

The logarithm in arithmetic geometry and homotopy theory

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Overview

- 1 History
- 2 Main identity
- 3 Deforming further

The logarithm

[N]othing is more tedious, fellow mathematicians, in the practice of the mathematical arts, than the great delays suffered in the tedium of lengthy multiplications and divisions, the finding of ratios, and in the extraction of square and cube roots [...] (Napier, *Mirifici Logarithmorum Canonis Descriptio*, 1614)

Napier, and immediately after him Briggs, spent 20 years (!) doing numerical calculations totalling almost 10 million entries in his tables. Before Napier, multiplication was reduced to addition through “prosthaphaeresis”, which meant using the formula $2 \cos(x) \cos(y) = \cos(x + y) + \cos(x - y)$ to quickly multiply large numbers using cos tables.

Of the logarithm, Laplace (1749-1827) said “by shortening the labors, [they] doubled the life of an astronomer”.

Napier's tables

Gr. 19						Gr. 19					
19	mil.	Sinus	Logarithmi	Difference	Logarithmi	19	mil.	Sinus	Logarithmi	Difference	Logarithmi
0	325682	11221830	10661613		560217	9455186	60	325682	11221830	10661613	560217
1	3258432	11213386	10651167		561219	9454239	59	3258432	11213386	10651167	561219
2	3261182	11204950	10642728		562222	9453291	58	3261182	11204950	10642728	562222
3	3263931	11196522	10633296		563226	9452342	57	3263931	11196522	10633296	563226
4	3266681	11188102	10623871		564231	9451392	56	3266681	11188102	10623871	564231
5	3269439	11179690	10614453		565237	9450441	55	3269439	11179690	10614453	565237
6	3272179	11171286	10605042		566244	9449490	54	3272179	11171286	10605042	566244
7	3274927	11162889	10595637		567252	9448538	53	3274927	11162889	10595637	567252
8	3277675	11154500	10586239		568261	9447585	52	3277675	11154500	10586239	568261
9	3280423	11146119	10576849		569270	9446631	51	3280423	11146119	10576849	569270
10	3283171	11137746	10567466		570280	9445676	50	3283171	11137746	10567466	570280
11	3285918	11129381	10558090		571291	9444720	49	3285918	11129381	10558090	571291
12	3288665	11121024	10548721		572303	9443764	48	3288665	11121024	10548721	572303
13	3291412	11112675	10539359		573316	9442807	47	3291412	11112675	10539359	573316
14	3294159	11104334	10530004		574330	9441849	46	3294159	11104334	10530004	574330
15	3296906	11095900	10520655		575345	9440890	45	3296906	11095900	10520655	575345
16	3299652	11087474	10511313		576361	9439931	44	3299652	11087474	10511313	576361
17	3302398	11079056	10501977		577377	9438971	43	3302398	11079056	10501977	577377
18	3305144	11070646	10492648		578392	9438010	42	3305144	11070646	10492648	578392
19	3307889	11062244	10483326		579408	9437048	41	3307889	11062244	10483326	579408
20	3310634	11053849	10474010		580423	9436085	40	3310634	11053849	10474010	580423
21	3313379	11045462	10464702		581440	9435122	39	3313379	11045462	10464702	581440
22	3316123	11037083	10455403		582458	9434158	38	3316123	11037083	10455403	582458
23	3318867	11028712	10446107		583476	9433193	37	3318867	11028712	10446107	583476
24	3321611	11020348	10436819		584495	9432227	36	3321611	11020348	10436819	584495
25	3324355	11011992	10427538		585515	9431260	35	3324355	11011992	10427538	585515
26	3327098	11003643	10418265		586536	9430293	34	3327098	11003643	10418265	586536
27	3329841	10995292	10408999		587557	9429325	33	3329841	10995292	10408999	587557
28	3332585	10986948	10399731		588579	9428356	32	3332585	10986948	10399731	588579
29	3335327	10978601	10390479		589602	9427386	31	3335327	10978601	10390479	589602
30	3338069	10970253	10381231		590626	9426415	30	3338069	10970253	10381231	590626

Gr. 19						Gr. 19					
19	mil.	Sinus	Logarithmi	Difference	Logarithmi	19	mil.	Sinus	Logarithmi	Difference	Logarithmi
31	3340811	10961902	10371982		591651	9425443	29	3340811	10961902	10371982	591651
32	3343553	10953550	10362766		592652	9424470	28	3343553	10953550	10362766	592652
33	3346294	10945195	10353548		593654	9423496	27	3346294	10945195	10353548	593654
34	3349035	10936837	10344327		594657	9422521	26	3349035	10936837	10344327	594657
35	3351776	10928476	10335092		595661	9421545	25	3351776	10928476	10335092	595661
36	3354516	10920112	10325844		596666	9420568	24	3354516	10920112	10325844	596666
37	3357256	10911746	10316582		597672	9419590	23	3357256	10911746	10316582	597672
38	3359996	10903378	10307317		598679	9418611	22	3359996	10903378	10307317	598679
39	3362736	10895009	10298049		599687	9417631	21	3362736	10895009	10298049	599687
40	3365475	10886638	10288778		600696	9416650	20	3365475	10886638	10288778	600696
41	3368214	10878266	10279504		601706	9415668	19	3368214	10878266	10279504	601706
42	3370953	10869893	10270227		602717	9414685	18	3370953	10869893	10270227	602717
43	3373692	10861519	10260948		603729	9413701	17	3373692	10861519	10260948	603729
44	3376431	10853144	10251666		604742	9412716	16	3376431	10853144	10251666	604742
45	3379170	10844768	10242381		605756	9411730	15	3379170	10844768	10242381	605756
46	3381909	10836391	10233094		606771	9410743	14	3381909	10836391	10233094	606771
47	3384648	10828013	10223805		607787	9409755	13	3384648	10828013	10223805	607787
48	3387387	10819634	10214514		608804	9408766	12	3387387	10819634	10214514	608804
49	3390126	10811254	10205221		609822	9407776	11	3390126	10811254	10205221	609822
50	3392865	10802873	10195926		610841	9406785	10	3392865	10802873	10195926	610841
51	3395604	10794491	10186629		611861	9405793	9	3395604	10794491	10186629	611861
52	3398343	10786108	10177329		612882	9404800	8	3398343	10786108	10177329	612882
53	3401082	10777724	10168027		613904	9403806	7	3401082	10777724	10168027	613904
54	3403821	10769339	10158722		614927	9402811	6	3403821	10769339	10158722	614927
55	3406560	10760953	10149414		615951	9401815	5	3406560	10760953	10149414	615951
56	3409300	10752566	10140103		616976	9400818	4	3409300	10752566	10140103	616976
57	3412039	10744178	10130789		618002	9399820	3	3412039	10744178	10130789	618002
58	3414778	10735789	10121472		619029	9398821	2	3414778	10735789	10121472	619029
59	3417517	10727399	10112152		620057	9397821	1	3417517	10727399	10112152	620057
60	3420256	10718999	10102829		621086	9396820	0	3420256	10718999	10102829	621086

History

Grégoire de Saint-Vincent and his student de Sarasa observed around 1647 that the area $A(x)$ between $[1, x]$ under a hyperbola $xy = 1$ satisfied $A(xy) = A(x) + A(y)$.

In 1668, Mercator realized that this meant $\log(x) = \int_1^x \frac{dx}{x}$, so that $\frac{1}{x} = 1 - (x-1) + (x-1)^2 - \dots$ can be used to write

$$\log(x) = \sum_{n \geq 1} (-1)^{n-1} \frac{(x-1)^n}{n}.$$

Newton had also discovered this in 1665 but did not publish it.

It took Euler to realize that if $y = \log_a(x)$, then $x = a^y$. It is amazing that logarithms were more well-studied than exponentials at this time!

Euler and interpolation

In 1749, Euler wrote a paper attempting to understand the base q logarithm $\ln_q$ by interpolation. His idea was that $\ln_q(q^n) = n$, so one can write down a series by interpolating between these values. He ends up with

$$s(x) = \sum_{n \geq 0} \frac{(1-x)(q-x) \cdots (q^{n-1}-x)}{q^{n(n-1)/2} - q^{n(n+1)/2}}.$$

It is an easy exercise that $s(q^n) = n$ for $n \geq 0$. Euler notes that this fails if $n < 0$. He also computes that when $q = 10$, $s(9) = 0.897778\dots$ (actually the calculation in his paper is wrong), which differs from $\log_{10}(9) = 0.954242\dots$

This failure of s to agree with $\log_{10}$ is not too hard to understand from the modern perspective (exercise!). But if q is very close to 1, this is a good approximation to $\ln_q$.

Euler and interpolation

Figure: Euler, *Consideratio quarumdam serierum, quae singularibus proprietatibus sunt praeditae* (1753). Euler Archive E190

CONSIDERATIO QVAVMDAM SERIERVM. 87

§. 2. Primo quidem patet, si quantitas x potestati cuiuspiam ipsius a aequalis capiatur, tum seriem alicubi ita obrumpi, vt omnes sequentes termini abeant in nihilum. Ponamus ergo in genere s pro summa seriei propositae, vt fit:

$$s = \frac{1-x}{1-x^2} + \frac{(1-x)(a-x)}{a-a^2} + \frac{(1-x)(a-x)(a^2-x)}{a^2-a^6} + \frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}} + \text{etc.}$$

Euler and interpolation

We can rewrite

$$\begin{aligned} s(x) &= \sum_{n \geq 0} \frac{(1-x)(q-x) \cdots (q^{n-1}-x)}{q^{n(n-1)/2} - q^{n(n+1)/2}} \\ &= \sum_{n \geq 0} (-1)^{n-1} q^{-n(n-1)/2} \frac{(x-1)(x-q) \cdots (x-q^{n-1})}{q^n - 1}. \end{aligned}$$

When q is close to 1, this computes $\ln_q(x) = \log(x)/\log(q)$. Motivated by this:

Definition

Let $[n]_q = \frac{q^n - 1}{q - 1}$ denote the q -integer, and define the q -logarithm as

$$\log_q(x) = \frac{q-1}{\log(q)} \log(x) = \sum_{n \geq 0} (-1)^{n-1} q^{-n(n-1)/2} \frac{(x-1)(x-q) \cdots (x-q^{n-1})}{[n]_q}.$$

Where art thou?

It will be helpful to be more precise about the world in which $\log_q$ lives. The expression $\frac{q-1}{\log(q)}\log(x)$, when viewed as a power series in x , naturally lives in the ring $\mathbf{Q}[[q-1, x-1]]$. This ring contains inverses to all $[n]_q = 1 + q + \cdots + q^{n-1}$, and Euler's interpolation formula tells us that in fact

$$\log_q(x) \in \mathbf{Z}[[q-1]] \left[\frac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q!} \right]^\wedge \subseteq \mathbf{Q}[[q-1, x-1]].$$

Here, $[n]_q! = [n]_q[n-1]_q\cdots[1]_q$. One should view the fraction appearing above as a q -deformation of the divided power $\frac{(x-1)^n}{n!}$.

Note how miraculous this is! The power series $\log_q(x)$ with *a priori* rational denominators now can be rewritten in such a way that no rational denominators appear, but instead only denominators in $[n]_q$ appear.

An audacious idea

Since $\log_q(x)$ is a “ q -deformation” of $\log(x)$, essentially obtained by replacing n by $[n]_q$, one is led to the following audacious idea:

Many important constructions/results in mathematics admit q -deformations, obtained roughly by replacing n by $[n]_q$.

As we have seen, this idea started off with Euler’s aforementioned work; later came Gauss’s work on “basic” hypergeometric functions; but this philosophy was taken really seriously by Eduard Heine, Frank Hilton Jackson, and James Rogers, around the turn of the 20th century. But, as happens so often, their work was not appreciated at the time.

q -derivative

One of Jackson's ideas was to define a q -analogue of the derivative. Namely,

Definition

Let ∂_x^q denote the q -derivative, given by

$$\partial_x^q f(x) = \frac{f(qx) - f(x)}{(q-1)x},$$

so that $\partial_x^q(x^n) = [n]_q x^{n-1}$.

Note that the q -derivative of $\log_q(x)$ is x^{-1} ; this is evident from the formula $\log_q(x) = \frac{q-1}{\log(q)} \log(x)$.

Just as with usual Taylor expansions, one can also define q -Taylor expansions; now, the expressions $\frac{(x-x_0)^n}{n!}$ when expanding around $x = x_0$ will be replaced by the expressions

$$\frac{(x-x_0)_q^n}{[n]_q!} := \frac{(x-x_0)(x-qx_0)\cdots(x-q^{n-1}x_0)}{[n]_q!}.$$

(This recovers Euler's interpolation formula (when $x_0 = 1$) for $\log_q(x)$.) This is because both $(x-x_0)^n$ and $(x-x_0)_q^n$ can be characterized by the properties:

- When $n = 0$, they are equal to 1;
- When $n = 1$, they are equal to $x - x_0$;
- Vanishes for $n \geq 1$ and $x = x_0$;
- The q -derivative of $(x-x_0)_q^n$ is equal to $[n]_q(x-x_0)_q^{n-1}$ (similarly for the derivative of $(x-x_0)^n$).

More generally, in my experience, characterizing q -deformations in terms of properties of their q -derivatives is a clean way to naturally produce the various factors of q which are ubiquitous in the theory (and which are hard to “guess”).

Baby case of main identity

The main identity this talk centers around is the following:

$$\begin{aligned}\sum_{n \geq 1} \frac{\log(x)^n}{n!} y^n &= \exp(y \log(x)) \\ &= x^y = (1 + (x - 1))^y \\ &= \sum_{n \geq 0} \binom{y}{n} (x - 1)^n.\end{aligned}$$

Easy enough! But there is an interesting q -deformation of this identity which I will state below. I wrote it in my thesis, and I would love to know if it had been recorded earlier. It is not particularly difficult, but is still interesting.

Main identity

Theorem

There is an equality

$$\sum_{n \geq 1} \frac{\log_q(x)(\log_q(x) - (q-1)) \cdots (\log_q(x) - (n-1)(q-1))}{n!} y^n$$

$$= \sum_{n \geq 0} q^{-n(n-1)/2} y(y - [1]_q) \cdots (y - [n-1]_q) \frac{(x-1)_q^n}{[n]_q!}.$$

When $q = 1$, this recovers the previous identity:

$$\sum_{n \geq 1} \frac{\log(x)^n}{n!} y^n = \sum_{n \geq 0} \binom{y}{n} (x-1)^n.$$

I will later describe a generalization of this identity, but for now I want to explain how it fits into the theory of crystalline and prismatic cohomology.

To motivate this connection, recall that $d\log(x) = \frac{dx}{x}$. This 1-form on $\mathbf{C}^\times$ provides a generator of $H_{\mathrm{dR}}^1(\mathbf{C}^\times) \cong \mathbf{C}$.

Actually, if k is a commutative ring and X is an algebraic variety over k , one can define *algebraic* de Rham cohomology $H_{\mathrm{dR}}^*(X/k)$ using polynomial differential forms (and this agrees with usual de Rham cohomology when $k = \mathbf{C}$ by Grothendieck).

Again, $d\log(x)$ generates the 1-forms $\Omega_{\mathbf{G}_m/k}^1$ on the multiplicative group $\mathbf{G}_m$. In the case of $\mathbf{G}_m$, algebraic de Rham cohomology is computed by the complex

$$\Omega_{\mathbf{G}_m/k}^* = \left[k[x^{\pm 1}] \xrightarrow{d} k[x^{\pm 1}] dx \right].$$

de Rham stacks

Grothendieck, Berthelot-Ogus, and later Simpson, Bhatt-Lurie, and Drinfeld realized that one can view the de Rham cohomology of X as the *coherent* cohomology of an algebraic stack ($\approx$ variety) X^{dR} obtained (informally) by imposing the equivalence relation that

“ x is close to y ” if $x - y$ admits divided powers, i.e., $\frac{(x-y)^n}{n!}$ are well-defined (and go to zero as $n \rightarrow \infty$).

Essentially, this is *forcing* the Poincaré lemma/the theory of Taylor expansions.

Note that admitting divided powers is a nontrivial criterion if the numbers $n!$ are not invertible! (And in fact the definition should really ask for divided power *structures*...)

Case of $\mathbf{G}_m$

For example, when $X = \mathbf{G}_m$, a point $x \in \mathbf{G}_m$ is “close to 1” if $\frac{(x-1)^n}{n!}$ are well-defined and go to zero as $n \rightarrow \infty$. This is neatly packaged using the group

$$\mathbf{G}_m^\# := \operatorname{Spf} k \left[x^{\pm 1}, \frac{(x-1)^n}{n!} \right]^\wedge.$$

From this point of view, the 1-form $d\log(x)$ comes from the homomorphism

$$\log : \mathbf{G}_m^\# \rightarrow \mathbf{G}_a, \quad x \mapsto \sum_{n \geq 1} (-1)^{n-1} \frac{(x-1)^n}{n!}.$$

Crystalline cohomology

Let us now fix a prime p . Then there is the miraculous fact, discovered by Berthelot-Ogus building on work of Grothendieck, that if X is a (smooth) algebraic variety over $\mathbf{Z}$, then $H_{\mathrm{dR}}^*(X/\mathbf{Z})_p^\wedge$ only depends (functorially!) on the mod p locus $X_{p=0} \subseteq X$.

This leads to the theory of *crystalline cohomology*. One can view Bhatt-Scholze's theory of *prismatic cohomology* as giving (among other things) a canonical Frobenius descent of crystalline cohomology.

Crystalline stacks

Bhatt-Lurie and Drinfeld realized that crystalline cohomology can *also* be viewed the *coherent* cohomology of an algebraic stack ($\approx$ variety) X^{crys} .

Let us return again to the case of $\mathbf{G}_m$. Explicitly, the prismatic theory tells us that $\mathbf{G}_m^{\text{crys}}$ is obtained from $\mathbf{G}_m^{(1)}$ (which just means the usual multiplicative group but with coordinate $x' = x^p$) by declaring that

“ $x' \in \mathbf{G}_m^{(1)}$ is close to 1” if there is some y with divided powers $\frac{y^n}{n!}$ such that $x' = \exp(py) = \sum_{n \geq 0} \frac{p^n y^n}{n!}$. (Think: p is “small”.)

This is neatly packaged using the group

$$\mathbf{G}_a^\# := \text{Spf } \mathbf{Z}_p \left[y, \frac{y^n}{n!} \right]^\wedge.$$

Just as with de Rham cohomology, $H_{\Delta}^1(\mathbf{G}_m/\mathbf{Z}_p)$ can be described from the above point of view via the obvious homomorphism (sometimes called the *prismatic logarithm*)

$$\text{can} : \mathbf{G}_a^\# \rightarrow \mathbf{G}_a, y \mapsto y.$$

Main identity

If crystalline and de Rham cohomology are to agree (as they must!) then we need to be able to compare the two homomorphisms

$$\begin{aligned}\log : \mathbf{G}_m^\# &\rightarrow \mathbf{G}_a, \\ \text{can} : \mathbf{G}_a^\# &\rightarrow \mathbf{G}_a\end{aligned}$$

whose sources implement the equivalence relations defining $\mathbf{G}_m^{\text{dR}}$ and $\mathbf{G}_m^{\text{crys}}$. The key point is that $\log$ *lifts along* can , i.e., it upgrades to a map $\mathbf{G}_m^\# \rightarrow \mathbf{G}_a^\#$.

Concretely, this is the claim that the divided powers $\frac{\log(x)^m}{m!}$ are all well-defined in $\mathbf{Z}_p[x^{\pm 1}, \frac{(x-1)^n}{n!}]$ for every m . How to prove this? Via our main identity!

Using the main identity

Our identity said

$$\sum_{m \geq 1} \frac{\log(x)^m}{m!} y^m = \sum_{m \geq 1} y(y-1) \cdots (y-(m-1)) \frac{(x-1)^m}{m!}.$$

So, extracting coefficients of y^m on the right-hand side gives an expression for the divided powers $\frac{\log(x)^m}{m!}$ in terms of $\frac{(x-1)^m}{m!}$, as desired!

(This plus a little more is enough to recover the relationship between prismatic/crystalline and de Rham cohomology.)

q -deformation?

We also had a q -deformed identity; does it also have a similar interpretation?
Recall the identity said

$$\begin{aligned} \sum_{n \geq 1} \frac{\log_q(x)(\log_q(x) - (q-1)) \cdots (\log_q(x) - (n-1)(q-1))}{n!} y^n \\ = \sum_{n \geq 0} q^{-n(n-1)/2} y(y - [1]_q) \cdots (y - [n-1]_q) \frac{(x-1)_q^n}{[n]_q!}. \end{aligned}$$

So the same trick tells us that $\log_q(x)$ admits $((q-1)$ -rescaled) binomial coefficients in the ring

$$\mathbf{Z}[[q-1]] \left[x^{\pm 1}, \frac{(x-1)_q^n}{[n]_q!} \right]^\wedge.$$

q -deformation!

Just as the fractions $\frac{(x-1)^n}{n!}$ controlled the de Rham cohomology of $\mathbf{G}_m$, its q -analogue $\frac{(x-1)_q^n}{[n]_q!}$ controls the q -de Rham cohomology of $\mathbf{G}_m$. Explicitly, this means the complex given by

$$\mathbf{Z}[[q-1]][x^{\pm 1}] \xrightarrow{\partial_x^q} \mathbf{Z}[[q-1]][x^{\pm 1}] dx,$$

where ∂_x^q is Jackson's derivative $\frac{f(qx)-f(x)}{(q-1)x}$.

That is, the q -de Rham complex is obtained from $\mathbf{G}_m$ by declaring

“ x is close to 1” if $x-1$ admits q -divided powers, i.e., $\frac{(x-1)_q^n}{[n]_q!}$ are well-defined (and go to zero as $n \rightarrow \infty$).

q -deformation!

Just as with $\mathbf{G}_m^{\text{crys}}$, the prismatic theory tells us that $\mathbf{G}_m^{\Delta}$ is obtained from $\mathbf{G}_m^{(1)}$ (which is just the usual multiplicative group but with coordinate $x' = x^p$) by declaring that

“ $x' \in \mathbf{G}_m^{(1)}$ is close to 1” if there is some y with rescaled binomial coefficients $\frac{y(y-(q-1)) \cdots (y-(n-1)(q-1))}{n!}$ such that $x' = q^{py/(q-1)}$. (Think: p is “small”.)

Now our observation that $\log_q(x)$ admits rescaled binomial coefficients can be used (in the same way as with crystalline cohomology) to show that

(Frobenius-twisted) prismatic cohomology of $\mathbf{G}_m$ is computed by q -de Rham cohomology.

This was in some sense baked into Bhatt-Scholze’s construction of prismatic cohomology, but I find it quite pretty that the main q -deformed identity “knows” about it.

Analogy

The analogy is that

crystalline cohomology	de Rham cohomology	(via $\log(x)$)
prismatic cohomology	q -de Rham cohomology	(via $\log_q(x)$)

Crystalline cohomology is actually a special case of prismatic cohomology; it can be thought of as the limit $q \rightarrow 1$

In fact, one can revisit many (but not yet *all*) different q -deformations throughout mathematics (like the theory of quantum groups) through prismatic cohomology and the explicit methods described above.

Underlying this (at least the way I think about it) is a more conceptual explanation of these relationships coming from homotopy theory; to describe it, it is convenient to place our discussion above into a broader framework.

Formal group laws

Consider the 2-variable polynomials $x + y$ and $x + y + (q - 1)xy$. It is easy to see that adding 1 to itself n times under this group law produces n and $[n]_q = \frac{q^n - 1}{q - 1}$ respectively. This suggests a generalization:

Definition

A (1-dimensional) formal group law (over a commutative ring R) is a 2-variable power series $x +_F y \in R[[x, y]]$ such that

- $x +_F y = y +_F x$;
- $x +_F 0 = x$;
- $x +_F (y +_F z) = (x +_F y) +_F z$.

Its homogenization will be denoted $x +_{\tilde{F}} y := \frac{(xt) +_F (yt)}{t}$ over $R[[t]]$.

Let $\langle n \rangle_F = 1 +_{\tilde{F}} \cdots +_{\tilde{F}} 1 \in R[[t]]$.

Examples

For example, $x +_F y = x + y$ is called the *additive* formal group law, and $x +_F y = x + y + xy$ is the *multiplicative* formal group law. One can also define

$$x +_F y = \frac{x + y}{1 + xy},$$

called the *hyperbolic* formal group law (since it is the addition law for $\tanh$). In this case, we have

$$\langle n \rangle_F = \frac{1}{t} \frac{(t+1)^n - (1-t)^n}{(t+1)^n + (1-t)^n}.$$

Another class of examples comes from elliptic curves, via the group law near the identity.

F-analogues

The philosophy I want to propose is:

Many important results in mathematics admit “F-deformations”, obtained roughly by replacing n by $\langle n \rangle_F$.

For example, one can define a generalization of the derivative (which recovers Jackson’s derivative in the multiplicative case) by declaring

$$\partial_x^F(x^n) = \langle n \rangle_F x^{n-1}$$

and extending linearly. For example, for the hyperbolic formal group law,

$$\partial_x^F(f(x)) = \frac{1}{tx} \frac{f((t+1)x) - f((1-t)x)}{f((t+1)x) + f((1-t)x)}.$$

In general, there is the operator identity

$$\partial_x^F x = (x \partial_x^F) +_{\tilde{F}} 1.$$

This “derivative” was explored in a project with Max Misterka.

F -analogues

One can also define certain polynomials $(x - x_0)_F^n$ characterized by the properties:

- When $n = 0$, it is equal to 1;
- When $n = 1$, it is equal to $x - x_0$;
- Vanishes if $n \geq 1$ and $x = x_0$;
- The F -derivative of $(x - x_0)_F^n$ is equal to $\langle n \rangle_F (x - x_0)_F^{n-1}$.

This recovers $(x - x_0)^n$ (resp. $(x - x_0)_q^n$) in the additive (resp. multiplicative) case. In the latter case, $t = q - 1$.

F -logarithm

Again, one can write down an analogue $\log_F$ of the q -logarithm, characterized by

$$\partial_x^F \log_F(x) = x^{-1}, \quad \log_F(1) = 0.$$

Explicitly,

$$\log_F(x) = \sum_{n \geq 1} \langle -1 \rangle_F \cdots \langle -n + 1 \rangle_F \frac{(x - 1)_F^n}{\langle n \rangle_F!}.$$

It satisfies

$$\log_F(xy) = \log_F(x) + \log_F(y),$$

and it can also be viewed as interpolating $\log(x)$, in the sense that there is a power series $\ell(t)$ such that $\log_F(x) = \frac{t}{\ell(t)} \log(x)$.

Main identity

Theorem

There is an equality

$$\sum_{n \geq 1} \beta_n(\log_F(x)) y^n = \sum_{n \geq 0} y(y - \tilde{F} \langle 1 \rangle_F) \cdots (y - \tilde{F} \langle n-1 \rangle_F) \frac{(x-1)_F^n}{\langle n \rangle_F!},$$

where β_n generate the algebra of distributions on the formal group $\tilde{F}$ (this generalizes divided powers and binomial coefficients).

This generalizes our main identity; again I want to emphasize how surprising this is: the left-hand side has rational denominators $(n!)$, but the right-hand side only has $\langle n \rangle_F!$ -denominators.

Just as before, we learn that $\log_F(x)$ admits “ β_n -coefficients” in the ring

$$R[[t]] \left[x^{\pm 1}, \frac{(x-1)_F^n}{\langle n \rangle_F!} \right]^\wedge.$$

Meaning?

In the additive and multiplicative cases, our identity, and in particular the consequence that $\log(x)$ admits divided powers (resp. $\log_q(x)$ admits binomial coefficients) implied a comparison between prismatic cohomology and:

de Rham cohomology	additive group law	(via $\log(x)$)
q -de Rham cohomology	multiplicative formal group law	(via $\log_q(x)$)

So how does our generalization of the main identity fit into this picture?

In recent work, Hahn-Raksit-Wilson, and followup work by myself, Hahn, Raksit, and Yuan, extend the theory of prismatic cohomology to *commutative ring spectra*. These objects are homotopical generalizations of commutative rings, obtained roughly by replacing “sets” by “homotopy types”.

On the other hand, Raksit, and independently I (in a “Koszul dual” form), realized that q -de Rham cohomology is in fact just de Rham cohomology (defined appropriately) for varieties over a particular ring spectrum $\mathbf{k}u$. (See also work of Wagner.) More generally, the “ F -de Rham complex” defined as

$$R\llbracket t \rrbracket[x^{\pm 1}] \xrightarrow{\partial_x^F} R\llbracket t \rrbracket[x^{\pm 1}]dx$$

is just a special case of de Rham cohomology for varieties over a suitable ring spectrum E_F .

Again, de Rham cohomology of $\mathbf{G}_m$ (thought of as a variety over E_F) is the coherent cohomology of an algebraic stack ($\approx$ variety) $\mathbf{G}_m^{F\mathrm{dR}}$ obtained (informally) by imposing the equivalence relation that

“ x is close to 1” if $x - 1$ admits F -divided powers, i.e., $\frac{(x-1)_F^n}{\langle n \rangle_F!}$ are well-defined (and go to zero as $n \rightarrow \infty$).

The relationship between formal group laws and ring spectra is very old (initiated by Quillen), and this is just one example of how the two stories relate.

Prismatic vs F -de Rham

We now have a generalization of prismatic cohomology, as well as a natural home for F -de Rham cohomology. Our main identity therefore suggests that:

(Frobenius-twisted) prismatic cohomology of $\mathbf{G}_m$ (viewed as a variety over a suitable ring spectrum) is computed by F -de Rham cohomology.

This is far from obvious from the Hahn-Raksit-Wilson definition of prismatic cohomology, even assuming the main identity.

But recent work of myself, Heuts, and Schlank shows that, at least, such a statement *is* true on the part of prismatic and F -de Rham cohomologies where Frobenius acts unipotently. Our proofs do not use the main identity, but that might change! The whole picture is very much in active development.

Conclusion

Although much of this picture is built on heavy machinery (commutative ring spectra, prismatic cohomology, ...), I hope the main identity around which our discussion was centered shows that this story reduces to some rather concrete mathematics; and also that it is a natural continuation of old threads explored by our predecessors.

There are many other ways in which replacing $\partial_x \rightsquigarrow \partial_x^F$ leads to interesting mathematics, like in Hodge theory in characteristic $p > 0$, geometric Langlands, Floer homotopy theory, ... But there are also some basic questions which I do not have a satisfactory answer to, like:

Question

$n = \#\{1, \dots, n\} = \#\mathbf{P}^{n-1}(\mathbf{F}_1)$ and $[n]_q = \#\mathbf{P}^{n-1}(\mathbf{F}_q)$. For a general formal group law F , what does $\langle n \rangle_F$ count?

Thank you for your time!