ANDERSON DUALITY FOR DERIVED STACKS (NOTES)

Abstract. In these notes, we will prove that many naturally occuring derived stacks in chromatic homotopy theory, which arise as even periodic refinements of Deligne-Mumford stacks, are Gorenstein (in the sense that their dualizing sheaves are line bundles).

1. Introduction

A similar analysis was undertaken by [GS17].

1.1. **Conventions.** If $\mathfrak{X}$ is a derived stack, then X will denote the underlying classical stack. If $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of derived stacks, then $f_0 : X \to Y$ will denote the underlying morphism of classical stacks. All Deligne-Mumford stacks will be assumed to have affine diagonal.

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2. The connective setting

A flat morphism $f : A \to B$ of classical (i.e., discrete) commutative rings is said to be Gorenstein if the relative dualizing module $\omega_{B/A} = \mathrm{Hom}_A(B, A)$ is an invertible B-module. This definition can, of course, be globalized: a flat morphism $f : X \to Y$ of classical Deligne-Mumford stacks is said to be Gorenstein if the relative dualizing complex $f^!\mathcal{O}_Y$ is in the Picard group $\mathrm{Pic}(X)$. This definition requires care, since the functor $f^!$ is not defined in general. Nonetheless, if f is proper, then there is an isomorphism $f_! \to f_*$ of functors, which allows us to regard $f^!$ as a right adjoint of f_*.

In the derived setting, we make a similar definition.

Definition 2.0.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a proper morphism of derived stacks. Then f is said to be *Gorenstein* if $f^!\mathcal{O}_{\mathfrak{Y}} \in \mathrm{Pic}(\mathfrak{X})$.

If $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is a morphism of affine derived schemes (not necessarily proper, i.e., finite), then f is said to be Gorenstein if $f^!\mathcal{O}_{\mathrm{Spec}\, A} = \underline{\mathrm{Map}}_A(B, A)$ is in the Picard group of B. An $\mathbf{E}_\infty$-ring A is said to be Gorenstein if the structure morphism $\mathrm{Spec}\, A \to \mathrm{Spec}\, S$ is Gorenstein.

There are numerous examples of such morphisms. For example:

Lemma 2.0.2. *Let G be a topological group, and let $f : A \to B$ be a G-Galois extension of $\mathbf{E}_\infty$-rings. Then f is Gorenstein.*

Proof. We need to show that $\underline{\mathrm{Map}}_A(B, A)$ is invertible as a B-module. We may base-change f to get a G-Galois extension $f' : B \to B \otimes_A B$; it suffices to check that $f^*\underline{\mathrm{Map}}_A(B, A)$ is invertible as a $B \otimes_A B$-module. However, $f^*\underline{\mathrm{Map}}_A(B, A) \simeq (B \otimes_A B) \otimes_B \underline{\mathrm{Map}}_A(B, A) \simeq \underline{\mathrm{Map}}_A(B, B)$, which, by [Rog08, Proposition 6.3.1], is equivalent to $B \wedge G_+$. Since $A \to B$ is G-Galois, we also have $B \otimes_A B \simeq \underline{\mathrm{Map}}(G_+, B)$, which shows that $f^*\underline{\mathrm{Map}}_A(B, A)$ is invertible as a $B \otimes_A B$-module, as desired. □

2.1. **Smooth morphisms are Gorenstein.** The goal of this section to prove that smooth morphisms are Gorenstein in the above sense. In the classical setting, this is a well-known result; here, however, the situation is complicated by the fact that there are two different notions of smoothness. We will recall these definitions in Section 2.1.1, and then prove our main results in Section 2.1.2.

2.1.1. *Smooth morphisms.* In [Lur18, §11.2], Lurie discusses two different notions of smoothness in the world of spectral algebraic geometry. We shall recall these definitions here.

Definition 2.1.1. Let $f : A \to B$ be a morphism of connective $\mathbf{E}_\infty$-rings such that $\pi_0 B$ is a finitely presented $\pi_0 A$-algebra. Then f is said to be *differentially smooth* if the cotangent complex $L_{B/A}$ is a projective B-module of finite rank.

The differential smoothness of a morphism can almost be checked on the level of ordinary commutative algebra, as the following result of Lurie's shows (see [Lur18, Corollary 11.2.2.8]):

Theorem 2.1.2. *A morphism $f : A \to B$ of connective $\mathbf{E}_\infty$-rings is differentially smooth if and only if the induced map $\pi_0 A \to \pi_0 A \otimes_A B$ is differentially smooth.*

The other notion of smoothness which we shall be concerned with is that of fiber smoothness.

Definition 2.1.3. Let $f : A \to B$ be a morphism of connective $\mathbf{E}_\infty$-rings. Then f is said to be *fiber smooth* if f is flat and almost of finite presentation, and for every residue field $B \to k$, the k-vector space $\pi_1(k \otimes_B L_{B/A})$ vanishes.

Then, we have (see [Lur18, Proposition 11.2.3.6]):

Theorem 2.1.4. *A morphism $f : A \to B$ of connective $\mathbf{E}_\infty$-rings is fiber smooth if and only if for every algebraically closed residue field $A \to k$, the $\mathbf{E}_\infty$-ring $k \otimes_A B$ is discrete and regular as a discrete k-algebra.*

Remark 2.1.5. Suppose A is an $\mathbf{E}_\infty$-ring such that every differentially smooth morphism $f : A \to B$ is fiber smooth. Then A is a $\mathbf{Q}$-algebra. The converse is also true, by [Lur18, Proposition 11.2.4.4]. For the other direction, consider the differentially smooth morphism $A \to A\{x\}$. By assumption, this map is fiber smooth. Let $A \to k$ be a residue field of A; then, the induced map $k \to k \otimes_A A\{x\} = k\{x\}$ is both differentially smooth and fiber smooth. It follows that $k\{x\}$ is flat over k. Since k is discrete, we conclude that $\bigoplus_{d,n\geq 0} \mathrm{H}_d(B\Sigma_n; k)$ vanishes for $d \neq 0$. This is impossible if the characteristic of k is nonzero, so every residue field of A has characteristic 0. This implies that A is a $\mathbf{Q}$-algebra.

2.1.2. *Gorenstein properties.* The goal of this section is to prove the following two results.

Theorem 2.1.6. *Let $f : A \to B$ be a fiber smooth morphism of connective $\mathbf{E}_\infty$-rings. Then f is Gorenstein.*

Proof. By Theorem 2.1.4, the morphism f is fiber smooth if and only if for every algebraically closed residue field $A \to k$, the $\mathbf{E}_\infty$-ring $k \otimes_A B$ is discrete and regular as a discrete k-algebra. In particular, $k \otimes_A B$ is Gorenstein over k. Since the morphism f is Gorenstein at every residue field, it follows from [Lur18, Proposition 6.6.6.7] that f is itself Gorenstein, as desired. □

Theorem 2.1.7. *Let A be a connective $\mathbf{E}_\infty$-ring. Then every differentially smooth morphism $f : A \to B$ of connective $\mathbf{E}_\infty$-rings is Gorenstein if and only if A is a $\mathbf{Q}$-algebra.*

Proof. Let f be as above, and assume that A is a $\mathbf{Q}$-algebra. By Theorem 2.1.2 and [Lur18, Proposition 6.6.6.7], it suffices to prove the desired result when A is an algebraically closed field k. Since A is a $\mathbf{Q}$-algebra, the field k is of characteristic 0. As B is differentially smooth over k, there exist elements $b_1, \cdots, b_n$ which generate the unit ideal in B such that there are étale morphisms $k\{x_1, \cdots, x_{m_i}\} \to B[b_i^{-1}]$ from the free $\mathbf{E}_\infty$-k-algebra on m_i generators. Suppose that $k\{x_1, \cdots, x_n\}$ is Gorenstein over k for every $n \geq 0$; it follows from the above observation that $B[b_i^{-1}]$ is étale for every $1 \leq i \leq n$. Since the property of being Gorenstein can be checked after passing to an étale cover, we conclude that B is itself Gorenstein.

We claim that the free $\mathbf{E}_\infty$-k-algebra $k\{x_1, \cdots, x_n\}$ on n generators is also a discrete $\mathbf{E}_\infty$-ring; this will imply that $k\{x_1, \cdots, x_n\} = k[x_1, \cdots, x_n]$, so the free $\mathbf{E}_\infty$-k-algebra is smooth, hence Gorenstein, over k, as desired. We will induct on n. When $n = 1$, we have $\pi_* k\{x\} = \bigoplus_{n \geq 0} \mathrm{H}_*(B\Sigma_n; k)$. Since k has characteristic 0, these homology groups vanish above degree 0, so $k\{x\}$ is discrete, and is in fact equivalent to the polynomial ring $k[x]$. The same argument proves the inductive step.

It remains to prove the converse. Let f be as above, and assume that f is Gorenstein. By reducing to the residue field of A, we may again assume that A is a discrete field k. Setting $B = k\{x\}$, we find that B must itself be Gorenstein. By [Lur18, Remark 6.6.5.5], this implies that B is m-truncated for some $m \gg 0$. In other words, there must exist some $m \gg 0$ such that $\mathrm{H}_d(B\Sigma_n; k) = 0$ for all $d \geq m$ and all nonnegative integers n. If k has characteristic $p > 0$, this is impossible: the group Σ_p has infinite cohomological dimension over k. It follows that every residue field of A must have characteristic 0, which implies that A is a $\mathbf{Q}$-algebra. □

We now record a few corollaries of Theorem 2.1.6. The first application is to a discussion of local complete intersection morphisms. The following definition is suggested by [Avr99, Theorem 4.12].

Definition 2.1.8. A morphism $A \to B$ of connective Noetherian $\mathbf{E}_\infty$-rings is said to be a *local complete intersection morphism* if $L_{B/A}$ is perfect and has Tor-amplitude in $[-1, 0]$.

Remark 2.1.9. Let $f : A \to B$ be a differentially smooth morphism. Then f is a local complete intersection morphism. Indeed, $L_{B/A}$ is a projective B-module of finite rank, and hence is perfect and flat by [Lur16, Lemma 7.2.2.14].

An immediate corollary of Theorem 2.1.7 and Remark 2.1.9 is the following.

Corollary 2.1.10. *Let A be a connective $\mathbf{E}_\infty$-ring. Then every local complete intersection morphism $f : A \to B$ of connective $\mathbf{E}_\infty$-rings is Gorenstein if and only if A is a $\mathbf{Q}$-algebra.*

We also have the following result.

Proposition 2.1.11. *Let $f : A \to B$ be a morphism of connective Noetherian $\mathbf{E}_\infty$-rings which is locally of finite presentation (so $L_{B/A}$ is perfect, by* [Lur16, Theorem 7.4.3.18]*). Then f is a local complete intersection morphism if and only if for every residue field $A \to k$, the induced map $k \to k \otimes_A B$ is a local complete intersection morphism.*

Proof. Indeed, assume that f is a local complete intersection morphism; then, $L_{B/A}$ has Tor-amplitude in $[-1, 0]$. Since $L_{k \otimes_A B/k} \simeq L_{B/A} \otimes_A k$, it follows that $L_{k \otimes_A B/k}$ is a k-module with Tor-dimension in $[-1, 0]$. Conversely, suppose that $L_{B/A} \otimes_A k$ is a perfect k-module with Tor-amplitude in $[-1, 0]$ for every residue field $A \to k$. In order to show that $L_{B/A}$ has Tor-amplitude in $[-1, 0]$, it suffices to prove the following statements.

(∗) Let A be a discrete Noetherian commutative ring, and let B be a connective $\mathbf{E}_\infty$-A-algebra which is of finite presentation over A. Let M be a perfect B-module. Then
  (1) Let $\mathfrak{p} \subseteq \pi_0 B$ be a prime ideal, and let $\mathfrak{q}$ denote its inverse image in A. Then $M_\mathfrak{p}$ has Tor-amplitude in $[m, n]$ as an A-module if and only if $\pi_d(M \otimes_A A/\mathfrak{q})_\mathfrak{p}$ vanishes for $d \notin [m, n]$.
  (2) Let $U \subseteq |\operatorname{Spec} \pi_0 B|$ denote the set of those prime ideals $\mathfrak{p} \in \operatorname{Spec} \pi_0 B$ such that $M_\mathfrak{p}$ has Tor-amplitude in $[m, n]$ as an A-module. Then U is open.

Let us first argue that $(*)$ is sufficient. We claim that a perfect module M over a connective Noetherian $\mathbf{E}_\infty$-ring A has Tor-amplitude in $[m,n]$ if and only if for every residue field $A \to k$, the k-module $M \otimes_A k$ has Tor-amplitude in $[m,n]$. One direction is clear. For the other direction, we can reduce to the case that A is discrete. Fix $m' \leq m$. The map $\pi_d(\tau_{\geq m'} M \otimes_A k) \to \pi_d(M \otimes_A k)$ is an isomorphism for $d \geq m'$. Therefore, $\pi_d(\tau_{\geq m'} M \otimes_A k) = 0$ for $m' \leq d \leq m$. This implies, by statement $(*)$, that $\tau_{\geq m'} M$ has Tor-amplitude $\geq m$ over A. It follows $\tau_{\geq m'} M$ is n-truncated, since A is discrete. In particular, M is n-truncated, so $M = \tau_{\geq m} M$ has Tor-amplitude $\geq m$. We can argue similarly to conclude that M has Tor-amplitude $\leq n$, which proves the desired claim.

We now prove statement $(*)$. We first prove (1); our proof follows that of [Mat86, Theorem 24.3]. Let $\kappa = A/\mathfrak{q}$. Clearly if $M_\mathfrak{p}$ has Tor-amplitude in $[m,n]$ as an A-module, then $\pi_d(M \otimes_A \kappa)_\mathfrak{p}$ vanishes for $d \notin [m,n]$. We will prove the converse. We need to show that for any discrete $A_\mathfrak{q}$-module N', the group $\pi_d(M_\mathfrak{p} \otimes_{A_\mathfrak{q}} N')$ vanishes for $d \notin [m,n]$. By writing N' as a filtered colimit of finitely generated discrete $A_\mathfrak{q}$-modules, we may reduce to the case when N' is finitely generated. Each such discrete $A_\mathfrak{q}$-module is an extension of modules of the form $A_\mathfrak{q}/I$ for $I \subseteq A_\mathfrak{q}$ a prime ideal, so we may assume that $N' = A_\mathfrak{q}/I$. We now prove the desired result by Noetherian induction on I. Fix $d \notin [m,n]$. When $I = \mathfrak{q}$ is the maximal ideal of $A_\mathfrak{q}$, the claim that $\pi_d(M_\mathfrak{p} \otimes_{A_\mathfrak{q}} N') = 0$ is simply our assumption that $\pi_d(M \otimes_A \kappa)_\mathfrak{p}$ vanishes. For the inductive step, suppose that I is not the maximal ideal of $A_\mathfrak{q}$. Let $a \in A_\mathfrak{q}$ be a nonzero element in the maximal ideal. Then $A_\mathfrak{q}/a$ is a finite extension of modules of the form $A_\mathfrak{q}/I'$ with $I' \supsetneq I$. By the inductive hypothesis, $\pi_d(M_\mathfrak{p} \otimes_{A_\mathfrak{q}} A_\mathfrak{q}/a) = 0$. Now, $\pi_d(M_\mathfrak{p} \otimes_{A_\mathfrak{q}} A_\mathfrak{q}/I)$ is finitely generated over the local ring $\pi_0 B_\mathfrak{p}$, since $M_\mathfrak{p} \otimes_{A_\mathfrak{q}} A_\mathfrak{q}/I$ is a perfect $B_\mathfrak{q}$-module. The map $A_\mathfrak{q} \to B_\mathfrak{p}$ sends a to an element of the maximal ideal $\mathfrak{p}$ of $\pi_0 B_\mathfrak{p}$. It follows from Nakayama's lemma that $\pi_d(M_\mathfrak{p} \otimes_{A_\mathfrak{q}} A_\mathfrak{q}/I)$ itself must vanish.

We now prove (2). Our proof uses the "topological Nagata criterion" (see [Mat86, Theorem 24.2]), which reduces us to proving:

(a) For $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} A$, if $\mathfrak{p} \in U$ and $\mathfrak{q} \subset \mathfrak{p}$, then $\mathfrak{q} \in U$.
(b) If $\mathfrak{p} \in U$, then U contains a nonempty open subset of $V(\mathfrak{p})$.

Statement (a) is obvious, since $M_\mathfrak{q}$ is a filtered colimit of copies of $M_\mathfrak{p}$, so $M_\mathfrak{q}$ has Tor-amplitude in $[m,n]$ if $M_\mathfrak{p}$ has Tor-amplitude in $[m,n]$. We now prove statement (b). Let $\mathfrak{p} \in U$, and let $\mathfrak{q}$ denote its inverse image in A. Let $N = M/\mathfrak{q}$. Then the assumption that $\mathfrak{p} \in U$ implies that $\pi_d N_\mathfrak{q} = 0$ for $d \notin [m,n]$. Each $\pi_d N$ is a finitely generated $\pi_0 B$-module, so there is $f \in \pi_0 B - \mathfrak{q}$ such that $\pi_d N\left[\frac{1}{f}\right] = 0$ for all $d \notin [m,n]$. By generic freeness (in the classical setting), we can also choose $g \in A/\mathfrak{q}$ such that $\pi_i N\left[\frac{1}{f}\right]\left[\frac{1}{g}\right]$ is a free $\pi_0 A/\mathfrak{q}\left[\frac{1}{g}\right]$-module for every $i \in [m,n]$. Let g' denote the image in $\pi_0 B$ of a lift of g to $\pi_0 A$.

It suffices to show that U contains the open subset of those prime ideals $\mathfrak{p}' \subseteq \mathfrak{p}$ (i.e., $\mathfrak{p}' \in V(\mathfrak{p})$) which do not contain f and g. Let $\mathfrak{p}'$ be such a prime ideal. By statement (1), we have to show that $\pi_d(M \otimes_A A/\mathfrak{q}')_{\mathfrak{p}'} = 0$ for each $d \notin [m,n]$, where $\mathfrak{q}' \subseteq \pi_0 A$ is the inverse image of $\mathfrak{p}'$. Now, $\pi_d(M \otimes_A A/\mathfrak{q}')_{\mathfrak{p}'} \cong \pi_d(N \otimes_{A/\mathfrak{q}} A/\mathfrak{q}')_{\mathfrak{p}'}$. Since $f, g \notin \mathfrak{p}'$, it suffices to show that $\pi_d\left(N\left[\frac{1}{fg'}\right] \otimes_{A/\mathfrak{q}\left[\frac{1}{g}\right]} A/\mathfrak{q}'\right)$ vanishes for $d \notin [m,n]$. The Künneth spectral sequence converging to this group has E_2-page $E_2^{s,d-s} = \operatorname{Tor}_s^{A/\mathfrak{q}\left[\frac{1}{g}\right]}\left(\pi_{d-s} N\left[\frac{1}{fg'}\right], A/\mathfrak{q}'\right)$. But f and g were chosen so that $\pi_i N\left[\frac{1}{fg'}\right]$ are free for all $i \in [m,n]$ and vanish for $i \notin [m,n]$, so $E_2^{s,t}$ vanishes in the desired range.

$\square$

Remark 2.1.12. Another proof of Proposition 2.1.11 can be provided when $\pi_0 A$ is Artinian. Indeed, the only nontrivial part is showing that if $L_{B/A} \otimes_A k$ is a perfect k-module with Tor-amplitude in $[-1, 0]$ for every residue field $A \to k$, then $L_{B/A}$ has Tor-amplitude in $[-1, 0]$. We will prove more generally that if $\pi_0 A$ is Artinian and M is any perfect A-module, then M has Tor-amplitude in $[m, n]$ if and only if $M \otimes_A k$ has Tor-amplitude in $[m, n]$ for all residue fields $A \to k$. One direction is easy. For the other direction, let N be a discrete A-module. Since $\pi_0 A$ is Artinian, the $\pi_0 A$-module N is an extension of the discrete modules $I^a N / I^{a+1} N$ as a varies. Since $M \otimes_A k$ has Tor-amplitude in $[m, n]$ and $I^a N / I^{a+1} N$ is a discrete k-module, we learn that $\pi_i(M \otimes_A I^a N / I^{a+1} N)$ vanishes for $i \notin [m, n]$. The collection of spectra with homotopy concentrated in $[m, n]$ is closed under extensions, so $\pi_i(M \otimes_A N)$ itself vanishes for $i \notin [m, n]$, as desired.

Corollary 2.1.13. *Let $f : A \to B$ be a morphism of connective Noetherian $\mathbf{E}_\infty$-rings which is locally of finite presentation. Then f is a local complete intersection morphism if and only if the map $\pi_0 A \to \pi_0 A \otimes_A B$ is a local complete intersection morphism.*

In light of these results, it is natural to ask if there is some well-behaved notion of "local complete intersection" morphisms of which fiber smooth morphisms are an example. The answer is indeed positive: a morphism $f : A \to B$ of connective Noetherian $\mathbf{E}_\infty$-rings can be said to be a fiber local complete intersection morphism if f is almost of finite presentation, and for every residue field $A \to k$, the tensor product $k \otimes_A B$ is a *discrete* local complete intersection k-algebra. If $\pi_0 f$ is a local complete intersection morphism and $L_{B/A}$ is perfect (so that f is locally of finite presentation by [Lur16, Theorem 7.4.3.18]), then f is a fiber local complete intersection morphism. Then, every fiber smooth morphism is a fiber local complete intersection morphism, and every fiber local complete intersection morphism is Gorenstein.

3. The even periodic setting

3.1. **Dualizing sheaves in the even periodic setting.** We recall some background from [Dev17]. To motivate this discussion, we prove the following proposition.

Proposition 3.1.1. *Let $\mathfrak{X}$ be a derived Deligne-Mumford stack which arises as the even periodic refinement of a flat map $X \to \mathcal{M}_{FG}$ from a locally Noetherian Deligne-Mumford stack X, and let $E = \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$. Then E is never Gorenstein over the sphere spectrum (in the sense of Definition 2.0.1) unless $E = 0$.*

We will rely on the following lemma in the course of the proof of Proposition 3.1.1.

Lemma 3.1.2. *In the setting of Proposition 3.1.1, the $\mathbf{E}_\infty$-ring E is L_n-local for some $n \gg 0$.*

Proof. This follows from the fact that X is Noetherian. Indeed, there is a descending sequence of closed substacks of $\mathcal{M}_{FG}$ given by $\mathcal{M}_{FG}^{\geq m}$, and each $\mathcal{M}_{FG}^{\geq m}$ is obtained from $\mathcal{M}_{FG}^{\geq m-1}$ by taking the substack corresponding to the vanishing locus of a regular element. As X is Noetherian, there is some n such that $X \times_{\mathcal{M}_{FG}} \mathcal{M}_{FG}^{\geq n}$ is empty. It suffices to show that if that $\operatorname{Spec} B \to X$ is a flat morphism, then the associated Landweber exact spectrum $\widetilde{B}$ is E_n-local. Indeed, since $A = \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is a homotopy limit of $\mathcal{O}_\mathfrak{X}(\operatorname{Spec} B \to X)$ over all étale maps $\operatorname{Spec} B \to X$, it follows from the fact that L_n-local spectra are closed under limits that $\Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is also L_n-local. □

Proof of Proposition 3.1.1. Recall that E is Gorenstein over the sphere spectrum if and only if $\underline{\operatorname{Map}}(E, S)$ is an element of $\operatorname{Pic}(E)$. In order to show that E is never Gorenstein over the sphere spectrum, it suffices to prove the following two statements.

- (∗) Let $I_{\mathbf{Q}/\mathbf{Z}}$ denote the Brown-Comenetz dualizing spectrum (recall that we p-localize everywhere, so this is the p-local Brown-Comenetz dualizing spectrum). Then $E \otimes I_{\mathbf{Q}/\mathbf{Z}}$ is contractible.

($*'$) The spectrum $\underline{\mathrm{Map}}(E, S)$ is contractible if and only if $E \otimes I_{\mathbf{Q}/\mathbf{Z}}$ is contractible.

We begin by proving ($*$). It follows from Lemma 3.1.2 that E is L_n-local for some $n \gg 0$. It therefore suffices to show that $E \otimes I_{\mathbf{Q}/\mathbf{Z}}$ is contractible for any L_n-local spectrum. We may therefore reduce to the case when $E = L_n S$. Since the Bousfield class of $L_n S$ is the same as the Bousfield class of the nth Morava E-theory E_n, it suffices to prove that $E_n \otimes I_{\mathbf{Q}/\mathbf{Z}}$ vanishes. The homotopy groups of $I_{\mathbf{Q}/\mathbf{Z}}$ are all bounded above and torsion, so $\langle \mathbf{F}_p \rangle \geq \langle I_{\mathbf{Q}/\mathbf{Z}} \rangle$, where $\mathbf{F}_p$ is regarded as a discrete $\mathbf{E}_\infty$-ring; it therefore suffices to prove that $E_n \otimes \mathbf{F}_p = 0$. The ring $\pi_*(E_n \otimes \mathbf{F}_p) = \mathrm{H}_*(E_n; \mathbf{F}_p)$ has two isomorphic formal groups defined over it: the additive formal group $\mathbf{G}_a$, and the formal group $\mathbf{G}$ base-changed from that of E_n. Picking a coordinate, we have $[p]_{\mathbf{G}_a}(x) = 0$. However, this is also equal to $[p]_{\mathbf{G}}(x) = \sum^{\mathbf{G}}_{0 \leq k \leq n} v_k x^{p^k}$, so we find that $v_k = 0$ for all $0 \leq k \leq n$. Since v_k is invertible in $\pi_*(E_n)$, and hence in $\mathrm{H}_*(E_n; \mathbf{F}_p)$, we conclude that $\mathrm{H}_*(E_n; \mathbf{F}_p) = 0$, as desired.

It remains to prove ($*'$). Since the canonical map $S \to \underline{\mathrm{Map}}(I_{\mathbf{Q}/\mathbf{Z}}, I_{\mathbf{Q}/\mathbf{Z}})$ is an equivalence, the spectrum $\underline{\mathrm{Map}}(E, S)$ is contractible if and only if $\underline{\mathrm{Map}}(E, \underline{\mathrm{Map}}(I_{\mathbf{Q}/\mathbf{Z}}, I_{\mathbf{Q}/\mathbf{Z}})) \simeq \underline{\mathrm{Map}}(E \otimes I_{\mathbf{Q}/\mathbf{Z}}, I_{\mathbf{Q}/\mathbf{Z}})$ is contractible. This occurs if and only if $\pi_* \underline{\mathrm{Map}}(E \otimes I_{\mathbf{Q}/\mathbf{Z}}, I_{\mathbf{Q}/\mathbf{Z}}) \cong \mathrm{Hom}(\pi_*(E \otimes I_{\mathbf{Q}/\mathbf{Z}}), \mathbf{Q}/\mathbf{Z}) = 0$, which in turn is possible if and only if $\pi_*(E \otimes I_{\mathbf{Q}/\mathbf{Z}}) = 0$, i.e., $E \otimes I_{\mathbf{Q}/\mathbf{Z}}$ is contractible. □

It follows that using Definition 2.0.1 to define the notion of a Gorenstein $\mathbf{E}_\infty$-ring fails to give anything interesting in the even periodic setting. Nonetheless, examples of what deserve to be called Gorenstein locally even-periodic $\mathbf{E}_\infty$-rings abound: we have $I_{\mathbf{Z}} KO \simeq \Sigma^4 KO \in \mathrm{Pic}(KO)$ (see [HS14]), and $I_{\mathbf{Z}}\mathrm{TMF} \simeq \Sigma^{21}\mathrm{TMF} \in \mathrm{Pic}(\mathrm{TMF})$, and analogously for Tmf (see [Sto12]). In this section, we recall a definition, and a few properties, of Gorenstein even periodic $\mathbf{E}_\infty$-rings.

We first define the notion of a dualizing sheaf.

Definition 3.1.3. Let $\mathfrak{X}$ be a locally Noetherian even periodic derived stack. A quasicoherent sheaf $\omega_{\mathfrak{X}}$ on $\mathfrak{X}$ is a *dualizing sheaf* if

1. The map $\mathcal{O}_{\mathfrak{X}} \to \underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ is an equivalence.
2. The functor $\mathbf{D}(\mathcal{F}) = \underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \omega_{\mathfrak{X}})$ gives an autoequivalence of the category of almost perfect quasicoherent sheaves on $\mathfrak{X}$ with itself.
3. For every étale map $f : \mathrm{Spec}\, R \to \mathfrak{X}$, the $\pi_0 R$-module $\pi_0 f^* \omega_{\mathfrak{X}}$ is a dualizing module for $\pi_0 R$.

We would like to understand when the structure sheaf (or some shift of it) of a derived stack $\mathfrak{X}$ is itself a dualizing complex. If this is the case, we say that $\mathfrak{X}$ is *Gorenstein*. Then, we have (see [Dev17, Remark 3.13 and Theorem 3.14]):

Theorem 3.1.4. *Let $\mathfrak{X}$ be a perfect locally Noetherian separated derived Deligne-Mumford stack which arises as the even-periodic refinement of a tame and flat map $X \to \mathcal{M}_{FG}$, where X has proper and finite global dimension. If $\mathfrak{X}$ is Gorenstein, then $f^! I_{\mathbf{Z}}$ is invertible, where $f : \mathfrak{X} \to$* $\mathrm{Spec}\, S$ *is the structure map.*

Remark 3.1.5. Suppose $\mathfrak{X}$ satisfies the conditions of Theorem 3.1.4. Let X denote the underlying stack of $\mathfrak{X}$. Then $\mathfrak{X}$ is Gorenstein if and only if X is Gorenstein.

We now prove the following result about Gorenstein morphisms in the even periodic setting.

Theorem 3.1.6. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a finite and flat morphism of Noetherian even periodic derived Deligne-Mumford stacks. Then f is Gorenstein if and only if f_0 is Gorenstein.*

Proof. Let $\mathcal{F}$ be a vector bundle on $\mathfrak{Y}$. We will first prove that the natural map $\pi_k f^! \mathcal{F} \to f_0^! \pi_k \mathcal{F}$ is an isomorphism. It suffices to check this isomorphism locally, so let $q : \mathrm{Spec}\, B \to Y$ be an étale cover, let $p : X \times_Y \mathrm{Spec}\, B \to X$ denote the pullback, and let $X \times_Y \mathrm{Spec}\, B$ be the pullback. Since $f_0 : X \to Y$ is finite, it is in particular affine, so the pullback is affine, i.e., $X \times_Y \operatorname{Spec} B \cong \operatorname{Spec} A$. There is a pullback square

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, equal] & X \times_Y \operatorname{Spec} B \arrow[r, "p"] \arrow[d, "g"] & X \arrow[d, "f_0"] \\ & \operatorname{Spec} B \arrow[r, "q"'] & Y \end{tikzcd}$$

Let $\widetilde{B} = \mathcal{O}_{\mathfrak{Y}}(\operatorname{Spec} B)$ be the even periodic lift of B coming from $\mathcal{O}_{\mathfrak{Y}}$, and let $\widetilde{q} : \operatorname{Spec} \widetilde{B} \to \mathfrak{Y}$ be the induced étale morphism. Similarly, let $\widetilde{A} = \mathcal{O}_{\mathfrak{X}}(\operatorname{Spec} A)$ be the even periodic lift of A coming from $\mathcal{O}_{\mathfrak{X}}$, and let $\widetilde{p} : \operatorname{Spec} \widetilde{A} \to \mathfrak{X}$ be the induced étale morphism. Denote by $\widetilde{g} : \operatorname{Spec} \widetilde{A} \to \operatorname{Spec} \widetilde{B}$ the induced morphism. It follows that

$$p^* f_0^! \pi_k \mathcal{F} \cong g^! q^* \pi_k \mathcal{F} \cong g^! \pi_k \widetilde{q}^* \mathcal{F} \cong \operatorname{Map}_B(A, \pi_k \widetilde{q}^* \mathcal{F}).$$

On the other hand, since there is a natural equivalence $\widetilde{p}^* f^! \mathcal{F} \simeq \widetilde{g}^! \widetilde{q}^* \mathcal{F}$ coming from [Lur18, Proposition 6.4.2.1], we learn that

$$p^* \pi_k f^! \mathcal{F} \cong \pi_k \widetilde{p}^* f^! \mathcal{F} \cong \pi_k \widetilde{g}^! \widetilde{q}^* \mathcal{F} \cong \pi_k \operatorname{Map}_{\widetilde{B}}(\widetilde{A}, \widetilde{q}^* \mathcal{F}).$$

Since g is finite and flat (and B is Noetherian, so that A is projective over B), this is isomorphic to $\operatorname{Map}_B(A, \pi_k \widetilde{q}^* \mathcal{F})$, as desired.

Let $\mathcal{F} = \mathcal{O}_{\mathfrak{Y}}$. It follows from even periodicity that $\pi_1 f^! \mathcal{O}_{\mathfrak{Y}} = 0$, and that $\pi_0 f^! \mathcal{O}_{\mathfrak{Y}} \cong f_0^! \mathcal{O}_Y$. Since f_0 is Gorenstein, this is a line bundle on X. We need to show that $f^! \mathcal{O}_{\mathfrak{Y}}$ is invertible. We claim that $(f^! \mathcal{O}_{\mathfrak{Y}})^\vee$ is an inverse. To show this, we need to check that the evaluation map $f^! \mathcal{O}_{\mathfrak{Y}} \otimes (f^! \mathcal{O}_{\mathfrak{Y}})^\vee \to \mathcal{O}_{\mathfrak{X}}$ is an equivalence. It suffices to check this on homotopy. By even periodicity, it suffices to check this on π_0. We compute that

$$\begin{aligned} \pi_k((f^! \mathcal{O}_{\mathfrak{Y}})^\vee) &\cong \operatorname{Hom}_X(\pi_0 f^! \mathcal{O}_{\mathfrak{Y}}, \pi_k \mathcal{O}_{\mathfrak{X}}) \cong \operatorname{Hom}_X(f_0^! \mathcal{O}_Y, \pi_k \mathcal{O}_{\mathfrak{X}}) \\ &\cong \begin{cases} (f_0^! \mathcal{O}_Y)^\vee \otimes \omega^{\otimes k/2} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd}, \end{cases} \end{aligned}$$

where the first isomorphism is because $f^! \mathcal{O}_{\mathfrak{Y}}$ is a vector bundle on $\mathfrak{X}$, and if $\mathcal{F}$ and $\mathcal{F}'$ are vector bundles on $\mathfrak{X}$, then $\pi_k \operatorname{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{F}') \cong \operatorname{Hom}_X(\pi_0 \mathcal{F}, \pi_k \mathcal{F}')$. It follows that π_0 of the evaluation map is exactly the map $f_0^! \mathcal{O}_Y \otimes (f_0^! \mathcal{O}_Y)^\vee \to \mathcal{O}_X$, which is an isomorphism because f_0 is Gorenstein.

Conversely, if f is Gorenstein, then the evaluation map $f^! \mathcal{O}_{\mathfrak{Y}} \otimes (f^! \mathcal{O}_{\mathfrak{Y}})^\vee \to \mathcal{O}_{\mathfrak{X}}$ is an equivalence, so taking π_0 and using the isomorphism $\pi_k f^! \mathcal{F}' \simeq f_0^! \pi_k \mathcal{F}'$, we learn that $f_0^! \mathcal{O}_Y$ is invertible, i.e., that f_0 is Gorenstein. $\square$

Corollary 3.1.7. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a finite and flat morphism of Noetherian even periodic derived Deligne-Mumford stacks. Assume that f_0 is Gorenstein. Then $\mathfrak{X}$ is Gorenstein if $\mathfrak{Y}$ is Gorenstein.*

Proof. By Theorem 3.1.6, the morphism f is itself Gorenstein. Moreover, the stack Y is Gorenstein since $\mathfrak{Y}$ is Gorenstein, so the stack X is Gorenstein. By Remark 3.1.5 and Theorem 3.1.4, the stack $\mathfrak{X}$ is Gorenstein, as desired. $\square$

Corollary 3.1.8. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a finite and flat morphism of Noetherian even periodic derived Deligne-Mumford stacks. Then the locus $U \subseteq |X|$ of points where f is Gorenstein (i.e., those points $x \in |X|$ for which $f^! \mathcal{O}_{\mathfrak{Y}}$ is invertible in an open neighborhood of x) is open.*

Proof. This follows from Theorem 3.1.6 and the fact that this result is true in the classical setting. $\square$

We now take the opportunity to compare the definition of Gorenstein even periodic $\mathbf{E}_\infty$-rings provided above in Definition 3.1.3 with the definitions provided in [BHV18]. We first recall the definitions provided in *loc. cit.*

Let A be a Noetherian $\mathbf{E}_\infty$-ring, and let $\mathfrak{p} \subseteq \pi_* A$ be a homogeneous prime ideal of degree d, generated by n elements. Denote by $E(A/\mathfrak{p})$ the injective hull of $\pi_* A/\mathfrak{p}$. Then, arguing as usual, we find that there is a Brown-Comenetz dualizing $A_\mathfrak{p}$-module $I_\mathfrak{p}$ with $\pi_* I_\mathfrak{p} \cong E(A/\mathfrak{p})$. Let $A/\mathfrak{p}^\infty := \Gamma_\mathfrak{p} A$ denote the $A_\mathfrak{p}$-module $\operatorname{colim}_k \Sigma^{-(kd+n)} A/\mathfrak{p}^s$.

Definition 3.1.9. The $\mathbf{E}_\infty$-ring A is said to be *absolute Gorenstein* if there is an A-module M such that for every homogeneous prime ideal $\mathfrak{p} \subseteq \pi_* A$ of degree d, there is an equivalence $M \otimes_A A/\mathfrak{p}^\infty \simeq \Sigma^{k+d} I_\mathfrak{p}$.

By [BHV18, Proposition 4.7], an $\mathbf{E}_\infty$-ring A such that $\pi_* A$ is Gorenstein, i.e., the localization $\pi_* A_\mathfrak{m}$ at each homogeneous maximal ideal $\mathfrak{m}$ is Gorenstein with shift d, is absolute Gorenstein; such an $\mathbf{E}_\infty$-ring is called *algebraically Gorenstein*. Then Theorem 3.1.4 and Remark 3.1.5 yield:

Corollary 3.1.10. *An even periodic Noetherian $\mathbf{E}_\infty$-ring A with $\pi_0 A$ of finite global dimension is Gorenstein in the sense of Definition 3.1.3 if and only if A is algebraically Gorenstein. In particular, every such $\mathbf{E}_\infty$-ring A is absolute Gorenstein.*

3.2. **Examples of Gorenstein even periodic stacks.**

Corollary 3.2.1. *Suppose $p : \mathfrak{X} \to \mathfrak{Y}$ is a finite flat cover of even periodic refinements of Deligne-Mumford stacks such that*

(1) *$\mathfrak{X}$ is a perfect Noetherian separated derived Deligne-Mumford stack which arises as the even-periodic refinement of a tame and flat map $X \to \mathcal{M}_{FG}$, and*
(2) *X is proper, Gorenstein, and has finite global dimension.*

Let $f : \mathfrak{Y} \to \operatorname{Spec} S$ denote the structure morphism. Then $f^! I_\mathbf{Z} \in \operatorname{Pic}(\mathfrak{Y})$ if and only if the morphism p_0 is Gorenstein.

Proof. Under our assumptions on X, Theorem 3.1.4 and Remark 3.1.5 show that $p^! f^! I_\mathbf{Z} \in \operatorname{Pic}(\mathfrak{X})$ if and only if X is Gorenstein. Since X is Gorenstein, we have $p^! f^! I_\mathbf{Z} \in \operatorname{Pic}(\mathfrak{X})$. Because p is finite flat, Grothendieck duality (as in [Lur18, Corollary 6.4.2.7]; note that p satisfies the assumptions there, since finite flat morphisms are exactly proper, flat, locally quasi-finite morphisms, which are locally of finite presentation) gives an equivalence $p^! f^! I_\mathbf{Z} = p^* f^! I_\mathbf{Z} \otimes p^! \mathcal{O}_\mathfrak{X}$. We have $f^! I_\mathbf{Z} \in \operatorname{Pic}(\mathfrak{Y})$ if and only if $p^! \mathcal{O}_\mathfrak{Y} \in \operatorname{Pic}(\mathfrak{X})$, i.e., if and only if p is Gorenstein. Since the assumptions of Theorem 3.1.6 are satisfied, it follows that p is Gorenstein if and only if p_0 is Gorenstein. □

We note the following lemma.

Lemma 3.2.2. *Let $\mathfrak{X}$ be an even periodic refinement of a flat and quasi-affine map $X \to \mathcal{M}_{FG}$ from a separated locally Noetherian Deligne-Mumford stack X. Then $\mathfrak{X}$ is a perfect stack.*

Proof. By [BZFN10, Proposition 3.9], we only need to show that $\operatorname{QCoh}(\mathfrak{X})$ is compactly generated and that the compact and dualizable objects coincide. This, however, is immediate from the main result of [MM15], which provides a symmetric monoidal equivalence $\operatorname{QCoh}(\mathfrak{X}) \simeq \operatorname{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X}))$. □

Let us now discuss some consequences of Corollary 3.2.1.

Corollary 3.2.3. *Invert n, and let $\Gamma \in \{\Gamma(n), \Gamma_1(n)\}$. Then the $\operatorname{Tmf}(\Gamma)$-module $I_\mathbf{Z}\operatorname{Tmf}(\Gamma)$ is in $\operatorname{Pic}(\operatorname{Tmf}(\Gamma))$.*

Proof. We will first prove the nonperiodic case. We begin with the case when Γ is not the full modular group; the reason for this restriction is that the map $\overline{\mathcal{M}}_{\mathrm{ell}} \to \mathcal{M}_{FG}$ from the compactified moduli stack is *not* tame. By [Con07, Theorem 3.2.7], the compactified moduli stacks $\overline{\mathcal{M}}_\Gamma$ are smooth and proper Deligne-Mumford stacks over $\operatorname{Spec}\mathbf{Z}[1/n]$. Moreover, again by [Mei17, Proposition 3.4], they are also tame for $n \geq 2$. By Lemma 3.2.2, $\overline{\mathcal{M}}_\Gamma^{\mathrm{der}}$ is perfect. By [Mei17, Example 3.1], we learn that the remaining cases for $\overline{\mathcal{M}}_\Gamma$ (for $n \geq 2$) are weighted projective stacks. By [BZFN10, Corollary 3.22], we learn that $\overline{\mathcal{M}}_\Gamma^{\mathrm{der}}$ is perfect even in these stacky cases. By Corollary 3.2.1, we learn that $f^! I_{\mathbf{Z}} \in \operatorname{Pic}(\overline{\mathcal{M}}_\Gamma^{\mathrm{der}})$. The map $\overline{\mathcal{M}_{\mathrm{ell}}} \to \mathcal{M}_{FG}$ is quasi-affine, so by the main result of [MM15], the global sections functor $\operatorname{QCoh}(\overline{\mathcal{M}}_\Gamma^{\mathrm{der}}) \to \operatorname{Mod}(\mathrm{Tmf}(\Gamma))$ is a symmetric monoidal equivalence of categories. Under this equivalence, $f^! I_{\mathbf{Z}}$ is sent to $I_{\mathbf{Z}}\mathrm{Tmf}(\Gamma)$, so Theorem 3.1.4 and Remark 3.1.5 finish off the proof of Corollary 3.2.3.

We now consider the case of $\overline{\mathcal{M}_{\mathrm{ell}}}$ itself. In order to prove the result in this case, it suffices to show (by Theorem 3.1.6) that the finite flat (but *not* étale) covering map $p : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}_{\mathrm{ell}}}[1/n]$ is Gorenstein. It suffices to prove that p is lci; but since $\overline{\mathcal{M}}_\Gamma$ and $\overline{\mathcal{M}_{\mathrm{ell}}}[1/n]$ are smooth stacks, any morphism between them is automatically lci. Using Corollary 3.2.1, we conclude that $I_{\mathbf{Z}}\mathrm{Tmf}[1/n] \in \operatorname{Pic}(\mathrm{Tmf}[1/n])$.

□

Remark 3.2.4. It follows from the proof of Corollary 3.2.3 that $\mathrm{Tmf}(\Gamma)$ is Spanier-Whitehead Gorenstein as a Tmf-module (this is a concrete translation of the condition that the map $\overline{\mathcal{M}}_\Gamma^{\mathrm{der}} \to \overline{\mathcal{M}_{\mathrm{ell}}}^{\mathrm{der}}[1/n]$ is Gorenstein).

Remark 3.2.5. In [Mei17], Meier proves that $I_{\mathbf{Z}}\mathrm{Tmf}_1(n)$ is equivalent to a shift of $\mathrm{Tmf}_1(n)$ if and only if n is in the set $S := \{1, \cdots, 8, 11, 14, 15, 23\}$. The content of Corollary 3.2.3 seems to be new; it shows that for $n \notin S$ (in particular, for arbitrarily large n), the element $I_{\mathbf{Z}}\mathrm{Tmf}_1(n)$ is an element of $\operatorname{Pic}(\mathrm{Tmf}_1(n))$ which is *not* a shift of $\mathrm{Tmf}_1(n)$. This naturally motivates the following question.

Question 3.2.6. What is $\operatorname{Pic}(\mathrm{Tmf}_1(n))$? The first case where $I_{\mathbf{Z}}\mathrm{Tmf}_1(n)$ is not a shift of $\mathrm{Tmf}_1(n)$ is the case $n = 9$; in this case, can we explicitly construct $I_{\mathbf{Z}}\mathrm{Tmf}_1(n)$ as a $\mathrm{Tmf}_1(n)$-module?

Remark 3.2.7. For $n \geq 5$, the stack $\overline{\mathcal{M}}_{\Gamma_1(n)}$ is a projective scheme, so $\operatorname{Pic}(\overline{\mathcal{M}}_{\Gamma_1(n)}) \simeq \operatorname{Pic}(X_1(n))$. For $1 \leq n \leq 10$ and $n = 12$, the curve $X_1(n)$ over $\mathbf{Q}$ has genus 0, so $\mathrm{Jac}_1(n)(\mathbf{Q}) = 0$, and $\operatorname{Pic}(X_1(n)_{\mathbf{Q}}) \cong \mathbf{Z}$.

Remark 3.2.8. Let $\overline{\mathcal{M}}_{\Gamma_0(n)}$ denote the moduli stack of generalized elliptic curves with a $\Gamma_0(n)$-structure, as defined by Deligne-Rapoport in [DR73]. However, this does not agree with the moduli stack of a similar type defined in [Con07] unless n is squarefree (see [Con07, Remark 4.1.5]). Although Hill and Lawson constructed in [HL16] a sheaf of $\mathbf{E}_\infty$-rings on the stack $\overline{\mathcal{M}}_{\Gamma_0(n)}$, we will only consider the associated spectrum when n is assumed to be squarefree. In this case, Corollary 3.2.3 reads: after inverting $\phi(n)$ or 6, the $\mathrm{Tmf}_0(n)$-module $I_{\mathbf{Z}}\mathrm{Tmf}_0(n)$ is in $\operatorname{Pic}(\mathrm{Tmf}_0(n))$; the proof is exactly the same.

We also have the following result.

Corollary 3.2.9. *Let* $\mathrm{Sh}(K^p)$ *denote one of the derived PEL Shimura varieties constructed by Behrens-Lawson in* [BL10]. *Suppose* $\mathrm{Sh}(K^p)$ *is proper, and let* $f : \mathrm{Sh}(K^p) \to \operatorname{Spec} S$ *denote the structure morphism. Then* $f^! I_{\mathbf{Z}} \in \operatorname{Pic}(\mathrm{Sh}(K^p))$.

Proof. By the discussion in [Kot92, Page 375], the Deligne-Mumford stacks $\mathrm{Sh}(K^p)$ have finite étale covers by smooth quasi-projective (Noetherian) schemes $\mathrm{Sh}(K'^p)$, where K'^p is a sufficiently small compact open subgroup of $GU(\mathbf{A}^{p,\infty})$. The map $\mathrm{Sh}(K'^p) \to \mathcal{M}_{FG}$ is flat (and clearly tame). Since the even periodic refinement of $\mathrm{Sh}(K'^p)$ is a quasicompact derived scheme with affine diagonal, it is a perfect stack. Moreover, $\mathrm{Sh}(K'^p)$ has finite global dimension (by [BL10, Corollary 7.3.3]). Since finite étale morphisms are Gorenstein, we can apply Corollary 3.2.1 to conclude that $f^! I_{\mathbf{Z}} \in \mathrm{Pic}(\mathrm{Sh}(K^p))$, where $f : \mathrm{Sh}(K^p) \to \operatorname{Spec} S$ is the structure morphism. $\square$

Using the formalism of [Lur09, Section 3.3], we can also prove the analogous result to Corollary 3.2.3 (for periodic TMF) in the genuine G-equivariant setting when G is a finite abelian group; this result is certainly well-known for G-equivariant KO.

Proposition 3.2.10. *Let G be a finite abelian group. If $\mathrm{TMF}(\Gamma)_G$ denotes the $\mathbf{E}_\infty$-ring of the G-fixed points of genuine G-equivariant topological modular forms, then $I_{\mathbf{Z}}\mathrm{TMF}(\Gamma)_G \in \mathrm{Pic}(\mathrm{TMF}(\Gamma)_G)$.*

Proof. Recall that if G is a compact abelian Lie group, and $\mathfrak{X}$ is a derived stack with a derived oriented p-divisible group $\mathbf{G}$ (with underlying p-divisible group $\mathbf{G}_0$) defined over it, then the genuine G-equivariant stack $\mathfrak{X}_G$ is defined to be $\mathrm{Hom}(G^\vee, \mathbf{G})$, where $G^\vee$ is the Pontryagin dual of G. Suppose $p : \mathfrak{Y} \to \mathfrak{X}$ is a morphism satisfying the conditions of Corollary 3.2.1 (so that $f^! I_{\mathbf{Z}} \in \mathrm{Pic}(\mathfrak{X})$). There is a pullback square:

$$\begin{tikzcd} \mathfrak{Y}_G \arrow[r] \arrow[d, "p_G"'] & \mathfrak{Y} \arrow[d, "p"] \\ \mathfrak{X}_G \arrow[r] & \mathfrak{X}. \end{tikzcd}$$

As Gorenstein morphisms are preserved under base change, the morphism p_G is Gorenstein. It suffices to prove the following two results.

(1) Let $g : \mathfrak{X}_G \to \operatorname{Spec} S$ denote the structure morphism; then $g^! I_{\mathbf{Z}} \in \mathrm{Pic}(\mathfrak{X}_G)$.
(2) There is an equivalence of symmetric monoidal stable presentable ∞-categories $\mathrm{QCoh}(\mathfrak{X}_G) \simeq \mathrm{Mod}(\Gamma(\mathfrak{X}_G, \mathcal{O}_{\mathfrak{X}_G}))$.

Part (1) follows from Corollary 3.2.1, once we check that $\mathfrak{Y}_G$ is a perfect Noetherian separated derived Deligne-Mumford stack arising as the even periodic refinement of a tame and flat map $Y_G \to \mathcal{M}_{FG}$, where Y_G is Gorenstein and has finite global dimension.

Since G is a finite abelian group, so is its Pontryagin dual $G^\vee$. Therefore, we can reduce to the case when G is cyclic. If G is of order k, then $\mathfrak{Y}_G = \mathbf{G}[k]$. Since the multiplication by k map is finite and flat, the underlying stack is $\mathbf{G}_0[k]$. Denote this stack by Y_G. Since the map $Y_G \to Y$ is flat and representable, the composite $Y_G \to Y \to \mathcal{M}_{FG}$ is tame. Then $\mathbf{G}[k]$ is an even periodic refinement of Y_G, so we need to show that $\mathbf{G}[k]$ is perfect, Noetherian, and separated. By [BZFN10, Proposition 3.21], we learn that $\mathbf{G}[k]$ is perfect. Since $\mathfrak{Y}_G \to \mathfrak{Y}$ is finite, we learn that Y_G is separated and Noetherian.

Finally, we need to show that $Y_G = \mathbf{G}_0[k]$ is Gorenstein and has finite global dimension. Since $\mathbf{G}_0[k] \to Y$ is finite and flat (and therefore, in particular, proper), it follows that Y_G is proper and has finite global dimension. Recall that the group scheme $\mathbf{G}_0$ over $\mathcal{M}_{\mathrm{ell}}$ is the p-divisible group associated to the universal smooth elliptic curve over $\mathcal{M}_{\mathrm{ell}}$; it follows that it suffices to show that $\mathcal{M}_{\Gamma_1(d)}$ is Gorenstein for all $d \mid k$. This is a consequence of [KM85, Theorem 5.1.1].

We now need to check part (2). Namely, by the main result of [MM15], we only need to show that if $\mathfrak{X}$ is an even-periodic refinement of a stack $f : X \to \mathcal{M}_{FG}$ such that f is quasi-affine, then the same is true for the derived stack $\mathfrak{X}_G$ for any finite abelian group. This follows from the fact that the map $\mathfrak{X}_G \to \mathfrak{X}$ is finite (hence affine), and that the composition of a quasi-affine map with an affine map remains quasi-affine. Note that in this case, the map $\mathfrak{X}_G \to \mathfrak{X}$ is always flat. To prove this, we can again reduce to the case when G is cyclic, say of order k; then, $\mathfrak{Y}_G = \mathbf{G}[k]$, so we need to show that $\mathbf{G}[k] \to \mathfrak{X}$ is flat. This follows from the fact that the multiplication-by-k map $[k] : \mathbf{G} \to \mathbf{G}$ is flat. $\square$

Remark 3.2.11. Note that statement (1) in the proof of Proposition 3.2.10 proves a stronger result, namely that if $g : (\mathcal{M}_{\mathrm{ell}})_G \to \operatorname{Spec} S$ is the structure morphism, then $g^! I_{\mathbf{Z}} \in \operatorname{Pic}(\mathfrak{X}_G)$.

We now consider an example coming from modular representation theory.

Proposition 3.2.12. *Let A be an abelian p-group, and let k be a field of characteristic p. Then $I_{\mathbf{Z}} k^{tA}$ is in* $\operatorname{Pic}(k^{tA})$.

Proof. Let $\mathbf{T}^n$ denote the n-torus. There is a $\mathbf{T}^n$-Galois extension $k^{t\mathbf{T}^n} \to k^{tA}$.

($*$) If G is a topological group and $f : A \to B$ is a G-Galois extension, then $I_{\mathbf{Z}} B \in \operatorname{Pic}(B)$ if $I_{\mathbf{Z}} A \in \operatorname{Pic}(A)$.

($*'$) $I_{\mathbf{Z}} k^{t\mathbf{T}^n}$ is an invertible $k^{t\mathbf{T}^n}$-module.

($*$) can be proved by arguing as in Corollary 3.2.1. We claim that there is an equivalence

$$\underline{\mathrm{Map}}(B, I_{\mathbf{Z}}) \simeq \underline{\mathrm{Map}}_A(B, A) \otimes_B (B \otimes_A \underline{\mathrm{Map}}(A, I_{\mathbf{Z}})) \simeq \underline{\mathrm{Map}}_A(B, A) \otimes_A \underline{\mathrm{Map}}(A, I_{\mathbf{Z}}).$$

Since B is dualizable as an A-module (by [Rog08, Proposition 6.2.1]), the B-module on the right is equivalent to $\underline{\mathrm{Map}}_A(B, \underline{\mathrm{Map}}(A, I_{\mathbf{Z}}))$ as B-modules, which in turn is equivalent to $\underline{\mathrm{Map}}(B, I_{\mathbf{Z}})$. Since $\underline{\mathrm{Map}}_A(B, A)$ is an invertible B-module by Lemma 2.0.2, and $\underline{\mathrm{Map}}(A, I_{\mathbf{Z}}) = I_{\mathbf{Z}} A$ is an invertible A-module, we conclude that $I_{\mathbf{Z}} B$ is an invertible B-module, as desired.

It remains to prove ($*'$). By the discussion in [Mat16, §9.2], there is an even periodic derived scheme $\mathfrak{X}$ whose underlying classical scheme is $\mathbf{P}^{n-1}_k$ such that the global sections functor supplies a symmetric monoidal equivalence $\operatorname{QCoh}(\mathfrak{X}) \to \operatorname{Mod}(k^{t\mathbf{T}^n})$. Since $\mathbf{P}^{n-1}_k$ is smooth (hence Gorenstein), Remark 3.1.5 and Theorem 3.1.4 show that $\mathfrak{X}$ is Gorenstein. It follows that $k^{t\mathbf{T}^n}$ is Gorenstein, i.e., that $I_{\mathbf{Z}} k^{t\mathbf{T}^n} \in \operatorname{Pic}(k^{t\mathbf{T}^n})$. $\square$

3.3. **Application: ambidexterity.**

In this section, we discuss an application of the above discussion to ambidexterity in $K(n)$-local stable homotopy theory. In [HL13], Hopkins and Lurie discuss a proof of the following result.

Theorem 3.3.1 (Hopkins-Lurie)**.** *Let X be a π-finite space, i.e., a Kan complex such that for every $x \in X$, the set $\pi_n(X, x)$ is finite for all n, and trivial for $n \gg 0$. Let $f : X \to L_{K(n)}\mathrm{Sp}$ denote any diagram; then there is a canonical homotopy equivalence* $\operatorname{colim} f \to \lim f$.

We shall be interested in one component of the proof of this result. Using [HL13, Corollary 4.4.23, Example 5.1.10], Theorem 3.3.1 can be reduced to proving the following proposition.

Proposition 3.3.2. *For each integer m and each prime ℓ, the $K(n)$-local spectrum $L_{K(n)}\Sigma^\infty_+ K(\mathbf{Z}/\ell, m)$ is Spanier-Whitehead self-dual in the symmetric monoidal ∞-category* $L_{K(n)}\mathrm{Sp}$.

Hopkins' and Lurie's proof of Proposition 3.3.2 involves knowing the entire computation of the Morava E-theory of Eilenberg-Maclane spaces. We shall provide a proof of a weaker analogue of Proposition 3.3.2 that "only" uses the Ravenel-Wilson computation of the Morava K-theory of Eilenberg-Maclane spaces.

Theorem 3.3.3. *Let $X = \Sigma^\infty_+ K(\mathbf{Z}/p, m)$. Then the $K(n)$-local Spanier-Whitehead dual of $L_{K(n)}X$ is in* $\operatorname{Pic}(L_{K(n)}X)$.

Proof. We need to show that $\underline{\mathrm{Map}}_{L_{K(n)}\mathrm{Sp}}(L_{K(n)}X, L_{K(n)}S)$ is in the Picard group of $L_{K(n)}X$. It suffices to check this after base-changing to E, a Morava E-theory at height n (which is unique if its residue field is required to be algebraically closed). In other words, we need to show that $E \widehat{\otimes} \underline{\mathrm{Map}}_{L_{K(n)}\mathrm{Sp}}(L_{K(n)}X, L_{K(n)}S)$ is equivalent to a shift of $E \widehat{\otimes} L_{K(n)}X$ by an element of the Picard group of $L_{K(n)}\operatorname{Mod}(E)$. The Picard group of $L_{K(n)}\operatorname{Mod}(E)$ is simply $\mathbf{Z}/2$. The

$K(n)$-local spectrum E is a dualizable object in $L_{K(n)}\mathrm{Sp}$ by [Str00, Proposition 16], so there are equivalences

$$E\widehat{\otimes}\underline{\mathrm{Map}}_{L_{K(n)}\mathrm{Sp}}(L_{K(n)}X, L_{K(n)}S) \simeq \underline{\mathrm{Map}}_{L_{K(n)}\mathrm{Sp}}(L_{K(n)}X, E) \simeq \underline{\mathrm{Map}}_{L_{K(n)}\mathrm{Mod}(E)}(E\widehat{\otimes}L_{K(n)}X, E) =: M.$$

In light of the above observations, it therefore suffices to prove that the map $\phi : E \to E\widehat{\otimes}L_{K(n)}X$ is Gorenstein.

By [HL13, Theorem 2.4.10] and [HS99, Proposition 8.4], the ring $E_*^\vee(L_{K(n)}X)$ is concentrated in even degrees and finitely generated over E_*. Since π_*E and $E_*^\vee(L_{K(n)}X)$ are even periodic, we can argue as in Theorem 3.1.6 to conclude that ϕ is Gorenstein if and only if the map $E_0 \to E_0^\vee(X)$ of Noetherian local rings is Gorenstein. It suffices to show that this map is Gorenstein after base-changing to the residue field $\kappa = K(n)_0$ of E_0, i.e., that the map $\kappa \to K(n)_0(X)$ is Gorenstein. This follows from the Ravenel-Wilson computation of $K(n)_0(X)$ (presented as [HL13, Theorem 2.4.10]) one consequence of which is that $K(n)_0(X)$ is a local complete intersection κ-algebra.

□

Remark 3.3.4. Proposition 3.3.2 would follow from Theorem 3.3.3 if we could prove that the $K(n)$-local Spanier-Whitehead dual of $L_{K(n)}X$ is in the image of the map $\mathrm{Pic}(L_{K(n)}\mathrm{Sp}) \to \mathrm{Pic}(L_{K(n)}X)$, but we do not know how to prove this from first principles.

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