

RAMIFICATION IN $\mathbf{E}_n$ -ALGEBRAS

SANATH K. DEVALAPURKAR AND ISHAN LEVY

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1. INTRODUCTION

Throughout this paper we work locally at a prime p . Inspired by the work of Schwänzl-Vogt-Waldhausen in [SVW99], the first author showed:

Theorem 1.1 ([Dev20, Theorem 1.3]). *Let $h \geq 1$ and $p^k > 2$. Then a $T(h)$ -local¹ $\mathbf{E}_\infty$ -ring cannot have a primitive p^k th root of unity ζ_{p^k} in its π_0 .*

This result can be viewed as saying that ramification in the homotopy rings of $\mathbf{E}_\infty$ -rings behaves quite differently at heights ≥ 1 than at height 0. Our goal in this note is to understand the extent to which this phenomenon continues for $\mathbf{E}_n$ -rings for $n < \infty$.

Given an even $\mathbf{E}_2$ -ring R , it is not hard to adjoin roots of unity as an $\mathbf{E}_1$ -ring. Namely, one starts with the polynomial ring $R[x]$, and use [Ang08, Corollary 3.2] to obtain an $\mathbf{E}_1$ -ring structure on $R[x]/\Phi_m(x)$, where Φ_m is the m^{th} cyclotomic polynomial. Thus the first m for which there might be obstructions to adding roots of unity to $\mathbf{E}_n$ -rings is $n = 2$. We show that there are indeed obstructions in this case.

Theorem A. *Let $h \geq 1$. There is a $k \geq 0$ depending on p, h such that a $T(h)$ -local $\mathbf{E}_2$ -ring cannot have ζ_{p^k} in π_0 .*

We obtain this result as a consequence of the more general result below, which obstructs perfectoid rings from mapping into the π_0 of nonzero $T(h)$ -local $\mathbf{E}_2$ -rings. Theorem A is a consequence of taking S to be $\mathbf{Z}_p^{\text{cyc}}$ the p -adic completion of the maximal p -cyclotomic extension of the integers:

Theorem B. *Let R be an $\mathbf{E}_2$ -ring with a map $S \rightarrow \pi_0 R$ where S is a perfectoid ring. Then there is a map of $\mathbf{E}_2$ -rings $S \rightarrow R$. In particular R is an $\mathbf{E}_1$ - S -algebra.*

Given Theorem A, we can make the following definition:

¹The version of this result for $K(h)$ and $T(h)$ are equivalent by the nilpotence theorem: see [LMMT24, Lemma 2.3].

Definition 1.2. Fix a prime p , a height $h > 0$, and an $n \geq 2$. The numbers $r_{p,h,n}$ to be the largest natural number such that there exists a $T(h)$ -local $\mathbf{E}_n$ -algebra with a $p^{r_{p,h,n}}$ -th root of unity in π_0 .

The numbers $r_{p,h,n}$ are nonincreasing in n , and measure the extent to which roots of unity must not exist in $T(h)$ -local $\mathbf{E}_n$ -algebras. We prove the following results about the numbers $r_{p,h,n}$:

Theorem C. *The following are true about the $r_{p,h,n}$:*

- $r_{p,1,4}$ is 0 for $p > 2$ and 1 for $p = 2$.
- Fix p and h . Then for $n \gg 0$, $r_{p,h,n}$ is 0 for $p > 2$ and 1 for $p = 2$.

At height $h = 1$, the above result is proven by using the existence of δ -operations $\pi_0(A) \rightarrow \pi_0(A/p^2)$. In fact, our proof also shows the more general obstruction to ramification in $\mathbf{E}_4$ -KU-algebras:

Theorem D. *Let A be a $K(1)$ -local $\mathbf{E}_4$ -KU-algebra. If there is a map from $\mathcal{O}_K \rightarrow \pi_0 A$ where $\mathcal{O}_K$ is the ring of integers of a ramified extension K of $\mathbf{Q}_p$, then $A = 0$.*

We also study in Section 3 obstructions to adjoining roots of unity in a strict way, for example showing in Corollary 3.5 that cyclotomic polynomial $\Phi_{p^k}(q)$ is not $\mathbf{E}_3$ -strict in $\mathrm{KU}[q]_p^\wedge$ for $p^k > 2$. This is an obstruction to constructing $\mathrm{KU}[\zeta_{p^k}]$ as an $\mathbf{E}_2$ -algebra in a strict way.

Despite these obstructions, we were surprised to learn that there are $K(1)$ -local $\mathbf{E}_2$ -algebras A with a p th root of unity.

Example 1.3. It follows from [Ang08, Theorem 5.17] that there is a $K(1)$ -local $\mathbf{E}_2$ -KU-algebra with π_0 isomorphic to $\mathbf{Z}_p[\zeta_p]$. This can be obtained from a Hochschild cohomology of a 2-periodic form of $K(1)$. We note [Ang08, Conjecture 5.20] describes an analog of this at higher height.

The above example shows that the number $r_{p,h,n}$ are interesting, and we believe it would be interesting to compute them.

Remark 1.4. One can attempt to study analogues of the statements discussed here for higher roots of unity in the sense of [CSY24, Definition 4.2]. In this case, recall that a primitive p^r th root of unity of height n for some $r \geq 1$ in a commutative algebra R in a presentably symmetric monoidal stable category is an $\mathbf{E}_\infty$ -map $\mathbb{1}[B^n \mathbf{Z}/p^r] \rightarrow R$ such that the pushout of commutative algebras of the span $\mathbb{1}[B^n \mathbf{Z}/p^{r-1}] \leftarrow \mathbb{1}[B^n \mathbf{Z}/p^r] \rightarrow R$ is 0.

We then claim that higher primitive roots of unity don't exist at the wrong height, i.e., if R is a $T(h)$ -local $\mathbf{E}_\infty$ -algebra with $h \neq n$ and a primitive p^r th root of unity, then $R = 0$. If $n > h$, then $B^n \mathbf{Z}/p^r$ is $T(h)$ -acyclic, so the pushout is R itself. If $n < h$, then because the $T(n)$ -local category is virtually $(\mathbf{F}_p, n)$ -orientable [BCSY24, Proposition 7.27], we can add a height n p^r th root of unity, we can identify this pushout with its Fourier dual [BCSY24, Theorem 7.33], which is $\mathbb{1}^{B^{h-n}((\mathbf{Z}/p^{r-1})^\vee)} \leftarrow \mathbb{1}^{B^{h-n}((\mathbf{Z}/p^r)^\vee)} \rightarrow R$. However, for $h - n \geq 1$, tensoring along the map $\mathbb{1}^{B^{h-n}((\mathbf{Z}/p^r)^\vee)} \rightarrow \mathbb{1}^{B^{h-n}((\mathbf{Z}/p^{r-1})^\vee)}$ is conservative, since the target ring has a $B^{h-n-1}(\mathbf{Z}/p)$ -action whose fixed points is the source, and fixed points commutes with colimits by ∞ -semiadditivity [CSY22].

We note that the notion of higher primitive roots of unity considered here is much stronger than what is considered in this paper, it is defined for $\mathbf{E}_\infty$ -algebras

and is a strict notion (see Section 3). It would be interesting to explore the extent to which this phenomenon persists for weaker notions of higher roots of unity.

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2. PERFECTOID RINGS IN π_0

In this section, we prove our main result:

Theorem B. *Let R be an $\mathbf{E}_2$ -ring with a map $S \rightarrow \pi_0 R$ where S is a perfectoid ring. Then there is a map of $\mathbf{E}_2$ -rings $S \rightarrow R$. In particular, R is an $\mathbf{E}_1$ - S -algebra.*

As a consequence, we deduce Theorem A.

Proof of Theorem A. Suppose that the conclusion of Theorem A is false. Then for each $k \geq 0$, there exists a $T(h)$ -local $\mathbf{E}_2$ -ring R_k such that $\pi_0 R_k$ contains a primitive p^k th root of unity ζ_{p^k} . Let R be an ultraproduct of the R_k . Then R has nonzero $T(h)$ -localization: this is because it is a filtered colimit of non-zero $T(h)$ -local rings given by products of the R_k , and $T(h)$ -local filtered colimits of nonzero $T(h)$ -local rings are nonzero (see the proof of [BSY26, Lemma A.11]). But $\pi_0(R)$ contains a system of primitive p th roots of unity, so there is a map $\mathbf{Z}_p^{\text{cyc}} \rightarrow \pi_0(R)$. Applying Theorem B to the perfectoid ring $\mathbf{Z}_p^{\text{cyc}}$, we obtain a contradiction, since $\mathbf{Z}_p^{\text{cyc}}$ is $T(h)$ -acyclic. $\square$

We now turn to proving Theorem B. Recall that there is a spherical Witt vector functor $\mathbb{W} : \text{Perf}_{\mathbf{F}_p} \rightarrow \text{CAlg}(\text{Sp}_p)$. To prove Theorem B, we first show that R lifts to an $\mathbf{E}_2$ -algebra over $\mathbb{W}((S/p)^b)$, which follows from the fact that spherical Witt vectors always act centrally on $\mathbf{E}_n$ -algebras (Proposition 2.2 below). Then we conclude by applying a perfectoid version of the Hopkins–Mahowald theorem due to Mao (Proposition 2.4), which implies that the existence of an $\mathbf{E}_2$ -map from S to an $\mathbf{E}_2$ - $\mathbb{W}((S/p)^b)$ -algebra can be checked on π_0 .

We begin with the proposition below, which is an analogue of [BSY26, Proposition 2.2] in the setting of $\mathbf{E}_n$ -algebras.

Proposition 2.1. *For $n \geq 2$, the composite $\mathbb{W} : \text{Perf}_{\mathbf{F}_p} \rightarrow \text{CAlg}(\text{Sp}_p) \rightarrow \text{Alg}_{\mathbf{E}_n}(\text{Sp}_p)$ is fully faithful with right adjoint $(-)^b$, sending R to $(\pi_0 R/p)^b$.*

Proof. We first show that for $n \geq 2$, $\mathbb{W}(A)$ is formally étale as a p -complete $\mathbf{E}_n$ -algebra, i.e. that the $\mathbf{E}_n$ -cotangent complex vanishes. To do this, it suffices by [Fra13, Theorem 2.26, Proposition 3.16] to show that $\int_{\mathbf{R}^n-0} \mathbb{W}(A) \cong \mathbb{W}(A)$. Because $\int_{\mathbf{R}^n-0} \mathbb{W}(A) \cong \mathbb{W}(A) \otimes_{\int_{\mathbf{R}^{n-1}-0} \mathbb{W}(A)} \mathbb{W}(A)$, it suffices to show this for $n = 2$, and since $\mathbb{W}$ is symmetric monoidal, it suffices to show this in $\text{Perf}_{\mathbf{F}_p}$ for A . This follows since

$$\int_{\mathbf{R}^2-0} A \cong A \otimes_{A \otimes A} A \cong A$$

It follows from the existence of the adjunction that $\mathbb{W}(A)$ is fully faithful, so it only remains to show the adjunction, i.e. produce a natural isomorphism

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_n}(\mathrm{Sp}_p)}(\mathbb{W}(A), R) \cong \mathrm{Map}_{\mathrm{Perf}_{\mathbf{F}_p}}(A, (\pi_0 R/p)^\flat)$$

To do this, since $\mathbb{W}(A)$ is connective, it suffices to assume that R is connective. Then since the Postnikov tower of R is a tower of square zero extensions and $\mathbb{W}(A)$ is formally étale, we reduce to the case R is discrete, where the result is now clear, since $\pi_0 \mathbb{W}(A) \cong \mathbb{W}(A)$. $\square$

The next key observation in proving Theorem B is that the counit map $\mathbb{W}(R^\flat) \rightarrow R$ is always central. Recall from [Lur16, Section 5.3] and [Fra13, Definition 3.1, Proposition 3.16] that the $\mathbf{E}_{n+1}$ -center of an $\mathbf{E}_n$ -algebra R is given by the endomorphism algebra $\mathrm{End}_{\int_{S^{n-1} \times \mathbf{R}} R}(R)$.

Proposition 2.2. *Given a p -complete $\mathbf{E}_n$ -algebra R with $n \geq 2$, R is an $\mathbf{E}_n$ - $\mathbb{W}(R^\flat)$ -algebra. Moreover, $\mathfrak{Z}_{\mathbf{E}_n}(R)^\flat \cong R^\flat$, where $\mathfrak{Z}_{\mathbf{E}_n}(R)$ is the $\mathbf{E}_{n+1}$ center of R .*

Proof. First we refine the $\mathbf{E}_n$ -algebra map $\mathbb{W}(R^\flat) \rightarrow R$ to an $\mathbf{E}_n$ -algebra map $\mathbb{W}(R^\flat) \rightarrow \mathfrak{Z}_n(R)$. To do this, it suffices by the universal property of $\mathfrak{Z}_n(R)$ ([Lur16, Definition 5.3.1.2]) to produce a diagram of $\mathbf{E}_n$ -algebras

$$\begin{array}{ccc} R & \xlongequal{\quad} & R \\ & \searrow_{1 \otimes R} & \nearrow \\ & \mathbb{W}(R^\flat) \otimes R & \end{array}$$

To do this, we consider the base-change along the multiplication map, which a priori is a map of $\mathbf{E}_{n-1}$ -algebras:

$$m : \mathbb{W}(R^\flat) \otimes R \rightarrow (\mathbb{W}(R^\flat) \otimes R) \otimes_{\mathbb{W}(R^\flat \otimes R^\flat)} \mathbb{W}(R^\flat).$$

Next, we note that the target of m is an idempotent algebra over the source: this follows because $\mathbb{W}$ takes surjective maps to idempotent algebras, and we are basechanging along such a map. Thus m canonically refines to an $\mathbf{E}_n$ -algebra map. Because $1 \otimes R$ can be written as an $\mathbf{E}_{n-1}$ -algebra map as $R \rightarrow R \otimes_{\mathbb{W}(R^\flat)} \mathbb{W}(R^\flat \otimes R^\flat)$ via the right unit, we find that $m \circ (1 \otimes R)$ is an equivalence². This gives the desired triangle.

Since we have an $\mathbf{E}_n$ -algebra map $\mathbb{W}(R^\flat) \rightarrow \mathfrak{Z}_n(R)$, by applying Proposition 2.1 it follows that the map $\mathbb{W}(R^\flat) \rightarrow \mathfrak{Z}_n(R)$ automatically refines to an $\mathbf{E}_{n+1}$ -map, since the mapping spaces in $\mathbf{E}_n$ and $\mathbf{E}_{n+1}$ -algebras are the same. Since R is an $\mathbf{E}_n$ - $\mathfrak{Z}_n(R)$ -algebra, it follows that it is also an $\mathbf{E}_n$ - $\mathbb{W}(R^\flat)$ -algebra.

To finish, we now show that $\mathfrak{Z}_n(R)^\flat \rightarrow R^\flat$ is an isomorphism. Surjectivity follows from the lift $\mathbb{W}(R^\flat) \rightarrow \mathfrak{Z}_n(R)$ we have already produced. For injectivity, if a is in the kernel, then we consider the map of $\mathbf{E}_{n+1}$ -algebras $g : \mathfrak{Z}_n(R) \rightarrow \mathfrak{Z}_n(R) \otimes_{\mathbb{W}(\mathbf{F}_p[x]^\sharp)} \mathbb{W}(\mathbf{F}_p)$ where x is sent to a . Here $\sharp$ denotes the colimit perfection. Since $\mathbb{W}(\mathbf{F}_p)$ is an idempotent algebra over $\mathbb{W}(\mathbf{F}_p[x]^\sharp)$, the target is an idempotent

²A subtlety is that it a priori is only the *identity* map as an $\mathbf{E}_{n-1}$ -ring map. But any equivalence is equivalent to the identity map, so this doesn't matter.

algebra over the source. Since a is in the kernel, we learn that R^b is an $\mathbf{E}_n$ - $\mathfrak{Z}_n(R) \otimes_{\mathbb{W}(\mathbf{F}_p[x]^\#)} \mathbb{W}(\mathbf{F}_p)$ -algebra, so that g has a section via the universal property of the center. It follows that $a = 0$. $\square$

Remark 2.3. The lift of R to an $\mathbf{E}_n$ - $\mathbb{W}(R^b)$ -algebra is essentially unique. Indeed, the space L of lifts is the pullback

$$\begin{array}{ccc} L & \xrightarrow{\quad \quad} & \mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_{n+1}}(\mathrm{Sp}_p)}(\mathbb{W}(R^b), \mathfrak{Z}_{\mathbf{E}_n}(R)) \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_n}(\mathrm{Sp}_p)}(\mathbb{W}(R^b), R) \end{array}$$

which is contractible since the right vertical map is an isomorphism as $\mathfrak{Z}_n(R)^b \cong R^b$.

The last ingredient for Theorem B is the perfectoid version of the Hopkins–Mahowald theorem, which we now explain. Let S be a perfectoid ring. Then the map $\mathbb{W}(S^b) \rightarrow S$ on π_0 is Fontaine’s map $\theta : A_{\mathrm{inf}}(S) \rightarrow S$. The kernel of θ is a principal ideal, with a generator ξ , giving $S = A_{\mathrm{inf}}(S)/\xi$ as discrete rings. This has the following lift to $\mathbf{E}_2$ -spectral algebraic geometry:

Proposition 2.4 ([Mao23, Theorem 1.13]). *Let S be a perfectoid ring. Then there is an isomorphism of $\mathbf{E}_2$ -algebras $\mathbb{W}(S^b) //_{\mathbf{E}_2} \xi \cong S$, where $//_{\mathbf{E}_2}$ is the $\mathbf{E}_2$ -quotient, i.e., the free $\mathbf{E}_2$ - $\mathbb{W}(S^b)$ -algebra equipped with a nullhomotopy of ξ .*

For $S = \mathbf{F}_p$, this is the Hopkins–Mahowald theorem [ACB19, Theorem 5.1].

Proof of Theorem B. Because we have a map $\pi_0 S \rightarrow \pi_0 R$, we have a map $\mathbb{W}(S) \rightarrow \mathbb{W}(R)$, making R an $\mathbf{E}_2$ - $\mathbb{W}(S)$ -algebra. Since d is sent to 0 in R , Proposition 2.4 produces an $\mathbf{E}_2$ -algebra map $S \rightarrow R$. $\square$

3. STRICT ELEMENTS

We now show that there are strong restrictions on $\mathbf{E}_n$ -algebras with *strict* roots of unity for $n > 1$. We first define the notion of an $\mathbf{E}_m$ -strict element of an $\mathbf{E}_n$ -algebra, which is inductive on $m - n$.

Definition 3.1. Let R be an $\mathbf{E}_n$ -algebra, $x \in \pi_0 R$, and $m \in \mathbf{N} \cup \infty$. We say x is an $\mathbf{E}_m$ -strict element if

- If $m \leq n$, we ask for the existence of an $\mathbf{E}_m$ -algebra map $\mathbb{S}[t] \rightarrow R$ sending the generator t to x .
- If $n < m < \infty$, we ask that x lift to an $\mathbf{E}_m$ -strict element of $\pi_0 \mathfrak{Z}_n R$.
- If $m = \infty$, we ask for x to be $\mathbf{E}_k$ -strict for all $k < \infty$ compatibly in k .

For example, Proposition 2.2 can be interpreted as saying that when x lifts to $\mathbb{W}(R^b)$, it is $\mathbf{E}_\infty$ -strict. If x is $\mathbf{E}_m$ -strict, then so are all of the powers of x (including negative powers, if they exist). There is also a variation of the notion of $\mathbf{E}_m$ -strict, where for an operad $\mathcal{O} \rightarrow \mathbf{E}_n$, we say that an element x in π_0 of an $\mathbf{E}_n$ -algebra R is $\mathcal{O}$ -strict if there is an $\mathcal{O}$ -algebra map $\mathbb{S}[t] \rightarrow R$ sending the generator t to x .

Proposition 3.2. *Suppose that $p^k \neq 1, 2$. Let R be a p -complete $\mathbf{E}_2$ -ring with a primitive p^k th root of unity ζ_{p^k} which is $\mathbf{E}_3$ -strict. Then R admits an $\mathbf{E}_2$ -map from $\mathbf{Z}_p[\zeta_{p^k}]$, and in particular is an $\mathbf{E}_1$ - $\mathbf{Z}_p[\zeta_{p^k}]$ -algebra.*

Proof. Since ζ_{p^k} is an $\mathbf{E}_3$ -strict element, there is an $\mathbf{E}_3$ -map $\mathbb{S}[q] \rightarrow \mathfrak{Z}_{\mathbf{E}_2} R$ which makes R into an $\mathbf{E}_2$ - $\mathbb{S}[q]$ -algebra. The claim follows once we show that the free $\mathbf{E}_2$ - $\mathbb{S}_p[q]$ -algebra with q a root of unity in π_0 is $\mathbf{Z}_p[\zeta_{p^k}]$. This is [Mao23, Theorem 1.19], but we quickly recall the argument for completeness. There is an $\mathbf{E}_2$ -map $\mathbb{S}_p[q] \parallel_{\mathbf{E}_2} \Phi_{p^k}(q) \rightarrow \mathbf{Z}_p[\zeta_{p^{n+1}}]$. Both target and source are complete with respect to $q - 1$, so it suffices to check that this map is an equivalence after base-changing along the map $\mathbb{S}_p[q] \rightarrow \mathbb{S}_p$ sending $q \mapsto 1$. After base-change, this becomes the map $\mathbb{S}_p \parallel_{\mathbf{E}_2} p \rightarrow \mathbf{F}_p$, which is an equivalence by the Hopkins–Mahowald theorem. $\square$

If R is an even $\mathbf{E}_2$ -algebra, there are no obstructions to any $x \in \pi_0 R$ being $\mathbf{E}_2$ -strict, since the cells of $\mathbb{S}[t]$ as an $\mathbf{E}_2$ -ring are in even degrees. As a consequence we can adjoin any algebraic equation³ $R[x]/(f)$ as a basechange $R[x] \otimes_{\mathbb{S}_p[f]} \mathbb{S}$, in particular allowing us to add roots of unity as an $\mathbf{E}_1$ -algebra (see also [Ang08, Theorem 3.1]).

Our next goal is to show using the Rezk logarithm [Rez06] that when R is a $K(1)$ -local $\mathbf{E}_\infty$ -ring, any unit which is $\mathbf{E}_2 \otimes \mathbf{A}_2$ -strict (for example an $\mathbf{E}_3$ -strict element) is $\mathbf{E}_\infty$ -strict. Recall that the π_0 of $K(1)$ -local $\mathbf{E}_\infty$ -rings naturally admits the structure of a δ -ring, with lift of Frobenius $\psi(x) = x^p + p\delta(x)$, and that $L_{K(1)}\mathbb{S}\{t\}$ is the free polynomial algebra on $\delta^i(t)$ for $i \geq 0$ [Hop14].

We first record a well known lemma. Let $\mathbb{S}\{t\}$ denote the free $\mathbf{E}_\infty$ -ring on a class t in degree 0.

Lemma 3.3. *There is an equivalence of $\mathbf{E}_\infty$ -rings $L_{K(1)}\mathbb{S}[t] \cong L_{K(1)}\mathbb{S}\{t\} \parallel_{\mathbf{E}_\infty} \delta(t)$. Here, the quotient on the right-hand side is taken in $K(1)$ -local $\mathbf{E}_\infty$ -rings.*

Proof. Since $\delta(t) = 0$ in $L_{K(1)}\mathbb{S}[t]$, there is an $\mathbf{E}_\infty$ -map $L_{K(1)}\mathbb{S}\{t\} \parallel_{\mathbf{E}_\infty} \delta(t) \rightarrow L_{K(1)}\mathbb{S}[t]$. To see this is an equivalence, McClure’s theorem [BMS86, Chapter IX] (see also [Hop14]) shows that $\pi_* L_{K(1)}\mathbb{S}\{t\}$ is a p -completed polynomial ring over $\pi_* L_{K(1)}\mathbb{S}$ on $\delta^i(t)$ for $i \geq 0$. Since $L_{K(1)}\mathbb{S}\{t\} \parallel_{\mathbf{E}_\infty} \delta(t) \cong L_{K(1)}\mathbb{S}\{t\} \hat{\otimes}_{L_{K(1)}\mathbb{S}\{\delta(t)\}} L_{K(1)}\mathbb{S}$, it follows from the p -completed bar spectral sequence that the map is an equivalence. $\square$

Proposition 3.4. *Let A be a $K(1)$ -local $\mathbf{E}_\infty$ - KU_p -algebra with $\pi_0(A)$ torsion free, and let $\xi \in \pi_0(A)^\times$. If ξ is $\mathbf{E}_2 \otimes \mathbf{A}_2$ -strict, then it is $\mathbf{E}_\infty$ -strict. In particular, $\delta(\xi) = 0$.*

Proof. By Lemma 3.3, it suffices to show that $\delta(\xi) = 0$ if ξ is $\mathbf{E}_2 \otimes \mathbf{A}_2$ -strict. We begin by observing that

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_2 \otimes \mathbf{A}_1}(\mathrm{Sp})}(\mathbb{S}[t], A) \cong \mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_2 \otimes \mathbf{A}_2}(\mathrm{S})}(\mathbf{N}, \Omega^\infty A)$$

where the target has the $\mathbf{E}_\infty$ structure coming from multiplication. Since ξ is in a unit, using $(-)_\xi$ to denote the component sending 1 to ξ , we get

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_2 \otimes \mathbf{A}_1}(\mathrm{Sp})}(\mathbb{S}[t], A)_\xi \cong \mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_2 \otimes \mathbf{A}_2}(\mathrm{S})}(\mathbf{Z}, \mathrm{GL}_1 A)_\xi \cong \mathrm{Map}_{\mathrm{Alg}_{\mathbf{A}_2}(\mathrm{S})}(B^2 \mathbf{Z}, B^2 \mathrm{GL}_1(A))_\xi$$

³One can also add multiple generators and relations at the same time.

In other words, we need to show that given a commutative diagram of the form

$$\begin{array}{ccc} \mathbf{CP}^\infty \times \mathbf{CP}^\infty & \xrightarrow{\xi, \xi} & \mathbf{B}^2\mathrm{GL}_1(A) \times \mathbf{B}^2\mathrm{GL}_1(A) \\ \downarrow \mu & & \downarrow \\ \mathbf{CP}^\infty & \xrightarrow{\xi} & \mathbf{B}^2\mathrm{GL}_1(A). \end{array}$$

$\delta(\xi) = 0$.

To understand this, we will use the $K(1)$ -local logarithm, which defines an infinite loop map $\ell : \mathbf{B}^2\mathrm{GL}_1(A) \rightarrow \Omega^\infty \Sigma^2 A$ since A is $K(1)$ -local. After composing with the logarithm, we obtain a diagram

$$\begin{array}{ccccc} \Sigma_+^\infty(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) & \xrightarrow{\ell(\xi), \ell(\xi)} & \Sigma^2(A \oplus A) & \xrightarrow{\sim} & A \oplus A \\ \downarrow \mu & & \downarrow & & \downarrow \\ \Sigma_+^\infty \mathbf{CP}^\infty & \xrightarrow{\ell(\xi)} & \Sigma^2 A & \xrightarrow{\sim} & A. \end{array}$$

The final column uses multiplication by β to identify $\Sigma^2 A \simeq A$. Since A is complex orientable and 2-periodic, a choice of orientation and periodicity generator gives an isomorphism $[\Sigma_+^\infty \mathbf{CP}^\infty, L_{K(1)} A] \cong \pi_0 L_{K(1)} A[[x]]$, to which $\ell(\xi)$ corresponds to a power series f . The commutativity of the diagram implies that f is a homomorphism from the formal group of A to the additive formal group.

Over a torsion free ring, up to rescaling the Lie algebra, there is a unique homomorphism between any two formal groups, which to the additive formal group is given by the logarithm. In the case of A , which is a KU_p -algebra, this is given by $f(x) = a \sum_{n \geq 1} \frac{x^n}{n}$. Since $\pi_0 A$ is p -torsion free, and A is p -complete, $\pi_0 A$ is p -adically separated, so $a = 0$ for f to be an integral power series, i.e $\ell(\xi) = 0$.

By [Rez06, Theorem 1.9], we have

$$0 = \ell(\xi) = \sum_{n \geq 1} (-1)^n \frac{p^{n-1}}{n} \left(\frac{\delta(\xi)}{\xi^p} \right)^n = \frac{1}{p} \log \left(\frac{\xi^p}{\psi(\xi)} \right).$$

Using that $p \in \pi_0(A)$ has divided powers, we can multiply by p and exponentiate to get

$$1 = \exp(p\ell(\xi)) = \sum_{n \geq 0} \ell(\xi)^n \frac{p^n}{n!} = \frac{\xi^p}{\psi(\xi)}$$

Thus $\psi(\xi) = \xi^p + p\delta(\xi) = \xi^p$ so $\delta(\xi) = 0$. $\square$

As discussed earlier, we can adjoin primitive p^k th roots of unity to even algebras (such as KU) as an $\mathbf{E}_1$ -ring by using the fact that the element $\Phi_{p^k}(q)$ is $\mathbf{E}_2$ -strict in $\mathrm{KU}[q]$. The following corollary shows that this method cannot be improved even to get an $\mathbf{E}_1 \otimes \mathbf{A}_2$ -structure, and nor can any other similar kind of strict quotient.

Corollary 3.5. *The element $\Phi_{p^k}(q)$ is not $\mathbf{E}_2 \otimes \mathbf{A}_2$ -strict in $\mathrm{KU}[q]_p^\wedge$ if $p^k \neq 1, 2$.*

Proof. If $\Phi_{p^k}(q)$ were $\mathbf{E}_2 \otimes \mathbf{A}_2$ -strict, it would be $\mathbf{E}_\infty$ -strict by Proposition 3.4. But then the tensor product $\mathbb{S} \otimes_{\mathbb{S}[t]} \mathrm{KU}[q]_p^\wedge$ would be a $K(1)$ -local $\mathbf{E}_\infty$ -ring having a primitive p^k th root of unity in π_0 , contradicting Theorem 1.1. $\square$

4. $K(1)$ -LOCAL $\mathbf{E}_4$ -KU-ALGEBRAS

The goal of this section is to prove Theorem C. The key point is Proposition 4.2 below, that $K(1)$ -local $\mathbf{E}_{2n+2}$ -algebras have a lift of Frobenius mod p^n . This makes the obstruction to roots of unity from Theorem 1.1 still hold, and it moreover generalizes to more general ramified extensions. We then use an ultraproduct argument to show that $r_{p,h,n}$ agrees with the number for $\mathbf{E}_\infty$ for n sufficiently large.

Proposition 4.1. *There is a square as below, where the vertical maps are isomorphisms, and the lower horizontal map is the p^n -Bockstein:*

$$\begin{array}{ccc} \mathrm{Conf}_p(\mathbf{R}^{2n+1}) \otimes \mathrm{KU} & \longrightarrow & B\Sigma_p \otimes \mathrm{KU} \\ \downarrow & & \downarrow \mathrm{tr} \\ \Sigma^{-1}\mathrm{KU}/p^n & \xrightarrow{\beta_{p^n}} & \mathrm{KU}. \end{array}$$

Proof. For $p > 2$, this follows from the commutative diagram on [BHK24, Section 6.2, Page 44]. For $p = 2$, note that $\mathrm{Conf}_2(\mathbf{R}^{2n+1}) \cong \mathbf{RP}^{2n}$, and the map to $B\Sigma_2 = \mathbf{RP}^\infty$ is the standard inclusion. Upon 2-completion, both $\mathrm{KU} \otimes \mathbf{RP}^{2n}$ and $\mathrm{KU} \otimes \mathbf{RP}^\infty$ are perfect KU_2 -modules, so it suffices to identify the KU_2 -linear dual of the map $\mathrm{KU}_2 \otimes \mathbf{RP}^{2n} \rightarrow (\mathrm{KU} \otimes \mathbf{RP}^\infty)_2$ with the map $\mathrm{KU}_2 \rightarrow \mathrm{KU}/2^n$. This follows from [Pow15, Corollary 4.17]. $\square$

Below, we use the map $p : X/p^n \rightarrow X/p^{n+1}$ for a spectrum X .

Proposition 4.2. *Suppose that R is a $K(1)$ -local $\mathbf{E}_{2n+2}$ -KU-algebra. Then there is a natural map*

$$\delta : \pi_0 R \rightarrow \pi_0(R/p^n)$$

such that

$$\begin{aligned} \delta(x+y) &= \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} x^i y^{p-i}, \\ \delta(xy) &= \delta(x)y^p + \delta(y)x^p + p\delta(x)\delta(y). \end{aligned}$$

In particular, the map $\phi : \pi_0 R \rightarrow \pi_0(R/p^{n+1})$ defined by $\phi(x) = x^p + p\delta(x)$, is a ring homomorphism lifting the Frobenius map.

Proof. We will define the map δ for all $K(1)$ -local $\mathbf{E}_{2n+1}$ -KU-algebras, and show that it satisfies the desired relations for all $\mathbf{E}_{2n+2}$ -algebras.

To define the map δ , it suffices to do this in the universal example of the free $\mathbf{E}_{2n+1}$ -KU-algebra on a class x in degree 0. From Proposition 4.1, we learn that in arity p , the map from the free $\mathbf{E}_{2n+1}$ -algebra to the free $\mathbf{E}_\infty$ -algebra is an isomorphism on π_0/p^n , allowing us to define δ as the unique lift of the associated $\mathbf{E}_\infty$ -operation.

For the relations, it suffices to prove them in the universal $\mathbf{E}_{2n+2}$ -algebra on two classes in degree 0. The relation for $\delta(x+y)$ is witnessed in the arity p part of this free algebra, which is $((\mathrm{KU} \oplus \mathrm{KU})^{\otimes p} \otimes (\mathrm{Conf}_p(\mathbf{R}^{2n+2}))_+))_{h\Sigma_p}$. We claim that the map

$$((\mathrm{KU} \oplus \mathrm{KU})^{\otimes p} \otimes (\mathrm{Conf}_p(\mathbf{R}^{2n+2}))_+))_{h\Sigma_p}/p^n \rightarrow ((\mathrm{KU} \oplus \mathrm{KU})^{\otimes p}_{h\Sigma_p})/p^n$$

is injective on the image of $\pi_0((\mathrm{KU} \oplus \mathrm{KU})^{\otimes p} \otimes (\mathrm{Conf}_p(\mathbf{R}^{2n+1}))_+))_{h\Sigma_p}/p^n$ in π_0 . The result then follows, since the target is the arity p part of the universal $\mathbf{E}_\infty$ -algebra on two generators, where the desired relation holds.

To see the claim, we note that (p -adically), $((\mathrm{KU} \oplus \mathrm{KU})^{\otimes p} \otimes (\mathrm{Conf}_p(\mathbf{R}^i))_+))_{h\Sigma_p}$ naturally is a sum of two copies of $\mathrm{KU}^{\otimes p} \otimes \mathrm{Conf}_p(\mathbf{R}^i)_{h\Sigma_p}$, and a sum of copies of $\mathrm{Conf}_p^{\mathrm{ord}}(\mathbf{R}^i)_+$ indexed by the nontrivial Σ_p orbits of 2^p . The claim then follows from the fact that the inclusion $\mathrm{Conf}_p^{\mathrm{ord}}(\mathbf{R}^i) \rightarrow \mathrm{Conf}_p^{\mathrm{ord}}(\mathbf{R}^{i+1})$ is null, so the image factors through a section from $\mathrm{Conf}_p^{\mathrm{ord}}(\mathbf{R}^\infty)$.

The proof of the relation for $\delta(xy)$ is nearly identical, except the relation takes place in arity $2p$ instead of p .

Once these relations are proven, it follows that φ is a ring map and a lift of Frobenius, for the same reason that a delta structure gives a lift of Frobenius. $\square$

Remark 4.3. The relations satisfied by δ are different if R is only an $\mathbf{E}_{2n+1}$ -KU-algebra, and can involve Browder brackets.

We sketch how this works for the relation $\delta(xy) = \delta(x)y^2 + \delta(y)x^2 + 2\delta(x)\delta(y)$ for $p = 2$ and $n = 1$. In this case, $\beta\delta$ lifts to a natural map $\pi_0(R) \rightarrow \pi_2(R/2)$ for an $\mathbf{E}_3$ -ku-algebra. Moreover, mod β , it is easy to see that this operation is the Dyer–Lashof operation Q_2 on $\mathbf{E}_3$ - $\mathbf{F}_2$ -algebras. The nontriviality of the Browder bracket term then follows from the fact that the Cartan formula for Q_2 of an $\mathbf{E}_3$ -algebra has a Browder bracket term: it is given by $Q_2(xy) = Q_2(x)y^2 + x^2Q_2(y) + x[x, y]y$ (see [Law19, Theorem 5.2]), using that Q_1 vanishes since the classes x, y are integral.

It is possible that arguments similar to ours are able to prove the analog of Theorem D for $K(1)$ -local $\mathbf{E}_3$ -KU-algebras, and we believe it would be interesting to work this out.

Using Proposition 4.2, we can prove Theorem D, which is the key input to Theorem C.

Theorem D. *Let A be a $K(1)$ -local $\mathbf{E}_4$ -KU-algebra. If there is a map from $\mathcal{O}_K \rightarrow \pi_0 A$ where $\mathcal{O}_K$ is the ring of integers of a ramified extension K of $\mathbf{Q}_p$, then $A = 0$.*

Proof. Let π be a uniformizer of $\mathcal{O}_K$, with valuation $\frac{1}{e}$ where $e > 1$ by assumption. Let a be an integer minimally chosen so that $\frac{ap}{e} \geq 1$. Note that $a < e$. Then $\varphi(\pi^a) \equiv \pi^{ap} \equiv 0 \pmod{p}$. On the other hand, $\varphi(\pi^e)|\varphi(p) = p$. It follows that $p\varphi(\pi^{e-a})|\varphi(\pi^e)|p$. But $\varphi(\pi^{e-a})$ is nilpotent in $\pi_0(A/p^2)$, so we learn that $p = 0$ in $\pi_0(A/p^2)$, and hence also in the subring $\pi_0(A)/p^2$. This means that $p = p^2x$ for some $x \in \pi_0(A)$. Applying δ , we obtain that $\delta(p) = \delta(p^2x)$ in $\pi_0(A/p)$. However $\delta(p^2x)$ is divisible by p and $\delta(p)$ is a unit, so we learn that $A/p = 0$, and hence $A = 0$. $\square$

Proof of Theorem C. The fact that $r_{p,1,4} = 0$ for $p > 2$ and $r_{2,1,4} = 1$ follows from Theorem D since $\mathbf{Z}_p[\zeta_p]$ and $\mathbf{Z}_2[\zeta_4]$ come from ramified extensions of $\mathbf{Q}_p$ and $\mathbf{Q}_2$ respectively.

We now show that if we fix p and h , then for $n \gg 0$, the number $r_{p,h,n}$ is 0 if $p > 2$ and is 1 if $p = 2$. If this were false, then there would be a sequence R_n of nonzero $T(h)$ -local $\mathbf{E}_n$ -algebras containing a p^{th} (resp. 4^{th} for $p = 2$) root of unity. The ultraproduct of these would then be a nonzero $T(h)$ -local $\mathbf{E}_\infty$ -algebra with such a root of unity, contradicting Theorem 1.1. $\square$

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FULD HALL, PRINCETON NJ 08540
 Email address: sdevalapurkar@gmail.com

FULD HALL, PRINCETON NJ 08540
 Email address: ishanl@ias.edu