Geometric Langlands duality with generalized coefficients

Sanath K. Devalapurkar

UChicago, (ex-)Harvard University

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Overview



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Geometric Langlands duality

Motivated by the number field/function field/manifolds analogy, Beilinson and Drinfeld proposed a geometric variant of Langlands duality, where number rings are replaced by Riemann surfaces. This relates the *topology* of a split reductive group G over \mathbf{Z} to the *algebraic geometry* of its "Langlands dual group" \check{G}_k . (E.g., $G = \mathrm{SL}_n$, $\check{G} = \mathrm{PGL}_n$.)

If Σ is a Riemann surface and k is a commutative ring, they proposed that there should be an equivalence

$$\operatorname{Shv}(\operatorname{Bun}_{G}(\Sigma); k) \simeq \operatorname{QCoh}(\operatorname{Loc}_{\check{G}_{k}}(\Sigma)).$$

Here, $\operatorname{Bun}_G(\Sigma)$ is the stack of (algebraic) *G*-bundles on Σ ; \check{G}_k is the *Langlands* dual group scheme, defined over k; and $\operatorname{Loc}_{\check{G}_k}(\Sigma)$ is the stack of \check{G}_k -local systems on Σ . (Not quite correct as stated...)

It is a very interesting conjecture which has generated a lot of deep and beautiful mathematics.

Geometric Satake

One way to approach the conjecture is to prove it "locally"; for example, replace Σ by a formal bubble, namely $\mathbb{B} := D \coprod_{D^\circ} D$ where D is a formal disk and D° is a formal punctured disk. Then

$$\operatorname{Bun}_{G}(\mathbb{B}) = G(\mathbb{O}) \setminus G(F) / G(\mathbb{O}),$$

where $G(F) = G(\mathbf{C}((t)))$ and $G(\mathcal{O}) = G(\mathbf{C}[[t]])$. The quotient $G(\mathcal{O}) \setminus G(F)$ is called the *affine Grassmannian*, and is denoted Gr_{G} .

In this case, the conjecture is a theorem of Bezrukavnikov-Finkelberg for $k = \mathbf{Q}$. (After using Koszul duality,) it states that there is an equivalence

$$\operatorname{Shv}(\operatorname{Gr}_{G}/G(\mathcal{O}); \mathbf{Q}) \simeq \operatorname{QCoh}(\check{\mathfrak{g}}_{\mathbf{Q}}^{*}[2]/\check{G}_{\mathbf{Q}}),$$

where $\check{g}_{\mathbf{Q}}^*$ is the *coadjoint representation*. This is called the (derived) geometric Satake equivalence. It is essentially geometric Langlands for $\Sigma = \mathbf{P}^1$.

Remarks

Assume from now that G is simply-laced and $\pi_1(G) = 0$ (i.e., isogenous to $\operatorname{SL}_n, \operatorname{Spin}_{2n}, E_6, E_7$, or E_8). Then $\check{G}_k = G_k/Z(G_k)$, and one can identify $\check{\mathfrak{g}}_k^* \cong \mathfrak{g}_k$. So we can rewrite:

$$\operatorname{Shv}(\operatorname{Gr}_{G}/G(\mathbb{O}); \mathbf{Q}) \simeq \operatorname{QCoh}(\mathfrak{g}_{\mathbf{Q}}[2]/\check{G}_{\mathbf{Q}}).$$

This is a *Fourier transform*: it sends the δ -sheaf at basepoint of Gr_{G} to the structure sheaf of $\mathfrak{g}_{\mathbf{Q}}[2]/\check{G}_{\mathbf{Q}}$. Taking endomorphisms, recover the well-known statement that $C^*(BG; \mathbf{Q}) \cong \operatorname{Sym}(\mathfrak{g}^*_{\mathbf{Q}}[-2])^{\check{G}_{\mathbf{Q}}}$. (But this is circular: this isomorphism is used in proving derived Satake.)

Quillen showed that there is a homotopy equivalence $Gr_G \simeq \Omega G$, and in fact the Satake equivalence also captures a lot of classical calculations about the equivariant (co)homology of the based loop space of G.

Goal

Goal

Understand what happens if k is replaced by a commutative ring *spectrum*.

To understand the form that the answer might take, we will consider the case when G is a torus T. (You could take $T = \mathbf{G}_m$, but this obscures some of the combinatorics.) In this case:

- Gr_T = $\Omega T = \pi_1(T)$ is just the lattice of cocharacters $\mathbf{G}_m \to T$, denoted $\mathbb{X}_*(T)$.
- The action of $T(\mathcal{O}) \simeq T$ on $\operatorname{Gr}_{\mathcal{T}}$ is trivial.

Together, these facts tell us that $Shv(Gr_T/T(\mathcal{O}); k)$ is a rather simple category.

Torus

Let us unwind:

$$\operatorname{Shv}(\operatorname{Gr}_{\mathcal{T}}/\mathcal{T}(\mathcal{O});k) \simeq \operatorname{Shv}(\mathbb{X}_*(\mathcal{T}) \times \mathcal{BT};k) \simeq \bigoplus_{\mathbb{X}_*(\mathcal{T})} \operatorname{Shv}(\mathcal{BT};k).$$

What do we mean by Shv(BT; k)? This should be the category of *T*-equivariant *k*-modules. So, we could either work:

• Borel-equivariantly, so $\operatorname{Shv}(BT; k) = \operatorname{Mod}_{C^*(BT;k)}^{\wedge}$. Thus

$$\operatorname{Shv}(BT;k) = \operatorname{QCoh}(\operatorname{Hom}(\mathbb{X}^*(T),\widehat{\mathbf{G}}_k^Q)),$$

where $\widehat{\mathbf{G}}_{k}^{Q} = \operatorname{Spf} C^{*}(BS^{1}; k)$ denotes the Quillen formal group over k.

• genuine-equivariantly (if k admits a genuine-equivariant refinement). So

$$\operatorname{Shv}(BT; k) = \operatorname{QCoh}(\operatorname{Hom}(\mathbb{X}^*(T), \mathbf{G}_k^Q)),$$

where \mathbf{G}_{k}^{Q} is a decompletion of the Quillen formal group.

Torus

If
$$\mathbf{H}_{k}^{\text{Spec}} := \widehat{\mathbf{G}}_{k}^{Q}$$
 or \mathbf{G}_{k}^{Q} , and $T_{\mathbf{H}_{k}^{\text{Spec}}} := \text{Hom}(\mathbb{X}^{*}(T), \mathbf{H}_{k}^{\text{Spec}})$, we find
 $\operatorname{Shv}(\operatorname{Gr}_{T}/T(\mathcal{O}); k) \simeq \bigoplus_{\mathbb{X}_{*}(T)} \operatorname{QCoh}(T_{\mathbf{H}_{k}^{\text{Spec}}}).$

Notice that if $\check{T}_k := \operatorname{Spec} k[\mathbb{X}_*(T)]$, then $\operatorname{Rep}(\check{T}_k) = \bigoplus_{\mathbb{X}_*(T)} \operatorname{Mod}_k$. The group scheme \check{T}_k is the Langlands dual torus defined over k. We find:

Satake equivalence for a torus

There is a k-linear equivalence

$$\operatorname{Shv}(\operatorname{Gr}_{T}/T(\mathcal{O});k) \simeq \operatorname{QCoh}(T_{\mathbf{H}^{\operatorname{Spec}}} \times B\check{T}_{k}).$$

Works for any compact abelian T. If T is finite, \check{T}_k is the Pontryagin dual, and the Satake equivalence becomes Hopkins-Kuhn-Ravenel character theory.

Other reductive groups

Given our success with tori, natural to wonder about the case of a general (split) reductive group G. Let $T \subseteq G$ be a maximal torus.

There is a theory of genuine-equivariant sheaves on topological stacks in development by Cnossen-Maegawa-Volpe and Konovalov-Perunov-Prikhodko. So one can make sense of $Shv(Gr_G/G(0); k)$.

We run into a problem on the Langlands dual side: what would replace \check{T}_k ? If k is an ordinary commutative ring, it is replaced by the Langlands dual group \check{G}_k defined over k: this is an algebraic group whose maximal torus is \check{T}_k .

If k is an arbitrary commutative ring spectrum, one needs to make sense of \check{G}_k as a group scheme over k. Is this even possible?

No-go

One cannot naturally lift ${\rm SL}_2$ to ${\rm ku}$ as an $\textbf{E}_4\text{-scheme:}$ power operations do not respect the relation $\det=1.$ (What about as an $\textbf{E}_3\text{-}$ or $\textbf{E}_2\text{-scheme?}$ I don't know.)

What to do?

Pretend that \check{G}_k exists over k, and that there was a Satake equivalence

 $\operatorname{Shv}(\operatorname{Gr}_G/G(\mathcal{O});k) \simeq \operatorname{QCoh}(\mathfrak{X}_k)$

for some spectral k-stack \mathfrak{X}_k having to do with \check{G}_k .

Suppose k is even. Any spectral k-stack X which is locally constructed from even affine k-schemes admits a degeneration to an ordinary graded $\pi_*(k)$ -stack X^{\heartsuit} , given by degenerating \mathcal{O}_X to $\pi_*\mathcal{O}_X$. (Just the even filtration.)

So, if there was a Satake equivalence as above, one would get a 1-parameter degeneration of $\operatorname{Shv}(\operatorname{Gr}_G/G(\mathcal{O}); k)$ into QCoh of $\mathfrak{X}_k^{\heartsuit}$.

Revised goal

Try to construct the $\pi_*(k)$ -stack $\mathfrak{X}_k^{\heartsuit}$ which \mathfrak{X}_k degenerates to, and actually prove that there is a 1-parameter degeneration

$$\operatorname{Shv}(\operatorname{Gr}_{G}/G(\mathcal{O});k) \rightsquigarrow \operatorname{QCoh}(\mathfrak{X}_{k}^{\heartsuit}).$$

Examples

We have two examples of the stack $\mathfrak{X}_k^{\heartsuit}$:

• k is an ordinary commutative ring: then Bezrukavnikov-Finkelberg tell us that

$$\mathfrak{X}_k^\heartsuit = \mathfrak{g}_k(2)/\check{G}_k$$

over $\pi_*(k) = k$.

• *G* is a torus *T*, and *k* arbitrary. Then $\mathfrak{X}_k = T_{\mathbf{H}_k^{\text{Spec}}} \times B\check{T}_k$. So, if **H** is the group scheme over $\pi_*(k)$ given by $(\widehat{\mathbf{G}}_k^Q)^{\heartsuit}$ or $(\mathbf{G}_k^Q)^{\heartsuit}$, then

$$\mathfrak{X}_{k}^{\heartsuit} = T_{\mathsf{H}} \times B\check{T}_{\pi_{*}(k)}$$

where $T_{\mathbf{H}} = \operatorname{Hom}(\mathbb{X}^*(T), \mathbf{H})$ and $\check{T}_{\pi_*(k)}$ denotes the *ordinary* group scheme given by the Langlands dual torus.

Note that $\mathbf{H} = \operatorname{Spf} \pi_*(k^{hS^1})$ in the Borel-equivariant case.

Adapting G to H

We will write $\mathfrak{X}_{k}^{\heartsuit}$ as $G_{\mathbf{H}}/\check{G}_{\pi_{*}(k)}$ for some stack $G_{\mathbf{H}}$ such that $G_{\mathbf{G}_{a}(2)} = \mathfrak{g}_{k}(2)$, and $T_{\mathbf{H}} = \operatorname{Hom}(\mathbb{X}^{*}(T), \mathbf{H})$. Here, $\check{G}_{\pi_{*}(k)}$ denotes the *ordinary* Langlands dual group, base-changed along $\mathbf{Z} \to \pi_{*}(k)$.

Definition (Fratila-Gunningham-Li, Moulinos-Robalo-Toen, Khan-Bouaziz, D., ...) Let X be a $\pi_*(k)$ -stack. The **H**-loop space $\mathcal{L}_{\mathbf{H}}(X)$ is defined using the Tannakian formalism as

$$\mathcal{L}_{\mathbf{H}}(X) := \operatorname{\mathsf{Fun}}_{\pi_*(k)}^{\otimes, L}(\operatorname{QCoh}(X)^{\otimes}, \operatorname{IndCoh}_0(\mathbf{H})^{\star}).$$

Here, $Coh_0(\mathbf{H})^*$ is the category of coherent sheaves on \mathbf{H} of length zero, with symmetric monoidal structure given by convolution.

If **H** is a formal group, then $\mathcal{L}_{\mathbf{H}}(X) = \operatorname{Map}(B\mathbf{H}^{\vee}, X)$ where \mathbf{H}^{\vee} is the Cartier dual of **H**.

Examples

When $X = BG_{\pi_*(k)}$, there is a map $\mathcal{L}_{\mathbf{H}}(BG_{\pi_*(k)}) \to BG_{\pi_*(k)}$. The pullback along $Spec(\pi_*(k)) \to BG_{\pi_*(k)}$ will be written $G_{\mathbf{H}}$. Here is a table of examples:

Н	G _H
$\mathbf{G}_{a}(2)$	g(2)
$\widehat{\mathbf{G}_{a}}(2)$	$\mathfrak{g}^\wedge_{\mathcal{N}}(2)$
G _m	G
$\widehat{\mathbf{G}_m}$	$G^{\wedge}_{\mathfrak{U}}$
E elliptic curve	$\operatorname{Bun}_{G}^{\operatorname{ss}}(E)^{\operatorname{triv}}$

For notational simplicity, I have dropped the subscript $\pi_*(k)$; everything is defined over this base. Here, \mathcal{N} is the cone of nilpotent elements, and \mathcal{U} is the cone of unipotent elements.

General conjecture

Conjecture (D.)

If k is even, G is simply-laced and simply-connected, then there is a 1-parameter degeneration

$$\operatorname{Shv}(\operatorname{Gr}_{G}/G(\mathcal{O}); k) \rightsquigarrow \operatorname{QCoh}^{\operatorname{gr}}(G_{\mathsf{H}}/\check{G}),$$

where the right-hand side is defined over $\pi_*(k)$. Think of as a sheafy version of the even filtration. (If k is not even, then work even-locally on k.)

One also work non-G-equivariantly: then there should be a 1-parameter degeneration

$$\operatorname{Shv}^{G(\mathcal{O})-\operatorname{cbl}}(\operatorname{Gr}_{G};k) \rightsquigarrow \operatorname{QCoh}^{\operatorname{gr}}(\mathcal{N}_{\mathsf{H}}/\check{G}),$$

where $\mathcal{N}_{\mathbf{H}}$ is the "**H**-nilpotent cone", given by central fiber of the invariant-theoretic quotient map $G_{\mathbf{H}} \to G_{\mathbf{H}} /\!\!/ \check{G}$.

General conjecture

If k is an ordinary commutative ring, the conjecture says (in the genuine equivariant setting)

$$\operatorname{Shv}(\operatorname{Gr}_G/G(\mathcal{O}); k) \rightsquigarrow \operatorname{QCoh}^{\operatorname{gr}}(\mathfrak{g}(2)/\check{G}).$$

View as integral refinement of Bezrukavnikov-Finkelberg. In the Borel-equivariant setting, get $\mathfrak{g}^{\wedge}_{\mathcal{N}}(2)/\check{G}$; renormalized version (see Arinkin-Gaitsgory).

On the other extreme, suppose G = 0 and k = S. Working even-locally on S, one obtains the 1-parameter degeneration via Adams-Novikov:

$$\operatorname{Shv}(*; \mathbb{S}) = \operatorname{Sp} \rightsquigarrow \operatorname{QCoh}^{\operatorname{gr}}(\mathcal{M}_{\operatorname{FG}}).$$

So one should think of the conjecture as mixing Langlands duality with Adams-Novikov phenomena.

A result

Here is a statement providing evidence for the conjecture (not quite correct as written).

Theorem (D.)

Suppose $k = \mathbf{Z}$, ku, KU, ko, j, KO, or elliptic cohomology. Also suppose G is not of type E_8 . Then there is a filtered category \mathbb{C}^{fil} over $\text{fil}_{ev}^*(k)$ whose:

- underlying k-linear category C is Shv(Gr_G/G(O); k);
- the associated graded $\operatorname{gr}_{\operatorname{ev}}^{\star}(k)$ -linear category $\operatorname{C}^{\operatorname{gr}}$ is equivalent to $\operatorname{QCoh}^{\operatorname{gr}}(G_{\operatorname{H}}/\check{G})$ upon base-change to any algebraically closed field under $\operatorname{gr}_{\operatorname{ev}}^{\star}(k)$ of sufficiently large characteristic.

When $G = GL_n$, one does not need to do this base-change. This case was previously considered by Cautis-Kamnitzer when k = KU.

Main tools: calculation of equivariant homology $\pi_* C^G_*(\Omega G; k)$ in terms of \check{G} ; and purity arguments using cellularity of Gr_G (Schubert filtration).

Philosophy + remarks

How should one think about the 1-parameter degeneration

 $\operatorname{Shv}^{\mathcal{G}(\mathcal{O})-\operatorname{cbl}}(\operatorname{Gr}_{\mathcal{G}};k) \rightsquigarrow \operatorname{QCoh}^{\operatorname{gr}}(\mathcal{N}_{\mathsf{H}}/\check{\mathcal{G}})?$

(Working with the non-equivariant version of the conjecture for simplicity.) Recall when G = 0 and k = S, this was supposed to be the degeneration of Sp to $\operatorname{QCoh}^{\operatorname{gr}}(\mathcal{M}_{\operatorname{FG}})$. This can be implemented through synthetic spectra, or equivalently (upon profinite completion) the category $\operatorname{SH}^{\operatorname{cell}}(\operatorname{Spec}(\mathbf{C}))$.

If X is a scheme over **C** equipped with a cellular stratification S (so each stratum is an affine space), let $SH^{S-cell}(X)$ be the category of motivic spectra over X whose !- and *-restriction to each stratum is cellular. Then (upon profinite completion) one gets a 1-parameter degeneration

$$\mathrm{SH}^{\mathrm{S-cell}}(X)[\tau^{-1}] \approx \mathrm{Shv}^{\mathrm{S-cell}}(X; \mathrm{Sp}) \rightsquigarrow \mathrm{SH}^{\mathrm{S-cell}}(X)_{\tau=0},$$

and the right-hand side is sometimes $\operatorname{QCoh}^{\operatorname{gr}}$ on some algebraic stack. Can view as a "relative" version of synthetic spectra. The conjectural degeneration above roughly corresponds to the case $X = \operatorname{Gr}_{\mathcal{G}}$ with the Schubert stratification.

Philosophy + remarks

Langlands duality with coefficients in an ordinary commutative ring k is of a "motivic nature", meaning roughly that the spectral side is ambivalent to the choice of k. If k is a ring spectrum, then the conjecture says instead that the spectral side depends on the choice of k essentially *only* through the corresponding 1-dimensional formal group **H** which controls Chern classes.

Note that in the stack $G_{\mathbf{H}}/\check{G}$, the "numerator" $G_{\mathbf{H}}$ depends on \mathbf{H} , so its fibers over $\operatorname{Spec}_{B\mathbf{G}_m}(\operatorname{gr}_{\mathrm{ev}}^{\star}(\mathbb{S})) \cong \mathcal{M}_{\mathrm{FG}}$ vary. But the "denominator" $B\check{G}$ is completely independent of the formal group \mathbf{H} : in fact, it is pulled back along the map $\mathcal{M}_{\mathrm{FG}} \to B\mathbf{G}_m$, so in a sense it is "defined over \mathbf{F}_1 ". This is in accordance with the motivic nature of Langlands duality.

Philosophy + remarks

Can also match objects under the degeneration: a *G*-space *X* defines a $\operatorname{Shv}(\operatorname{Gr}_G/G(\mathcal{O}); k)$ -module category; describing its degeneration in terms of $G_{\mathbf{H}}/\check{G}$ can often be very interesting. If *k* is an ordinary commutative ring, this is the content of *relative Langlands duality* (Ben-Zvi–Sakellaridis–Venkatesh). Here is an example:

Theorem (D.; here $X = PGL_2/\mathbf{G}_m$)

There is a 1-parameter degeneration

 $\operatorname{Shv}(\operatorname{PGL}_2(\mathcal{O})\backslash\operatorname{PGL}_2(F)/\mathbf{G}_m(F);\operatorname{ku}) \rightsquigarrow \operatorname{QCoh}^{\operatorname{gr}}(T^*_{\beta}(\mathbf{A}^2)/\operatorname{SL}_2),$

where $T^*_{\beta}(\mathbf{A}^2)$ is the scheme of pairs $(u, v) \in \mathbf{A}^2 \oplus (\mathbf{A}^2)^*$ such that $1 + \beta \langle u, v \rangle$ is a unit. The action of $\mathbf{Z}/2 = N_{\mathrm{PGL}_2}(\mathbf{G}_m)/\mathbf{G}_m$ on the left-hand side identifies with (a β -deformation of) the symplectic Fourier transform.

Upon base-change along $ku \rightarrow \textbf{Z},$ get a geometrization of spherical harmonics.

Loop rotation

The category $\operatorname{Shv}(\operatorname{Gr}_G/G(\mathcal{O}); k)$ is an $\mathbf{E}_3 \rtimes S^1$ -monoidal category. I'll ignore the \mathbf{E}_3 -structure, and focus on the S^1 -action: this comes from *loop-rotation*. E.g., under the homotopy equivalence between Gr_G and $\Omega^2 BG = \operatorname{Map}_*(S^2, BG)$, the S^1 -action rotates S^2 . One can therefore consider the k^{hS^1} -linear category $\operatorname{Shv}_{S^1}(\operatorname{Gr}_G/G(\mathcal{O}); k)$.

Theorem (Bezrukavnikov-Finkelberg)

There is a $\mathbf{Q}^{hS^1} = \mathbf{Q}[\hbar]$ -linear equivalence

 $\operatorname{Shv}_{\mathcal{S}^1}(\operatorname{Gr}_{\mathcal{G}}/\mathcal{G}(\mathcal{O}); \mathbf{Q})[\hbar^{-1}] \simeq U(\check{\mathfrak{g}})\operatorname{-mod}(\operatorname{Rep}(\check{\mathcal{G}}))[\hbar^{\pm 1}].$

Here, $U(\tilde{\mathfrak{g}})$ is the universal enveloping algebra of \check{G} .

Without loop rotation, the right-hand side was $\operatorname{QCoh}(\check{\mathfrak{g}}^*[2]/\check{G})$. So, adding loop-rotation amounts to *deformation quantizing* $\check{\mathfrak{g}}^*$ to $U(\check{\mathfrak{g}})$. (There is a much more general story about $\mathbf{E}_3 \rtimes S^1$ -algebras and deformation quantizations, via $\operatorname{fil}^*_{\operatorname{ev}} C^*(\operatorname{Conf}_n(\mathbf{R}^3)_{hS^1}; \mathbb{S})$; for another time!)

Torus

What happens when we add in loop-rotation equivariance for other commutative ring spectra k? When G = T is a torus, the *T*-action on $\operatorname{Gr}_{T} = \Omega T$ is trivial; but it is **not** loop-rotation equivariantly trivial. This is for the same reason that the S^1 -action on Hochschild homology is interesting. In general (working Borel-equivariantly for simplicity), one finds:

Theorem (D.)

Suppose k is even, so that $\pi_*(k^{hS^1}) \cong \pi_*(k)[\hbar]^{\wedge}$. Let $T = \mathbf{G}_m$ for simplicity, so $\check{T} = \mathbf{G}_m$ too. Then there is a 1-parameter degeneration

$$\operatorname{Shv}_{\mathcal{S}^1}(\operatorname{Gr}_{\mathcal{T}}/\mathcal{T}(\mathcal{O});k) \rightsquigarrow \mathcal{D}^{\mathsf{H}}_{\check{\mathcal{T}}}\operatorname{-mod}(\operatorname{Rep}(\check{\mathcal{T}} \times \check{\mathcal{T}})),$$

where $\mathcal{D}_{\check{\tau}}^{\mathsf{H}}$ is the associative ('**H**-Weyl") $\pi_*(k)$ -algebra defined by

$$\mathcal{D}_{\check{T}}^{\mathsf{H}} := \pi_*(k)[\hbar] \langle x^{\pm 1}, \nabla_x^{\mathsf{H}} \rangle^{\wedge} / (\nabla_x^{\mathsf{H}} x = (x \nabla_x^{\mathsf{H}}) +_{\mathsf{H}} \hbar).$$

Calculation is Koszul dual to an unpublished result of Arpon Raksit about the even filtration on $HC^{-}((\mathbf{G}_m)_k/k)$. Can rephrase in terms of \mathbf{E}_2 -Hochschild cohomology.

Torus

The algebra $\mathcal{D}_{\vec{T}}^{\mathsf{H}}$ on the preceding slide is just the usual Weyl algebra of \check{T} when k is an ordinary commutative ring; and it recovers the *q*-Weyl algebra when $k = \mathrm{ku}$. I will remark that the preceding result could be rewritten as

 $\operatorname{Shv}_{S^1}(\operatorname{Gr}_T/T(\mathcal{O}); k) \rightsquigarrow U_{\mathsf{H}}(\check{T})\operatorname{-mod}(\operatorname{Rep}(\check{T})),$

where $U_{\mathbf{H}}(\check{T}) = (\mathcal{D}_{\check{T}}^{\mathbf{H}})^{\check{T}}$ is isomorphic to $\pi_*(k)[\hbar, \nabla_x^{\mathbf{H}}]^{\wedge}$.

One can view $U_{\mathbf{H}}(\check{\mathcal{T}})$ as an analogue of the enveloping algebra $U(\check{\mathfrak{t}})$.

What about other G? Let's for simplicity take $G = PGL_2$, so $\check{G} = SL_2$, and ask: what is the analogue of $U(\mathfrak{sl}_2)$ which deformation quantizes $(PGL_2)_{H}$?

$G = \mathrm{PGL}_2$

(Vague) conjecture (D.)

The category $\text{Shv}_{S^1}(\text{Gr}_{PGL_2}/\text{PGL}_2(0); k)$ is related to modules over the associative algebra

$$U_{\mathsf{H}}(\mathrm{SL}_2) := \pi_*(k)[\hbar] \langle e, f, h \rangle^{\wedge} / I,$$

where I is given by the relations

$$eh = (h -_{\mathbf{H}} \hbar)e,$$

$$fh = (h +_{\mathbf{H}} \hbar)f,$$

$$ef - fe = h(\overline{h} +_{\mathbf{H}} \hbar) - \overline{h}(h +_{\mathbf{H}} \hbar).$$

Here, \overline{h} is the inverse of h in **H**.

I'm close to being able to prove such a statement, but cannot yet; relations above come from calculations with $\operatorname{Gr}_{\operatorname{PGL}_2}$. When $k = \mathbb{Z}[1/2]$, get $U(\mathfrak{sl}_2)$; when $k = \operatorname{ku}$, get essentially the quantum group $U_q(\operatorname{SL}_2)$ (where $q = 1 + \beta\hbar$).

Remarks

I find the algebra $U_{H}(SL_2)$ very beautiful. Its representation theory is similar to that of $U(\mathfrak{sl}_2)$ and of the quantum group. Also, it has a central "Casimir" element

$$c := fe - \overline{h}(h +_{\mathbf{H}} \hbar),$$

and there is an isomorphism

$$U_{\mathsf{H}}(\mathrm{SL}_2)/c \cong R\Gamma(\mathbf{P}^1; \mathcal{D}_{\mathbf{P}^1}^{\mathsf{H}}).$$

This is exactly like in Beilinson-Bernstein. One can also generalize $U_{\mathbf{H}}(\mathrm{SL}_2)$ to $U_{\mathbf{H}}(\check{G})$ for other \check{G} , via an **H**-deformation of the Serre relations in $U(\check{\mathfrak{g}})$.

I don't (yet?) know how to relate $U_{\mathbf{H}}(\check{G})$ to $\operatorname{Shv}_{S^1}(\operatorname{Gr}_G/G(\mathfrak{O}); k)$. It should nevertheless be interesting to study $U_{\mathbf{H}}(\check{G})$ independently, e.g., in the context of Lusztig-Williamson's "philosophy of generations". In general, I think that there is a lot about representation theory that the combination of chromatic homotopy theory + geometry can be used to uncover.

Thank you!

Devalapurka	