22 Loop groups and intertwining of positive-energy representations

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We will give an introduction to the representation theory of loop groups of compact Lie groups: we will discuss what positive energy representations are, why they exist, how to construct them (via a Schur–Weyl style construction and a Borel–Weil style construction), and how to show that they don't depend on choices. Motivation will come from both mathematics and quantum mechanics.

The theory of positive-energy representations of loop groups is modeled on the representation theory of compact Lie groups. Some parts of the talk will make more sense if you are familiar with the compact Lie group story, but this is not a requirement: in this section, we try to emphasize the "big picture" over details, and we hope that this choice makes it readable for you. Likewise, we will not assume any familiarity with loop groups or infinite-dimensional topology, nor will we dig into those details.

In §22.1, we state the main theorem (Theorem 22.1.1) and discuss some motivation for caring about representations of loop groups. In §22.2, we begin thinking about projective representations of loop groups and the corresponding central extensions. In §22.3, we provide an extended proof sketch of Theorem 22.1.1, and discuss some connections to physics. Finally, in § 22.4, we discuss how this relates to differential cohomology. There are two ways to lift the construction of central extensions of loop groups to differential cohomology; one follows the Chern–Weil story we've used several times already in this part, and the other more closely resembles the story we told about off-diagonal Deligne cohomology and the Virasoro algebra in Chapter 17.

22.1 Overview

The objective of this chapter is to explain the following theorem of Pressley–Segal [PS86, Theorem 13.4.2]:

22.1.1 Theorem. *Let G be a simply connected compact Lie group. Then any positive energy representation E of the loop group $\mathrm{L}G$ admits a projective intertwining action of $\mathrm{Diff}^+(\mathrm{S}^1)$.*

If this means nothing to you, that's okay: the goal of this talk is to explain all the components of this theorem (§ 22.2) and sketch a proof (§ 22.3). Then, in § 22.4, we discuss how the representation theory of loop groups is related to differential cohomology.

Here's a rough sketch of what Theorem 22.1.1 is about. The representation theory of a semisimple compact Lie group G is very well-behaved: the Peter–Weyl theorem [PW27] allows one to provide any finite-dimensional G-representation with a G-invariant Hermitian inner product, and this inner product decomposes the representation into a direct sum of irreducibles. Moreover, the irreducibles are in bijection with dominant weights, where by the Borel–Weil theorem (see [Ser54]), the representation associated to a dominant weight is given as the global sections of a line bundle associated to a homogeneous space of G (a particular flag variety).

Most representations of loop groups will not satisfy analogues of this property, so we'd like to hone down on the ones which do. These are the "positive energy representations"; these essentially satisfy properties necessary to be able to write down highest/lowest weight vectors. Theorem 22.1.1 then states that positive energy representations are preserved under reparametrizations of the circle (which give automorphisms of the loop group $\mathrm{L}G$). One can therefore think of Theorem 22.1.1 as a consistency result.

Before proceeding, I'd like to give some motivation for caring about the representation theory of loop groups.

(1) One motivation comes from the connection between representation theory and homotopy theory. The Atiyah–Segal completion theorem [Ati61, Theorem 7.2; AH61, §4.8; AS69, Theorem 2.1] relates representations of a compact Lie group G to G-equivariant K-theory, and likewise the representation theory of the loop group $\mathrm{L}G$ is related to (twisted) G-equivariant elliptic cohomology. This has been explored in [Bry90; Dev96; Liu96; And00; And03; Gro07; Lur09a; Gan14; Lau16; Kit19; Rez20; BT21].

(2) Another motivation comes from the hope that geometry on the free loop space $\mathrm{L}M$ of a manifold M is supposed to correspond to correspond to "higher-dimensional geometry" over M. For instance, if M has a Riemannian metric, one can think of the scalar curvature of $\mathrm{L}M$ at a loop as the integral of the Ricci curvature of g over the loop. Similarly, spin structures on M are closely related to orientations on $\mathrm{L}M$ [Wit85; Ati85, §3; Wit88; McL92, §2; ST05, Theorem 9; Wal16b, Corollary E, §1.2], and string structures on M are closely related to spin structures on $\mathrm{L}M$ [Kil87; NW13, Theorem 6.9].[26]

In light of this hope, it is rather pacifying to have a strong analogy between representation theory of compact Lie groups and of loop groups. In fact, all of these motivations are related by a story that still seems to be mysterious at the moment.

There's also motivation from physics for studying the representation theory of loop groups. The wavefunction of a free particle on the circle S^1 must be an L^2-function on S^1 (because the probability of finding the particle somewhere on the circle is 1). There is an action of the loop group LU_1 on $\mathrm{L}^2(\mathrm{S}^1;\mathbb{C})$ given by pointwise multiplication (a pair $\gamma : \mathrm{S}^1 \to \mathrm{U}_1$ and $f \in \mathrm{L}^2(\mathrm{S}^1;\mathbb{C})$ is sent to the L^2-function $f_\gamma(z) = \gamma(z)f(z)$). In particular, LU_1 gives a lot of automorphisms of the Hilbert space $\mathrm{L}^2(\mathrm{S}^1;\mathbb{C})$; this is relevant to quantum mechanics, where observables are (Hermitian) operators on the Hilbert space of states. Having a particularly (mathematically) natural source of symmetries is useful. In [Seg85], Segal in fact says: "In fact it is not much of an exaggeration to say that the mathematics of two-dimensional quantum field theory is almost the same thing as the representation theory of loop groups".

22.2 Representations of loop groups

22.2.1 Definition. Let G be a compact connected Lie group. The loop group $\mathrm{L}G \coloneqq \mathrm{C}^\infty(\mathrm{S}^1, G)$ is the group of smooth unbased loops in G.

[26] There are a number of other works providing additional proofs of this fact or pointing out subtleties in the definitions, including [PW88; CP89; McL92, §3; KY98; ST05; KM13b; Wal15; Cap16; Wal16a; Kri20].

If G is positive-dimensional, LG is not finite-dimensional. A fair amount of the theory of finite-dimensional manifolds generalizes to infinite-dimensional spaces locally modeled by nice classes of topological vector spaces, and in this sense LG is an infinite-dimensional Lie group, in fact quite a nice one. Reading this chapter does not require any additional familiarity with infinite-dimensional topology, but if you're interested, you can learn more in [Ham82b; Mil84; PS86, §3.1]

There will be a lot of circles floating around, and so we will distinguish these by subscripts. Some of these will be denoted by $\mathbb{T}$, for "torus".

22.2.2 Remark (Classification of compact Lie groups)**.** We quickly review the classification of compact Lie groups. This may clarify the generality in which some of the results in this section hold.

- Let G be a compact Lie group and $G_0 \subset G$ denote the connected component containing the identity. Then there is a short exact sequence $1 \to G_0 \to G \to \pi_0(G) \to 1$.
- Let G be a compact, connected Lie group. Then there is a short exact sequence $1 \to F \to \tilde{G} \to G \to 1$, where F is finite and $\tilde{G}$ is a product of a torus $\mathbb{T}^n$ and a simply connected group.
- Let G be a compact, connected, simply connected Lie group. Then G is a product of *simple* simply connected Lie groups.
- Let G be a compact, simply connected, simple Lie group. Then G is isomorphic to one of SU_n, Spin_n, Sp_n, G_2, F_4, E_6, E_7, or E_8.

Most of the results in this section require G to be connected and simply connected; a few will also require G to be simple. In particular, when G is simple, $\mathrm{H}^4(\mathrm{B}G;\mathbb{Z}) \cong \mathbb{Z}$.[27]

22.2.3 Remark. The loop group LG is an infinite-dimensional Lie group, and it has an action of S^1 by rotation. We will denote this "rotation" circle by $\mathbb{T}_{\mathrm{rot}}$. This action will turn out to be very useful shortly.

The action of $\mathbb{T}_{\mathrm{rot}}$ allows one to consider the semidirect product $\mathrm{L}G \rtimes \mathbb{T}_{\mathrm{rot}}$. The following proposition is then an exercise in manipulating symbols:

22.2.4 Proposition. *An action of* $\mathrm{L}G \rtimes \mathbb{T}_{\mathrm{rot}}$ *on a vector space* V *is the same data as an action* R *of* $\mathbb{T}_{\mathrm{rot}}$ *on* V *and an action* U *of* LG *on* V *satisfying*

$$R_\theta U_\gamma R_\theta^{-1} = U_{R_\theta \gamma}.$$

Most interesting representations U of LG on a vector space V are not, strictly speaking, representations: instead of $U_\gamma U_{\gamma'} = U_{\gamma\gamma'}$, they satisfy the weaker condition that

$$U_\gamma U_{\gamma'} = c(\gamma,\gamma')U_{\gamma\gamma'}, \tag{22.2.5}$$

[27] This isomorphism can be made canonical by specifying that under the Chern–Weil map, the Killing form $B\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ defines a positive element of $\mathrm{H}^4_{\mathrm{dR}}(\mathrm{B}G) \cong \mathbb{R}$.

where $c(\gamma,\gamma') \in \mathbb{C}^\times$. This is precisely:

22.2.6 Definition. A *projective representation* of $\mathrm{L}G$ on a Hilbert space V is a continuous homomorphism $\mathrm{L}G \to \mathrm{PU}(V)$.

22.2.7 Remark. Why Hilbert spaces? From a mathematical perspective, this is because Hilbert spaces are well-behaved infinite-dimensional vector spaces. From a physical perspective, this is because Hilbert spaces are spaces of states. In fact, this also explains why most interesting representations are projective: the state of a quantum system is not a vector in the Hilbert space, but rather a vector in the projectivization of the Hilbert space. This corresponds to the statement that shifting the wavefunction by a phase does not affect physical observations.

Assume V is an infinite-dimensional, separable Hilbert space. Then $\mathrm{PU}(V)$ is a $K(\mathbb{Z},2)$, so projective representations determine cohomology classes in $H^2(\mathrm{L}G;\mathbb{Z})$.

22.2.8 Lemma. *When G is compact and simply connected,* $H^2(\mathrm{L}G;\mathbb{Z}) \cong H^3(G;\mathbb{Z})$.

Proof. Since G is simply connected, $\pi_1(G) = 0$, and π_2 vanishes for any Lie group. Therefore the Hurewicz theorem identifies $\pi_3(G)$ and $H_3(G;\mathbb{Z})$. Let ΩG denote the based loop space of G, i.e. the subspace of $\mathrm{L}G$ consisting of loops beginning and ending at the identity. Essentially by definition, there is an isomorphism $\pi_k(G) \to \pi_{k-1}(\Omega G)$ for $k > 1$, so we learn $\pi_1(\Omega G) = 0$ and $\pi_2(\Omega G) \cong \pi_3(G)$.

To get to $\mathrm{L}G$, we use that as topological spaces, $\mathrm{L}G \cong G \times \Omega G$ [PS86, §4.4]. Thus $\pi_1(\mathrm{L}G) = 0$ and $\pi_2(\mathrm{L}G) \cong \pi_3(G)$, and the Hurewicz and universal coefficient theorems allow us to conclude. □

Another way to construct this isomorphism is as follows: there is an evaluation map $\mathrm{ev}\colon S^1 \times \mathrm{L}G \to G$ sending $(x,\ell) \mapsto \ell(x)$; then the isomorphism in Lemma 22.2.8 is: pull back by ev, then integrate in the S^1 direction.

It turns out that when G is compact and simply connected, every class in $H^2(\mathrm{L}G;\mathbb{Z})$ arises from a projective representation as above [PS86, Theorem 4.4.1]. There is a central extension[28]

$$1 \to \mathbb{T}_{\mathrm{cent}} \to \mathrm{U}(V) \to \mathrm{PU}(V) \to 1, \tag{22.2.9}$$

and so any projective representation ρ of $\mathrm{L}G$ determines a central extension by pulling (22.2.9) back:

$$1 \to \mathbb{T}_{\mathrm{cent}} \to \widetilde{\mathrm{L}G}_\rho \to \mathrm{L}G \to 1. \tag{22.2.10}$$

Conversely, any central extension of $\mathrm{L}G$ gives rise to a projective representation of $\mathrm{L}G$. In particular:

[28] This central extension is also a fiber bundle, and by Kuiper's theorem [Kui65], the total space $\mathrm{U}(V)$ is contractible (see also [DD63, Lemme 3; AS04, Proposition A2.1]). This fiber bundle is homotopy equivalent to two other interesting fiber bundles: the universal principal U_1-bundle $\mathrm{U}_1 \to E\mathrm{U}_1 \to B\mathrm{U}_1$, and the loop space-path space bundle $\Omega K(\mathbb{Z},2) \to P\,K(\mathbb{Z},2) \to K(\mathbb{Z},2)$.

22.2.11 Definition. Let G be a simple and simply connected compact Lie group. The *universal central extension* $\widetilde{\mathrm{L}G}$ of $\mathrm{L}G$ is the central extension corresponding to the generator of $\mathrm{H}^2(\mathrm{L}G;\mathbb{Z}) \cong \mathbb{Z}$.

We first met universal central extensions in a different context, in §17.3.

The following result is key.

22.2.12 Theorem [PS86, Theorem 4.4.1]**.** *Let G be simply connected. Then there is a unique action of* $\mathrm{Diff}^+(\mathbb{T}_{\mathrm{rot}})$ *on* $\widetilde{\mathrm{L}G}$ *which covers the action on* $\mathrm{L}G$. *Moreover,* $\widetilde{\mathrm{L}G}$ *deserves to be called "universal", because there is a unique map of extensions from* $\widetilde{\mathrm{L}G}$ *to any other central extension of* $\mathrm{L}G$.

22.2.13 Remark. As a consequence, the action of $\mathbb{T}_{\mathrm{rot}}$ on $\mathrm{L}G$ lifts canonically to $\widetilde{\mathrm{L}G}$. Every projective unitary representation of $\mathrm{L}G$ with an intertwining action of $\mathbb{T}_{\mathrm{rot}}$ is equivalently a unitary representation of $\widetilde{\mathrm{L}G} \rtimes \mathbb{T}_{\mathrm{rot}}$. For the remainder of this talk, we will assume G is simply connected and abusively say write "representation of $\mathrm{L}G$" to mean a representation of $\widetilde{\mathrm{L}G} \rtimes \mathbb{T}_{\mathrm{rot}}$.

22.2.14 Notation. It is a little inconvenient to constantly keep writing $\widetilde{\mathrm{L}G} \rtimes \mathbb{T}_{\mathrm{rot}}$, so we will henceforth denote it by $\widetilde{\mathrm{L}G}^+$. The subgroup $\mathbb{T}_{\mathrm{rot}}$ of $\widetilde{\mathrm{L}G}^+$ is also known as the "energy circle" (for reasons to be explained below).

One of the nice properties of tori is that their representations take on a particularly simple form, thanks to the magic of Fourier series. The action of S^1 on a finite-dimensional vector space is the same data as a $\mathbb{Z}$-grading. The case of topological vector spaces is slightly more subtle: if S^1 acts on a topological vector space V, then one can consider the closed "weight" subspace V_n of V where the action of S^1 is by the character[29] $z \mapsto z^{-n}$. Then the direct sum $\bigoplus_{n\in\mathbb{Z}} V_n$ is a dense subspace of V; it is known as the subspace of *finite energy* vectors in V. This is simply the usual weight decomposition adapted to the topological setting.

22.2.15 Definition. The action of S^1 on a topological vector space V is said to satisfy the *positive energy condition* if the weight subspace $V_n = 0$ for $n < 0$. Equivalently, the action of S^1 is represented by $e^{-iA\theta}$, where A is an operator with positive spectrum.

22.2.16 Remark. The motivation for this definition comes from quantum mechanics: the wavefunction of a free particle on a circle is e^{inx} (up to normalization), and requiring that the energy (which is essentially the weight n) to be positive is mandated by physics.

22.2.17 Definition. A representation of $\mathrm{L}G$ (which, recall, means a representation of $\widetilde{\mathrm{L}G}^+$) is said to satisfy the *positive energy condition* if it satisfies the positive energy condition when viewed as a representation of the energy/central circle $\mathbb{T}_{\mathrm{rot}}$.

22.2.18 Remark. It doesn't make sense for a representation of $\mathrm{L}G$ to be positive energy if you take "representation of $\mathrm{L}G$" to mean a literal representation of $\mathrm{L}G$; one needs to interpret that phrase as meaning a representation of $\widetilde{\mathrm{L}G}^+$.

[29] Some conventions are different: the action might be by $z \mapsto z^n$. We're following [PS86].

We can now see the utility of Theorem 22.1.1: the positive energy condition involves the canonical parametrization of the circle, and to ensure that our definition would agree with that of an alien civilization's, we should ensure that the pullback f^*V of any positive energy representation V of $\mathrm{L}G$ along an orientation-preserving diffeomorphism $f \in \mathrm{Diff}^+(\mathbb{T}_{\mathrm{rot}})$ is another positive energy representation. That is precisely the content of Theorem 22.1.1.

At the beginning of this chapter, we said that positive energy representations of loop groups satisfy analogues of many properties of representations of compact Lie groups. To make that statement precise, we need to introduce some definitions that impose sanity conditions on the representations we want to study.

22.2.19 Definition. Let V be a representation of a topological group G (possibly infinite-dimensional). Then V is said to be:

- *irreducible* if it has no closed G-invariant subspace;
- *smooth* if the following condition is satisfied: let V_{sm} denote the subspace of vectors $v \in V$ such that the orbit map $G \to V$ sending g to gv is continuous; then V_{sm} is dense in V.

Two G-representations V and W are *essentially equivalent* if there is a continuous G-equivariant map $V \to W$ which is injective and has dense image.

22.2.20 Warning. Essential equivalence is *not* an equivalence relation!

The representation theory of compact Lie groups is really nice: every finite-dimensional complex representation of a compact Lie group G is semisimple (i.e. it is a direct sum of irreducible representations), and unitary, and extends to a representation of the complexification $G_{\mathbb{C}}$ of G.[30] These properties have analogues for positive energy representations of loop groups.

22.2.21 Theorem [PS86, Theorem 9.3.1]**.** *Let V be a smooth positive energy representation of* $\mathrm{L}G$*. Then up to essential equivalence:*

- *V is completely reducible into a discrete direct sum of irreducible representations,*
- *V is unitary,*
- *V extends to a holomorphic projective representation of* $\mathrm{L}(G_{\mathbb{C}})$*, and*
- *V admits a projective intertwining action of* $\mathrm{Diff}^+(\mathrm{S}^1)$*, where this* S^1 *is the energy/rotation circle. (This is Theorem 22.1.1.)*

The proof of this result takes up the bulk of the second part of Pressley–Segal.

22.2.22 Remark. The group G includes into $\mathrm{L}G$ as the subgroup of constant loops. Let G be simple and simply connected. If T is a maximal torus of G, then one has $\mathbb{T}_{\mathrm{rot}} \times T \times \mathbb{T}_{\mathrm{cent}} \subseteq$

[30] A complexification of a real Lie group G is a complex Lie group, generally noncompact, whose Lie algebra is isomorphic to $\mathfrak{g} \otimes \mathbb{C}$. When G is compact, $G_{\mathbb{C}}$ is unique up to isomorphism.

$\widetilde{\mathrm{L}G}^+$. Consequently, if V is a representation of $\widetilde{\mathrm{L}G}^+$, then V can be decomposed (up to essential equivalence) as a $\mathbb{T}_{\mathrm{rot}} \times T \times \mathbb{T}_{\mathrm{cent}}$-representation:

$$V = \bigoplus_{(n,\lambda,h)\in\mathbb{T}_{\mathrm{rot}}^\vee\times T^\vee\times\mathbb{T}_{\mathrm{cent}}^\vee} V_{(n,\lambda,h)} \,. \tag{22.2.23}$$

Here, n is the energy of V; λ is a weight of V (regarded as a representation of T); and h is a character of $\mathbb{T}_{\mathrm{cent}}$. The notation $(-)^\vee \coloneqq \mathrm{Hom}(-,\mathbb{C}^\times)$ denotes the character dual: because $\mathbb{T}_{\mathrm{rot}} \times T \times \mathbb{T}_{\mathrm{cent}}$ is a compact abelian group, its unitary representations are direct sums of one-dimensional representations. Therefore as a $\mathbb{T}_{\mathrm{rot}} \times T \times \mathbb{T}_{\mathrm{cent}}$-representation, V splits as a direct sum of one-dimensional representations, which are indexed by the character dual $(\mathbb{T}_{\mathrm{rot}} \times T \times \mathbb{T}_{\mathrm{cent}})^\vee = \mathbb{T}_{\mathrm{rot}}^\vee \times T^\vee \times \mathbb{T}_{\mathrm{cent}}^\vee$.

If V is irreducible, then $\mathbb{T}_{\mathrm{cent}}$ must act by scalars by Schur's lemma, and so only one value of h can occur; this is called the *level* of V. It turns out that if V is a smooth positive energy representation, then each weight space $V_{n,\lambda,h}$ is finite-dimensional. In fact, a representation of $\mathrm{L}G$ of level h is the same as a representation of $\widetilde{\mathrm{L}G}_h \rtimes \mathbb{T}_{\mathrm{rot}}$, where $\widetilde{\mathrm{L}G}_h$ is the central extension of $\mathrm{L}G$ corresponding to $h \in \mathbb{Z} \cong \mathrm{H}^2(\mathrm{L}G;\mathbb{Z})$.

22.2.24 Remark. By Remark 22.2.22, an irreducible positive energy representation V of $\mathrm{L}G$ is uniquely determined by the level h and its lowest energy subspace V_0: the representation V is generated as a $\widetilde{\mathrm{L}G}^+$-representation by V_0.

22.2.25 Remark. Since G is simply connected, there are transgression isomorphisms

$$\mathrm{H}^4(\mathrm{B}G;\mathbb{Z}) \to \mathrm{H}^3(G;\mathbb{Z}) \to \mathrm{H}^2(\mathrm{L}G;\mathbb{Z}) \,,$$

meaning we can understand the level as (up to homotopy) a map $\mathrm{B}G \to \mathrm{K}(\mathbb{Z},4)$. This $\mathrm{K}(\mathbb{Z},4)$ is closely tied to the twisting $\mathrm{K}(\mathbb{Z},4) \to \mathrm{BGL}_1(\mathrm{tmf})$ of tmf constructed in [ABG10, Theorem 1.1]: see [And00; Gro07; BT21].

As a side note, we observe the following:

22.2.26 Proposition. *Let V be a smooth positive energy representation of* $\mathrm{L}G$. *Then V is irreducible as a representation of* $\widetilde{\mathrm{L}G}$.

Proof. Assume V is not irreducible as a $\widetilde{\mathrm{L}G}$-representation. Projection onto a proper $\widetilde{\mathrm{L}G}$-invariant summand defines a bounded self-adjoint operator $T : V \to V$ which commutes with $\widetilde{\mathrm{L}G}$, but (by hypothesis) not with the action of $\mathbb{T}_{\mathrm{rot}}$. Choose $R \in \mathbb{T}_{\mathrm{rot}}$; then define for each $n \in \mathbb{Z}$ the bounded operator

$$T_n = \int_{\mathbb{T}_{\mathrm{rot}}} z^n R_z T R_z^{-1}\,\mathrm{d}z \,. \tag{22.2.27}$$

T_n commutes with the action of $\widetilde{\mathrm{L}G}$, and T_n sends the weight space V_m to V_{m+n}. Because T does not commute with $\mathbb{T}_{\mathrm{rot}}$, the operator T_n must be nontrivial for at least one $n < 0$. Suppose that m is the lowest energy of V (i.e., the smallest m such that the weight space $V_m \neq 0$).[31] Then $T_n(V_m) = 0$ if $n < 0$. Since V is irreducible as a representation of $\widetilde{L}G^+$, it is generated as a representation by V_m. But then $T_n(V) = 0$ for all $n < 0$. The adjoint to T_n is T_{-n}, and so $T_n(V) = 0$ for all $n \neq 0$.

This implies that T commutes with the action of $\mathbb{T}_{\mathrm{rot}}$, which is a contradiction: the T_n are the Fourier coefficients of the loop $S^1 \to \mathrm{End}(V)$ sending z to $R_z T R_z^{-1}$, so we find that this loop must be constant. Consequently, T must commute with the action of $\mathbb{T}_{\mathrm{rot}}$, as desired. ☐

22.3 A proof sketch of Theorem 22.1.1

The goal of this section is to go through the proof of Theorem 22.1.1. As with all proofs in representation theory, we may first reduce to the irreducible case, thanks to the first part of Theorem 22.2.21.

22.3.1 Observation. Recall that Schur–Weyl duality sets up a one-to-one correspondence between representations of SU_n and representations of the symmetric groups, by studying the decomposition of the tensor power $V^{\otimes d}$ of the standard representation V under the action of Σ_d.

One may hope that some analogue of Observation 22.3.1 is true for representations of loop groups: suppose we could construct a giant representation of $L\mathrm{SU}_n$ whose h-fold tensor product contains all the irreducible positive energy representations of level h, such that this big representation admits an intertwining action of $\mathrm{Diff}^+(S^1)$. Then (with a little bit of work), we would obtain an intertwining action of $\mathrm{Diff}^+(S^1)$ on all irreducible positive representations of $L\mathrm{SU}_n$, which would prove Theorem 22.1.1 in this particular case. We would like to then reduce from the case of a general G to the case of SU_n. The Peter–Weyl theorem says that a simply connected G is a closed subgroup of SU_n for some n, suggesting that a technique like this might work.

Pressley–Segal's approach is similar, but not the same.

- Their base case consists not just of $L\mathrm{SU}_n$, but the loop groups of all simply connected, simply laced compact Lie groups.[32] In [PS86, Lemma 13.4.4], they extend from simply laced groups to all simply connected Lie groups; the reason they cannot just use an embedding $j\colon G \hookrightarrow \mathrm{SU}_n$ is that, given a representation V of $\widetilde{L}G$, Pressley–Segal need not just the embedding j, but also the condition that there is an irreducible representation V' of the bigger group with V a summand in j^*V'.

- Now assume G is simply connected and simply laced. Instead of constructing a huge tensor product, Pressley–Segal reduce to the case of level 1 representations in a different way. Let $m_n\colon LG \to LG$ be the map precomposing a loop $S^1 \to G$ with the n^{th}-power map $S^1 \to S^1$. Then [PS86, Proposition 9.3.9] every irreducible representation V of $\widetilde{L}G$ is

[31] Because V is positive energy, $m \geq 0$ — but that doesn't matter for now.

[32] Recall that G is simply laced if all its nonzero roots have the same length; in other words, if the Dynkin diagram of G does not have multiple edges (so the Dynkin diagram is of ADE type). The simple, simply connected, simply laced Lie groups are SU_n for all n, Spin_n for n even, E_6, E_7, and E_8.

contained in $m_h^* F$ for some level 1 representation F. This allows Pressley–Segal to carry the $\mathrm{Diff}^+(\mathrm{S}^1)$-action from F to V.

- Finally, when G is simply laced and F is level 1, Pressley–Segal construct the $\mathrm{Diff}^+(\mathrm{S}^1)$-action directly using the "blip construction" [PS86, §13.2, §13.3].

22.3.2 Remark. Pressley–Segal write that "one hopes that a more satisfactory proof of Theorem 22.1.1 can be found," [PS86, p. 271], so perhaps there's a proof out there that more closely resembles the Schur–Weyl-style argument.

Now we will see how the story goes for LSU_n.

22.3.3 Construction. Let $G = \mathrm{SU}_n$. Define $H := \mathrm{L}^2(\mathrm{S}^1, V)$, where V is the standard representation. Let $\mathrm{Har}^2(\mathrm{S}^1, V) \subseteq H$ denote the *Hardy space* of L^2-functions on S^1 with only nonnegative Fourier coefficients, and let P denote orthogonal projection of H onto $\mathrm{Har}^2(\mathrm{S}^1, V)$. Then $H = PH \oplus P^\perp H$. The *Fock space* Fock_P is the Hilbert space completion of the alternating algebra:

$$\mathrm{Fock}_P = \widehat{\Lambda}(PH \oplus \overline{P^\perp H}) \cong \widehat{\bigoplus_{i,j\geq 0}} \Lambda^i(PH) \oplus \Lambda^j(\overline{P^\perp H})\,. \tag{22.3.4}$$

Here $\overline{V}$ denotes the complex conjugate vector space to V, and $\widehat{\Lambda}$ and $\widehat{\bigoplus}$ denote Hilbert space completions. The Fock space turns out to be the "giant representation" we were after: it's the fundamental representation of LSU_n.

22.3.5 Remark (The Fock space in physics)**.** The process of building a Fock space out of a Hilbert space H, as in (22.3.4), has a quantum-mechanical interpretation. Suppose that H is the space of states describing the mechanics of a particle: for example, $\mathrm{L}^2(\mathrm{S}^1, \mathbb{C})$ corresponds to a particle moving on a circle. The corresponding Fock space is the space of states for systems with any number of particles. In Construction 22.3.3, we used the alternating algebra, which means that the particles are fermions: the relation $f \wedge f = 0$ is the Pauli exclusion principle, imposing that two fermions cannot be in the same state. For a bosonic many-body system, one would use the (Hilbert space completion of the) symmetric algebra. The process of building a Fock space from a single-particle Hilbert space is called second quantization.

In our setting, $\mathrm{L}^2(\mathrm{S}^1, V)$ corresponds to a system with a fermion moving on a circle, together with some kind of G-symmetry. The subspace $\Lambda^i(PH) \oplus \Lambda^j(\overline{P^\perp H})$ consists of i fermionic particles and j fermionic antiparticles. This explains why we take the conjugate space to $P^\perp H$: it is so that the antiparticles have positive energy.

A loop on G acts on H by pointwise multiplication, and $f \in \mathrm{Diff}^+(\mathrm{S}^1)$ acts on H by sending $\xi\colon \mathrm{S}^1 \to V$ to $\xi(f^{-1}(z)) \cdot |(f^{-1})'(z)|^{1/2}$. (The square root factor is a normalization factor to ensure unitarity of the action.) In fact, this gives an action of $LG \rtimes \mathrm{Diff}^+(\mathrm{S}^1)$ on H, and one can ask when this descends to a projective representation of $LG \rtimes \mathrm{Diff}^+(\mathrm{S}^1)$ on the Fock space Fock_P. Segal wrote down a *quantization condition* for when a unitary operator on H descends to a projective transformation of Fock_P: namely, u descends to Fock_P if and only if the commutator

$[u, P]$ is Hilbert–Schmidt.[33] One checks that the action of $\mathrm{L}G \rtimes \mathrm{Diff}^+(\mathrm{S}^1)$ on H satisfies Segal's quantization criterion, and so descends to a projective representation of $\mathrm{L}G \rtimes \mathrm{Diff}^+(\mathrm{S}^1)$ on the Fock space Fock_P.

Almost by definition, the action of $\mathrm{S}^1 = \mathbb{T}_{\mathrm{rot}}$ on Fock_P is of positive energy, and so Fock_P is a representation of positive energy. It turns out that:

22.3.6 Theorem [PS86, Section 10.6; Was98, Chapter I.5]**.** *The irreducible summands of* $\mathrm{Fock}_P^{\otimes h}$ *give all the irreducible positive energy representations of* LSU_n *of level h.*

We will expand on this construction of the irreducible level h representations of LSU_n in Chapter 23, when we discuss the Segal–Sugawara construction.

The first reduction comes from:

22.3.7 Lemma [PS86, Lemma 13.4.3]**.** *Let V and W be positive energy representations of* $\widetilde{\mathrm{L}}G$*. Suppose that V is irreducible, and that $V \oplus W$ admits an intertwining action of* $\mathrm{Diff}^+(\mathrm{S}^1)$*. Then V admits an intertwining action of* $\mathrm{Diff}^+(\mathrm{S}^1)$*.*

We will prove this shortly; first, we will indicate how to use this to prove the general case.

22.3.8 Remark. It suffices to prove by Lemma 22.3.7 that for every irreducible positive energy representation V of $\mathrm{L}G$, there is some G' and an embedding $i \colon \mathrm{L}G \to \mathrm{L}G'$ where Theorem 22.1.1 is true for G', and an irreducible representation V' of $\mathrm{L}G'$ such that V is a summand of i^*V'.

To use this reduction, we first need to establish that Theorem 22.1.1 is true for a class of Lie groups G. In fact:

22.3.9 Theorem. *Theorem 22.1.1 is true if G is simple, simply connected, and simply laced.*

The proof of this result is quite similar to that of Theorem 22.3.6: one constructs the analogue of the Fock space for $\mathrm{L}G$ (which, like in the SU_n case, has an intertwining action of $\mathrm{Diff}^+(\mathrm{S}^1)$), and then shows that every irreducible positive energy representation is a summand of some twist of this representation of $\mathrm{L}G$. See [PS86, §13.4] for more details.

22.3.10 Construction. Let ΩG denote the *based* loop space of G, regarded as the homogeneous quotient $\mathrm{L}G/G \simeq \mathrm{L}G_{\mathbb{C}} / \mathrm{L}^+ G_{\mathbb{C}}$. Since G is simple any simply connected,

$$\mathrm{H}^2(\Omega G; \mathbb{Z}) \cong \mathrm{H}^3(G; \mathbb{Z}) \cong \mathbb{Z} \,,$$

so every integer gives rise to a complex line bundle on ΩG. The holomorphic sections Γ of the line bundle corresponding to the generator is called the *basic representation* of $\mathrm{L}G$.[34]

22.3.11 Example. If $G = \mathrm{SU}_n$, Γ is the Fock space described above.

Then:

[33] Recall that a bounded operator A on a Hilbert space is *Hilbert–Schmidt* if $\mathrm{tr}(A^*A)$ is finite.

[34] Of course, the abelian group $\mathbb{Z}$ has two generators. Here we have a canonical one: as discussed above, we have a canonical generator for $\mathrm{H}^4(\mathrm{B}G; \mathbb{Z})$, hence $\mathrm{H}^3(G; \mathbb{Z})$ via transgression, and therefore also for $\mathrm{H}^2(\Omega G; \mathbb{Z})$.

22.3.12 Proposition [PS86, Proposition 9.3.9]**.** *Let G be a simple, simply connected, and simply laced Lie group. Then any irreducible positive energy representation of level h of* LG *is a summand in $i_h^*\Gamma$, where $i_h : \mathrm{L}G \to \mathrm{L}G$ is the map induced by the degree h map $\mathrm{S}^1 \to \mathrm{S}^1$.*

The level 1 representation Γ admits an intertwining action of $\mathrm{Diff}^+(\mathrm{S}^1)$ via the "blip construction." We will not go into the details here; see [PS86, §13.3]. Assuming this, combining proposition 22.3.12 with Lemma 22.3.7 shows that Theorem 22.1.1 is true for LG when G is simply laced (and simple and simply connected).

According to Remark 22.3.8, it now suffices to show:

22.3.13 Proposition. *For every irreducible positive energy representation V of* LG*, there is a simply laced G' and an embedding $i : \mathrm{L}G \to \mathrm{L}G'$, as well as an irreducible representation V' of* LG' *such that V is a summand of i^*V'.*

This is proved in [PS86, Lemma 13.4.4] in the following manner.

One first classifies all the irreducible representations of LG. Using the loop group analogue of Schur–Weyl duality worked well when $G = \mathrm{SU}_n$, but that won't do in the general case. Instead, one utilizes a loop group analogue of Borel–Weil (see [Seg85, Section 4.2]). Recall how this works for finite-dimensional, compact Lie groups: fix a maximal torus T of G, and then, for every antidominant weight λ of T (i.e., $\langle h_\alpha, \lambda\rangle \leq 0$ for every positive root α), there is an associated line bundle $\mathcal{L}_\lambda$ on $G/T \cong G_{\mathbb{C}}/B^+$. The space of holomorphic sections of $\mathcal{L}_\lambda$ is an irreducible representation of G of lowest weight λ, and all irreducible representations of G arise this way.

In the loop group case, one again begins by fixing a maximal torus T of G (one should think of $\mathbb{T}_{\mathrm{rot}} \times T \times \mathbb{T}_{\mathrm{cent}}$ as a maximal torus of LG). Consider the homogeneous space LG/T. There is a fiber sequence

$$G/T \to \mathrm{L}G/T \to \Omega G, \tag{22.3.14}$$

and the set of isomorphism classes of complex line bundles on LG/T is

$$\mathrm{H}^2(\mathrm{L}G/T;\mathbb{Z}) \cong \mathrm{H}^2(\Omega G;\mathbb{Z}) \oplus \mathrm{H}^2(G/T;\mathbb{Z}) = \mathbb{Z} \oplus \widehat{T}, \tag{22.3.15}$$

where $\widehat{T}$ is the character group of T. You can prove this using the Serre spectral sequence, which as usual is easier because G is simple and simply connected. Anyways, we learn that line bundles on LG/T are indexed by $(h,\lambda) \in \mathbb{Z} \oplus \widehat{T}$.

22.3.16 Theorem (Borel–Weil for loop groups [PS86, Theorem 9.3.5])**.** *One has:*

- *The space $\Gamma(\mathcal{L}_{h,\lambda})$ of holomorphic sections is zero or irreducible of positive energy of level h; moreover, every projective irreducible representation of* LG *arises this way.*

- *The space $\Gamma(\mathcal{L}_{h,\lambda})$ is nonzero if and only if (h,λ) is antidominant,*[35] *i.e.,*

$$0 \geq \lambda(h_\alpha) \geq -\frac{h}{2}\langle h_\alpha, h_\alpha\rangle$$

for each positive coroot h_α of G. (In particular, λ is antidominant as a weight of $T \subseteq G$.)

The upshot is that irreducible representations correspond to antidominant weights. To prove Proposition 22.3.13, it suffices to show that all antidominant weights of LG are restrictions of antidominant weights of LG' for some simply laced G'. The argument now proceeds case-by-case, as G ranges over all simple simply connected simply laced compact Lie groups. The proof is not very enlightening, so we will not go into more detail here.

22.3.17 Remark (Relationship with Wess–Zumino–Witten theory)**.** Segal [Seg04] studies the theory of positive energy representations of LG from a different perspective, that of conformal field theory. Specifically, the category of level h positive energy representations of LG has the structure of a *modular tensor category* Given a modular tensor category C, one can build

(1) a 3-dimensional topological field theory Z_{C} [RT90; RT91; Wal91; BK01; KL01; BDSV15], and

(2) a 2-dimensional conformal field theory [MS89].

These two theories are related: the 2d CFT is a boundary theory for the 3d TFT [Wit89; FT14]. When C is the category of level h representations of LG, the TFT is Chern–Simons theory (see Remark 21.2.7) and the CFT is the Wess–Zumino–Witten model (see Remark 21.2.12).[36]

You do not need Theorem 22.1.1 to construct the modular tensor category structure on $\mathsf{Rep}_k(LG)$, and the TFT and CFT provide a very large amount of data associated to that structure. It may be possible to coax Theorem 22.1.1 out of that extra structure. For example, Segal [Seg04, §12] discusses this for abelian Lie groups.

22.4 OK, but what does this have to do with differential cohomology?

There is differential cohomology hiding in the background of the story of central extensions of loop groups. There are two ways in which it appears: one which is related to the story of on-diagonal differential characteristic classes built from Chern–Weil theory, and another which relates central extensions to off-diagonal Deligne cohomology similarly to the discussion of the Virasoro group in Chapter 17. This, together with the appearance of $\mathrm{Diff}^+(S^1)$ in the representation theory of loop groups, suggests that loop groups and the Virasoro group should interact somehow, as we will see in the next chapter.

[35] Recall that if G is the simply laced group SU_n, then the weight lattice is $\bigoplus_{1\leq i\leq n+1}\mathbb{Z}\chi_i/\mathbb{Z}\sum_i \chi_i$, and the roots are $\chi_i - \chi_j$ with $i \neq j$. The positive roots, corresponding to the usual Borel subgroup of upper-triangular matrices, are $\chi_i - \chi_j$ for $i < j$. Therefore, $(h, \lambda = \lambda_1, \cdots, \lambda_n)$ is antidominant if λ is antidominant, i.e., $\lambda_1 \leq \cdots \leq \lambda_n$, and if $\lambda_n - \lambda_1 \leq h$.

[36] One might wonder if every modular tensor category arises in this way, as a category of positive-energy representations of a loop group. This is the Moore–Seiberg conjecture, and is open at the time of writing. See, e.g., [HRW08].

22.4.a The on-diagonal story

Suppose G is simple and simply connected, so that $\mathrm{H}^4(\mathrm{B}G;\mathbb{Z})$, $\mathrm{H}^3(G;\mathbb{Z})$, and $\mathrm{H}^2(\mathrm{L}G;\mathbb{Z})$ are all isomorphic to $\mathbb{Z}$, and the transgression maps

$$\mathrm{H}^4(\mathrm{B}G;\mathbb{Z}) \to \mathrm{H}^3(G;\mathbb{Z}) \to \mathrm{H}^2(\mathrm{L}G;\mathbb{Z})$$

are isomorphisms. The level h canonically refines to $\hat{h} \in \hat{\mathrm{H}}^4(\mathrm{B}_\nabla G;\mathbb{Z})$ (Theorem 13.1.1), and the transgression map refines to a map $\hat{\mathrm{H}}^4(\mathrm{B}_\nabla G;\mathbb{Z}) \to \hat{\mathrm{H}}^3(G;\mathbb{Z})$ [CJM+05, §3], as we discussed in Remark 19.3.12. Does the story continue to a differential refinement $\hat{\mathrm{H}}^3(G;\mathbb{Z}) \to \hat{\mathrm{H}}^2(\mathrm{L}G;\mathbb{Z})$? That is, a projective representation $\mathrm{L}G \to \mathrm{PU}(V)$ determines a central extension $\widetilde{\mathrm{L}G}$ of $\mathrm{L}G$, which is a principal $\mathbb{T}$-bundle over $\mathrm{L}G$. Does this $\mathbb{T}$-bundle come with a canonical connection?

Of course, this is a loaded question, and we'll see that the answer is yes. But first, a (relatively) down-to-Earth plausibility argument. Given a central extension

$$1 \to \mathbb{T}_{\mathrm{cent}} \to \widetilde{\mathrm{L}G} \to \mathrm{L}G \to 1, \tag{22.4.1a}$$

we can differentiate it to obtain a central extension of Lie algebras

$$0 \to \mathbb{R} \to \widetilde{\mathrm{L}\mathfrak{g}} \to \mathrm{L}\mathfrak{g} \to 0. \tag{22.4.1b}$$

Recall from Remark 17.1.6 that the central extension (22.4.1b) can be described by a cocycle for the Lie algebra cohomology group $\mathrm{H}^2_{\mathrm{Lie}}(\mathrm{L}\mathfrak{g};\mathbb{R})$. Cocycles are alternating maps $\omega\colon\ \mathrm{L}\mathfrak{g}\times\mathrm{L}\mathfrak{g} \to \mathbb{R}$ satisfying the cocycle condition (17.1.7). Choose a cocycle ω; then, $\widetilde{\mathrm{L}\mathfrak{g}}$ is the vector space $\mathrm{L}\mathfrak{g}\oplus\mathbb{R}$ with the Lie bracket

$$[(\xi,a),(\eta,b)] \coloneqq ([\xi,\eta],\omega(\xi,\eta)). \tag{22.4.2}$$

For example, an element of $\mathrm{H}^4(\mathrm{B}G;\mathbb{R})$ corresponds via the Chern–Weil machine to an invariant symmetric bilinear form $\langle -,-\rangle\colon\ \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$, and it defines a degree-2 Lie algebra cocycle for $\mathrm{L}\mathfrak{g}$ by [PS86, §4.2]

$$\omega(\xi,\eta) \coloneqq \frac{1}{2\pi}\int_{S^1} \langle \xi(\theta),\eta'(\theta)\rangle\,\mathrm{d}\theta. \tag{22.4.3}$$

Suppose that ω comes from a central extension of $\mathrm{L}G$ which is a principal $\mathbb{T}$-bundle $\pi\colon\ \widetilde{\mathrm{L}G} \to \mathrm{L}G$. Then $T\widetilde{\mathrm{L}G}$ fits into a short exact sequence

$$0 \to T\mathbb{T} \to T\widetilde{\mathrm{L}G} \to \pi^*T\mathrm{L}G \to 0. \tag{22.4.4}$$

At the identity of $\widetilde{\mathrm{L}G}$ this is (22.4.1b), and left translation carries this identification to every tangent space. The data of ω includes a splitting of (22.4.1b), and left translation turns this into a splitting of (22.4.4). A connection on $\pi\colon\ \widetilde{\mathrm{L}G} \to \mathrm{L}G$ is a $\mathbb{T}$-invariant splitting, and since $\mathbb{T}$ acts trivially on its Lie algebra, we have just built a connection with curvature ω. Thus the class of (22.4.1b) in $\mathrm{H}^2(\mathrm{L}G;\mathbb{Z})$ refines to a class in $\hat{\mathrm{H}}^2(\mathrm{L}G;\mathbb{Z})$. Pressley–Segal [PS86, Theorem 4.4.1] show that this is a necessary and sufficient condition on ω for any compact, simply connected Lie group G, and that ω determines the extension.[37]

22.4.5 Remark. It may be possible to do this "all at once" by finding a canonical connection A on the principal $\mathbb{T}$-bundle $\pi\colon \mathrm{U}(V)\to\mathrm{PU}(V)$ where V is an infinite-dimensional separable Hilbert space; this would lift the tautological class $c_1(\mathrm{U}(V))\in\mathrm{H}^2(\mathrm{PU}(V);\mathbb{Z})=\mathrm{H}^2(\mathrm{K}(\mathbb{Z},2);\mathbb{Z})$ to $\hat{c}_1(\mathrm{U}(V),A)\in\hat{\mathrm{H}}^2(\mathrm{PU}(V);\mathbb{Z})$. Then a projective representation would pull back $\hat{c}_1(\mathrm{U}(V),A)$ (and A) to $\mathrm{L}G$.

To summarize a little differently, given $\hat{h}\in\hat{\mathrm{H}}^4(\mathrm{B}_\nabla G;\mathbb{Z})$, we can obtain a Chern–Weil form $\langle -,-\rangle$, hence a cocycle $\omega\in\mathrm{H}^2_{\mathrm{Lie}}(\mathrm{L}\mathfrak{g};\mathbb{R})$. Because $\mathrm{curv}(\hat{h})$ satisfies an integrality condition, so does ω, which turns out to be the same condition needed to define a central extension $\widetilde{\mathrm{L}G}\to\mathrm{L}G$ with a connection. That is, we built a map $\hat{\mathrm{H}}^4(\mathrm{B}_\nabla G;\mathbb{Z})\to\hat{\mathrm{H}}^2(\mathrm{L}G;\mathbb{Z})$. We would like to describe it more directly.

The first step is the transgression map $\hat{\mathrm{H}}^4(\mathrm{B}_\nabla G;\mathbb{Z})\to\hat{\mathrm{H}}^3(\mathrm{B}G;\mathbb{Z})$ constructed by [CJM+05, §3]. To get from 3 to 2, Gawędzki [Gaw88, §3] constructs for any closed manifold M a transgression map

$$\hat{\mathrm{H}}^3(M;\mathbb{Z})\to\hat{\mathrm{H}}^2(\mathrm{L}M;\mathbb{Z}) \tag{22.4.6}$$

from the perspective that differential cohomology is isomorphic to the hypercohomology of the Deligne complex[38]

$$0\to\mathbb{Z}\to\Omega^0\to\cdots\to\Omega^{n-1}\to 0\,.$$

Another option is to construct the transgression as follows: first pull back by the evaluation map $\mathrm{S}^1\times\mathrm{L}M\to M$, then integrate over the S^1 factor using the map we constructed in Chapter 9.

22.4.b The off-diagonal story

In Chapter 17, we saw in Corollary 17.3.3 that central extensions of a Lie group Γ (possibly infinite-dimensional) which are principal $\mathbb{T}$-bundles are classified by $\mathrm{H}^3(\mathrm{B}_\bullet\Gamma;\mathbb{Z}(1))$. The central extensions of loop groups we constructed in this chapter are principal $\mathbb{T}$-bundles. Therefore there is in principle a way to start with a class $h\in\mathrm{H}^4(\mathrm{B}G;\mathbb{Z})$ and obtain a class $\phi(h)\in\mathrm{H}^3(\mathrm{B}_\bullet\mathrm{L}G;\mathbb{Z}(1))$, and that is what we are going to do next.

Recall that truncating defines a map of complexes of sheaves of abelian groups $\mathbb{Z}(n)\to\mathbb{Z}$, inducing for us a map

$$\mathrm{H}^4(\mathrm{B}_\bullet G;\mathbb{Z}(2))\to\mathrm{H}^4(\mathrm{B}_\bullet G;\mathbb{Z})\xrightarrow{\simeq}\mathrm{H}^4(\mathrm{B}G;\mathbb{Z}). \tag{22.4.7}$$

22.4.8 Lemma. *For G a compact Lie group,* (22.4.7) *is an isomorphism.*

[37] When G is not simply connected, the theorem is not quite as nice: see [PS86, Theorem 4.6.9] and [Wal17].

[38] Gawędzki actually works with a different complex, namely $0\to\mathbb{T}\to i\Omega^1\to\cdots\to i\Omega^{n-1}\to 0$, where the map $\mathbb{T}\to i\Omega^1$ is $\mathrm{d}\circ\log$. This is equivalent to $\Sigma\mathbb{Z}(n)$ [BM94, Remark 3.6], and the proof is a straightforward generalization of Lemma 17.3.1.

Proof. Recall from Corollary 16.2.5 that (22.4.7) is part of the pullback square

$$
\begin{array}{ccc}
\mathrm{H}^4(\mathrm{B}_\bullet G;\mathbb{Z}(2)) & \xrightarrow{(22.4.7)} & \mathrm{H}^4(\mathrm{B}G;\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathrm{Sym}^2(\mathfrak{g}^\vee)^G & \longrightarrow & \mathrm{H}^4(\mathrm{B}G;\mathbb{R}),
\end{array}
\tag{22.4.9}
$$

where the bottom map is the Chern–Weil map. Since G is compact, the Chern–Weil map is an isomorphism, so (22.4.7) is as well. □

Therefore our level $h \in \mathrm{H}^4(\mathrm{B}G;\mathbb{Z})$ is equivalent data to an off-diagonal characteristic class $\tilde{h} \in \mathrm{H}^4(\mathrm{B}_\bullet G;\mathbb{Z}(2))$. The next step is the construction of yet another transgression map, this time due to Brylinski–McLaughlin [BM94, §5, on p. 618]:

$$\mathrm{H}^4(\mathrm{B}_\bullet G;\mathbb{Z}(2)) \longrightarrow \mathrm{H}^3(\mathrm{B}_\bullet \mathrm{L}G;\mathbb{Z}(1)). \tag{22.4.10}$$

Their construction models elements of these two differential cohomology groups simplicially: they identify $\mathrm{H}^4(\mathrm{B}_\bullet G;\mathbb{Z}(2))$ as the abelian group of equivalence classes of gerbes with a connective structure over a simplicial manifold model for $\mathrm{B}_\bullet G$, and $\mathrm{H}^3(\mathrm{B}_\bullet \mathrm{L}G;\mathbb{Z}(1))$ as equivalence classes of line bundles over a simplicial model for $\mathrm{B}_\bullet \mathrm{L}G$ (*ibid.*, Theorem 5.7).

We have obtained some class in $\mathrm{H}^3(\mathrm{B}_\bullet \mathrm{L}G;\mathbb{Z}(1))$ from a level $h \in \mathrm{H}^4(\mathrm{B}G;\mathbb{Z})$, hence some central extension. That this coincides with the central extension obtained from h by the other methods in this chapter is due to Brylinski–McLaughlin (*ibid.*, §5). See also Brylinski [Bry08, §6.5] for related discussion and Waldorf [Wal10, §3.1] for another construction of this transgression map.