class records all the "Smith" information about W. We will therefore refer to the Smith homomorphism as capping with the Euler class or as the map of Thom spectra interchangeably.

The dual version of Proposition III.54 also holds.

Proposition III.59. *Let $\mathcal{R}$ be a ring spectrum. Then the pullback map on $\mathcal{R}$-cohomology $\mathrm{sm}_W^*\colon \mathcal{R}^*(X^{V\oplus W}) \to \mathcal{R}^*(X^V)$ is equal to the cup product with $e^{\mathcal{R}}(W)$.*

Remark III.60. The symmetry breaking long exact sequence from §II is cohomological in nature: it is given by applying $I_{\mathbb{Z}}MT\xi$-cohomology to sm_W. However, Proposition III.59 does not apply: the Smith homomorphism there cannot be described as taking the product with an $I_{\mathbb{Z}}\mathcal{R}$-Euler class. This is because if $\mathcal{R}$ is a ring spectrum, $I_{\mathbb{Z}}\mathcal{R}$ usually admits no ring spectrum structure. However, $I_{\mathbb{Z}}\mathcal{R}$ is an $\mathcal{R}$-module, so we do learn from Proposition III.59 that this Smith homomorphism is the cup product with $e^{\mathcal{R}}(W)$ using the $\mathcal{R}$-module structure. For example, when we study fermionic invertible phases, we will typically choose $\mathcal{R} = MTSpin$.

Let us review the standard story that "the Euler class is the pullback of the Thom class to the zero section." First we review orientations and Thom classes. For simplicity, we will define them only for vector bundles, though the story generalizes to virtual bundles and much more.

Definition III.61. Let W be a vector bundle of rank r on X. Fix $\mathcal{R}$ an $\mathbb{E}_1$-algebra in spectra and let Mod_R be the ∞-category of $\mathcal{R}$-module spectra. An *$\mathcal{R}$-orientation* of W is a natural isomorphism ϕ of functors between

$$\mathcal{R}^W : X \xrightarrow{W} BO(r) \to \mathsf{Sp} \xrightarrow{-\wedge\mathcal{R}} \mathsf{Mod}_{\mathcal{R}} \tag{III.62}$$

and the constant functor valued in $\Sigma^r\mathcal{R}$. An $\mathcal{R}$-orientation of a manifold M means an $\mathcal{R}$-orientation of TM.

Remark III.63. The map $z^\vee$ from (III.47b) is similar to an orientation on $-W$, in the sense of Ando-Blumberg-Gepner-Hopkins-Rezk, except that $z^\vee$ is in general non-invertible and between different suspensions of the sphere spectrum.

An $\mathcal{R}$-orientation ϕ on W induces an equivalence

$$\mathrm{colim}_X \mathcal{R}^W \simeq \Sigma^\infty_+ \mathrm{Th}(X;W) \wedge \mathcal{R} \simeq X \wedge \Sigma^n\mathcal{R} \simeq \Sigma^n\Sigma^\infty_+ X \wedge \mathcal{R}. \tag{III.64}$$

Definition III.65. The composite

$$U\colon \Sigma^\infty_+\mathrm{Th}(X;W) = X^W \to \Sigma^\infty_+\mathrm{Th}(X;W) \wedge \mathcal{R} \simeq \Sigma^n\Sigma^\infty_+ X \wedge \mathcal{R} \to \Sigma^n\mathcal{R} \tag{III.66}$$

is the *Thom class*. Often we think of U through its homotopy class, which lives in $\mathcal{R}^n(\mathrm{Th}(X;W))$.

Given a $\mathcal{R}$-orientation on W, we can also define the (untwisted) Euler class of W. This is a standard definition (e.g. [Bec70, §13]).

Definition III.67. Given an $\mathcal{R}$-orientation, we have a natural isomorphism of functors $X \to \mathsf{Mod}_R$

$$R^{-W} \simeq \Sigma^{-n}\underline{\mathcal{R}}, \tag{III.68}$$

where $\Sigma^{-n}\underline{\mathcal{R}}$ is the constant functor valued in $\Sigma^{-n}\mathcal{R}$. The composite

$$\Sigma^{-n}X \longrightarrow \Sigma^{-n}X \wedge \mathcal{R} \simeq X^{-W} \wedge \mathcal{R} \xrightarrow{\mathrm{sm}_W} X \wedge \mathcal{R} \to \mathcal{R} \tag{III.69}$$

is called the (untwisted) *Euler class* of W.

Unlike the twisted Euler class, this untwisted Euler class depends on the $\mathcal{R}$-orientation.

Finally, we can prove that our definition of the Euler class, Definition III.50, coincides with the more standard Definition III.67 where they overlap (i.e. when there is an $\mathcal{R}$-orientation chosen on V).

Lemma III.70. *Suppose W is $\mathcal{R}$-oriented, and let $U \in \mathcal{R}^r(\mathrm{Th}(X;W))$ denote the Thom class. Then $e^{\mathcal{W}}(W) = z_W^*U$, where $z_W\colon X \to \mathrm{Th}(X;W)$ is the inclusion as the zero section.*

Proof. After suspending, the zero section map becomes the Smith map. Therefore it suffices to show that the following diagram commutes.

$$\begin{tikzcd}
\Sigma^\infty_+ X \arrow[r, "-\wedge\mathcal{R}"] \arrow[d, "\mathrm{sm}_W"] & \Sigma^\infty_+ X \wedge \mathcal{R} \arrow[r, "\simeq"] \arrow[d, "\mathrm{sm}_W \wedge \mathrm{id}_{\mathcal{R}}"] & \Sigma^n X^{-W} \wedge \mathcal{R} \arrow[d, "\mathrm{sm}_W"] \\
\Sigma^\infty_+ \mathrm{Th}(X; W) \simeq X^W \arrow[r, "-\wedge\mathcal{R}"] & X^W \wedge \mathcal{R} \arrow[r, "\simeq"] & \Sigma^n X \wedge \mathcal{R}.
\end{tikzcd} \tag{III.71}$$

Here the equivalences in the right square are the ones induced by the orientation ϕ.

The left-hand square commutes because smashing with $\mathcal{R}$ is a functor. The right-hand square commutes because the following diagram commutes in $\mathsf{Fun}(X, \mathsf{Mod}_R)$:

$$\begin{tikzcd}
\underline{\mathcal{R}} \arrow[r, "(\phi\wedge\mathcal{R}^{-W})"] \arrow[d, "z^\vee\wedge\mathcal{R}"] & \Sigma^n \mathcal{R}^{-W} \arrow[d, "z^\vee\wedge\mathcal{R}\wedge\mathcal{R}^W"] \\
\mathcal{R}^W \arrow[r, "\phi"] & \Sigma^n \underline{\mathcal{R}},
\end{tikzcd} \tag{III.72}$$

which follows from naturality. Recall that $z^\vee : \underline{\mathcal{R}} \to \mathcal{R}^W$ is the map of spherical fibrations over X that induces the Smith map. ☐

4. *Smith homomorphisms defined via Atiyah-Poincaré dual of the generalized Euler classes*

Now equipped with the theory of Euler classes, we can give another alternate definition of the Smith homomorphism. Fix $\xi \colon B \to B\mathrm{O}$, $V \to X$ of rank r_V, and $W \to X$ of rank r_W as in Definition III.35. Recall that by Corollary III.32, a class $c \in \Omega^\xi_n(X^{V-r_V})$ can be represented by a closed n-manifold M with an (X, V)-twisted ξ-structure, which includes the data of a map $f \colon M \to X$.

In this subsubsection, we assume that $MT\xi$ is a ring spectrum.

Definition III.73. The *Smith homomorphism* associated to ξ, V, and W is the homomorphism

$$\mathrm{sm}_W : \Omega^\xi_n(X^{V-r_V}) \longrightarrow \Omega^\xi_{n-r_W}(X^{V\oplus W-r_V-r_W}) \tag{III.74}$$

sending the class $[M]$ to the Poincaré dual of the cobordism Euler class $e^{MT\xi}(f^*W)$.

We will show this abstractly. But let us first recall Atiyah dualities. There is the standard notion of duals in any symmetric monoidal category C [Lin78, DP80, DM82]. Here for C we take the homotopy category of spectra, which is monoidal with respect to the smash product $\wedge$. If A, B have duals $A^\vee, B^\vee$, then a morphism $f \colon A \to B$ induces a dual morphism, which we write $f^\vee \colon B^\vee \to A^\vee$.

Theorem III.75 (Atiyah duality [Ati61b, Proposition 3.2 and Theorem 3.3])**.** *Let M be a compact manifold; then $(M/\partial M)^\vee \simeq M^{-TM}$. If M is closed and $V \to M$ is a virtual vector bundle, then $(M^V)^\vee \simeq M^{-TM-V}$.*

Furthermore, dual spectra provide isomorphisms between homology and cohomology groups: let X be a spectrum with a dual $X^\vee$; then, for any spectrum $\mathcal{R}$, we have a canonical isomorphism

$$\mathcal{R}_*(X) \xrightarrow{\cong} \mathcal{R}^{-*}(X^\vee). \tag{III.76}$$

We call two classes $\alpha \in \mathcal{R}_*(X)$ and $\beta \in \mathcal{R}^{-*}(X^\vee)$ *Atiyah-Poincaré dual* if $\alpha \mapsto \beta$ under the isomorphism (III.76).

Furthermore, this is functorial: given a map $f\colon X \to Y$ of dualizable spectra, let $f^\vee\colon Y^\vee \to X^\vee$ be the dual map. We have a commutative square:

$$\begin{tikzcd}
\mathcal{R}_*(X) \arrow[r, "f_*"] \arrow[d, "\simeq"] & \mathcal{R}_*(Y). \arrow[d, "\simeq"] \\
\mathcal{R}^{-*}(X^\vee) \arrow[r, "(f^\vee)^*"] & \mathcal{R}^{-*}(Y^\vee).
\end{tikzcd} \tag{III.77}$$

Let $\Omega_*^{\mathrm{fr}}(X)$ denote the stably framed bordism of X, i.e. the bordism groups of manifolds with a map to X and a trivialization of the stable tangent bundle (or equivalently, the stable normal bundle). The Pontrjagin-Thom theorem identifies these bordism groups with the stable homotopy groups of X. We learn a neat fact:

Lemma III.78. *Let M be a closed compact d-dimensional manifold. Then M defines a canonical class in $\Omega_d^{\mathrm{fr}}(M, -TM) = \mathbb{S}_0(M^{-TM})$. This is the Atiyah-Poincaré dual to the Euler class for the trivial bundle $e^{\mathbb{S}}(0) \in \mathbb{S}^0(M)$.*

Proof. The Euler class is represented by $e\colon M_+ \to S^0$ by taking $+$ to the basepoint of S^0 and the entire M to the other point. On the other hand, given an embedding $\iota\colon M \to \mathbb{R}^N$, let ν be the normal bundle. Then $\Sigma_+^\infty \mathrm{Th}(M;\nu) \simeq \Sigma^{-N} M^{-TM}$. By the Pontraygin-Thom construction, the tautological class $[M] \in \Omega_d^{\mathrm{fr}}(M, -TM)$ comes from the Pontrjagin collapse map $S^N = (\mathbb{R}^N)^+ \to \mathrm{Th}(M;\nu)$, where $(-)^+$ is the one point compactification.

The result follows from the finite-dimensional description of the evaluation and co-evaluation map of M and M^{-TM} [Ati61a]: we have an evaluation map $S^N \to M_+ \wedge \mathrm{Th}(M;\nu)$, representing $\mathbb{S} \to M \wedge M^{-TM}$. The composite $S^N \to M_+ \wedge \mathrm{Th}(M;\nu) \xrightarrow{e} S^0 \wedge \mathrm{Th}(M;\nu) = \mathrm{Th}(M;\nu)$ is precisely the Pontrjagin-Thom collapse map. ☐

Now we see how Atiyah duality interacts with Smith homomorphisms on compact manifolds:

Lemma III.79. *Fix a closed compact manifold M. Given a virtual bundle $V \to M$, and a vector bundle $W \to M$, then the Atiyah dual $(sm_W)^\vee$ of the Smith map*

$$sm_W\colon M^V \longrightarrow M^{V\oplus W} \tag{III.80}$$

is the Smith map associated to $-TM - V - W$:

$$sm_W\colon M^{-TM-V-W} \longrightarrow M^{-TM-V}. \tag{III.81}$$

Proof. Let us do the case $V = 0$; the general case follows in the same way. First we give a space-level description of the Atiyah dual map. Consider the manifold with boundary $D_M(W)$, the disc bundle of W. Its tangent bundle is $T(D_M(W)) \cong TM \oplus W$, where we are implicitly pulling back W to $D_M(W)$. Now consider an embedding $\mu_D\colon D_M(W) \to \mathbb{R}^N$. Then M, sitting as the zero section, also gets an embedding $\mu_M\colon M \to D_M(W) \to \mathbb{R}^N$.

Let ν_D, resp. ν_M be the normal bundle of μ_D. resp. μ_M. As virtual bundles,

$$\nu_D \cong \mathbb{R}^N - TM - W \tag{III.82a}$$
$$\nu_M \cong \mathbb{R}^N - TM. \tag{III.82b}$$

Note that $\nu_M = \nu_D \oplus W$. Now let $N_D(\mu)$ be a tubular neighborhood of $D_M(W)$ and $N_M(\mu)$ the same for M. $N_D(\mu)$ and $N_M(\mu)$ are diffeomorphic to μ_D, resp. μ_M.

Using the standard Pontrjagin-Thom collapse argument, the open embedding $i\colon N_M(\mu) \to N_D(\mu)$ induces a map of one-point compactifications $i^+\colon N_D(\mu)^+ \to N_M(\mu)^+$. By Proposition III.9, we can write this as $\mathrm{Th}(D_M(W); \nu_D) \simeq \mathrm{Th}(M; \nu_D) \to \mathrm{Th}(M; \nu_D)$. Recall that $D_M(W)$ is homotopically equivalent to W.

After suspending to spectra, Equation (III.82) gives a map

$$\Sigma^n M^{-TM-W} \longrightarrow \Sigma^n M^{-TM}. \tag{III.83}$$

This is the Atiyah dual map of the Smith map.

To show this is the Smith map for $-TM - V - W$ as claimed, notice that the composite $\mathrm{Th}(M; \nu_D) \to \mathrm{Th}(D_M(W); \nu_D) \to \mathrm{Th}(M; \nu_D \oplus W)$ is induced by the inclusion of disk bundles, a.k.a. the Smith homomorphism on Thom spaces, which suspends to the Smith map on Thom spectra. □

Lemma III.84. *Let W be a rank r_W vector bundle on a closed compact d-manifold M, and let $[M] \in \Omega_d^{\mathrm{fr}}(M, -TM)$ be the tautological class. Then $(\mathrm{sm}_W)_*([M]) \in \Omega_d^{\mathrm{fr}}(M, -TM + W) = \mathbb{S}_0(M^{-TM+W})$ is the Atiyah-Poincaré dual of the Euler class $e^{\mathbb{S}}(W) \in \Omega_{\mathrm{fr}}^{rw-d}(M, -W)$.*

Proof. By Equation (III.77), $\mathrm{sm}_{W*}([M])$ is Atiyah-Poincaré dual to $((\mathrm{sm}_W)^\vee)^*(e^{\mathbb{S}}(0))$, where 0 denotes the zero vector bundle. By Lemma III.79, $(\mathrm{sm}_W)^\vee$ is still sm_W. By Proposition III.51, $(\mathrm{sm}_W)^*(e^{\mathbb{S}}(0))$ is $e^{\mathbb{S}}(W)$. □

Now we can collect our prize: we show that Definitions III.44 and III.73 are equivalent definitions of the Smith homomorphism. In other words, the Smith homomorphism as we first defined it is the same as the map taking the Poincaré dual of the Euler class, as it is often described in the literature.

Corollary III.85. *Let $V \to X$ be a virtual vector bundle and $W \to X$ be a rank r_W vector bundle. Choose a bordism class in $\Omega_d^{\mathrm{fr}}(X, V)$ (i.e. (X, V)-twisted framed bordism) and let M be a closed manifold representative of that class. Let $[N] \in \Omega_d^{\mathrm{fr}}(M, -TM \oplus W)$ be the Atiyah-Poincaré dual of the Euler class $e^{\mathbb{S}}(W|_M)$. Then the image of $[N]$ in $\Omega_d^{\mathrm{fr}}(X, V \oplus W)$ is $\mathrm{sm}_W([M])$.*

Proof. Since M has a (X, V)-twisted framing, the map $M \to X$ Thomifies to a map $f\colon M^{-TM} \to X^V$. The Smith map is functorial, so we get a commutative square:

$$\begin{tikzcd}
M^{-TM} \arrow[r, "f"] \arrow[d, "\mathrm{sm}_{W|_M}"] & X^V \arrow[d, "\mathrm{sm}_W"] \\
M^{-TM+W} \arrow[r, "f"] & X^{V\oplus W}.
\end{tikzcd} \tag{III.86}$$

Furthermore, $[M] \in \Omega_d^{\mathrm{fr}}(X, V)$ is the f-pushforward of the tautological class in $\Omega_d^{\mathrm{fr}}(M, -TM)$. The result now follows from Lemma III.84. □

Remark III.87. This tells us that given a bordism class represented by M, $\mathrm{sm}_W([M])$ is represented by a manifold N that is Atiyah-Poincaré dual (in the bordism homology theory) to the twisted cobordism Euler class of M.

5. *Smith fiber sequence*

In this section we extend the Smith map into a fiber sequence, which allows us to derive a long exact sequence of bordism groups and, dually, the long exact sequence of field theories in Section II. The Smith homomorphism defines only the defect map Def_ρ from §II B by Anderson-dualizing—it is what we do in this subsection that allows us to produce Res_ρ (§II A) and Ind_ρ (§II C).

For any vector bundle $E \to X$ of rank r, let E also denote the classifying map $X \to B\mathrm{O}(r)$. Which of these two things we mean by E will be clear from context.

In this section, we will write $S_X(E)$ and $D_X(E)$ for the sphere, resp. disc bundles of E, because there will be places where it will help to remember which base space we work over. Finally,

Theorem III.88. *Let V, W be real vector bundles over X. Then there is a cofiber sequence in pointed spaces:*

$$S_X(W)^V \to X^V \to X^{W\oplus V}. \tag{III.89}$$

Similarly, if V is a virtual bundle, we have a (co)fiber sequence in spectra:

$$S_X(W)^V \to X^V \to X^{V\oplus W}. \tag{III.90}$$

Proof. We will do the case where V is an actual vector bundle; the virtual bundle case is analogous. Given an r-dimensional vector space W, we have a cofiber sequence in pointed spaces:

$$S(W)_+ \to D(W)_+ \simeq S^0 \to S^W. \tag{III.91}$$

Now since $\mathrm{Aut}(W) \cong \mathrm{O}(r)$ acts on W, we can upgrade (III.91) to a cofiber sequence of spaces with $\mathrm{O}(r)$-actions; equivalently, (III.91) is a cofiber sequence of functors $B\mathrm{O}(r) \to \mathsf{Top}_*$. Pulling back to X via the map $X \to B\mathrm{O}(r)$ classifying W, we get a cofiber sequence of functors $X \to \mathsf{Top}_*$. Now smash with S^V: we get a cofiber sequence of the form

$$S(W)_+ \wedge S^V \to D(W)_+ \wedge S^V \to S^W \wedge S^V \simeq S^{V\oplus W}. \tag{III.92}$$

This cofiber sequence is in the category of functors $X \to \mathsf{Top}_*$.

Since taking the colimit over X preserves cofiber sequences, it is sufficient to show that the colimit of (III.92) over X is

$$S_X(W)^V \longrightarrow X^V \longrightarrow X^{V\oplus W}. \tag{III.93}$$

For the last term $S^{V\oplus W}$ in (III.92), this follows directly from the definition of the Thom spectrum (Definition III.14).

For $\mathrm{colim}_X(D(W)_+ \wedge S^V)$, note that $D(W)_+ \simeq S^0$, so $D(W)_+ \wedge S^V \simeq S^V$, so Definition III.14 once again tells us the colimit is X^V. It also follows that the map $X^V \to X^{V\oplus W}$ on colimits is the Smith map.

Lastly, the colimit of $S(W)_+$ over X is the associated sphere bundle $S_X(W)$. It follows that the colimit of $S(W)_+ \wedge S^V$ over X is equivalent to the colimit of (the pullback of) S^V over $S_X(W)$, which is $S_X(W)^V$. □

Remark III.94. Everything here is functorial, so given a map $Y \to X$, then we get maps between cofiber sequences and therefore a map of long exact sequences of homotopy groups.

Corollary III.95. *Applying π_* to the fiber sequence, we get a long exact sequence of bordism groups:*

$$\cdots \longrightarrow \Omega_k^\xi(S_X(W)^V) \longrightarrow \Omega_k^\xi(X^V) \longrightarrow \Omega_{k-r}^\xi(X^{V+W-r}) \longrightarrow \Omega_{k-1}^\xi(S_X(W)^V) \longrightarrow \cdots \quad \text{(III.96)}$$

Though written as bordism groups of Thom spectra, these are also twisted ξ-bordism groups thanks to Corollary III.32. We work through an explicit example long exact sequence on the level of manifold generators in Appendix A.

Corollary III.97. *Applying $I_\mathbb{Z}$ to the cofiber sequence* (III.90)*, we obtain the following long exact sequence of Anderson-dualized bordism groups, or in light of Theorem III.26, groups of invertible field theories:*

$$\cdots \longrightarrow \Omega_\xi^{k-r}(X^{V+W-r}) \xrightarrow{\alpha} \Omega_\xi^k(X^V) \xrightarrow{\beta} \Omega_\xi^k(S_X(W)^V) \xrightarrow{\gamma} \Omega_\xi^{k-r+1}(X^{V+W-r}) \longrightarrow \cdots \quad \text{(III.98)}$$

The long exact sequence (III.98) is our mathematical model for the symmetry-breaking long exact sequence from §II. Specifically, α corresponds to Def_ρ, β to Res_ρ, and γ to Ind_ρ.

Remark III.99. Suppose $X = BG$ for a compact Lie group G and $W \to X$ is the vector bundle associated to an orthogonal representation of G, such that G acts transitively on the unit sphere in G. Then the sphere bundle has a particularly simple form: if G_v is the stabilizer subgroup for a point $v \in S(W)$, then the bundle map $S_X(W) \to X$ is homotopy equivalent to the map $BG_v \to BG$ induced by the inclusion $G_v \hookrightarrow G$. Thus the obstruction for an invertible field theory to be in the image of the Anderson-dualized Smith homomorphism is its restriction from manifolds with G-bundles (and some sort of tangential structure) to manifolds with G_v-bundles (and the corresponding tangential structure).

There is a special case where $S_X(V)$ is particularly simple. Let $X = BG$ and V a G representation where G acts transitively on the sphere. Then $S_X(V) \simeq BG_v$ where G_v is the fix-point subgroup of a point v on the sphere. Therefore the anomaly obstruction lives in BG_v^ξ, where by ξ we the tangential structure restricted to H. Physically, this means that if G acts transitively on the sphere of the representation, then the symmetry breaking anomaly obstruction is equivalent to the vanishing of the G_v anomaly. This is because if we can gap the theory while preserving the G_v symmetry, then we can use the G symmetry to gap it on the whole sphere. See the dicussion at the end of Section II A.

Remark III.100 (Smith and Gysin long exact sequences). The reader looking at the type signatures of (III.96) and (III.98) might notice that they resemble Gysin sequences: long exact sequences involving (co)homology groups of the base space and total space of a sphere bundle, especially because one of the maps can be interpreted as a product with an Euler class. And indeed, if one takes ordinary homology or cohomology, the Smith long exact sequence becomes the Gysin long exact sequence, as can be verified by comparing the three maps in the long exact sequence.

Thus, the Smith long exact sequence can be thought of as the generalization of the Gysin long exact sequence to arbitrary vector bundle twists of generalized cohomology theories.

D. Periodicity of twists and shearing

The goal of this subsection is to provide tools for working with twists of tangential structures. We are interested in collections of similar twists over the same base space; this provides an organizing principle for different Smith homomorphisms that we will use many times in the next subsection.

Definition III.101. Fix a space X, a virtual vector bundle $V \to X$ of rank r_V, a vector bundle $W \to X$ of rank r_W, and a tangential structure ξ. The *family of Smith homomorphisms* associated to this data is the set of Smith homomorphisms

$$\mathrm{sm}_W \colon \Omega_n^\xi(X^{V-r_V+k(W-r_W)}) \longrightarrow \Omega_{n-r_W}^\xi(X^{V-r_V+(k+1)(W-r_W)}) \tag{III.102}$$

for $k \in \mathbb{Z}$, i.e. the Smith homomorphisms from $(X, V \oplus kW)$-twisted ξ-bordism to $(X, V \oplus (k+1)W)$-twisted ξ-bordism.

If there is some $\ell \in \mathbb{Z}$ and an identification of $(X, V \oplus kW)$-twisted ξ-structures with $(X, V \oplus (k+\ell)W)$-twisted tangential structures for all k that commutes with the Smith homomorphisms (III.102), we say this Smith family is *periodic* with period the smallest positive such ℓ.

This definition may seem too specific to be very applicable, but we will soon see many examples of periodic families.

The main new result in this section is Proposition III.108, providing a way to calculate the periodicity of a family of Smith homomorphisms. We also review the theory of shearing in and around Lemma III.115, which is a convenient way to split the Thom spectra for a wide class of twisted bordism theories, and is an essential step in identifying the terms in Smith long exact sequences. Our perspective on shearing follows [DY23a, §1], so see there for some more details; see also [FH21, Bea17, Ste22, DDHM23] for additional approaches.

Definition III.103. Let $\xi \colon B \to B\mathrm{O}$ be a tangential structure. *Two-out-of-three data* for ξ is the data of:

- for each pair of ξ-structured virtual vector bundles $V, W \to X$, a natural ξ-structure on $V \oplus W$; and
- for each ξ-structured virtual vector bundle $V \to X$, a natural ξ-structure on $-V \to X$.

The reason for this name is that, given this data, a ξ-structure on any two of V, W, and $V \oplus W$ induces a ξ-structure on the third. Unfortunately, this is sometimes called a “two-out-of-three property.”

Example III.104. The tangential structures O, SO, Spin^c, Spin, String, U, SU, and Sp all have two-out-of-three data. Pin$^\pm$ and Pin^c do not.

If M and N are manifolds, $T(M \times N) \cong p_1^*(TM) \oplus p_2^*(TN)$, where p_1 and p_2 are the projections of $M \times N$ onto M, resp. N, so two-out-of-three data induces a ring structure on Ω_*^ξ given by the direct product of manifolds. More abstractly, this data makes B into a grouplike E_∞-space and ξ into an E_∞-map, where $B\mathrm{O}$ has the direct sum E_∞-structure. This implies by work of Ando-Blumberg-Gepner [ABG18, Theorem 1.7] that $MT\xi$ is an E_∞-ring spectrum.

For R an E_∞-ring spectrum, May [May77, §III.2] defines a grouplike E_∞-space $\mathrm{GL}_1(R)$, and Ando-Blumberg-Gepner-Hopkins-Rezk [ABG$^+$14a, ABG$^+$14b] associate to a map $f\colon X \to B\mathrm{GL}_1(R)$, which we call a *twist* of R, a Thom spectrum $Mf \in \mathsf{Mod}_R$. The f-twisted R-homology groups of X are by definition the homotopy groups of Mf [ABG$^+$14a, Definition 2.27]. Homotopy-equivalent twists induce equivalent Thom spectra. All of this generalizes our discussion around Definition III.14, for which $R = \mathbb{S}$.

Example III.105 (Vector bundle twists)**.** We have been using (rank-zero virtual) vector bundles to define twists of bordism theories, and these two notions of twist are compatible: rank-zero virtual vector bundles $V \to X$ are classified by maps $f_V\colon X \to B\mathrm{O}$, and the J-homomorphism is a map $B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S})$; then, if ξ is a tangential structure with two-out-of-three data, the unit map $e\colon \mathbb{S} \to MT\xi$ induces a map $e\colon B\mathrm{GL}_1(\mathbb{S}) \to B\mathrm{GL}_1(MT\xi)$. The Thom spectrum for (X, V)-twisted ξ-bordism, as we characterized it in Corollary III.32, is naturally equivalent to the Thom spectrum $M(e \circ J \circ f_V)$ of the map

$$X \xrightarrow{f_V} B\mathrm{O} \xrightarrow{J} B\mathrm{GL}_1(\mathbb{S}) \xrightarrow{e} B\mathrm{GL}_1(MT\xi). \tag{III.106}$$

This is a combination of theorems of Lewis [LMSM86, Chapter IX] and Ando-Blumberg-Gepner-Hopkins-Rezk (see [ABG$^+$14a, Corollary 3.24] and [ABG$^+$14b, §1.2]).

Beardsley [Bea17, Theorem 1] established a canonical null-homotopy of the map

$$e \circ J \circ \xi\colon B \to B\mathrm{GL}_1(MT\xi), \tag{III.107}$$

so $e \circ J$ factors through the cofiber $B\mathrm{O}/B$.[14] In other words, the homotopy type of the Thom spectrum for (X, V)-twisted ξ-bordism depends only on the image of V in $B\mathrm{O}/B$. And the key slogan is that the orders of elements in $[X, B\mathrm{O}/B]$ control the periodicity of families of Smith homomorphisms for twisted ξ-bordism; the group structure on $[X, B\mathrm{O}/B]$ uses the fact that $B\mathrm{O}/B$ is the cofiber of a map of grouplike E_∞-spaces, hence is also a grouplike E_∞-space, so homotopy classes of maps into $B\mathrm{O}/B$ naturally form abelian groups.

Proposition III.108. *Let $V \to X$ be a vector bundle and ϵ be the order of the image of $V - \mathrm{rank}(V)$, interpreted as an element of the abelian group $[X, B\mathrm{O}]$, under the homomorphism $[X, B\mathrm{O}] \to [X, B\mathrm{O}/B]$. If ϵ is finite, the Smith homomorphism family of (X, kV)-twisted ξ-bordism is ϵ-periodic.*

Proof. The image $\overline{f_V}$ of the classifying map $f_V\colon X \to B\mathrm{O}$ in $[X, B\mathrm{O}/B]$ satisfies $(k+\epsilon)\overline{f_V} = k\overline{f_V}$. Since the homotopy type of the Thom spectrum for (X, W)-twisted ξ-bordism only depends on $\overline{f_W} \in [X, B\mathrm{O}/B]$, this implies that the notions of (X, kV)-twisted ξ-bordism and $(X, (k+\epsilon)V)$-twisted spin bordism coincide, so the Smith family $\{(X, kV) : k \in \mathbb{Z}\}$ is ϵ-periodic. ☐

Though Proposition III.108 seems abstract, it lends itself readily to examples.

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[14] Beardsley's proof is more abstract, more general, and more powerful than this statement: see [DY23a, Lemma 1.13] for a simpler proof of just this part of Beardsley's theorem.

Example III.109 (Unoriented bordism families are 1-periodic)**.** Proposition III.108 implies that when $\xi = \mathrm{id}\colon B\mathrm{O} \to B\mathrm{O}$, the periodicity of a Smith family of (X, kV)-twisted unoriented bordism divides the exponent of $[X, B\mathrm{O}/B\mathrm{O}] = 0$. In other words, all Smith families of twisted unoriented bordism are 1-periodic.

We will see some examples of Smith families for unoriented bordism in Examples III.127, III.144, and III.161.

Example III.110 (Oriented bordism families are 2-periodic)**.** Because $B\mathrm{SO}$ is the fiber of $w_1\colon B\mathrm{O} \to K(\mathbb{Z}/2, 1)$, and the Whitney sum formula implies w_1 is a map of E_∞-spaces, the cofiber $B\mathrm{O}/B\mathrm{SO}$ is equivalent to $K(\mathbb{Z}/2, 1)$ as grouplike E_∞-spaces. Thus for all spaces X, $[X, B\mathrm{O}/B\mathrm{SO}]$ is annihilated by 2, so all Smith families for twisted oriented bordism are 2-periodic (or 1-periodic).

We will see some examples of Smith families for oriented bordism in Examples III.129, III.136, III.144, III.157, and III.161.

Example III.111 (Complex and spin^c bordism families are 2-periodic)**.** If V is a real vector bundle, then $V \oplus V$ has a canonical complex structure (think of this bundle as $V \oplus iV$), and therefore also a canonical spin^c structure. Therefore for any map $f\colon X \to B\mathrm{O}$, $2f$ lifts to $B\mathrm{U}$ and to $B\mathrm{Spin}^c$. Therefore the image of the map $[X, B\mathrm{O}] \to [X, B\mathrm{O}/B\mathrm{U}]$ has exponent 2 (and likewise for Spin^c), so by Proposition III.108 all Smith families of complex and spin^c bordism are at most 2-periodic.

For examples of Smith families for spin^c bordism, see Examples III.134, III.144, and III.161 and Footnote 22.

Example III.112 (Spin bordism families are 4-periodic)**.** $B\mathrm{O}/B\mathrm{Spin}$ is not equivalent to a product of Eilenberg-Mac Lane spaces even as an E_1-space [DY23a, Lemma 1.37], so we cannot reuse the strategy of (III.110). However, there is a cofiber sequence of grouplike E_∞-spaces (heuristically, an extension of abelian ∞-groups) [DY23a, §1.2.3]

$$K(\mathbb{Z}/2, 2) \longrightarrow B\mathrm{Spin}/B\mathrm{O} \longrightarrow K(\mathbb{Z}/2, 1), \tag{III.113}$$

inducing a long exact sequence on $[X, –]$. Since $[X, K(\mathbb{Z}/2, 2)]$ and $[X, K(\mathbb{Z}/2, 1)]$ both have exponent at most 2 for any X, exactness implies $[X, B\mathrm{O}/B\mathrm{Spin}]$ has exponent at most 4. Thus using Proposition III.108 we conclude that all twisted spin bordism Smith families are at most 4-periodic; in fact, Example III.131 has period exactly 4, which implies (III.113) does not split. One could also argue 4-periodicity similarly to Example III.111.

If we restrict to oriented vector bundles, we can do better, as periodicity is controlled by maps into $B\mathrm{SO}/B\mathrm{Spin} \simeq K(\mathbb{Z}/2, 2)$ (the argument is similar to $B\mathrm{O}/B\mathrm{SO} \simeq K(\mathbb{Z}/2, 1)$ from Example III.110). Therefore we conclude that twisted spin Smith families using an oriented vector bundle are 2-periodic.

We will discuss several examples of 1-, 2-, and 4-periodic Smith families for spin bordism in Examples III.131, III.148, III.155, III.157, III.161, and III.164.

Example III.114 (String families need not be periodic)**.** As grouplike E_∞-spaces, $B\mathrm{O}/B\mathrm{String}$ is an extension of $B\mathrm{O}/B\mathrm{Spin}$ by $B\mathrm{Spin}/B\mathrm{String} \simeq K(\mathbb{Z}, 4)$ (see [DY23a, §1.2.4]); since $B\mathrm{O}/B\mathrm{Spin}$ is itself an extension of $K(\mathbb{Z}/2, 1)$ by $K(\mathbb{Z}/2, 2)$, if X is a 3-connected space, $[X, B\mathrm{O}/B\mathrm{String}] \cong H^4(X; \mathbb{Z})$. Thus for a general space X, Smith families for twisted string bordism do not have finite period.

In special cases, though, there can still be periodicity results: for example, because $H^*(B\mathbb{Z}/2;\mathbb{Z})$ is 2-torsion in positive degrees and $[B\mathbb{Z}/2, B\mathrm{O}/B\mathrm{Spin}]$ has exponent 4, the long exact sequence associated to the cofiber sequence $K(\mathbb{Z},4) \to B\mathrm{O}/B\mathrm{String} \to B\mathrm{O}/B\mathrm{Spin}$ implies $[B\mathbb{Z}/2, B\mathrm{O}/B\mathrm{String}]$ has exponent at most 8, implying that all Smith families of $(B\mathbb{Z}/2, V)$-twisted string bordism are at most 8-periodic; an 8-periodic example appears in Example III.133.

In the rest of this subsection, we discuss how to use this perspective to concretely identify examples of twists of ξ-bordism for the tangential structures SO, Spin^c, Spin, and String.

Lemma III.115 (Shearing [ABG$^+$14b, §1.2])**.** *If a twist $f\colon X \to B\mathrm{GL}_1(M\xi)$ factors through a map $g_V\colon X \to B\mathrm{O}$ classifying a rank-zero virtual vector bundle $V \to X$ as in* (III.106)*, then $Mf \simeq MT\xi \wedge X^V$.*

We will use this lemma as follows: first, for the four tangential structures $\xi\colon BG \to B\mathrm{O}$ mentioned above, we compute the homotopy type of $B\mathrm{O}/BG$ and understand the map $B\mathrm{O} \to B\mathrm{O}/BG$, to recognize when a map $X \to B\mathrm{O}/BG$ comes from a (virtual rank-zero) vector bundle $V \to X$. In that situation, Lemma III.115 describes the corresponding twisted ξ-bordism groups as $\Omega_*^\xi(X^V)$, so we can use the Smith homomorphism tools we developed in this paper.

Example III.116 (Twists of oriented bordism)**.** Recall from Example III.110 that $B\mathrm{O}/B\mathrm{SO} \simeq K(\mathbb{Z}/2,1)$; the argument there implies the map $B\mathrm{O} \to B\mathrm{O}/B\mathrm{SO} \xrightarrow{\simeq} K(\mathbb{Z}/2,1)$ is the first Stiefel-Whitney class. Given a map $a\colon X \to B\mathrm{O}/B\mathrm{SO}$, the Thom spectrum of the corresponding twist $f_a\colon X \to B\mathrm{GL}_1(\mathit{MTSO})$ of MTSO is the bordism spectrum whose homotopy groups are the bordism groups of manifolds M with a map $\phi\colon M \to X$ and a trivialization of $w_1(M) - \phi^*(a)$.[15]

Every class $a \in H^1(X;\mathbb{Z}/2)$ is the first Stiefel-Whitney class of some line bundle $L_a \to X$, so for any twist $f\colon X \to B\mathrm{GL}_1(\mathit{MTSO})$ described by a map $f_a\colon X \xrightarrow{a} K(\mathbb{Z}/2,1) \simeq B\mathrm{O}/B\mathrm{SO} \to B\mathrm{GL}_1(\mathit{MTSO})$, there is a homotopy equivalence

$$Mf \xrightarrow{\simeq} \mathit{MTSO} \wedge X^{L_a - 1}. \tag{III.117}$$

For example, unoriented bordism is an example of such a twist: every manifold M has a canonical map to $K(\mathbb{Z}/2,1)$, given by $w_1(M)$, and $w_1(M) - w_1(M)$ has a canonical trivialization. Therefore unoriented bordism is twisted oriented bordism for the twist $K(\mathbb{Z}/2,1) \xrightarrow{\simeq} B\mathrm{O}/B\mathrm{SO}$, and Lemma III.115 implies $\mathit{MTO} \simeq \mathit{MTSO} \wedge (K(\mathbb{Z}/2,1))^{\sigma-1}$, where $\sigma \to B\mathbb{Z}/2 \simeq K(\mathbb{Z}/2,1)$ is the tautological line bundle; this is a theorem of Atiyah [Ati61a, Proposition 4.1].

For another example of how to use Lemma III.115, let $\mathcal{W}$ denote the Thom spectrum for the notion of bordism of manifolds M equipped with a lift of $w_1(M)$ to a class $\alpha \in H^1(M;\mathbb{Z})$. The class α is equivalent to a map $\phi\colon M \to B\mathbb{Z} = S^1$, and $\alpha = \phi^*x$, where $x \in H^1(S^1;\mathbb{Z})$ is the generator; rephrased in this way, the condition that $\alpha \bmod 2 = w_1(M)$ is equivalent to a trivialization of $w_1(M) - \phi^*(x \bmod 2)$. Therefore $\mathcal{W}$-bordism is twisted oriented bordism for $(S^1, x \bmod 2)$, and as $x \bmod 2$ is w_1 of the Möbius bundle $\sigma \to S^1$, we learn from Lemma III.115 that $\mathcal{W} \simeq \mathit{MTSO} \wedge (S^1)^{\sigma-1}$. This is also due to Atiyah [Ati61a, §4].

Example III.118 (Twists of spin^c bordism)**.** There is an equivalence of spaces, but not E_1-spaces, $B\mathrm{O}/B\mathrm{Spin}^c \simeq K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3)$ [DY23a, Proposition 1.20, Lemma 1.30], and the map

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[15] Strictly speaking, what one trivializes is $w_1(\nu) - \phi^*(a)$, where $\nu \to M$ is the stable normal bundle, but there is a canonical identification of $w_1(M)$ and $w_1(\nu)$. This nuance will matter for spin structures.

$B\mathrm{O} \to B\mathrm{O}/B\mathrm{Spin}^c$ is picked out by $(w_1, \beta(w_2))$, where β is the integral Bockstein. The fact that $\beta(w_2)$ is not linear in the direct sum of vector bundles is why this decomposition of $B\mathrm{O}/B\mathrm{Spin}^c$ does not respect the E_1-structure.

Given data $a \in H^1(X;\mathbb{Z}/2)$ and $c \in H^3(X;\mathbb{Z})$, if $Mf_{a,c}$ is the Thom spectrum for the corresponding twist

$$f_{a,c}\colon X \xrightarrow{(a,c)} K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3) \simeq B\mathrm{O}/B\mathrm{Spin}^c \longrightarrow B\mathrm{GL}_1(\mathit{MTSpin}^c), \tag{III.119}$$

then the homotopy groups of $Mf_{a,c}$ are the bordism groups of manifolds M with maps $\phi\colon M \to X$ and trivializations of $w_1(M) - \phi^*(a)$ and $\beta(w_2(M)) - \phi^*(c)$; the proof is essentially the same as Hebestreit-Joachim's [HJ20, Corollary 3.3.8] (Footnote 15 still applies: what appears is the stable normal bundle, but the characteristic classes are the same). If there is a (rank-zero, virtual) vector bundle $V \to X$ with $w_1(V) = a$ and $\beta(w_2(V)) = c$, then Lemma III.115 implies $Mf_{a,c} \simeq \mathit{MTSpin}^c \wedge X^V$ and we can invoke the Smith homomorphism on V.

For example, a pin^c structure on a manifold M is a trivialization of $\beta(w_2(M))$ (i.e. the spin^c condition without the trivialization of w_1). Thus a pin^c structure is equivalent to a twisted spin^c structure where $X = B\mathbb{Z}/2$, a is the generator of $H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$, and $c = 0$: as in Example III.116, $w_1(M)$ gives us a canonical map to $B\mathbb{Z}/2$, and there is a canonical trivialization of $w_1(M) - w_1(M)$; and $c = 0$ means this twisted spin^c condition does not modify $\beta(w_2)$. So this twisted spin^c condition is $\beta(w_2) = 0$ and w_1 is arbitrary, i.e. a pin^c structure. And if $\sigma \to B\mathbb{Z}/2$ is the tautological line bundle, $w_1(\sigma) = a$ and $\beta(w_2(\sigma)) = 0 = c$, so Lemma III.115 implies $\mathit{MTPin}^c \simeq \mathit{MTSpin}^c \wedge (B\mathbb{Z}/2)^{\sigma-1}$, reproving a theorem of Bahri-Gilkey [BG87a, BG87b].

Other examples of twists of spin^c bordism which can be realized by vector bundles include the spin-U(2) bordism of Davighi-Lohitsiri [DL20, DL21] and the tangential structure corresponding to Stehouwer's alternate class AI fermionic groups [Ste22, §2.2].

Not every choice of (a, c) can be realized by a vector bundle; for example, $\beta(w_2)$ is always 2-torsion, but c need not be. There are also examples with 2-torsion c, as a consequence of work of Gunawardena-Kahn-Thomas [GKT89, §2].

Example III.120 (Twists of spin bordism)**.** The most commonly studied examples of twisted ξ-bordism in mathematical physics are twists of spin bordism. The story is closely analogous to Example III.118, with $K(\mathbb{Z},3)$ replaced with $K(\mathbb{Z}/2,2)$, and the map $B\mathrm{O} \to B\mathrm{O}/B\mathrm{Spin} \simeq K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)$ is (w_1, w_2). Given classes $a \in H^1(X;\mathbb{Z}/2)$ and $b \in H^2(X;\mathbb{Z}/2)$, the homotopy groups of the Thom spectrum of the corresponding twist $f_{a,b}\colon X \to B\mathrm{GL}_1(\mathit{MTSpin})$ are the bordism groups of manifolds M with maps $\phi\colon M \to X$ and trivializations of $w_1(\nu) - \phi^*(a)$ and $w_2(\nu) - \phi^*(b)$ [HJ20, Corollary 3.3.8], where $\nu \to M$ is the stable normal bundle. Now, unlike in Footnote 15, the distinction between TM and ν matters: $w_1(TM) = w_1(\nu)$, but $w_2(TM) + w_1(TM)^2 = w_2(\nu)$, providing a formula for the nontrivial transition from tangential to normal data. If $a = w_1(V)$ and $b = w_2(V)$ for a rank-zero virtual vector bundle $V \to X$, Lemma III.115 implies $Mf_{a,b} \simeq \mathit{MTSpin} \wedge X^V$. See [DY23a, §1.2.3] for more information.

Many commonly studied tangential structures arise as vector bundle twists of spin structures.

1. A pin$^-$ structure is a trivialization of $w_2(M) + w_1(M)^2$, with no condition on w_1. Thus this is equivalent to a trivialization of $w_2(\nu)$. Like in Examples III.116 and III.118, we can ask for a map $\phi\colon M \to B\mathbb{Z}/2$ and a trivialization of $w_1(\nu) - \phi^*(a)$, where $a \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$ is the generator, and this is no data at all; then we also want to impose $w_2(\nu) = 0$. So pin$^-$ bordism is the Thom spectrum of the twist $f_{a,0}\colon B\mathbb{Z}/2 \to B\mathrm{GL}_1(\mathit{MTSpin})$. The classes a

and 0 are w_1 and w_2 of $\sigma \to B\mathbb{Z}/2$, so we learn that $\mathit{MTPin}^- \simeq \mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{\sigma-1}$, a splitting first written down by Peterson [Pet68, §7].

2. A pin$^+$ structure is a trivialization of $w_2(M)$, with no condition on w_1. Switching to the stable normal bundle, we want a trivialization of $w_2(\nu) + w_1(\nu)^2$. Just as for pin$^-$ structures, pick a map $\phi\colon M \to B\mathbb{Z}/2$ and ask for a trivialization of $w_1(\nu) - \phi^*(a)$, which is no data; then we want to trivialize $w_2(\nu) + \phi^*(a^2)$. Thus pin$^+$ bordism is the Thom spectrum of the twist $f_{a,a^2}\colon B\mathbb{Z}/2 \to B\mathrm{GL}_1(\mathit{MTSpin})$. The classes a and a^2 are w_1, resp. w_2 of the virtual vector bundle $-\sigma$, so Lemma III.115 tells us $\mathit{MTPin}^+ \simeq \mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{1-\sigma}$, a result of Stolz [Sto88, §8].[16]
3. A spin^c structure is data of a trivialization of $w_1(TM)$ and a class $c_1 \in H^2(M;\mathbb{Z})$ such that $c_1 \bmod 2 = w_2(TM)$; in this case there is no difference between $w_2(TM)$ and $w_2(\nu)$. This is a twisted spin structure where $X = B\mathrm{U}(1) = K(\mathbb{Z}, 2)$, $a = 0$, and b is the generator of $H^2(K(\mathbb{Z}, 2); \mathbb{Z}/2) \cong \mathbb{Z}/2$. As 0, resp. b are the first and second Stiefel-Whitney classes of the tautological complex line bundle $L \to B\mathrm{U}(1)$, Lemma III.115 implies $\mathit{MTSpin}^c \simeq \mathit{MTSpin} \wedge (B\mathrm{U}(1))^{L-2}$, which is known due to Bahri-Gilkey [BG87a, BG87b].
4. A spin-$\mathbb{Z}/2k$ structure on a manifold M is data of a principal $\mathbb{Z}/k$-bundle $P \to M$ together with trivializations of $w_1(M)$ and $w_2(M) - w_2(V_P)$, where V is the standard one-dimensional complex representation of $\mathbb{Z}/k$ as rotations and $V_P \to M$ is the associated complex line bundle to P. Thus, analogous to the spin^c argument above, this structure is a twisted spin structure for $X = B\mathbb{Z}/k$, $a = 0$, and $b = w_2(V)$, and Lemma III.115 implies $MT(\text{Spin-}\mathbb{Z}/2k) \simeq \mathit{MTSpin} \wedge (B\mathbb{Z}/k)^{V-2}$, reproving a theorem of Campbell [Cam17, §7.9].
5. A spin^h structure is data of a trivialization of $w_1(M)$ and a rank-3 oriented vector bundle $E \to M$ and a trivialization of $w_2(M) - w_2(E)$. Again tangential vs. normal does not matter here, and one can use the same line of reasoning to show that spin^h structures are twisted spin structures for $X = B\mathrm{SO}_3$, $a = 0$, and $b = w_2$. As these are w_1, resp. w_2 of the tautological vector bundle $V \to B\mathrm{SO}_3$, Lemma III.115 tells us $\mathit{MTSpin}^h \simeq \mathit{MTSpin} \wedge (B\mathrm{SO}_3)^{V-3}$, which is due to Freed-Hopkins [FH21, §10].

There are many more examples of vector bundle twists of spin bordism, including the examples in, e.g., [FH21, Guo18, DL20, GOP$^+$20, WW20a, DL21, Ste22, DDHM23]. But one can find twists of spin bordism not described by vector bundle twists, even in physically motivated examples: see [DY22, Theorem 4.2] for an example where $X = B\mathrm{SU}_8/\{\pm 1\}$, with a few more examples given in [DY23a, §3.1]. The Smith-theoretic techniques in our paper do not apply in those situations.

Example III.121 (James periodicity as Smith periodicity)**.** James periodicity [Jam59] is a classical result in homotopy theory that the homotopy types of the *stunted projective spaces* $\mathbb{RP}^n_k := \mathbb{RP}^n/\mathbb{RP}^k$ (here $k < n$) are periodic, with periodicity dependent on n and k. There are also results for the analogously defined stunted complex and quaternionic projective spaces $\mathbb{CP}^n_k := \mathbb{CP}^n/\mathbb{CP}^k$ and $\mathbb{HP}^n_k := \mathbb{HP}^n/\mathbb{HP}^k$. These periodicities can be thought of in terms of periodic Smith families for framed bordism—or conversely, the periodicities in the previous several examples can be thought of as generalizations of James periodicity over other ring spectra than $\mathbb{S}$.

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[16] As $[B\mathbb{Z}/2, BO/B\mathrm{Spin}]$ has exponent 4 by Example III.112, $[1-\sigma] = [3\sigma - 3]$, so the reader who prefers to avoid virtual vector bundles can write $\mathit{MTPin}^+ \simeq \mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{3\sigma-3}$.

Proposition III.108 is the engine behind our periodicity results; its key idea is that vector bundles inducing equivalent maps to $B\mathrm{GL}_1(R)$ have equivalent R-module Thom spectra. For framed bordism, where $R = \mathbb{S}$, we therefore should look at the image of the homomorphism $[X, BO] \to [X, B\mathrm{GL}_1(\mathbb{S})]$; following Atiyah [Ati61b, §1], this image is typically denoted $J(X)$. Atiyah (*ibid.*, Lemma 2.5) proves that if $V, W \to X$ have equal images in $J(X)$, then $X^V \simeq X^W$.[17] Therefore we can obtain framed bordism Smith periodicities, or equivalences of Thom spectra, by calculating the groups $J(X)$. Atiyah (*ibid.*, Proposition 1.5) shows that when X is a finite CW complex, $J(X)$ is a finite group, implying the existence of many framed Smith families.

For James periodicity specifically, choose $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Stunted projective spaces are Thom spectra: if $L \to F\mathbb{P}^k$ denotes the tautological (real, complex, or quaternionic) line bundle, there is an equivalence $\Sigma^\infty F\mathbb{P}^n_k \simeq (F\mathbb{P}^k)^{(n-k)L}$ [Ati61b, Proposition 4.3], reducing the proof of James periodicity to the computation of the order of L in $J(F\mathbb{P}^k)$. For example, for $F = \mathbb{R}$ Adams calculates the order of L in $J(\mathbb{RP}^k)$ in [Ada62, Theorem 7.4] and [Ada65, Example 6.3] to be $2^{\phi(k)}$, where $\phi(k)$ is the number of integers s with $0 < s \le k$ and $s \equiv 0$, 1, 2, or 4 mod 8. Therefore for all k and n, there is a homotopy equivalence

$$\Sigma^\infty \mathbb{RP}_k^{n+2^{\phi(k)}} \xrightarrow{\simeq} \Sigma^\infty \Sigma^{2^{\phi(k)}} \mathbb{RP}^n_k. \tag{III.122}$$

(and in fact this is true even before applying Σ^∞ [Mah65]). Additional computations in J-groups of $F\mathbb{P}^k$ are done by Adams-Walker [AW65], Lam [Lam72], Federer-Gitler [FG73, FG77], Sigrist [Sig75], Walker [Wal81], Crabb-Knapp [CK88], Dibağ [Dib99, Dib03], Obiedat [Obi01], and Randal-Williams [RW23, §5.3].

Remark III.123. There are many other tangential structures ξ that one can study twists of. See [SSS09, Sat10, Sat11a, Sat11b, Sat12, SSS12, Sat15, SW15, SW18, LSW20, SY21, DY23a] for more examples.

E. Examples of Smith cofiber sequences

In this section, we implement the discussion from the previous section for some commonly studied vector bundles. We find many previously studied Smith homomorphisms, and also identify a few other well-known cofiber sequences, including Wood's sequences, Wall's sequence, and the cofiber sequences associated to the Hopf maps and to transfer maps, as Smith homomorphisms (Examples III.136 and III.139). We include this Pokédex of examples in part to illustrate what kinds of Smith cofiber sequences are out there; in part to make contact with preexisting literature; and in part to illustrate how to put theorems such as Theorem III.88 into practice to explicitly write down Smith cofiber sequences.

1. *Twisting by real line bundles*

Our first family of examples use the tautological line bundle $\sigma \to B\mathbb{Z}/2$; its sphere bundle is the tautological $\mathbb{Z}/2$-bundle $E\mathbb{Z}/2 \to B\mathbb{Z}/2$, whose total space is contractible. Therefore by

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[17] See Held-Sjerve [HS73, Theorem 1.2] for a partial converse to this result.

Theorem III.88, for any $k \in \mathbb{Z}$, we have a cofiber sequence

$$\mathbb{S} \longrightarrow (B\mathbb{Z}/2)^{k(\sigma-1)} \xrightarrow{\mathrm{sm}_\sigma} \Sigma(B\mathbb{Z}/2)^{(k+1)(\sigma-1)}, \tag{III.124}$$

where sm_σ is the Smith homomorphism associated to σ. When $k = 0$, this is especially nice: the middle spectrum is $\Sigma^\infty_+ B\mathbb{Z}/2 \simeq \mathbb{S} \vee \Sigma B\mathbb{Z}/2$ and the map $\mathbb{S} \to \mathbb{S} \vee \Sigma B\mathbb{Z}/2$ is the inclusion of the first factor of the wedge sum, leading to a Smith *isomorphism* $\mathrm{sm}_\sigma \colon \Sigma^\infty B\mathbb{Z}/2 \xrightarrow{\cong} (B\mathbb{Z}/2)^\sigma$. This equivalence is well-known; see Kochman [Koc96, Lemma 2.6.5] for a proof.

Remark III.125. The Thom spectrum $(B\mathbb{Z}/2)^{k\sigma}$ is often denoted in the homotopy theory literature by $\mathbb{RP}^\infty_k$, so that $(B\mathbb{Z}/2)^{k(\sigma-1)}$ can be identified with its desuspension $\Sigma^{-k}\mathbb{RP}^\infty_k$. One justification for this notation stems from (III.124): suspending it k times gives a cofiber sequence

$$\mathbb{S}^k \longrightarrow \mathbb{RP}^\infty_k \longrightarrow \mathbb{RP}^\infty_{k+1}, \tag{III.126}$$

which exhibits $\mathbb{RP}^\infty_{k+1}$ as the spectrum obtained by crushing the bottom cell of $\mathbb{RP}^\infty_k$.

Example III.127. Smash (III.124) with MTO. As every virtual bundle has a unique MTO-orientation, this cofiber sequence simplifies to

$$MTO \longrightarrow MTO \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{sm}_\sigma} MTO \wedge \Sigma(B\mathbb{Z}/2)_+. \tag{III.128}$$

This was the first Smith homomorphism studied; it was defined and named the Smith homomorphism by Conner-Floyd [CF64, Theorem 26.1]. Thom's celebrated calculation of Ω_*^{O} implies that MTO is a sum of shifts of $H\mathbb{Z}/2$; on each of these copies, the Smith map (III.128) is the cap product with the nonzero element of $H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$.

Stong [Sto69, Proposition 5] and Uchida [Uch70] study related examples, where one smashes (III.128) with spaces X; they identify the fiber $MTO \wedge X$ and show that the long exact sequence of homotopy groups splits. Their papers are among the earliest examples identifying the Smith long exact sequence.[18]

Example III.129. Smash (III.124) with $MTSO$. Since σ is not orientable, but 2σ is oriented (see Example III.110), we obtain a 2-periodic series of codimension-1 Smith homomorphisms between the oriented bordism of $B\mathbb{Z}/2$ and $(B\mathbb{Z}/2, \sigma)$-twisted oriented bordism. The latter can be identified with unoriented bordism: a $(B\mathbb{Z}/2, \sigma)$-twisted orientation on V is data of a line bundle on L and an orientation of $V \oplus L$, which is no data at all: this identifies $L \cong \mathrm{Det}(V)^* \cong \mathrm{Det}(V)$ up to a contractible space of choices, and $V \oplus \mathrm{Det}(V)$ is canonically oriented. So every vector bundle has a canonical $(B\mathbb{Z}/2, \sigma)$-twisted orientation.

Therefore by Theorem III.88 we obtain a 2-periodic sequence of codimension-1 Smith homomorphisms:

$$MTSO \longrightarrow MTSO \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MTO \tag{III.130a}$$

$$MTSO \longrightarrow MTO \xrightarrow{\mathrm{sm}_\sigma} \Sigma MTSO \wedge (B\mathbb{Z}/2)_+. \tag{III.130b}$$

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[18] At the time, it was common to think of $\Omega_*^{\mathrm{O}}(B\mathbb{Z}/2)$ as the bordism groups of manifolds M equipped with a free involution τ, rather than manifolds with a principal $\mathbb{Z}/2$-bundle; Stong and Uchida's results are phrased in that language. To pass between these perspectives, rewrite (M, τ) as the principal $\mathbb{Z}/2$-bundle $M \to M/\tau$; in the other direction, take the deck transformation involution of the total space of a principal $\mathbb{Z}/2$-bundle.

These maps are obtained by taking smooth representatives of Poincaré duals of w_1 either of the manifold (when the domain is Ω_*^{O}) or of the principal $\mathbb{Z}/2$-bundle (when the domain is $\Omega_*^{\mathrm{SO}}(B\mathbb{Z}/2)$). We work out the long exact sequences corresponding to the Anderson duals of (III.130) in §II D 2 in low degrees.

These Smith homomorphisms were first introduced by Komiya [Kom72, §5]; see also Shibata [Shi73, Proposition 2.1]. See Córdova-Ohmori-Shao-Yan [COSY20, Appendix A], Hason-Komargodski-Thorngren [HKT20a, §4.4], and Fidkowski-Haah-Hastings [FHH20] for applications of these Smith homomorphisms to physics. The splitting of the $k = 0$ case of (III.124) implies a homotopy equivalence $MTSO \wedge B\mathbb{Z}/2 \xrightarrow{\cong} \Sigma MTO$, a theorem of Atiyah [Ati61a, Proposition 4.1].

Example III.131. Some of the coolest examples of this kind come about by smashing (III.124) with *MTSpin*. As we discussed in Example III.112, the periodicity of this family is 1, 2, or 4; a Whitney sum formula calculation shows that $k\sigma$ is spin iff k is a multiple of 4, and therefore this Smith family is 4-periodic. The corresponding $(B\mathbb{Z}/2, k\sigma)$-twisted spin bordism groups can be identified with H-bordism for certain Lie groups H, as discussed in Example III.120; specifically,

1. a $(B\mathbb{Z}/2, \sigma)$-twisted spin structure is equivalent to a pin^- structure;
2. a $(B\mathbb{Z}/2, 2\sigma)$-twisted spin structure is equivalent to an H structure, where $H = \mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4$; and
3. a $(B\mathbb{Z}/2, 3\sigma)$-twisted spin structure is equivalent to a pin^+ structure.

Using Theorem III.88 once again, the 4-periodic sequence of codimension-1 Smith homomorphisms takes the form

$$MTSpin \longrightarrow MTSpin \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MTPin^- \tag{III.132a}$$

$$MTSpin \longrightarrow MTPin^- \xrightarrow{\mathrm{sm}_\sigma} \Sigma MT(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4) \tag{III.132b}$$

$$MTSpin \longrightarrow MT(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4) \xrightarrow{\mathrm{sm}_\sigma} \Sigma MTPin^+ \tag{III.132c}$$

$$MTSpin \longrightarrow MTPin^+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma MTSpin \wedge (B\mathbb{Z}/2)_+, \tag{III.132d}$$

with each sm_σ obtained by taking a smooth representative of a Poincaré dual of w_1 of the manifold or of a associated principal $\mathbb{Z}/2$-bundle, like in (III.130).

The splitting of the $k = 0$ Smith homomorphism in (III.124) gives us an equivalence $MTSpin \wedge B\mathbb{Z}/2 \simeq MTPin^-$, a theorem of Peterson [Pet68, §7].

This family of Smith homomorphisms has been discussed in the literature before. The piece involving $\mathrm{Spin} \times \mathbb{Z}/2$ and Pin^- was used by Peterson [Pet68, §7] and Anderson-Brown-Peterson [ABP69], who say that it was already "well-known." The long exact sequence corresponding to (III.132c) appears in [Gia73b, Theorem 3.1], where it is attributed to Stong. The Smith homomorphism sm_σ in (III.132d) appears in Kreck [Kre84, §4]. The composition of two maps in (III.132) in a row to go between pin^+ and pin^- bordism appears in Kirby-Taylor [KT90a, Lemma 7]. We use the Anderson dual of (III.132a) to study symmetry breaking in §II C 3, and work out the corresponding long exact sequences for all four Smith homomorphisms in §II D 3.

The full family appears more recently in work of Hambleton-Su [HS13, §4.C], Kapustin-Thorngren-Turzillo-Wang [KTTW15, §8], Tachikawa-Yonekura [TY19, §3.1], Hason-Komargodski-Thorngren [HKT20a, §4.4], and Wan-Wang-Zheng [WWZ20, §6.7].

Example III.133. As we discussed in Example III.114, for a general vector bundle $V \to X$, there is no guarantee that kV has a string structure. However, on $B\mathbb{Z}/2$, $k\sigma$ has a string structure iff $k \equiv 0 \bmod 8$, so there is an eight-periodic family of codimension-1 Smith homomorphisms between bordism groups of manifolds with $(B\mathbb{Z}/2, k\sigma)$-twisted string structures for various k.[19]

In Example III.131, the four twisted spin structures turned out to be equivalent to G-structures for four Lie groups G. An analogous result is true here, but in the world of 2-groups, because the string group is a Lie 2-group [SP11]. One can show that for each $k \in \mathbb{Z}/8$, there is a Lie 2-group $\mathbb{G}[k]$ and a map $\xi\colon B\mathbb{G}[k] \to B\mathrm{O}$ such that $\mathbb{G}[k]$-structures on a smooth manifolds are naturally equivalent to $(B\mathbb{Z}/2, k\sigma)$-twisted string structures. These Lie 2-groups $\mathbb{G}[k]$ are extensions of $\mathrm{Spin} \times \mathbb{Z}/2$, $\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4$, and $\mathrm{Pin}^\pm$ by $B\mathrm{U}(1)$; such extensions of a compact Lie group G by $B\mathrm{U}(1)$ are classified by $H^4(BG;\mathbb{Z})$ [SP11, Wei22], and the $\mathbb{G}[k]$ 2-groups' extension classes are λ of various spin vector bundles over $B\mathrm{Spin} \times B\mathbb{Z}/2$, $B(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/4)$, and $B\mathrm{Pin}^\pm$. For example, $\mathbb{G}[4] = \mathrm{String} \times_{B\mathrm{U}(1)} \mathsf{sLine}$, where sLine is the abelian Lie 2-group of Hermitian super lines.

Example III.134. If one smashes (III.124) with MTSpin^c, one obtains a very similar story to Example III.129: twice any vector bundle is complex, hence spin^c, and $(B\mathbb{Z}/2, \sigma)$-twisted spin^c bordism is naturally identified with pin^c bordism, as we discussed in Example III.118. So taking Poincaré duals of w_1 as in Example III.129 defines a 2-periodic sequence of codimension-1 Smith homomorphisms

$$\mathit{MTSpin}^c \longrightarrow \mathit{MTSpin}^c \wedge (B\mathbb{Z}/2)_+ \xrightarrow{\mathrm{sm}_\sigma} \Sigma \mathit{MTPin}^c \tag{III.135a}$$

$$\mathit{MTSpin}^c \longrightarrow \mathit{MTPin}^c \xrightarrow{\mathrm{sm}_\sigma} \Sigma \mathit{MTSpin}^c \wedge (B\mathbb{Z}/2)_+. \tag{III.135b}$$

To our knowledge, these long exact sequences first appear in Hambleton-Su [HS13, §4.C].

We also obtain an equivalence $\mathit{MTSpin}^c \wedge B\mathbb{Z}/2 \xrightarrow{\simeq} \Sigma \mathit{MTPin}^c$, which was first observed by Bahri-Gilkey [BG87a, §3]. See Shiozaki-Shapourian-Ryu [SSR17, §E.1] and Kobayashi [Kob21, §IV] for applications in condensed-matter physics and [DYY] for an application of a closely related Smith long exact sequence.

Example III.136. Pull back (III.124) along the map $B\mathbb{Z} \to B\mathbb{Z}/2$, i.e. $S^1 = \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^\infty$. The sphere bundle of $\sigma \to \mathbb{RP}^1$ is not contractible: it is the double cover $S^1 \to \mathbb{RP}^1$, and its Thom space is $\mathbb{RP}^2$. Therefore we obtain from Theorem III.88 a cofiber sequence $\Sigma^\infty_+ S^1 \to \Sigma^\infty \mathbb{RP}^2 \to \Sigma^{1+\infty}_+ \mathbb{RP}^1$, which is a rotated version of the multiplication-by-2 cofiber sequence

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \longrightarrow \Sigma^{-1+\infty} \mathbb{RP}^2. \tag{III.137}$$

The same story applies to the complex, quaternionic, and octonionic Hopf fibrations: their cofibers are the respective projective planes $\Sigma^{-2+\infty}\mathbb{CP}^2$, $\Sigma^{-4+\infty}\mathbb{HP}^2$, and $\Sigma^{-8+\infty}\mathbb{OP}^2$, and in each case the map to the cofiber is a Smith homomorphism for the tautological line bundle over the respective projective line (which is a sphere). In the case of the complex Hopf fibration, after smashing with ko or KO, one obtains the Wood cofiber sequences [Woo63] $\Sigma KO \xrightarrow{\eta} KO \to KU$ and $\Sigma ko \xrightarrow{\eta} ko \to ku$ as rotated versions of Smith cofiber sequences.

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[19] To prove the claimed fact about string structures on $k\sigma$, first use the Whitney sum formula to show that $w_1(k\sigma)$, $w_2(k\sigma)$, and $w_4(k\sigma)$ all vanish iff $k \equiv 0 \bmod 8$. The reduction mod 2 map $H^4(B\mathbb{Z}/2;\mathbb{Z}) \to H^4(B\mathbb{Z}/2;\mathbb{Z}/2)$ is an isomorphism, so the string obstruction $\lambda(k\sigma)$ vanishes iff its mod 2 reduction does, and $\lambda \bmod 2 = w_4$.

Smash (III.137) with $MTSO$ and you obtain Wall's cofiber sequence [Wal60, Theorem 3]

$$MTSO \xrightarrow{2} MTSO \longrightarrow \mathcal{W}, \tag{III.138}$$

where $\mathcal{W}$ is the Thom spectrum whose homotopy groups are the bordism groups of manifolds with an integral lift of w_1. This follows from Atiyah's identification of $\mathcal{W} \simeq \Sigma^{-1}MTSO \wedge \mathbb{RP}^2$ [Ati61a, §4], but it is also easy to directly check that an integral lift of w_1 is equivalent data to a $(\mathbb{RP}^1, \sigma)$-twisted orientation, using that $\mathbb{RP}^1$ is a $B\mathbb{Z}$.

It is also interesting to smash (III.137) with $MTSpin$; we work out the induced long exact sequence of bordism groups in low degrees in Figure 5, and this long exact sequence also appears in [DYY].

Example III.139. Let $\pi\colon E \to B$ be a principal $\mathbb{Z}/2$-bundle and $L \coloneqq E \times_{\mathbb{Z}/2} \mathbb{R} \to B$ be the associated line bundle. Then we have a Smith homomorphism $\mathrm{sm}_L\colon B^{-L} \to \Sigma^\infty_+ B$. The fiber is the Thom spectrum of the pullback of L to its sphere bundle; the sphere bundle is E and $\pi^*(L)$ is trivial, so Theorem III.88 gives us a cofiber sequence

$$B^{-L} \xrightarrow{\mathrm{sm}_L} \Sigma^\infty_+ B \xrightarrow{\tau} \Sigma^\infty_+ E. \tag{III.140}$$

Lemma III.141. *The map τ in (III.140) is the Becker-Gottlieb transfer [Rou72, KP72, BG75] for π.*

Proof. It suffices to work universally with the Smith cofiber sequence $(B\mathbb{Z}/2)^{-\sigma} \to \Sigma^\infty_+ B\mathbb{Z}/2 \to \Sigma^\infty_+ E\mathbb{Z}/2$, i.e. $(\mathbb{RP}^\infty)^{-\sigma} \to \Sigma^\infty_+ \mathbb{RP}^\infty \to \mathbb{S}$, and to show that the latter map is the transfer for $E\mathbb{Z}/2 \to B\mathbb{Z}/2$.

This transfer map admits the following description: consider the map of $\mathbb{Z}/2$-spectra[20] $f\colon \mathbb{S} \to \Sigma^{1-\sigma}(\mathbb{Z}/2)_+$, whose cofiber is $\mathbb{S}^{-\sigma}$. Upon taking homotopy orbits, we obtain a map $f_{h\mathbb{Z}/2}\colon \Sigma^\infty_+ \mathbb{RP}^\infty \to \mathbb{S}$, and this is the transfer map.

If G is a finite group and $V \in RO(G)$, there is a natural equivalence of spectra $(\mathbb{S}^V)_{hG} \simeq (BG)^V$.[21] And taking homotopy orbits of G-spectra preserves cofiber sequences, so the fiber of the transfer $f_{h\mathbb{Z}/2}$ is the map $(\mathbb{RP}^\infty)^{-\sigma} \to \Sigma^\infty_+ \mathbb{RP}^\infty$ given by the "inclusion" of virtual representations $-\sigma \hookrightarrow 0$, which is the Smith homomorphism we began with. ☐

In the case that B is a finite CW complex, one can prove Lemma III.141 more classically by adapting Cusick's calculation [Cus85, Corollary 2.11] identifying the cofibers of transfer maps for double covers.

Remark III.142. For another example along the lines of (III.140), Morisugi [Mor09, Theorem 1.3] shows that the cofibers of certain Smith homomorphisms over compact Lie groups can be described as Becker-Schultz transfer maps [BS74, §4]. And Uchida [Uch69], motivated by the study of immersions, works out the Smith long exact sequences of a few special cases of Example III.139, where $E = B\mathrm{O}(k) \times B\mathrm{O}(k)$ and $B = B(\mathrm{O}(1) \ltimes (\mathrm{O}(k)^{\times 2}))$, where $\mathrm{O}(1)$ acts on $\mathrm{O}(k)^{\times 2}$ by swapping the two factors.

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[20] This fact, and our argument using it, works for both Borel and genuine $\mathbb{Z}/2$-spectra.

[21] One quick way to prove this uses the Ando-Blumberg-Gepner-Hopkins-Rezk approach to Thom spectra [ABG+14a, ABG+14b]: both $(\mathbb{S}^V)_{hG}$ and $(BG)^V$ are both the colimit of the pt/G-shaped diagram whose value on pt is $\mathbb{S}^V$ and whose value on the morphism set G encodes the G-action on $\mathbb{S}^V$ [ABG+14a, Theorem 1.17]. It is also possible to prove this more classically by working with Thom spaces.

Now we consider the analogous family of examples arising from the tautological complex line bundle $L \to B\mathrm{U}(1)$. Its sphere bundle is $E\mathrm{U}(1) \to B\mathrm{U}(1)$, which is contractible, so just like in (III.124), we have for any $k \in \mathbb{Z}$ a cofiber sequence

$$\mathbb{S} \longrightarrow (B\mathrm{U}(1))^{k(L-2)} \xrightarrow{\mathrm{sm}_L} \Sigma^2(B\mathrm{U}(1))^{(k+1)(L-2)}. \tag{III.143}$$

Again, when $k = 0$, this sequence splits, yielding another Smith isomorphism $\Sigma^\infty B\mathrm{U}(1) \xrightarrow{\cong} (B\mathrm{U}(1))^L$. This equivalence is well-known, e.g. [Ada74, Example 2.1].

Example III.144. Let G be one of O, SO, Spin^c, or U; then the tautological line bundle over $B\mathrm{U}(1)$ has a G-structure, and MTG is an E_∞-ring spectrum and we can make sense of G-orientations. The G-orientation on L untwists the Thom spectrum, so smashing (III.143) with MTG has a similar effect to Example III.127: the result is a cofiber sequence

$$MTG \longrightarrow MTG \wedge (B\mathrm{U}(1))_+ \xrightarrow{\mathrm{sm}_L} \Sigma^2 MTG \wedge (B\mathrm{U}(1))_+. \tag{III.145}$$

For $G = \mathrm{U}$, this Smith homomorphism was first studied by Conner-Floyd [CF66, §5].

Lemma III.146. *For $G = \mathrm{O}$, SO, Spin^c, or* U*,*

$$MTG \wedge (B\mathrm{U}(1))_+ \simeq \bigvee_{k \geq 0} \Sigma^{2k} MTG. \tag{III.147}$$

Proof. The zeroth step is splitting off the basepoint: $MTG \wedge (B\mathrm{U}(1))_+ \simeq MTG \vee MTG \wedge (B\mathrm{U}(1))$. As noted above, $\Sigma^\infty B\mathrm{U}(1) \simeq (B\mathrm{U}(1))^L$, and we have a Thom isomorphism $MTG \wedge (B\mathrm{U}(1))^L \simeq MTG \wedge \Sigma^2(B\mathrm{U}(1))_+$. We are now in the same situation as at the beginning of the proof, but shifted up by 2, and we carry on in a similar way. ☐

Example III.148. Smash (III.143) with $MTSpin$; the bundle $L \to B\mathrm{U}(1)$ is oriented but not spin, so $2L$ is spin, and therefore we obtain a 2-periodic, codimension-2 family of Smith homomorphisms between the spin bordism of $B\mathrm{U}(1)$ and $(B\mathrm{U}(1), L)$-twisted spin bordism. A $(B\mathrm{U}(1), L)$-twisted spin structure is equivalent data to a spin^c structure, as we discussed in Example III.120, so this Smith family takes the form

$$MTSpin \longrightarrow MTSpin^c \xrightarrow{\mathrm{sm}_L} \Sigma^2 MTSpin \wedge (B\mathrm{U}(1))_+ \tag{III.149a}$$

$$MTSpin \longrightarrow MTSpin \wedge (B\mathrm{U}(1))_+ \xrightarrow{\mathrm{sm}_L} \Sigma^2 MTSpin^c. \tag{III.149b}$$

The long exact sequence arising from (III.149a) was identified by Kirby-Taylor [KT90b, Corollary 6.12, Remark 6.14]. The splitting of (III.143) when $k = 0$ leads to an equivalence $MTSpin \wedge B\mathrm{U}(1) \simeq \Sigma^2 MTSpin^c$, a theorem due to Stong [Sto68, Chapter XI].

We use the symmetry breaking long exact sequences corresponding to (III.149), i.e. the long exact sequences on cohomology for the Anderson dual cofiber sequences to (III.149), several times in this paper, including §II B 1, §II C 1 and §II C 4 where we apply it to symmetry breaking, and in §II D 1, where we explicitly calculate the groups and maps in the long exact sequences in low dimensions.

It would be interesting to study analogues of this example for pin^c or pin$^{\tilde{c}\pm}$ bordism and applications to invertible phases. Kirby-Taylor [KT90b, Remark 6.15] consider two additional analogues of (III.149a), including a Smith long exact sequence for G-bordism where $G \coloneqq \mathrm{Spin}\times_{\{\pm 1\}} \mathrm{O}(2)$. Guillou-Marin [GM80] and Stehouwer [Ste22, §4] compute G-bordism groups in low dimensions, and G-bordism also appears in [DDHM22, DDHM23, DYY]. In addition, Hambleton-Kreck-Teichner [HKT94, §2] study a pin$^-$ and pin^c analogue of Example III.148.

Example III.150. Pull back (III.143) along the inclusion $\mathbb{Z}/k \hookrightarrow \mathrm{U}(1)$, giving us Smith homomorphisms $(B\mathbb{Z}/k)^{k(L-2)} \to \Sigma^2(B\mathbb{Z}/k)^{(k+1)(L-2)}$, where L is the complex line bundle induced by the rotation representation of $\mathbb{Z}/k$ on $\mathbb{C}$. Recall from Theorem III.88 the fiber sequence

$$S(V_2)^{V_1} \to X^{V_1} \to X^{V_1\oplus V_2}. \tag{III.151}$$

For this example, we start with $X = B\mathbb{Z}/n$, $V_2 = i^*L - 2$ (for L as in the previous example and 2 the trivial complex line bundle), and $V_1 = k(i^*L - 2)$. We can compute the sphere bundle $S(i^*L)$ by fitting it into a pullback square:

$$\begin{tikzcd} S(i^*L) \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & S(L) \simeq * \arrow[d] \\ B\mathbb{Z}/n \arrow[r] & B\mathrm{U}(1). \end{tikzcd} \tag{III.152}$$

As noted above, $S(L)$ is contractible as it is the total space of the universal fibration. Therefore, the other three corners of the square form a fiber sequence. To compute the fiber of $B\mathbb{Z}/n \to B\mathrm{U}(1)$, we notice that applying the classifying space functor to the short exact sequence $\mathbb{Z}/n \hookrightarrow \mathrm{U}(1) \overset{\times n}{\to} \mathrm{U}(1)$ gives a fibration $B\mathbb{Z}/n \to B\mathrm{U}(1) \to B\mathrm{U}(1)$. Then, recognizing the map $B\mathrm{U}(1) \to B\mathrm{U}(1)$ as the classifying map for a principal $\mathrm{U}(1)$-bundle over $\mathrm{U}(1)$ with total space $B\mathbb{Z}/n$, we conclude that the fiber of the map $B\mathbb{Z}/n \to B\mathrm{U}(1)$ is exactly $\mathrm{U}(1)$. So, $S(i^*L) \simeq S^1$.

Next, we need to pull back V_1 along the projection $p\colon S(i^*L) \to B\mathbb{Z}/n$. We have that $p^*(k(i^*L - 1)) \cong \bigoplus_k p^*(i^*L)$. Since L is oriented as a real vector bundle, its pullbacks are as well, so p^*i^*L is oriented when considered as a real vector bundle over S^1, and thus it is the trivial 2-plane bundle.

Therefore, we recognize the Thom spectrum $S(i^*L)^{kp^*(i^*L)}$ as

$$\begin{aligned} S(i^*L)^{kp^*(i^*L)} &\simeq (S^1)^k \\ &\simeq \Sigma^{2k}\mathrm{Th}(S^1; 0) \\ &\simeq \Sigma^{2k}(\Sigma^\infty_+ S^1) \\ &\simeq \Sigma^{2k}(\Sigma^\infty S^1 \oplus \Sigma^\infty S^0) \\ &\simeq \Sigma^{2k+1}\mathbb{S} \vee \Sigma^{2k}\mathbb{S}. \end{aligned}$$

Thus for each $k \geq 0$ we have a Smith cofiber sequence

$$\Sigma^{2k+1}\mathbb{S} \vee \Sigma^{2k}\mathbb{S} \longrightarrow (B\mathbb{Z}/n)^{k\cdot i^*L} \longrightarrow B\mathbb{Z}/n^{(k+1)i^*L}. \tag{III.153}$$

Finally, we place V_1 in virtual dimension zero by taking $V_1 = k(i^*L - 2)$, to be consistent with the other examples in this section, and obtain the cofiber sequence

$$\Sigma\mathbb{S} \vee \mathbb{S} \longrightarrow (B\mathbb{Z}/n)^{k(i^*L-2)} \xrightarrow{\mathrm{sm}_L} \Sigma^2(B\mathbb{Z}/n)^{(k+1)(i^*L-2)}. \tag{III.154}$$

Example III.155. Smash (III.154) with MTSpin. Like in Example III.148, i^*L is oriented but not spin, and $2i^*L$ is spin, so we obtain a 2-periodic, codimension-2 family of Smith homomorphisms between the spin bordism of $B\mathbb{Z}/n$ and $(B\mathbb{Z}/n, i^*L)$-twisted spin bordism. Campbell [Cam17, §7.9] identifies the latter as bordism for the tangential structure $\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/2n$, explicitly giving us Smith cofiber sequences

$$\Sigma\mathit{MTSpin} \vee \mathit{MTSpin} \longrightarrow \mathit{MT}(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/2k) \xrightarrow{\mathrm{sm}_{i^*L}} \Sigma^2\mathit{MTSpin} \wedge (B\mathbb{Z}/k)_+ \tag{III.156a}$$

$$\Sigma\mathit{MTSpin} \vee \mathit{MTSpin} \longrightarrow \mathit{MTSpin} \wedge (B\mathbb{Z}/k)_+ \xrightarrow{\mathrm{sm}_{i^*L}} \mathit{MT}(\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/2k). \tag{III.156b}$$

$\mathrm{Spin} \times_{\{\pm 1\}} \mathbb{Z}/2k$ bordism appears in the mathematical physics literature in [Cam17, GEM19, Hsi18, Li19, GOP+20, DDHM22, Deb21, DL21, DDHM23, HTY22, DYY]; the case $k = 2$ also appears in [Gia73a, HKT20a, TY19, FH20, MV21]. The Smith homomorphisms in (III.156) for $n = 4$ appear in [DDHM23]. We work out the Anderson-dualized long exact sequences corresponding to (III.156) for $n = 3$ and $n = 4$ in low degrees in §II D 4, resp. §II D 5.

Example III.157. We elaborate on Example III.155 when $n = 2$. The rotation representation is isomorphic to 2σ, where σ denotes the real sign representation; we will also let σ denote the associated bundle over $B\mathbb{Z}/2$.

Everything in Example III.155 still works for $n = 2$, but now we have more options: we can start with an odd number of copies of σ. In this case, the fiber of the Smith map is the Thom spectrum of the Möbius bundle $(\sigma - 1) \to \mathrm{U}(1)$; one can directly check that the Thom space of σ is $\mathbb{RP}^2$, so the Thom spectrum of $\sigma - 1$ is $\Sigma^{-1+\infty}\mathbb{RP}^2$. Therefore we have a Smith cofiber sequence

$$\Sigma^{-1+\infty}\mathbb{RP}^2 \longrightarrow (B\mathbb{Z}/2)^{(2k-1)(\sigma-1)} \xrightarrow{\mathrm{sm}_{2\sigma}} \Sigma^2(B\mathbb{Z}/2)^{(2k+1)(\sigma-1)}. \tag{III.158}$$

Out of all the examples we have studied in this section, this is the first one where the pullback of V_2 to the sphere bundle is nontrivial.

As usual, we smash (III.158) with various bordism spectra. The map $\mathrm{sm}_{2\sigma}$ is the composition of two iterations of sm_σ from (III.124), so some of the resulting cofiber sequences look familiar from that perspective. We only discuss a few examples, but plenty more are out there.

- If we smash (III.158) with MTSO, we obtain a cofiber sequence first discussed by Atiyah [Ati61a, (4.3)]:

  $$\mathcal{W} \longrightarrow \mathit{MTO} \xrightarrow{\mathrm{sm}_{2\sigma}} \Sigma^2\mathit{MTO}, \tag{III.159}$$

  where $\mathcal{W}$ is Wall's bordism spectrum (see Example III.136). Here we use the identifications $\Sigma\mathit{MTO} \simeq \mathit{MTSO} \wedge B\mathbb{Z}/2$ and $\mathcal{W} \simeq \mathit{MTSO} \wedge \Sigma^{-1}\mathbb{RP}^2$, both due to Atiyah [Ati61a, §4], that we discussed in Examples III.129 and III.136, respectively.

- If we instead smash (III.158) with MTSpin, we obtain a cofiber sequence

  $$\mathit{MTSpin} \wedge \Sigma^{-1}\mathbb{RP}^2 \longrightarrow \mathit{MTPin}^{\pm} \xrightarrow{\mathrm{sm}_{2\sigma}} \Sigma^2\mathit{MTPin}^{\mp}, \tag{III.160}$$

which was first constructed by Kirby-Taylor [KT90a, Lemma 7]. Here we have used the identifications of pin$^+$, resp. pin$^-$ bordism as $(B\mathbb{Z}/2, 3\sigma)$, resp. $(B\mathbb{Z}/2, \sigma)$-twisted spin bordism that we discussed in Example III.131. In Figure 6, we calculate the long exact sequence on bordism groups corresponding to (III.160) (specifically, the pin$^-$ to pin$^+$ case) in low degrees. See [DDHM23] for an application of a related but different Smith homomorphism in physics.

The Smith homomorphism in (III.160) is the composition of two of the Smith homomorphisms in the 4-periodic collection of Example III.131, where we go from pin$^+$ to $\mathrm{Spin}\times\mathbb{Z}/2$ to pin$^-$, or from pin$^-$ to $\mathrm{Spin}\times_{\{\pm 1\}}\mathbb{Z}/4$ to pin$^+$. The other two compositions, which exchange the spin bordism of $B\mathbb{Z}/2$ with $\mathrm{Spin}\times_{\{\pm 1\}}\mathbb{Z}/4$ bordism, are (III.156) for $n=2$.

3. *A few other miscellaneous examples*

In this section, we record some more examples that do not arise from real or complex line bundles.

Example III.161. Like our previous examples over $B\mathbb{Z}/2$ and $B\mathrm{U}(1)$, we can study Smith homomorphisms for the tautological quaternionic line bundle $V\to B\mathrm{SU}(2)$. Once again, the sphere bundle of V is contractible, as it is $E\mathrm{SU}(2)\to B\mathrm{SU}(2)$, so we obtain Smith cofiber sequences like in (III.124) and (III.143):

$$\mathbb{S}\longrightarrow (B\mathrm{SU}(2))^{k(V-4)}\xrightarrow{\mathrm{sm}_V}\Sigma^4(B\mathrm{SU}(2))^{(k+1)(V-4)}. \tag{III.162}$$

For $k=0$, this sequence splits, yielding a third Smith isomorphism $\Sigma^\infty B\mathrm{SU}(2)\overset{\cong}{\to}(B\mathrm{SU}(2))^V$. This equivalence is well-known, e.g. [Tam97, §2].

This bundle has a G-structure for G including O, SO, Spin, Spin^c, U, SU, and Sp, and in all of these cases, smashing with MTG produces Smith homomorphisms similar to those in Examples III.127 and III.144. The proof of Lemma III.146 still works in this setting, and for these G we obtain splittings

$$MTG\wedge(B\mathrm{SU}(2))_+\simeq\bigvee_{k\geq 0}\Sigma^{4k}MTG. \tag{III.163}$$

When $G=\mathrm{Spin}$, (III.163) is closely related to the splitting in Corollary B.19.

The Smith map (III.162), after smashing with $MTSp$, was studied by Landweber [Lan68, §5].

Example III.164. Consider the Smith homomorphisms coming from the tautological rank-3 vector bundle $V\to B\mathrm{SO}(3)$. Then, like in Example III.168, the one-point compactification of $\mathfrak{so}(3)/\mathfrak{u}_1$ is isomorphic to $\mathrm{SO}(3)/\mathrm{U}(1)\cong S^2$. Since $\mathfrak{so}(3)/\mathfrak{u}_1\oplus\mathbb{R}$ is isomorphic to the defining representation V of $\mathrm{SO}(3)$, we obtain a cofiber sequence of spectra

$$(B\mathrm{U}(1))^{k(L-2)}\longrightarrow(B\mathrm{SO}(3))^{k(V-3)}\longrightarrow\Sigma^3(B\mathrm{SO}(3))^{(k+1)(V-3)}. \tag{III.165}$$

We are most interested in smashing this sequence with $MTSpin$.[22] Note that V is not spin, but

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[22] It is also interesting to smash (III.165) with $MTSpin^c$: in this case one obtains a codimension-3, 2-periodic family

because V is oriented, $2V$ is spin; therefore we obtain a 2-periodic family of codimension-3 Smith homomorphisms exchanging the spin bordism of $B\mathrm{SO}(3)$ and $(B\mathrm{SO}(3), V)$-twisted spin bordism. Freed-Hopkins [FH21, (10.20)] identify $(B\mathrm{SO}(3), V)$-twisted spin bordism with bordism for the group $G^0 \coloneqq \mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}(2)$, which is in various sources called spin^h bordism, spin^q bordism, spin-SU(2) bordism, or G^0 bordism.[23] The fiber we've seen before in Example III.148: spin^c bordism when k is odd in (III.165), and the spin bordism of $B\mathrm{U}(1)$ when k is even.

In summary, we have two Smith cofiber sequences

$$\mathit{MTSpin}^c \longrightarrow \mathit{MTSpin}^h \xrightarrow{\mathrm{sm}_V} \Sigma^3 \mathit{MTSpin} \wedge (B\mathrm{SO}(3))_+ \tag{III.166a}$$

$$\mathit{MTSpin} \wedge (B\mathrm{U}(1))_+ \longrightarrow \mathit{MTSpin} \wedge (B\mathrm{SO}(3))_+ \xrightarrow{\mathrm{sm}_V} \Sigma^3 \mathit{MTSpin}^h. \tag{III.166b}$$

The long exact sequence of bordism groups associated to (III.166a) appears in Theorem B.2 as an example where one must use the cobordism Euler class to calculate the Smith homomorphism: ordinary cohomology Euler classes give the wrong answer. We also discuss the symmetry breaking implications of the Anderson duals of (III.166) in §II C 2, and work the Anderson-dualized long exact sequences out in low degrees in §II D 6. Other works studying anomalies of spin^h QFTs include [FH21, WW19, WWW19, WWZ20, DL20, WW20b, BCD22, DY22, WY22, DYY].

Remark III.167. Freed-Hopkins [FH21] also study two unoriented analogues of spin^h structures, called pin$^{h\pm}$ or $G^\pm$ structures, corresponding to the groups $\mathrm{Pin}^\pm \times_{\{\pm 1\}} \mathrm{SU}(2)$. It would be interesting to work out analogues of the Smith homomorphisms such as the ones in Examples III.131 and III.164 for pin$^{h\pm}$ structures and apply them to symmetry breaking. Pin$^{h\pm}$ manifolds are also studied in [BC18, GPW18, LS19, AM21, DYY].

Example III.168. If we pull Example III.164 back to $B\mathrm{SU}(2)$, we obtain a Smith long exact sequence which makes an appearance both in §II D 6 and in Appendix B.

The tautological quaternionic line bundle over $B\mathrm{SU}(2)$ is *not* isomorphic to the bundle associated to $\mathfrak{su}_2 \oplus \mathbb{R}$, where $\mathfrak{su}_2$ is the adjoint representation of SU(2). Rather, since $\mathfrak{su}_2 \cong \mathbb{R} \oplus \mathfrak{su}_2/\mathfrak{u}_1$, the map $B\mathrm{U}(1) \to B\mathrm{SU}(2)$ exhibits $B\mathrm{U}(1)$ as the unit sphere bundle in the adjoint representation of SU(2). It follows that there is a cofiber sequence

$$B\mathrm{U}(1) \longrightarrow B\mathrm{SU}(2) \xrightarrow{\mathrm{sm}_V} \Sigma^3 (B\mathrm{SU}(2))^{V-3}, \tag{III.169}$$

where $V \to B\mathrm{SU}(2)$ is the vector bundle associated to $\mathfrak{su}(2)$. We claim the first map is induced by the inclusion of a maximal torus into SU(2). To see that the sphere bundle is $B\mathrm{U}(1)$ as claimed, identify $\mathrm{SU}(2) \to \mathrm{SO}(3)$ with $\mathrm{Spin}(3) \to \mathrm{SO}(3)$ and $U(1) \to \mathrm{SU}(2)$ with $\mathrm{Spin}(2) \to \mathrm{Spin}(3)$; by the third isomorphism theorem, $\mathrm{Spin}(3)/\mathrm{Spin}(2) \cong \mathrm{SO}(3)/\mathrm{SO}(2)$, and in Example III.164 we identified that quotient with the unit sphere inside $\mathbb{R}^3$. Therefore taking associated bundles, we end up with $B\mathrm{Spin}(2)$ as the fiber in (III.169).

Since SU(2) is simply connected, $B\mathrm{SU}(2)$ is 2-connected and therefore all of its vector bundles

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of Smith homomorphisms exchanging the spin^c bordism of $B\mathrm{SO}(3)$ with "spin-U(2) bordism," i.e. bordism of the group $\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{U}(2) \cong \mathrm{Spin}^c \times_{\{\pm 1\}} \mathrm{SU}(2)$. Davighi-Lohisiri [DL20, DL21] introduced Spin-U(2) bordism and calculated it in low dimensions; spin-U(2) structures also appear in Seiberg-Witten theory (e.g. [FL02, DW19]) under the name *spin^u structures.*

[23] To the best of our knowledge, spin^h structures were first studied in [BFF78] in the context of quantum gravity; they have also been applied to Seiberg-Witten theory [OT96], index theory, e.g. in [May65, Nag95, Bär99, FH21, Che17], almost quaternionic geometry, e.g. in [Nag95, Bär99, AM21], immersion problems [Bär99, AM21], and the study of invertible field theories [FH21, BC18, WWW19, DY22]. See [Law23] for a review.

admit spin structures. Thus, when we smash (III.169) with *MTSpin*, we obtain a cofiber sequence

$$\mathit{MTSpin}\wedge(B\mathrm{U}(1))_+\longrightarrow\mathit{MTSpin}\wedge(B\mathrm{SU}(2))_+\xrightarrow{\mathrm{sm}_V}\Sigma^3\mathit{MTSpin}\wedge(B\mathrm{SU}(2))_+. \tag{III.170}$$

The Anderson dual of (III.170) appears in a physics application in §II D 6. The Thom spectrum $(B\mathrm{SU}(2))^{\mathfrak{su}(2)}$ is known as James' "quasiprojective space" (see [Jam76]).

The same Thom isomorphism applies for any *MTSpin*-oriented ring spectrum, such as *MTSO* or *ko*; if we used *ko* instead of *MTSpin* in (III.170), we would obtain the cofiber sequence in (B.25).

Example III.171. In §II B 2, we study the SBLES in twisted spin bordism corresponding to the vector bundle $2L\to B\mathrm{U}(1)$, where L denotes the tautological bundle. Since $2L$ is spin, we obtain a one-periodic family of Smith homomorphisms of the form

$$S(2L)\longrightarrow B\mathrm{U}(1)\xrightarrow{\mathrm{sm}_{2L}}\Sigma^4(B\mathrm{U}(1))^{2L-4}. \tag{III.172a}$$

The new wrinkle is showing that $S(2L)\to B\mathrm{U}(1)$ is homotopy equivalent to the map $S^2\to B\mathrm{U}(1)$ given by the inclusion of the 2-skeleton. But this is not so hard: using the long exact sequence in cohomology associated to the cofiber sequence, one learns that if C is the cofiber of sm_{2L}, $\widetilde{H}^*(C;\mathbb{Z})$ vanishes except in degree 3, where it is $\mathbb{Z}$; this characterizes S^3, so the fiber, which is the total space of the sphere bundle, is S^2. Stably this splits as $\mathbb{S}\vee\Sigma^2\mathbb{S}$, so our cofiber sequence is

$$\mathbb{S}\vee\Sigma^2\mathbb{S}\longrightarrow\Sigma^\infty_+B\mathrm{U}(1)\xrightarrow{\mathrm{sm}_{2L}}(B\mathrm{U}(1))^{2L-4}. \tag{III.172b}$$

For (II B 2), we smash this with *MTSpin*.

This cofiber sequence is a complexified version of (III.154). One therefore wonders what happens if we consider it within its family

$$\mathrm{sm}_{2L}\colon(B\mathrm{U}(1))^{kL-2k}\longrightarrow\Sigma^4(B\mathrm{U}(1))^{(k+1)L-2k-2}. \tag{III.173}$$

If we smash with *MTSpin*, this is a 2-periodic family: it only matters whether k is odd or even. For k even we reduce to (III.172b) above; for k odd, we have a very similar cofiber sequence, but the sphere bundle does not split: we obtain for the fiber $(\mathbb{CP}^1)^{\mathcal{O}(-1)-2}\simeq\mathbb{CP}^2$:

$$\mathit{MTSpin}\wedge\mathbb{CP}^2\longrightarrow\mathit{MTSpin}^c\longrightarrow\Sigma^4\mathit{MTSpin}^c, \tag{III.174}$$

using the identification $\mathit{MTSpin}\wedge(B\mathrm{U}(1))^{L-2}\simeq\mathit{MTSpin}^c$ from Example III.120. This is the complex analogue of (III.160).

Remark III.175. There is a related example where one uses $L\oplus L^*\to B\mathrm{U}(1)$ instead of $2L$; the corresponding long exact sequence in twisted SU-bordism by Conner-Floyd [CF66, §§6, 14, 17]. When L is odd, the third term in the long exact sequence, corresponding to the sphere bundle, is the bordism of manifolds with *c_1-aspherical structures* or *complex Wall structures*, first introduced by Conner-Floyd [CF66], and also discussed by Stong [Sto68, Chapter VIII]. Complex Wall bordism plays an important role in the calculation of Ω_*^{SU} via the Adams-Novikov spectral sequence [Nov67, §7], and has also been studied in the context of complex orientations [Buh72, CP21, Che22].

Example III.176. The unit sphere bundle to the tautological bundle $V_{n+1}\to B\mathrm{O}(n+1)$ is homotopy equivalent to the map $B\mathrm{O}(n)\to B\mathrm{O}(n+1)$. This is because $S^n\cong\mathrm{O}(n+1)/\mathrm{O}(n)$, so the unit sphere bundle can be described by the mixing construction

$$S^n \times_{\mathrm{O}(n+1)} E\mathrm{O}(n+1) \cong (\mathrm{O}(n+1)/\mathrm{O}(n)) \times_{\mathrm{O}(n+1)} E\mathrm{O}(n+1) \cong E\mathrm{O}(n+1)/\mathrm{O}(n) \cong B\mathrm{O}(n). \tag{III.177}$$

More generally, if $\xi_{n+1}\colon B_{n+1} \to B\mathrm{O}(n+1)$ is an unstable tangential structure and $\xi_n\colon B_n \to B\mathrm{O}(n)$ is the pullback of ξ_{n+1} by $B\mathrm{O}(n) \to B\mathrm{O}(n+1)$, the sphere bundle of $\xi_{n+1}^* V_{n+1}$ is the pullback of $S(V_{n+1}) = B\mathrm{O}(n)$ by ξ_{n+1}, which is ξ_n. If you then pull $\xi_{n+1}^* V_{n+1}$ back across $B_n \to B_{n+1}$, it splits as $V_n \oplus \underline{\mathbb{R}}$, so there is a Smith cofiber sequence

$$\Sigma^{-1} B_n^{n-V_n} \longrightarrow B_{n+1}^{n+1-V_{n+1}} \longrightarrow \Sigma^\infty_+ B_{n+1}. \tag{III.178}$$

This cofiber sequence is due to Galatius-Madsen-Tillmann-Weiss [GMTW09, (3.3), §5]. The spectrum $\Sigma^n B_n^{n-\xi_n^* V_n}$ is often denoted $MT\xi_n$.

IV. DISCUSSION

In this paper, we have presented a long exact sequence in symmetry breaking, relating three maps: the residual family anomaly which captures the equivariant family anomaly when we move around the order parameter space and which gives the obstruction to having a local ρ-defect, the defect anomaly map which reconstructs the bulk anomaly from that of the ρ-defect, and the index map which describes the anomaly of the ρ-defect in an anomaly-free equivariant family on a sphere and describes how the different symmetry breaking patterns are distinguished by their ρ-defects. The kernel of each map is the image of the next, connecting anomaly matching formulas for a given group and representation in all dimensions.

Under Freed-Hopkins' bordism-theoretic classification of reflection-positive invertible field theories [FH21], we have identified our symmetry breaking long exact sequence with the long exact sequence associated with a cofiber sequence of Anderson duals of Thom spectra, whose bordism-theoretic dual is the most general form of the Smith homomorphism. We use this identification to study interesting examples in both math and physics, and to make computations.

There are a few directions for future work we think are promising. The first is to better understand how to formulate the twisted symmetry G_ρ on the lattice. We can use the CPT symmetry to obtain this symmetry in Lorentz invariant, unitary theories, as described in [HKT20a]. However, on the lattice there may be no such symmetry and it is not clear how to proceed.

One approach which seems fruitful is to make contact with the recent anomaly approaches to Lieb-Schultz-Mattis (LSM) theorems [PWJZ17, YJVR18, ET20]. In particular, we can think of the reconstruction of the bulk G_ρ anomaly from the G_ρ anomaly of the ρ-defect as a pure point-group LSM theorem. Indeed, in this case the LSM map of [ET20] (see Appendix I there) is given by cup with the Euler class of ρ and agrees with the defect anomaly map we computed. It seems that the two anomalies are related by the crystalline equivalence principle, which we intend to revisit in future work.

The SBLES is a convenient tool for computing classifications of anomalies in different symmetry classes, since different symmetry breaking patterns can be combined to obtain more constraints on the classification group in terms of lower dimensional groups, and the maps are often determined by exactness. This approach is complementary to the "decorated domain wall" methods [CLV14], which are mathematically formalized as an Atiyah-Hirzebruch spectral sequence [GJF19, WNC21, TW21, SXG23]. In these methods, low dimensional invertible phases are glued together to form higher dimensional ones, allowing one to bootstrap the classification, simply knowing the gluing rules. These rules however, known as the spectral sequence differentials, have still not been completely computed. However, the physical interpretation of these differentials (see for instance [SXG23]) matches the index map we have defined, and it seems possible that all differentials may be computable in terms of it. This is a direction we are currently exploring.

Another interesting direction is what happens in the absence of unitarity. The mathematical backbone of our work generalizes nicely to the nonunitary case: Freed-Hopkins-Teleman [FHT10] classify invertible topological field theories in the absence of a reflection positivity structure using unstable Madsen-Tillmann spectra, and the Smith long exact sequence generalizes to this case (see, e.g., Example III.176). Anomalies of nonunitary theories are not so well-studied, but some examples appear in [CL21, HTY22], and the fact that the Smith long exact sequence generalizes suggests our methods do too.

From there one could ask: the appearance of Madsen-Tillmann spectra in the classification of invertible TFTs is due to theorems of Galatius-Madsen-Tillmann-Weiss [GMTW09], Nguyen [Ngu17], and Schommer-Pries [SP17] establishing Madsen-Tillmann spectra as classifying spectra for bordism (higher) categories. Can one lift the Smith homomorphism to a morphism of bordism categories? This is a question in pure mathematics whose affirmative answer would suggest a generalization of our methods to noninvertible TFTs, and therefore to the symmetry breaking of noninvertible symmetries of field theories, as studied in, e.g., [LTL+21, ABC+23, CHZ23, DAC23, DY23b].

Appendix A: The Long Exact Sequence in Bordism

In this paper, we have examined a long exact sequence of field theories (the SBLES of Section II) induced by a Smith map (as discussed in Section III). In this appendix, we examine the Anderson dual picture to the SBLES, which is the long exact sequence in bordism, and we work out one example for concreteness.

Assume that we begin with a d-dimensional manifold M with an (X, V)-twisted ξ-structure, where V has rank r, and that we want to perform an SSB process corresponding to an order parameter transforming in some representation ρ. We will abuse notation and also write ρ for the k-dimensional vector bundle over X associated to this representation. The long exact sequence of bordism groups[24] corresponding to this setup is

$$\cdots \to \Omega_d^\xi(S(\rho)^{p^*V-r}) \xrightarrow{p} \Omega_d^\xi(X^{V-r}) \xrightarrow{\mathrm{sm}_\rho} \Omega_{d-k}^\xi(X^{V+\rho-k-r}) \xrightarrow{\delta} \Omega_{d-1}^\xi(S(\rho)^{V-r}) \to \cdots \tag{A.1}$$

We write $p\colon S(\rho) \to X$ for the projection, and as usual we interpret the bordism groups of these Thom spectra as twisted ξ-bordism groups.

Compared to the SBLES at the beginning of Section II, p is dual to the restriction map Res_ρ, sm_ρ is dual to the defect map Def_ρ, and the connecting map δ is dual to the index map Ind_ρ. In the first line, $\Omega_D(S(\rho)^{p^*V-r})$ is the bordism group of d-manifolds M equipped with a map

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[24] Note that we use subscripts for the dimension of the bordism group, in contrast to the superscripts used in Section II to denote the group of field theories. This is consistent with the fact that bordism forms a homology theory, while groups of invertible TFTs form a cohomology theory. For the same reason, in the bordism setting, the maps in the long exact sequence go in the opposite direction. Invertible TFTs are discussed in Section I A 4 and Section III B.

$f\colon M\to S(\rho)$, the sphere bundle of ρ, together with a ξ-structure on $TM\oplus f^*p^*V$. $\Omega_d(X^{V-r})$ is the bordism group of d-manifolds equipped with a map to X with the analogous twisted ξ-structure, and $\Omega_{d-k}(X,-\eta+\rho)$ is the bordism group of $(d-k)$-manifolds M equipped with a map f to X with a ξ-structure on $TM\oplus f^*V\oplus f^*\rho$. The map sm_ρ, the Smith homomorphism, lowers the dimension by k and twists the tangential structure condition by ρ.

Tangential structures are discussed in detail in §I A 3 and §III A. Smith homomorphisms are discussed in §III C and §II B as well as in the references.

In this informal discussion, we will neglect to show that our maps are well-defined at the level of bordism. Instead, we will just describe what each map does to a manifold, starting from the leftmost group.

1. p: Let M be a closed d-manifold M with a map $h\colon M\to S(\rho)$ such that $TM\oplus V$ has a ξ-structure, so that M represents a bordism class in $\Omega_d^\xi(S(\rho)^{V-r})$. The image of M under p is represented by the same manifold M with an (X,V)-twisted ξ-structure given by the composition with the projection. That is, equip M with the map $M\xrightarrow{h}S(\rho)\xrightarrow{p}X$.

2. sm_ρ: Now let M be a closed d-manifold equipped with a map $f\colon M\to X$ such that $TM\oplus f^*\eta$ has a ξ-structure. Let $s\colon M\to\rho$ be a generic section of the vector bundle ρ, so it is transverse to the zero section s_0. Then, the intersection $N:=s(M)\pitchfork s_0(M)$ is a $d-k$-dimensional manifold. Let δ be the composite $g\colon N\hookrightarrow M\to X$. Since the normal bundle ν to N satisfies $\nu\cong f^*\rho|_N=g^*\rho$, $TM|_N\cong TN\oplus\nu\cong TN\oplus g^*\rho$, and hence N carries an $(X,V+\rho)$-twisted ξ-structure coming from the (X,V)-twisted ξ-structure on M. Write $\mathrm{sm}_\rho\colon M\mapsto N$.

3. δ: This is the connecting map. Start with a closed $d-k$ manifold N with $(X,V+\rho)$-twisted ξ-structure given by, as above, $g\colon N\to X$ and a ξ-structure on $TN\oplus g^*V\oplus g^*\rho$. Consider the sphere bundle $S(g^*\rho)$ of ρ restricted to N: it has a map to $S(\rho)$ given by inclusion.

   We claim that $S(g^*\rho)$ is the image under δ of N, but it remains to show that $S(g^*\rho)$ has the appropriate tangential structure. This will be a corollary of a general splitting result of tangent bundles of sphere bundles.

   Lemma A.2. *For any vector bundle $p\colon V\to B$, there is an isomorphism of vector bundles, canonical up to a contractible space of choices,*

   $$TS(V)\oplus\underline{\mathbb{R}}\xrightarrow{\cong}\pi^*(TB)\oplus\pi^*(V). \tag{A.3}$$

   Proof. Choose a metric and connection on V; both of these are contractible choices. For any fiber bundle $\pi\colon E\to B$ of smooth manifolds, the choice of connection splits TE as a direct sum of the horizontal subbundle, which is isomorphic to $\pi^*(TB)$, and the vertical tangent bundle $T_vE=\ker(\pi_*)$, which when pulled back to a fiber is the tangent bundle of that fiber.

   Let ν be the normal bundle of $S(V)\hookrightarrow V$. Then there is a canonical isomorphism $T_vS(V)\oplus\nu\cong\pi^*(V)$, which is a parametrized version of the standard isomorphism $TS^n\oplus\nu_{S^n\hookrightarrow\mathbb{R}^{n+1}}\cong\underline{\mathbb{R}}^{n+1}$. Combining this with the previous paragraph,

   $$TS(V)\oplus\nu\cong\pi^*(TB)\oplus T_vS(V)\oplus\nu\cong\pi^*(TB)\oplus\pi^*(V), \tag{A.4}$$

   and the fiberwise outward unit normal vector field trivializes ν. □

If we analyze the vertical and horizontal pieces of the tangent bundle to $S(g^*\rho)$, as explained in Lemma A.2, we find that $T(S(g^*\rho)) \oplus \underline{\mathbb{R}} \cong p^*TN \oplus p^*g^*\rho$. Then, we can pull back the relationship describing the tangential structure of N to see that $p^*TN \oplus p^*g^*\rho \oplus p^*g^*\eta$ over $S(g^*\rho)$ has a ξ-structure. So, $T(S(g^*\rho)) \oplus \underline{\mathbb{R}} \oplus p^*g^*V$ has a ξ-structure, and thus $S(g^*\rho)$ has a $(S(\rho), V)$-twisted ξ-structure.

1. Example Long Exact Sequence: $\mathrm{Pin}^- \rightsquigarrow \mathrm{Pin}^+$

Let us go through the long exact sequence of bordism groups for the Smith map III.160. In this case, the Smith homomorphism is a map

$$\mathrm{sm}_{2\sigma}\colon \Omega_d^{\mathrm{Pin}^-} \longrightarrow \Omega_{d-2}^{\mathrm{Pin}^+} \tag{A.5}$$

between the bordism group of d-dimensional pin$^-$ manifolds to the bordism group of $(d-2)$-dimensional pin$^+$ manifolds, described by sending a pin$^-$ manifold M to any closed submanifold N whose homology class is Poincaré dual to $w_1(M)^2$. Alternatively, in view of Definition III.35, we could define $\mathrm{sm}_{2\sigma}$ by choosing a section s of the pullback of 2σ to M transverse to the zero section, then letting N be the zero locus of s. Recall from Example III.120 that a pin$^-$ structure is a trivialization of $w_1(M)^2 + w_2(M)$, while a pin$^+$ structure on M is equivalent to a trivialization of $w_2(M)$. Equivalently, a pin$^-$ manifold M admits a spin structure on $TM \oplus \det(M)$, while a pin$^+$ manifold M admits a spin structure on $TM \oplus 3\det(M)$. These conditions mean that if N is Poincaré dual to $w_1(M)^2$ inside a pin$^-$ manifold M, then N acquires a pin$^+$ structure.

Using the techniques explained in III E, we can find another sequence of bordism groups which, in degrees $d+1$, resp. d measures the extent to which this Smith homomorphism is injective, resp. surjective. The relevant family of bordism groups is $\widetilde{\Omega}_{*+1}^{\mathrm{Spin}}(\mathbb{RP}^2)$: the bordism groups of spin manifolds X equipped with maps $f\colon X \to \mathbb{RP}^2$, modulo the subgroup for which f is null-homotopic. Equivalently, we may consider the twisted bordism groups $\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$. Elements of this group are represented by manifolds N with maps $f\colon N \to \mathbb{RP}^1$ such that $TN \oplus f^*\sigma$ is spin.

We next describe the other two maps that appear alongside $\mathrm{sm}_{2\sigma}$ in the bordism long exact sequence and provide several lemmas that help us understand the geometry.

Definition A.6. Define a map $p\colon \Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \to \Omega_*^{\mathrm{Pin}^-}$ by sending $(N, f\colon N \to \mathbb{RP}^1)$ to N.

Lemma A.7. *If N has an $(\mathbb{RP}^1, \sigma)$-twisted spin structure, then N has a canonical pin$^-$ structure (so the map p lands in pin$^-$ bordism as claimed).*

Proof. The orientation of $TN \oplus f^*\sigma$ is equivalent data to an isomorphism $\mathrm{Det}(TN) \xrightarrow{\cong} f^*\sigma$, so we obtain a spin structure on $TN \oplus \mathrm{Det}(TN)$, i.e. a pin$^-$ structure. ☐

In addition to $\mathrm{sm}_{2\sigma}$ and p, we will use a third map $\delta\colon \Omega_*^{\mathrm{Pin}^+} \to \Omega_{*+1}^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$. The map δ sends a pin$^+$ manifold M to the total space of the sphere bundle $S(2\mathrm{Det}(TM))$. The key to δ is showing $S(2\mathrm{Det}(TM))$ has a $(\mathbb{RP}^1, \sigma)$-twisted spin structure; in particular, we must cook up a map to $\mathbb{RP}^1$.

We care a lot about $S(2\mathrm{Det}(TM))$ in this section because it pulls back from the fiber of the Smith map, the sphere bundle of $2\sigma \to B\mathbb{Z}/2$.

Definition A.8. Given a pin$^+$ manifold M, choose a metric on $\mathrm{Det}(TM)$ (a contractible choice); then, given $x \in M$ and $p, q \in \sigma_x$ with $\sqrt{|p^2| + |q^2|} = 1$, so that $(x, p, q) \in S(2\mathrm{Det}(TM))$, the two sections of $\pi^*(2\mathrm{Det}(TM))$

$$\begin{aligned}(x,p,q) &\mapsto (p,q)\\ (x,p,q) &\mapsto (-q,p)\end{aligned} \tag{A.9}$$

are everywhere linearly independent, so $\pi^*(2\mathrm{Det}(TM))$ is canonically trivial. This allows us to define a map $\varphi_M\colon S(2\mathrm{Det}(TM)) \to \mathbb{RP}^1$: given $(x,p,q) \in S(2\mathrm{Det}(TM))$ as above, $(p,q) \in (\pi^*(2\mathrm{Det}(TM)))_{(x,p,q)}$, which is canonically identified with $\mathbb{R}^2$; then send (p,q) to its image $[p:q] \in \mathbb{RP}^1$ (using that p and q are never both 0).

Definition A.10. Let $\delta\colon \Omega_*^{\mathrm{Pin}^+} \to \Omega_{*+1}^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ be the map sending $M \mapsto (S(2\mathrm{Det}(TM)), \varphi_M)$, where φ_M is defined above in Definition A.8.

If $\sigma \to \mathbb{RP}^1$ is the Möbius bundle, then $\varphi_M^*(\sigma) = \pi^*(\mathrm{Det}(TM))$.

Lemma A.11. *If M is pin$^+$, $(S(2\mathrm{Det}(TM)), \varphi_M)$ has a canonical $(\mathbb{RP}^1, \sigma)$-twisted spin structure, up to a contractible space of choices, so that δ lands in $\Omega_{*+1}^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ as claimed.*

Proof. Plugging in $V = 2\mathrm{Det}(TM)$ to Lemma A.2, we learn

$$TS(2\mathrm{Det}(TM)) \oplus \underline{\mathbb{R}} \cong \pi^*(TM) \oplus 2\pi^*(\mathrm{Det}(TM)). \tag{A.12a}$$

Since $\varphi_M^*(\sigma) \cong \pi^*(\mathrm{Det}(TM))$,

$$TS(2\mathrm{Det}(TM)) \oplus \varphi_M^*(\sigma) \oplus \underline{\mathbb{R}} \cong \pi^*(TM) \oplus 3\pi^*(\mathrm{Det}(TM)). \tag{A.12b}$$

Since M is pin$^+$, the right-hand-side of (A.12b) is spin, so the left-hand side is too; by two-out-of-three, this means $TS(2\mathrm{Det}(TM)) \oplus \varphi_M^*(\sigma)$ is also spin. ☐

The maps $\mathrm{sm}_{2\sigma}$, p, and δ assemble into a long exact sequence in bordism, as we will draw in Figure 6. But to write out this long exact sequence, we need to know the relevant bordism groups in low dimensions. Giambalvo [Gia73b, §2, §3] computes $\Omega_k^{\mathrm{Pin}^+}$ for $k \le 12$, more than good enough for us, and gives generating manifolds in all degrees we need except $k = 2, 3$ (though see [KT90a] for a correction); the rest were given by Kirby-Taylor [KT90b, Proposition 3.9, Theorem 5.1]. Anderson-Brown-Peterson [ABP69, Theorem 5.1] computed pin$^-$ bordism groups, with generating manifolds again described by Giambalvo [Gia73b, Theorem 3.4] and Kirby-Taylor [KT90b, Theorem 2.1]. However, the twisted spin bordism of $\mathbb{RP}^1$ is less well-documented, so we calculate it here, using another Smith homomorphism.

Lemma A.13. *There is an abelian group A of order 4 such that*

$$\Omega_k^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \cong \begin{cases} \mathbb{Z}/2, & k = 0,1,3,4\\ A, & k = 2\\ 0, & k = 5. \end{cases} \tag{A.14}$$

Proof. We may start the computation of $\Omega^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ using the observation of Kirby and Taylor [KT90b] that the degree two map

$$\mathbb{S} \xrightarrow{\cdot 2} \mathbb{S} \longrightarrow \Sigma_+^{\infty-1}\mathbb{RP}^2 \tag{A.15}$$

of Example III.136 induces multiplication by two on spin bordism.[25] Taking the spin bordism long exact sequence of A.15 and inputting the spin bordism of a point, we may deduce the groups $\Omega^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma)$ in low dimensions, up to one ambiguity, as indicated in Figure 5. ☐

$$\begin{tikzcd}
\ast & \Omega_*^{\mathrm{Spin}} & \Omega_*^{\mathrm{Spin}} & \Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma) \\
5 & 0 & 0 & 0 \\
4 & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \\
3 & 0 & 0 & \mathbb{Z}/2 \arrow[dll, out=-30, in=150] \\
2 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & A \arrow[dll, out=-30, in=150] \\
1 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[dll, out=-30, in=150] \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2
\end{tikzcd}$$

FIG. 5: Long exact sequence in spin bordism partially determining $\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma)$

Remark A.16. To address the question as to whether A is isomorphic to $\mathbb{Z}/4$ or $\mathbb{Z}/2\oplus\mathbb{Z}/2$, one could appeal to geometric arguments or an Adams spectral sequence calculation, but it turns out that the Smith long exact sequence that we will study in Figure 6 provides a cleaner argument that $A\cong\mathbb{Z}/4$.

We will provide some explicit descriptions of the interesting maps in this sequence using knowledge of the generators of each bordism group, which for *pin*$^+$ and *pin*$^-$ may be found in [KT90b]. For the twisted spin bordism of $\mathbb{RP}^1$, we use what we learned in Lemma A.13.

(a) $\ast=0$: The group $\Omega_0^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma)\cong\mathbb{Z}/2$ is generated by the class of the point equipped with the inclusion i into $\mathbb{RP}^1$. The condition of $T\mathrm{pt}\oplus i^*\sigma$ being spin is satisfied since $i^*\sigma$ is trivial. The map f forgets i, so sends this generator to the point with its pin$^-$ structure, which is a generator of $\Omega_0^{\mathrm{Pin}^-}\cong\mathbb{Z}/2$.

(b) $\ast=1$: Consider the circle with spin structure induced from its Lie group framing, denoted S^1_{nb}, equipped with the degree two map $\phi\colon S^1\to S^1\simeq\mathbb{RP}^1$. If $x\in H^1(\mathbb{RP}^1;\mathbb{Z}/2)$ is the generator, we have

$$w(TS^1\oplus\phi^*\sigma)=w(TS^1)\phi^*w(\sigma)=(1)(1+2\phi^*(x))=1, \tag{A.17}$$

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[25] Note that $\Omega_*^{\mathrm{Spin}}(\Sigma_+^{\infty-1}\mathbb{RP}^2)\cong\Omega_{*+1}^{\mathrm{Spin}}(\mathbb{RP}^2)\cong\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma)$, so this sequence indeed includes the twisted bordism groups we need. One can begin to justify the first isomorphism by using that the Thom space of $\sigma\to\mathbb{RP}^1$ is $\mathbb{RP}^2$.

$$\begin{tikzcd}
\ast & \Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma) & \Omega_*^{\mathrm{Pin}^-} & \Omega_{*-2}^{\mathrm{Pin}^+} \\
6 & 0 & \mathbb{Z}/16 \arrow[r, "\text{(g)}"] & \mathbb{Z}/16 \\
5 & 0 & 0 & \mathbb{Z}/2 \arrow[dll, "\text{(f)}"'] \\
4 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 \arrow[dll, "\text{(e)}"'] \\
3 & \mathbb{Z}/2 & 0 & 0 \\
2 & \mathbb{Z}/4 \arrow[r, "\text{(c)}"] & \mathbb{Z}/8 \arrow[r, "\text{(d)}"] & \mathbb{Z}/2 \\
1 & \mathbb{Z}/2 \arrow[r, "\text{(b)}"] & \mathbb{Z}/2 & 0 \\
0 & \mathbb{Z}/2 \arrow[r, "\text{(a)}"] & \mathbb{Z}/2 & 0
\end{tikzcd}$$

FIG. 6: Bordism Long Exact Sequence for $\mathrm{Pin}^- \rightsquigarrow \mathrm{Pin}^+$

so (S^1_{nb}, ϕ) has an $(\mathbb{RP}^1, \sigma)$-twisted spin structure. The map p forgets ϕ, so sends the bordism class of (S^1_{nb}, ϕ) to S^1_{nb}, which generates $\Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}/2$ [KT90b, Theorem 2.1].

(c) $* = 2$ (part 1): Exactness of the Smith long exact sequence at $\Omega_2^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \cong A$ implies that A maps injectively to $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$, so $A \cong \mathbb{Z}/4$, and we have resolved the extension problem from Lemma A.13.

The Klein bottle K is an S^1-bundle over $\mathbb{RP}^1$, with the monodromy of the fiber S^1 around the base given by reflection. Therefore $K = S(\sigma \oplus \underline{\mathbb{R}})$ as S^1-bundles over $\mathbb{RP}^1$. Let $\pi\colon K \to \mathbb{RP}^1$ be the bundle map; then Lemma A.2 defines an isomorphism $TK \oplus \underline{\mathbb{R}} \cong \pi^*(\sigma) \oplus \underline{\mathbb{R}}^2$ (using the Lie group trivialization of $T\mathbb{RP}^1$). The Möbius bundle σ represents the nonzero class in $[\mathbb{RP}^1, B\mathrm{O}_1] = \pi_1(B\mathrm{O}_1) \cong \mathbb{Z}/2$, so 2σ is trivializable, and in particular spin, meaning that (K, π) admits an $(\mathbb{RP}^1, \sigma)$-twisted spin structure (in fact, it admits 4).

That (K, π) generates $\Omega_2^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma)$ depends on which of the four $(\mathbb{RP}^1, \sigma)$-twisted spin structures one chooses. Specifically, each $(\mathbb{RP}^1, \sigma)$-twisted spin structure restricts to a spin structure on the fiber S^1, and we need this to be the spin structure on S^1 induced by the Lie group framing. Two of the four $(\mathbb{RP}^1, \sigma)$-twisted spin structures satisfy this. To then see that either of these two Klein bottles generates, one can play with the Smith long exact sequence from Example III.136

$$\cdots \longrightarrow \Omega_k^{\mathrm{Spin}} \xrightarrow{\cdot 2} \Omega_k^{\mathrm{Spin}} \longrightarrow \Omega_k^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \xrightarrow{\mathrm{sm}_\sigma} \Omega_{k-1}^{\mathrm{Spin}} \longrightarrow \cdots \tag{A.18}$$

to see that $\mathrm{sm}_\sigma\colon \Omega_2^{\mathrm{Spin}}(\mathbb{RP}^1, \sigma) \to \Omega_1^{\mathrm{Spin}}$ is the unique surjective map $\mathbb{Z}/4 \to \mathbb{Z}/2$; the

Poincaré dual to $w_1(\sigma)$ is represented by the fiber S^1 in K, which we chose to have the Lie group spin structure, so $\mathrm{sm}_\sigma(K,\pi) = S^1_{nb}$, which generates Ω_1^{Spin}, implying (K,π) generates $\Omega_2^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma)$.

Now take $f(K,\pi)$, which amounts to forgetting π and finding the pin$^-$ bordism class of K. The Arf-Brown-Kervaire invariant is a complete invariant $\Omega_2^{\mathrm{Pin}^-} \overset{\cong}{\to} \mathbb{Z}/8$ [Bro71, KT90b], so it suffices to compute this invariant on K, as has been explicitly worked out in [Tur20, §II.D]. Our choice of the nonbounding spin structure on the fiber implies that the Arf-Brown-Kervaire map $\Omega_2^{\mathrm{Pin}^-} \overset{\cong}{\to} \mathbb{Z}/8$ sends $[K] \mapsto \pm 2$, so $f\colon \mathbb{Z}/4 \to \mathbb{Z}/8$ sends $1 \mapsto 2$, as required by exactness.

(d) $* = 2$ (part 2): There are two pin$^-$ structures on $\mathbb{RP}^2$, and both are generators of $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ [KT90b, §3]. Pick either of these pin$^-$ structures; the class $w_2(\sigma) \in H^2(\mathbb{RP}^2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ is a generator, and the Smith homomorphism $\Omega_2^{\mathrm{Pin}^-} \to \Omega_0^{\mathrm{Pin}^+}$ maps the input $\mathbb{RP}^2$ to the Poincaré dual of $w_2(2\sigma)$. The class $PD(w_2(2\sigma))$ is $1 \in H_0(\mathbb{RP}^2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ and is represented by a single pin$^+$ point. The class of the point also corresponds to the zero-dimensional intersection of the zero section and a generic section of 2σ.

(e) $* = 4 \to 3$: $\Omega_2^{\mathrm{Pin}^+} \cong \mathbb{Z}/2$ is generated by the Klein bottle K, where as before we need the nonbounding spin structure on the S^1 fiber of K. The connecting map δ sends K to $S(2\mathrm{Det}(K))$; we saw above in part (c) that $\mathrm{Det}(K) \cong \sigma$ and 2σ is trivialized over K, so $S(2\mathrm{Det}(K)) \cong S^1 \times K$. Tracking the (twisted) spin structures through this argument, one sees that we obtain the nonbounding spin structure on S^1, so $g(K) = [S^1_{nb} \times K] \in \Omega_3^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma) \cong \mathbb{Z}/2$, and $[S^1_{nb} \times K]$ is indeed the generator.[26]

(f) $* = 5 \to 4$: $\Omega_3^{\mathrm{Pin}^+} \cong \mathbb{Z}/2$ is generated by $S^1_{\mathrm{nb}} \times K$ [KT90b, §5], and $\Omega_4^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma) \cong \mathbb{Z}/2$ is generated by $S^1_{nb} \times S^1_{nb} \times K$, with the map to $\mathbb{RP}^1$ induced from the fiber bundle $K \to \mathbb{RP}^1$ from part (c).[27] Thus the story is the same as in (e), crossed with S^1_{nb}.

(g) $* = 6$: The group $\Omega_6^{\mathrm{Pin}^-}$ is generated by $\mathbb{RP}^6$ with either of its two pin$^+$ structures, while $\Omega_4^{\mathrm{Pin}^+}$ is generated by $\mathbb{RP}^4$ with either of its two pin$^-$ structures. Since the normal bundle to $\mathbb{RP}^4$ inside $\mathbb{RP}^6$ is indeed the restriction of 2σ, $\mathbb{RP}^4$ represents the Poincaré dual homology class to $e(2\sigma)$ and is the image of the Smith homomorphism applied to $\mathbb{RP}^6$.

Appendix B: Why we use the cobordism, rather than the cohomology, Euler class

The Smith homomorphism is often defined by taking a Poincaré dual of the $\mathbb{Z}$- or $\mathbb{Z}/2$-cohomology Euler class of a vector bundle $V \to X$, for example in [KTTW15, COSY20, HKT20a]. However, in Definition III.73, we used a different and more abstract definition: the Smith homomorphism for twisted ξ-bordism should use the (possibly twisted) ξ-cobordism Euler class. The purpose of this appendix is to explain that additional effort: we will walk through a concrete, low-dimensional

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[26] Another way to see this is that because the connecting morphism in the Smith long exact sequence is obtained from a map of spectra by taking homotopy groups, the connecting morphism commutes with the $\pi_*(\mathbb{S})$-actions on $\Omega_*^{\mathrm{Pin}^+}$ and $\Omega_*^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma)$. The Pontrjagin-Thom theorem identifies this $\pi_*(\mathbb{S})$-action on bordism groups with taking products with stably framed manifolds; focusing specifically on the nonzero element of $\pi_1(\mathbb{S})$, which is represented by the bordism class of S^1_{nb}. Thus, since $\times S^1_{nb}\colon \Omega_2^{\mathrm{Pin}^+} \to \Omega_3^{\mathrm{Pin}^+}$ is an isomorphism [KT90b, §5] and the Smith maps $\Omega_{k-2}^{\mathrm{Pin}^+} \to \Omega_k^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma)$ are isomorphisms for $k = 3, 4$ as we saw in the long exact sequence, then $\times S^1_{nb}\colon \Omega_3^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma) \to \Omega_4^{\mathrm{Spin}}(\mathbb{RP}^1,\sigma)$ is also an isomorphism.

[27] Another choice of generator is the K3 surface with trivial map to $\mathbb{RP}^1$, as follows from (A.18). The complicated topology of the K3 surface makes this generator harder to work with explicitly.

example where the cohomological Euler class does not produce a well-defined Smith homomorphism, and show that the cobordism Euler class does suffice.

Recall that a spin^h structure is a $(B\mathrm{SO}(3), V_3)$-twisted spin structure, where $V_3 \to B\mathrm{SO}(3)$ is the tautological representation. Then, as we discussed in (III.166a), there is a Smith homomorphism

$$\mathrm{sm}_V \colon \Omega_k^{\mathrm{Spin}^h} \to \Omega_{k-3}^{\mathrm{Spin}}(B\mathrm{SO}(3)). \tag{B.1}$$

Theorem B.2. *Give S^4 the spin^h structure whose SO_3-bundle is classified by either map $S^4 \to B\mathrm{SO}(3)$ whose homotopy class generates $\pi_4(B\mathrm{SO}(3)) \cong \mathbb{Z}$.*

1. *Exactness forces $\mathrm{sm}_V(S^4)$ to be the bordism class of S^1_{nb} with constant map to $B\mathrm{SO}(3)$) in $\Omega_1^{\mathrm{Spin}}(B\mathrm{SO}(3))$.*
2. *$e(V) \in H^3(S^4;\mathbb{Z}) = 0$, and there is no way to assign every smooth representative of the Poincaré dual of $e(V)$ a spin structure whose bordism class equals that of S^1_{nb}.*
3. *The spin cobordism Euler class of V is nonzero, and all smooth representatives of its Poincaré dual have the spin bordism class of S^1_{nb} and a constant map to $B\mathrm{SO}(3)$.*

This is why we use cobordism Euler classes.

We work with $\xi = \mathrm{Spin}$ and its twists throughout this appendix; see Remark B.38 for other tangential structures. Let *ko* denote the connective real K-theory spectrum; work of Anderson-Brown-Peterson [ABP67] shows that the Atiyah-Bott-Shapiro map $\mathit{MTSpin} \to \mathit{ko}$ [ABS64] is 7-connected, meaning that as long as we restrict to manifolds of dimension 7 and below, we may replace twisted spin bordism with twisted *ko*-homology; in particular, we will work with *ko*-cohomology Euler classes.

Another consequence of the Atiyah-Bott-Shapiro map is that vector bundles with spin structure are oriented for *ko*-cohomology, meaning that if $V \to X$ is a spin vector bundle, the Euler class $e^{ko}(V)$ that a priori lives in $ko^r(X^{V-r})$ in fact can be passed by the Thom isomorphism to $e^{ko}(V) \in ko^r(X)$.

Recall the exceptional isomorphism $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$, and recall that $ko^* \cong \mathbb{Z}[\eta, v, w]/(2\eta, \eta^3, 2v, 4w - v^2)$ with $|\eta| = -1$, $|v| = -4$, and $|w| = -8$.[28]

The following result is stated without proof by Davis-Mahowald [DM79, §2]; see Bruner-Greenlees [BG10, Theorem 5.3.1] for a proof.

Proposition B.3. *There is an isomorphism of ko^*-modules $ko^*(B\mathrm{Sp}(1)) \cong ko^*[[p_1^{\mathbb{H}}]]$ with $|p_1^{\mathbb{H}}| = 4$.*

The class $p_1^{\mathbb{H}}$ is called the *first symplectic ko-Pontrjagin class*. The specific isomorphism in Proposition B.3 can be fixed uniquely by requiring that the image of $p_1^{\mathbb{H}}$ under $ko \to H\mathbb{Z}$ is the usual first symplectic Pontrjagin class, which is positive on the tautological quaternionic line bundle over $\mathbb{HP}^1$.

Given a spin vector bundle $V \to X$, let $\mathcal{S}_V \to X$ be the associated spinor bundle, which is the quaternionic line bundle associated to the accidental isomorphism $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$.

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[28] The negative grading is a feature of generalized cohomology: for any spectrum E, $E^k(\mathrm{pt}) = E_{-k}(\mathrm{pt}) = \pi_{-k}(E)$.

Theorem B.4. *Let $V \to X$ be a rank-3 vector bundle with spin structure. Then $e^{\mathbb{Z}}(V) \in H^3(X;\mathbb{Z})$ and $e^{\mathbb{Z}/2}(V) \in H^3(X;\mathbb{Z}/2)$ both vanish, and*

$$e^{ko}(V) = \eta p_1^{\mathbb{H}}(\mathcal{S}_V) \in ko^3(X). \tag{B.5}$$

Remark B.6. Theorem B.4 is new as far as we know. It is a subtle result in that several standard techniques for computing *ko*-Euler classes do not provide any information.

1. Analogous to the formula for *ku*-Euler classes of complex vector bundles, there is a formula for *ko*-Euler classes of quaternionic vector bundles (see, e.g., Davis-Mahowald [DM79]), but a rank-3 vector bundle cannot be quaternionic.
2. For non-quaternionic vector bundles, one could compare with Euler classes in *ku*-cohomology or ordinary cohomology, as Davis-Mahowald (*ibid.*, §2) do, but $H^3(B\mathrm{Spin}(3);\mathbb{Z}) = 0$, so comparing with the $\mathbb{Z}$-cohomological Euler class provides no information. Moreover, $ku^*(B\mathrm{Spin}(3))$ is a free ku^*-algebra on generators in even degrees [BG10, Theorem 5.3.1], so $ku^3(B\mathrm{Spin}(3)) = 0$, and therefore we can learn nothing even by comparing to *ku*.
3. It is more fruitful to compare to *KO*-Euler classes, understood in many cases (see [Cra91, Corollary 3.37(i)] and [FH00, Footnote 13]), but not in rank 3.
4. The use of the splitting principle to compute Euler classes is stymied by the fact that maximal tori in Spin(3) can be conjugated into the usual embedding Spin(2) → Spin(3), so the pullback of the Euler class to the maximal torus vanishes, as the pulled-back vector bundle will have a nonvanishing section.

Taking Theorem B.4 for granted now, let us dig into Theorem B.2.

Proof of Theorem B.2 assuming Theorem B.4. Recall from (III.166a) that the Smith homomorphism $\mathrm{sm}_V \colon \Omega_k^{\mathrm{Spin}^h} \to \Omega_{k-3}^{\mathrm{Spin}}(B\mathrm{SO}(3))$ belongs to a long exact sequence whose third term is spin^c bordism:

$$\cdots \to \Omega_2^{\mathrm{Spin}}(B\mathrm{SO}(3)) \to \Omega_4^{\mathrm{Spin}^c} \to \Omega_4^{\mathrm{Spin}^h} \xrightarrow{\mathrm{sm}_V} \Omega_1^{\mathrm{Spin}}(B\mathrm{SO}(3)) \to \dots \tag{B.7}$$

From Stong [Sto68, Chapter XI] we know $\Omega_4^{\mathrm{Spin}^c} \cong \mathbb{Z}^2$, from Freed-Hopkins [FH21, Theorem 9.97] we know $\Omega_4^{\mathrm{Spin}^h} \cong \mathbb{Z}^2$, and from Wan-Wang [WW19, §5.5.3] we know $\Omega_1^{\mathrm{Spin}}(B\mathrm{SO}(3)) \cong \mathbb{Z}/2$ and $\Omega_2^{\mathrm{Spin}}(B\mathrm{SO}(3))$ is torsion. Plugging this into (B.7), we see that sm_V is surjective.

Wan-Wang's argument implies that the map $\Omega_1^{\mathrm{Spin}} \to \Omega_1^{\mathrm{Spin}}(B\mathrm{SO}(3))$ choosing the trivial SO(3)-bundle is an isomorphism, so the generator of $\Omega_1^{\mathrm{Spin}}(B\mathrm{SO}(3))$ is any nonbounding spin 1-manifold with trivial SO(3)-bundle. Hu [Hu, §3.5] shows that $\mathbb{CP}^2$ and S^4 generate $\Omega_4^{\mathrm{Spin}^h}$, where $\mathbb{CP}^2$ has spin^h structure induced from its spin^c structure via the standard inclusion U(1) ≅ SO(2) → SO(3), and S^4 has spin^h structure whose principal SO(3)-bundle $V \to S^4$ is induced from the tautological quaternionic line bundle on $\mathbb{HP}^1 \cong S^4$: this has an associated Sp(1)-bundle, and we quotient by $\{\pm 1\}$ to get an SO(3)-bundle. In particular, $\mathbb{CP}^2$ is in the image of $\Omega_4^{\mathrm{Spin}^c} \to \Omega_4^{\mathrm{Spin}^h}$, so because sm_V is surjective, $\mathrm{sm}_V(S^4, V)$ must be S^1_{nb} with trivial map to $B\mathrm{SO}(3)$, proving the first part of the theorem.

Because $H^3(S^4;\mathbb{Z})$ and $H^3(S^4;\mathbb{Z}/2)$ both vanish, the $\mathbb{Z}$ and $\mathbb{Z}/2$ cohomology Euler classes of V are zero. Therefore any null-homologous 1-manifold in S^4 (i.e. any closed, oriented 1-manifold mapping to S^4) is a smooth representative of the Poincaré dual of $e(V)$. Most of these manifolds, such as the standard $S^1 \subset S^4$, can be given a nonbounding spin structure, but the empty submanifold cannot, even though it is Poincaré dual to $e(V)$. This proves the second part of the theorem.

As discussed above, the Atiyah-Bott-Shapiro map is 7-connected, and therefore for discussing degree-3 spin cobordism of S^4, we may use *ko*-cohomology without losing information. The Atiyah-Hirzebruch spectral sequence quickly implies

$$ko^*(S^4) \cong ko^*[z]/(z^2),\ |z| = 4. \tag{B.8}$$

In particular, $ko^3(S^4) \cong \mathbb{Z}/2$, generated by ηz.

The spinor bundle of V is the quaternionic line bundle associated to the identification $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$. Since V came from the identification $S^4 \cong \mathbb{HP}^1$, the spinor bundle of V is the tautological quaternionic line bundle $L_{\mathbb{H}} \to \mathbb{HP}^1$. This is classified by the inclusion $j\colon \mathbb{HP}^1 \to \mathbb{HP}^\infty \simeq B\mathrm{Sp}(1)$ as the 4-skeleton; considering the map of Atiyah-Hirzebruch spectral sequences for *ko*-cohomology induced by j shows that $p_1^{\mathbb{H}} \in ko^4(B\mathrm{Sp}(1))$ pulls back by j to $z \in ko^4(S^4)$. Thus by Theorem B.4, $e^{ko}(V) = \eta z \neq 0$ in $ko^3(S^4)$.

Because $ko^3(S^4)$ has only one nonzero element, its Poincaré dual must be the nonzero element x of $ko_1(S^4) \cong \mathbb{Z}/2$. Pulling back to spin bordism, the same argument we made for $B\mathrm{SO}(3)$ shows that the smooth representatives of x are precisely the nonbounding spin 1-manifolds with null-bordant map to S^4 — and composing with the map $S^4 \to B\mathrm{SO}(3)$ classifying V, we have shown that every smooth representative of the Poincaré dual of $e^{ko}(V)$ (hence also the spin cobordism Euler class) represents the image of (S^4, V) under the Smith homomorphism. ☐

The rest of this appendix is devoted to proving Theorem B.4.

Lemma B.9. *Let X be a CW complex with finitely many cells in each dimension, and whose cells are concentrated solely in even degrees. Suppose that the images of the attaching maps of X in ko-homology are never of the form $w^s\eta$ times any other class, where $s > 0$. Then there is an equivalence of ko-module spectra from $ko \wedge X_+$ to a sum of shifts of copies of ko and ku.*

Here *ku* is a *ko*-module in the usual way, i.e. through the complexification map $c\colon ko \to ku$. In essence, this is downstream from the way in which $\mathbb{C}$ is an $\mathbb{R}$-module.

Proof. It suffices to prove this when X is a finite-dimensional CW complex, and then take the colimit. Thus we may induct on the dimension of X, as the result is vacuously true when X is 0-dimensional.

If X is n-dimensional (so n is even), with $(n-2)$-skeleton X', then X is the cofiber of the map

$$\bigvee_{i=1}^{N} S^{2n-1} \longrightarrow X', \tag{B.10a}$$

classified by $(f_1, \dots, f_N) \in \pi_{2n-1}(X')$, which attaches the n-cells of X. Smash with *ko* and apply the inductive assumption to deduce that $ko \wedge X$ is the cofiber of a map of *ko*-modules

$$(ko \wedge f_1, \dots, ko \wedge f_N)\colon \bigvee_{i=1}^{N} \Sigma^{2n-1} ko \longrightarrow \bigvee_{i\in\mathcal{I}} \Sigma^{2i} ko \vee \bigvee_{j\in\mathcal{J}} \Sigma^{2j} ku. \tag{B.10b}$$

A map of ko-modules $\Sigma^\ell ko \to M$ is equivalent data to a map of spectra $\Sigma^\ell \mathbb{S} \to M$. Therefore the homotopy class of each $ko \wedge f_i$ is an element of

$$\bigoplus_{i\in\mathcal{I}} \pi_{2i-(2n-1)} ko \oplus \bigoplus_{j\in\mathcal{J}} \pi_{2j-(2n-1)} ku, \tag{B.11}$$

and knowledge of these classes for $1 \le i \le N$ suffices to recover $ko \wedge X$ as the cofiber. Moreover, we can compute the cofiber by attaching one sphere at a time, computing the cofiber, and continuing.

The first observation is that $ko \wedge f_i$ is trivial on the $\Sigma^{2j} ku$ summands, because the odd-degree homotopy groups of ku vanish. And on the $\Sigma^{2i} ko$ summands, our only nonzero choices are $w^s\eta$, where $w \in \pi_8(ko)$ is the Bott class. By assumption, $w^s\eta$ does not occur for $s > 0$, so we only need to check the cofibers of 0 and η. The cofiber of $0\colon \Sigma^k ko \to ko$ is $ko \vee \Sigma^{2k+1} ko$, and Wood's theorem implies the cofiber of $\eta\colon \Sigma ko \to ko$ is ku. □

We will want to know the specific factors in the decomposition promised by Lemma B.9.

Definition B.12. Let $\mathcal{A}$ denote the mod 2 Steenrod algebra and $\mathcal{A}(1) \coloneqq \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle \subset \mathcal{A}$, which acts on the $\mathbb{Z}/2$-cohomology of any space. Since $\mathcal{A}(1)$ is $\mathbb{Z}$-graded ($|\mathrm{Sq}^i| = i$), we consider only $\mathbb{Z}$-graded $\mathcal{A}(1)$-modules. Then, consider the following two $\mathcal{A}(1)$-modules.

1. $\mathbb{Z}/2$ in degree 0 with trivial $\mathcal{A}(1)$-action.
2. $C\eta$, which consists of two $\mathbb{Z}/2$ summands in degrees 0 and 2, with a nontrivial Sq^2-action from the former to the latter.

If $k \in \mathbb{Z}$ and M is an $\mathcal{A}(1)$-module, we will let $\Sigma^k M$ (a *suspension* or *shift* of M) denote the same ungraded $\mathcal{A}(1)$-module with the grading of each homogeneous element increased by k. For example, this means that $C\eta \cong \Sigma^{-2} \widetilde{H}^*(\mathbb{CP}^2; \mathbb{Z}/2)$ as $\mathcal{A}(1)$-modules.

Lemma B.13. *With X as in Lemma B.9, there is an $\mathcal{A}(1)$-module isomorphism from $H^*(X;\mathbb{Z}/2)$ to a sum of shifts of $\mathbb{Z}/2$ and $C\eta$.*

Proof. Since X only has cells in even degrees, $H^*(X;\mathbb{Z}/2)$ is concentrated in even degrees, meaning Sq^1 acts trivially on $H^*(X;\mathbb{Z}/2)$. Thus the problem reduces to how Sq^2 can act; the Adem relation $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1$ means that $\mathrm{Sq}^2\mathrm{Sq}^2$ acts trivially on $H^*(X;\mathbb{Z}/2)$. Therefore if $\mathrm{Sq}^2(x) \ne 0$ for any $x \in H^k(X;\mathbb{Z}/2)$, then x and $\mathrm{Sq}^2(x)$ generate a $\Sigma^k C\eta \subset H^*(X;\mathbb{Z}/2)$, and this is a direct summand, because x cannot be Sq^1 or Sq^2 of anything. After doing this for all x which Sq^2 acts nontrivially on, the result is a direct sum of shifts of the trivial $\mathcal{A}(1)$-module $\mathbb{Z}/2$. □

Corollary B.14. *Let X be as in Lemma B.9. If the decomposition of $H^*(X;\mathbb{Z}/2)$ from Lemma B.13 is of the form*

$$H^*(X;\mathbb{Z}/2) \cong \bigoplus_{i\in\mathcal{I}} \Sigma^{m_i}\mathbb{Z}/2 \oplus \bigoplus_{j\in\mathcal{J}} \Sigma^{m_j} C\eta, \tag{B.15a}$$

then there is an equivalence of ko-modules

$$ko \wedge X_+ \simeq \bigvee_{i\in\mathcal{I}} \Sigma^{m_i} ko \vee \bigvee_{j\in\mathcal{J}} \Sigma^{m_j} ku. \tag{B.15b}$$

Proof. By Lemma B.9, we know there are n_k, n_ℓ such that

$$ko \wedge X_+ \simeq \bigvee_{k\in\mathcal{K}} \Sigma^{n_k} ko \vee \bigvee_{\ell\in\mathcal{L}} \Sigma^{n_\ell} ku; \tag{B.16}$$

now we need to match this data to the data coming from cohomology in (B.15a).

Stong [Sto63] showed $H^*(ko;\mathbb{Z}/2) \cong \mathcal{A}\otimes_{\mathcal{A}(1)}\mathbb{Z}/2$ and Adams [Ada61] showed $H^*(ku;\mathbb{Z}/2) \cong \mathcal{A}\otimes_{\mathcal{E}(1)}\mathbb{Z}/2$, where $\mathcal{E}(1) := \langle \mathrm{Sq}^1, \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1\rangle$; there is an isomorphism $\mathcal{E}(1) \cong \mathcal{A}(1)\otimes_{\mathcal{E}(1)}\mathbb{Z}/2$ [BC18, Example 4.5.6], so

$$\mathcal{A}\otimes_{\mathcal{A}(1)} C\eta \cong \mathcal{A}\otimes_{\mathcal{A}(1)}\mathcal{A}(1)\otimes_{\mathcal{E}(1)}\mathbb{Z}/2 \cong \mathcal{A}\otimes_{\mathcal{E}(1)}\mathbb{Z}/2 \cong H^*(ku;\mathbb{Z}/2). \tag{B.17}$$

Therefore (B.16) implies

$$H^*(ko\wedge X_+;\mathbb{Z}/2) \cong \mathcal{A}\otimes_{\mathcal{A}(1)}\left(\bigoplus_{k\in\mathcal{K}} \Sigma^{n_k}\mathbb{Z}/2 \oplus \bigoplus_{\ell\in\mathcal{L}} \Sigma^{n_\ell} C\eta\right), \tag{B.18a}$$

and the Künneth formula and Stong's result above implies

$$H^*(ko\wedge X;\mathbb{Z}/2) \cong \mathcal{A}\otimes_{\mathcal{A}(1)} H^*(X;\mathbb{Z}/2). \tag{B.18b}$$

We conclude by plugging (B.15a) into (B.18b) and comparing with (B.18a); a priori information could be lost by tensoring with $\mathcal{A}$, but this tensor product respects direct sums and $\mathcal{A}\otimes_{\mathcal{A}(1)}\mathbb{Z}/2$ and $\mathcal{A}\otimes_{\mathcal{A}(1)} C\eta$ are not isomorphic, so no information is lost. ☐

Corollary B.19. *As ko-modules,*

$$ko \wedge (B\mathrm{U}(1))_+ \simeq ko \vee \bigvee_{n\ge 0} \Sigma^{4n+2} ku \tag{B.20a}$$

$$ko \wedge (B\mathrm{Sp}(1))_+ \simeq \bigvee_{n\ge 0} \Sigma^{4n} ko. \tag{B.20b}$$

See (III.163) for a related but different splitting result.

Proof. Once we have shown that $B\mathrm{U}(1)$ and $B\mathrm{Sp}(1)$ satisfy the hypothesis of Lemma B.9, the result follows from Corollary B.14 together with the understanding of $H^*(B\mathrm{U}(1);\mathbb{Z}/2)$ and $H^*(B\mathrm{Sp}(1);\mathbb{Z}/2)$. The latter is a trivial $\mathcal{A}(1)$-module (i.e. a sum of shifts of $\mathbb{Z}/2$) for degree reasons, and the $\mathcal{A}(1)$-module structure on $H^*(B\mathrm{U}(1);\mathbb{Z}/2)$ is computed in [BC18, Example 3.4.2 and Figure 4] to be a direct sum of $\mathbb{Z}/2$ and a $\Sigma^{4n+2}C\eta$ for each $n \ge 0$.

Thus all we have left to do is verify the hypotheses of Lemma B.9. The standard CW decomposition of $B\mathrm{U}(1) \simeq \mathbb{CP}^\infty$ has a k-cell in every nonnegative even degree k, attached to the $(k-2)$-cell (for $k > 0$) by the map $\eta \in \pi_1(\mathbb{S}) \cong \mathbb{Z}/2$, which satisfies the hypothesis, as it maps to the class we call η in ko_1. For $B\mathrm{Sp}(1) \simeq \mathbb{HP}^\infty$, the standard CW decomposition has a k-cell in each degree $k \equiv 0 \bmod 4$, attached to the $(k-4)$-cell (again $k > 0$) by the map $\nu \in \pi_3(\mathbb{S})$. The image of ν in $ko_3 \cong 0$ vanishes for degree reasons, and so the hypothesis of Lemma B.9 is met. ☐

Definition B.21. Recall the complexification map $c\colon ko \to ku$. The cofiber of c is a map $R\colon ku \to \Sigma^2 ko$, denoted *realification.*

As $ku \not\simeq ko \vee \Sigma^2 ko$, the third map in the cofiber sequence begun by c and R must be nontrivial in

$$\pi_0 \mathrm{Map}_{ko}(\Sigma ko, ko) \cong \mathrm{Map}_{\mathbb{S}}(\Sigma \mathbb{S}, ko) \cong \pi_1 ko \cong \mathbb{Z}/2, \tag{B.22}$$

so must be the nontrivial class, namely the Hopf map $\eta\colon \Sigma^{ko} \to ko$. That is, we have found the *Wood cofiber sequence*

$$ko \xrightarrow{c} ku \xrightarrow{R} \Sigma^2 ko \xrightarrow{\eta} \Sigma ko \longrightarrow \cdots \tag{B.23}$$

which we identified as a Smith cofiber sequence in Example III.136.

Recall from Example III.176 that the unit sphere bundle inside the tautological rank-3 vector bundle $V_3 \to B\mathrm{Spin}(3)$ is homotopy equivalent to the map $B\mathrm{Spin}(2) \to B\mathrm{Spin}(3)$, which can be identified via accidental isomorphisms to the map $B\mathrm{U}(1) \to B\mathrm{Sp}(1)$ given by the inclusion of a maximal torus. Choose for concreteness the standard maximal torus, given by the map $\mathrm{U}(1) \to \mathrm{SU}(2) \cong \mathrm{Sp}(1)$ defined by

$$i\colon z \mapsto \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}. \tag{B.24}$$

Thus there is a Smith cofiber sequence

$$ko \wedge (B\mathrm{U}(1))_+ \xrightarrow{i_*} ko \wedge (B\mathrm{Sp}(1))_+ \xrightarrow{\frown e^{ko}(V)} ko \wedge \Sigma^3 (B\mathrm{Sp}(1))^{V_3 - 3}, \tag{B.25}$$

which is the cofiber sequence in Example III.168 smashed with ko.[29] This sequence is also studied, and placed in context, in Example III.168.

Since $V_3 \to B\mathrm{Sp}(1)$ is spin, the Thom isomorphism identifies the third term in this sequence with $\Sigma^3 ko \wedge (B\mathrm{Sp}(1))_+$.

Proposition B.26. *The identifications in Corollary B.19 may be chosen to produce the following identifications of ko-module homomorphisms.*

1. *The map $i_*\colon ko \wedge (B\mathrm{U}(1))_+ \to ko \wedge (B\mathrm{Sp}(1))_+$ is the direct sum of the maps*

$$\Sigma^{4n+2} R\colon \Sigma^{4n+2} ku \longrightarrow \Sigma^{4n} ko, \tag{B.27a}$$

   together with the identity $ko \to ko$ on the basepoint.

2. *The fiber of i_*, which is a map $y\colon ko \wedge \Sigma^2 (B\mathrm{Sp}(1))_+ \to ko \wedge (B\mathrm{U}(1))_+$, is the direct sum of the maps*

$$\Sigma^{4n+2} c\colon \Sigma^{4n+2} ko \longrightarrow \Sigma^{4n+2} ku. \tag{B.27b}$$

3. *The map $\frown e^{ko}(V)\colon ko \wedge (B\mathrm{Sp}(1))_+ \to \Sigma^3 ko \wedge (B\mathrm{Sp}(1))_+$ is the direct sum of the maps*

$$\Sigma^{4n-1} \eta\colon \Sigma^{4n} ko \longrightarrow \Sigma^{4n-1} ko, \tag{B.27c}$$

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[29] In §II D 6, we calculate the Anderson dual long exact sequence in low degrees.

together with the zero map on the copy of ko in degree 0.

Proof. Using the Wood cofiber sequence (B.23), any one of these three results implies the other two; we will prove (2).

Restricted to $\Sigma^{4k+2}ko$, y is a map

$$y|_{\Sigma^{4k+2}ko}\colon \Sigma^{4k+2}ko \longrightarrow ko \vee \bigvee_{\ell\geq 0} \Sigma^{4\ell+2}ku. \tag{B.28}$$

We will show that it is possible to choose the equivalences in (B.19) to make y "diagonal", i.e. after composing to the projection onto each summand of (B.28) *except* $\Sigma^{4k+2}ku$, $y|_{\Sigma^{4k+2}ko}$ is trivial. We know that the "diagonal terms," i.e. the maps obtained by restricting y to $\Sigma^{4k+2}ko$ and then projecting to the $\Sigma^{4k+2}ku$ summand in the codomain, must be $\pm c$, because this is the only choice compatible with base change along $ko \to H\mathbb{Z}$ inducing maps on $\mathbb{Z}$ cohomology which are isomorphisms in those degrees: this is because

$$\pi_0\mathrm{Map}_{ko}(\Sigma^{4k+2}ko, \Sigma^{4k+2}ku) \cong \pi_0\mathrm{Map}_{\mathbb{S}}(\mathbb{S}, ku) \cong \pi_0 ku \cong \mathbb{Z} \tag{B.29}$$

and c is a generator; thus we must obtain either c or $-c$ on the equal-degree summand.

The map out of $\Sigma^{4k+2}ko$ is trivial when projected to the ko in degree 0, because we need that $\Sigma^0 ko$ summand to map to the degree-0 ko summand in the cofiber $ko \wedge (B\mathrm{Sp}(1))_+$, because that map arose from a basepoint-preserving map of spaces. In the rest of the proof, we will address the $\Sigma^{4\ell+2}ku$ summands.

A map of ko-modules $\Sigma^m ko \to \Sigma^n ku$ is equivalent data to a map of spectra $\Sigma^m\mathbb{S} \to \Sigma^n ku$, which is classified by $\pi_n(ku)$. Since ku is connective, all "off-diagonal terms" vanish unless $4k+2 \geq 4\ell+2$; therefore for our $\Sigma^{4k+2}ko$ summand we may restrict to the map

$$y\colon \Sigma^2 ko \vee \cdots \vee \Sigma^{4k+2}ko \longrightarrow \Sigma^2 ku \vee \cdots \vee \Sigma^{4k+2}ku. \tag{B.30}$$

We may therefore describe y as a $(k+1)\times(k+1)$ matrix. Connectivity of ku implies this matrix is upper triangular.

We saw in (B.29) that if $m \geq \ell$, then $\pi_0\mathrm{Map}_{ko}(\Sigma^{4m+2}ko, \Sigma^{4\ell+2}ku) \cong \pi_{2(m-\ell)}ku \cong \mathbb{Z}$; tracing through the identifications there, we learn that this $\mathbb{Z}$ of maps is the set of scalar multiples of the map $b^{2(m-\ell)}c$, where $b\colon \Sigma^2 ku \to ku$ is the connective version of the Bott periodicity map. Therefore there are integers λ_{ij} for $1 \leq i < j \leq k+1$ such that the map (B.30) is given by the following upper triangular matrix:

$$\begin{bmatrix} \pm c & \lambda_{12}b^2c & \lambda_{13}b^4c & \cdots & \lambda_{1(k+1)}b^{2k}c \\ & \pm c & \lambda_{23}b^2c & \cdots & \lambda_{2(k+1)}b^{2k-2}c \\ & & \ddots & \ddots & \vdots \\ & & & \pm c & \lambda_{k(k+1)}b^2c \\ & & & & \pm c \end{bmatrix}. \tag{B.31}$$

This matrix can clearly be row-reduced over ko_* to $c\cdot\mathrm{Id}$, and the requisite row operations correspond to automorphisms of $ko \vee \cdots \vee \Sigma^{4k+2}ko$. The row operations are compatible with adding on more summands by increasing k, so we may conclude. ☐

By Corollary B.19, there is an isomorphism $\varphi\colon ko_*(B\mathrm{Sp}(1)) \xrightarrow{\cong} ko_*[x]$, where $|x| = 4$. Here we use polynomial notation only for conciseness; we have not defined any ring structure on $ko_*(B\mathrm{Sp}(1))$.

Lemma B.32. *Recall $ko^*(B\mathrm{Sp}(1)) \cong ko^*[[p_1^{\mathbb{H}}]]$ from Proposition B.3. The isomorphism φ can be chosen such that the $ko^*(B\mathrm{Sp}(1))$-module structure on $ko_*(B\mathrm{Sp}(1))$ is the one uniquely specified by*

$$p_1^{\mathbb{H}} \frown x^k = x^{k-1}. \tag{B.33}$$

Proof. This follows directly from the analogous statement for the $\mathbb{Z}$ homology and cohomology of $B\mathrm{Sp}(1)$, where it is standard. ☐

Finally, we can calculate the ko-Euler class!

Proof of Theorem B.4. It suffices to work universally with the tautological bundle $V_3 \to B\mathrm{Spin}(3)$; the spinor bundle is the tautological quaternionic line bundle associated to $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$, and so $p_1^{\mathbb{H}}(\mathcal{S}_{V_3})$ is the class we called $p_1^{\mathbb{H}} \in ko^4(B\mathrm{Sp}(1))$ in Proposition B.3.

By Proposition B.26,

$$e^{ko}(V_3) \frown x^k = \eta x^{k-1}, \tag{B.34}$$

where we define $x^{-1} = 0$ for convenience.[30] A general element of $ko^3(B\mathrm{Sp}(1))$ is of the form

$$\sum_{k\geq 0} \eta (p_1^{\mathbb{H}})^k w^{k-1}. \tag{B.35}$$

We know how η and w^{k-1} act on $ko_*(B\mathrm{Sp}(1))$ because the ko-theory cap product is linear over ko^*. We know how $p_1^{\mathbb{H}}$ acts on $ko_*(B\mathrm{Sp}(1))$ thanks to Lemma B.32. Using these, we can see that the only class of the form (B.35) whose cap product matches that of $e^{ko}(V_3)$ in (B.34) is $\eta p_1^{\mathbb{H}}$.

Finally, we have to check that $e^{\mathbb{Z}}(V_3)$ and $e^{\mathbb{Z}/2}(V_3)$ both vanish. $B\mathrm{Sp}(1)$ is 3-connected, so $H^3(B\mathrm{Sp}(1);\mathbb{Z})$ and $H^3(B\mathrm{Sp}(1);\mathbb{Z}/2)$ both vanish. ☐

Remark B.36 (Euler classes of low-rank spin vector bundles)*.* For $2 \leq n \leq 6$, $\mathrm{Spin}(n)$ participates in an accidental isomorphism[31] with a Lie group satisfying Lemma B.9, and one can run a similar argument to compute ko-Euler classes of other low-rank vector bundles.

1. If L is a real line bundle with spin structure, $e^{ko}(L) = 0$, because e^{ko} pulls back from the twisted Euler class over $B\mathrm{SO}(1) = *$. The image of this fact in KO-theory is due to Crabb [Cra91, Corollary 3.37(i)].

2. If V_2 has rank 2, one can use the accidental isomorphism $\mathrm{Spin}(2) \cong \mathrm{U}(1)$ and the fact that the map $c\colon ko^*(B\mathrm{U}(1)) \to ku^*(B\mathrm{U}(1))$ is injective [BG10, §5.2] to show that $e^{ko}(V_2)$ is determined by $e^{ku}(V)$, hence also by $e^K(V)$, the image in periodic K-theory. In particular, V_2 acquires the structure of a complex line bundle, and there is a formula for the K-theory Euler classes of complex vector bundles, e.g. in Bott [Bot69, (7.2)].

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[30] As we have not been careful about explicit choices of isomorphisms, there could be a sign factor in the choice of x^k, but since $2\eta = 0$, the possible sign error goes away.

[31] “There are no mistakes, just happy little accidental isomorphisms.”

4. If $V_4 \to X$ has rank 4, its spinor bundle factors as $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, where the two factors $\mathcal{S}^\pm$ are quaternionic line bundles associated to the two factors of $\phi\colon \mathrm{Spin}(4) \xrightarrow{\cong} \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. There is a choice of ϕ such that
$$e^{ko}(V_4) = p_1^{\mathbb{H}}(\mathcal{S}^+) - p_1^{\mathbb{H}}(\mathcal{S}^-) \in ko^4(X). \tag{B.37}$$

5. There is an accidental isomorphism $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$, and $ko^*(B\mathrm{Sp}(2)) \cong ko^*[[p_1^{\mathbb{H}}, p_2^{\mathbb{H}}]]$ with $|p_1^{\mathbb{H}}| = 4$ and $|p_2^{\mathbb{H}}| = 8$ (see [DM79, §2] or [BG10, Theorem 5.3.5]). Therefore $ko^5(B\mathrm{Sp}(2)) \cong 0$, so for any rank-5 spin vector bundle V_5, $e^{ko}(V_5) = 0$. The image of this fact in KO-theory is due to Crabb [Cra91, Corollary 3.37(i)].

Remark B.38. We saw above that for twisted spin bordism, the ko-theoretic Euler class suffices. For other tangential structures, one may need more or less information.

- Unoriented bordism decomposes as a sum of shifts of mod 2 homology, and this splitting is compatible with the Smith homomorphism. Therefore in this setting, one can use the $\mathbb{Z}/2$-cohomology Euler class.

- Wall [Wal60] showed that $MTSO$, localized at 2, splits as a sum of shifts of $H\mathbb{Z}$ and $H\mathbb{Z}/2$. Therefore when one studies Smith homomorphisms for twisted oriented bordism, the $\mathbb{Z}$-cohomology Euler class will be accurate up to odd-primary torsion. On odd-primary torsion, oriented and spin bordism coincide, so in that setting one can use ko-Euler classes for twisted oriented bordism.

- Analogously to spin and ko, one can use ku-theory Euler classes for twisted spin^c bordism Smith homomorphisms.

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