Exceptional isomorphisms

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There are many, many, low-dimensional coincidences in group theory; they often admit numerous interpretations, and such interpretations often play a profound role as base cases of more general facts. In this talk, I would like to describe some such low-dimensional coincidences arising from the theory of complex Lie groups: in particular, I will only talk about the isomorphisms $\mathfrak{sl}_2 \cong \mathfrak{so}_3$, $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \cong \mathfrak{so}_4$, $\mathfrak{sl}_4 \cong \mathfrak{so}_6$, and $\mathfrak{sp}_4 \cong \mathfrak{so}_5$. One can, of course, use modern language to state and prove these results with "little effort", but I want to try to emphasize the classical way in which these ideas came about (using modern terminology, but minimal modern techniques). I'm also going to avoid talking about spinors, even though it can be used to prove a lot of these isomorphisms. I found Helgason's "Sophus Lie, the mathematician", and Hawkins' "Emergence of the Theory of Lie Groups" to be very enjoyable references.

1. Riemann sphere

One of the most famous "exceptional isomorphisms" is the identification of SL_2 with Sp_2, and also with a double cover of SO_3; or, a weaker statement is the isomorphisms $\mathfrak{sl}_2 \cong \mathfrak{so}_3 \cong \mathfrak{sp}_2$. Let us list a few proofs, all of which are "the same" in some sense:

- The corresponding Lie algebras are classified by their Dynkin diagrams, which just has a single vertex; so $\mathfrak{sl}_2 \cong \mathfrak{so}_3 \cong \mathfrak{sp}_2$.
- The Lie algebra $\mathfrak{sl}_2$ admits a Killing form $\mathrm{Tr}(AB)$, which is preserved by the adjoint action of SL_2. Since the Killing form is a quadratic form, we get a map $\mathrm{SL}_2 \to \mathrm{O}_3$, which lands in SO_3 because elements of SL_2 have determinant 1. The action of the center μ_2 of SL_2 is trivial, so we get a map $\mathrm{SL}_2/\mu_2 \to \mathrm{SO}_3$, which gives the desired isomorphism. Similarly, one can observe that $\mathfrak{sl}_2$ with its Killing form identifies with the imaginary (complex) quaternions with its norm; so one obtains a map $\mathrm{SL}_2 \to \mathrm{GL}_1(\mathbf{H} \otimes_{\mathbf{R}} \mathbf{C}) = \mathrm{Sp}_2$, which is an isomorphism.
- The group SL_2 is cut out inside $\mathbf{C}^4$ by the condition that the determinant is one. If we identify $\mathbf{C}^4$ with $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H}$, then the determinant of a matrix identifies with the norm on the (complexified) quaternions. Conjugation on quaternions defines a map $\mathbf{C}^4 \to \mathrm{End}_{\mathbf{C}}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H})$ sending $v \mapsto (w \mapsto vwv^{-1})$. Such a map is invertible if and only if $v \in \mathbf{C}^4$ has unit norm; so we get a map $\mathrm{SL}_2 \to \mathrm{GL}_1(\mathbf{H} \otimes_{\mathbf{R}} \mathbf{C}) \cong \mathrm{Sp}_2$.

  In order for the endomorphism associated to an element $v \in \mathbf{C}^4$ to preserve the norm on $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H}$, we need v to have unit norm. In this case, it also preserves the imaginary quaternions. Since the automorphisms of the imaginary quaternions are given by SO_3, we obtain a map $\mathrm{SL}_2 \to \mathrm{SO}_3$. Since negating a unit quaternion does not affect conjugation, this descends to the desired isomorphism $\mathrm{SL}_2/\mu_2 \to \mathrm{SO}_3$.

Way back, people did not think about groups of symmetries as above. Instead, it was more common to understand (what we now call) the associated homogeneous spaces. In the present case, this means that one observes that SL_2, SO_3, and Sp_2 all have the Borel subgroup B of SL_2 sitting inside, and therefore one gets identifications

$$\mathrm{SL}_2/B \cong \mathrm{SO}_3/B \cong \mathrm{Sp}_2/B.$$

The geometry behind these isomorphisms can be understood as follows.

- A point of $\mathrm{SL}_2/B \cong \mathrm{GL}_2/B$ is the data of a line in $\mathbf{C}^2$, i.e., identifies with $\mathbf{P}^1$.
- If V is a vector space, $T^*V = V \oplus V^*$ is the associated symplectic vector space with symplectic form ω, and P is the parabolic subgroup of $\mathrm{Sp}(T^*V)$ with Levi quotient $\mathrm{GL}(V)$, one can identify $\mathrm{Sp}(T^*V)/P$ with the space of Lagrangians in T^*V, i.e., maximal subspaces of T^*V on which ω vanishes. To see the identification with the above description, just note that if ℓ is a line in $\mathbf{C}^2$, it is a Lagrangian for the standard symplectic form on $\mathbf{C}^2 \cong T^*$.
- If (V, q) is a quadratic space and $v \in V$ has $q(v) = 1$, then $\mathrm{SO}(V)/\mathrm{SO}(v^\perp)$ identifies with the hyperboloid $\{w \in V | q(w) = 1\}$. Therefore, SO_3/B is a sphere. The identification between points of this sphere with lines in $\mathbf{C}^2$ is given by stereographic projection.

2. Four-space

There is an isomorphism $\mathrm{SO}_3 \times \mathrm{SO}_3 \cong \mathrm{SO}_4/\mu_2$, which again (geometrically) boils down to a bunch of other geometric identifications.

- The Dynkin diagrams of $\mathfrak{so}_3 \times \mathfrak{so}_3$ and $\mathfrak{so}_4$ can be identified with $A_1 \times A_1$ and D_2; both are simply two disjoint vertices, hence are isomorphic.
- Let us use the identification of the previous section to replace SO_3 by SL_2/μ_2. Consider the map $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_4$ sending $p, q \in \mathrm{SL}_2 \times \mathrm{SL}_2$ (viewed as unit quaternions) to the automorphism $v \mapsto pvq^{-1}$. Again, this automorphism preserves norms, so it lands in SO_4. Its kernel consists of μ_2 embedded diagonally into $\mathrm{SL}_2 \times \mathrm{SL}_2$. The resulting map $(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2 \to \mathrm{SO}_4$ is the desired isomorphism. From this, one gets an isomorphism $(\mathrm{SL}_2 \times \mathrm{SL}_2)/(\mu_2 \times \mu_2) \cong \mathrm{SO}_3 \times \mathrm{SO}_3 \to \mathrm{SO}_4/\mu_2$, as desired.

In terms of homogeneous spaces, this can be thought of as follows. The isomorphism $(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2 \xrightarrow{\sim} \mathrm{SO}_4$ sends the diagonal copy of $\mathrm{SL}_2/\mu_2 = \mathrm{SO}_3$ to the standard embedding of SO_3 into SO_4. Therefore, we get an isomorphism

$$((\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2)/(\mathrm{SL}_2/\mu_2) \xrightarrow{\sim} \mathrm{SO}_4/\mathrm{SO}_3.$$

Again, $\mathrm{SO}_4/\mathrm{SO}_3$ can be viewed as a quadric in $\mathbf{C}^4$. The left-hand side can be identified with $(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mathrm{SL}_2^{\mathrm{diag}}$, which is just SL_2. Therefore, this lets one identify SL_2 as a quadric in $\mathbf{C}^4$; this *is* the determinant locus.

3. The Klein correspondence and Lie's line-sphere transformation

3.1. Line geometry. One of my goals in this talk was to define Lie's line-sphere transformation, and explain some historical context for this construction. The story starts off with Plücker's "line geometry" (which gave rise to what we now call the Plücker embedding). Namely, let us consider the geometry of (complex) projective 3-space. Let us write $X = [x_0 : \cdots : x_3]$ and $Y = [y_0 : \cdots : y_3]$ as points of $\mathbf{P}^3$. Let $p_{ij} = x_i y_j - x_j y_i$, so that up to rescaling, the p_{ij} are the six homogeneous coordinates of the line joining X and Y. One has the famous identity

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0,$$

whose generalizations have driven a lot of modern mathematics. This equation defines a quadric in $\mathbf{P}^5$, which is called the *Klein quadric*. I will denote this quadric by Q_4. In other words, it defines an embedding

$$\{\ell \subseteq \mathbf{P}^3\} = \mathrm{Gr}_2(\mathbf{C}^4) \hookrightarrow \mathbf{P}^5 = \mathbf{P}(\wedge^2\mathbf{C}^4),\ \ell \mapsto \wedge^2\ell.$$

In other words, if we view $\mathbf{P}^5$ as $\mathbf{P}(\wedge^2\mathbf{C}^4)$, the Klein quadric is cut out by the locus of "decomposable" vectors $v \wedge w$.

This embedding fully describes the geometry of lines in $\mathbf{P}^3$, or equivalently, the geometry of the quadric in $\mathbf{P}^5$. Here are some results that Klein and Plücker showed:

- Two lines in $\mathbf{P}^3$ intersect if and only if the associated points on the Klein quadric Q_4 lie on the same line. Indeed, say $V_1, V_2 \subseteq \mathbf{C}^4$ intersect in a 1-dimensional subspace, and let e be a basis vector of this subspace. Extend e to a basis $\{e, x_1\}$ of V_1 and $\{e, x_2\}$ of V_2; then the corresponding line in $\mathbf{P}(\wedge^2\mathbf{C}^4) = \mathbf{P}^5$ connecting $\wedge^2 V_1$ and $\wedge^2 V_2$ is $\mathbf{P}(\mathbf{C}\{e \wedge x_1, e \wedge x_2\})$.[1]
- The set of lines in $\mathbf{P}^3$ going through a fixed point of $\mathbf{P}^3$ can be identified with a 2-plane in Q_4; planes in Q_4 which arise in this way are called *α-planes*. Indeed, a point $[x] \in \mathbf{P}^3$ lies on the line $\mathbf{P}(V) \subseteq \mathbf{P}^3$ if $x \in V$; so we can write $V = \mathbf{C}\{x, v\}$, and the associated point of Q_4 is $[x \wedge v]$. If we extend x to a basis $\{x, e_1, e_2, e_3\}$ of $\mathbf{C}^4$, then $[x \wedge v]$ lies in the plane $\mathbf{P}(\mathbf{C}\{x \wedge e_1, x \wedge e_2, x \wedge e_3\}) \cong \mathbf{P}^2$ in Q_4.
- The set of lines $\ell \in \mathbf{P}^3$ contained in a fixed hyperplane $H \subseteq \mathbf{P}^3$ can also be identified with a 2-plane in Q_4; planes in Q_4 arising in this way are called *β-planes*. This is essentially the projective dual to the above statement. Indeed, lines in $\mathbf{P}^3$ contained in H correspond to lines in $(\mathbf{P}^3)^\vee$ through the point $H^\vee$, hence (by the preceding point) identifies with a 2-plane in Q_4.

In fact, all planes in Q_4 are either α-planes or β-planes. To see this, take a plane H in $\mathrm{Gr}_2(\mathbf{C}^4)$, and let $\ell_1, \ell_2, \ell_3 \subseteq \mathbf{P}^3$ be three points in H which do not lie on the same line. Now, the three lines joining these points all lie in H, hence (by the first bullet above) the lines ℓ_1, ℓ_2, and ℓ_3 pairwise intersect. Now, either they all intersect in the same point, or they intersect in three different points. In the first case, H is an α-plane. In the second case, let's assume that the lines ℓ_1, ℓ_2, and ℓ_3 intersect in $[x], [y], [z] \in \mathbf{P}^3$. Then $\ell_1 = [x \wedge z], \ell_2 = [y \wedge z]$, and $\ell_3 = [x \wedge y]$, so $\ell_1, \ell_2, \ell_3 \in \mathbf{P}(\mathbf{C}\{x, y, z\})$.

[1] This line lies in the Klein quadric because $\lambda_1(e \wedge x_1) + \lambda_2(e \wedge x_2) = e \wedge (\lambda_1 x_1 + \lambda_2 x_2)$ is decomposable.

Finally, let us make a brief note about intersecting α- and β-planes (one could view this as using the geometry of lines in $\mathbf{P}^3$ to understand planes in Q_4). Two distinct α-planes intersect in a point (there is a unique line going through two points in $\mathbf{P}^3$!); two distinct β-planes also intersect in a point (by duality, or by the fact that two distinct hyperplanes in $\mathbf{P}^3$ intersect in a unique line!); and finally, an α- and β-plane intersect in either a point or a line. (Suppose the α-plane is represented by a point $[x] \in \mathbf{P}^3$, and the β-plane is represented by a plane $H \subseteq \mathbf{P}^3$. Either $[x] \in H$, in which case the intersection is cut out by a linear condition in the β-plane, hence is a line; or $[x] \notin H$, in which case there is no line contained in H which can contain x, so the intersection is empty.)

One can summarize these results as follows:

Theorem (Klein, Plücker)**.** *The space of planes in the Klein quadric $Q_4 \subseteq \mathbf{P}^5$ has two connected components, and each is isomorphic to $\mathbf{P}^3$.*

One can find a retelling of this story in Example 22.7 of Harris' "Algebraic geometry".

3.2. The exceptional isomorphism. The "Klein correspondence" can be understood using the exceptional isomorphism $\mathrm{SL}_4/\mu_2 \cong \mathrm{SO}_6$. Here are some explanations of this isomorphism:

- Again, the "simplest" explanation is via the Dynkin diagram: for SL_4, the diagram is A_3, while the Dynkin diagram for SO_6 is D_3; both look like •–•–•. The obvious identification between them gives the isomorphism of Lie algebras $\mathfrak{sl}_4 \cong \mathfrak{so}_6$.
- The action of SL_4 on $\mathbf{C}^4$ defines an action on $\wedge^2\mathbf{C}^4 \cong \mathbf{C}^6$, which defines a map $\mathrm{SL}_4 \to \mathrm{GL}_6$. There is a *symmetric* bilinear form on $\wedge^2\mathbf{C}^4$ given by $(v\wedge w, x\wedge y) = v\wedge w\wedge x\wedge y$, and since SL_4 acts trivially on $\wedge^4\mathbf{C}^4 = \mathbf{C}$, its action on $\wedge^2\mathbf{C}^4$ preserves this symmetric bilinear form. Therefore, we obtain a map $\mathrm{SL}_4 \to \mathrm{SO}_6$. This map kills the subgroup μ_2 (which acts trivially on $\wedge^2\mathbf{C}^4$), and the resulting map $\mathrm{SL}_4/\mu_2 \to \mathrm{SO}_6$ is an isomorphism.

The way this exceptional isomorphism is related to the Klein correspondence is as follows. The space $\mathrm{Gr}_2(\mathbf{C}^4)$ can be identified with SL_4/P, where P is the parabolic given by the partition $[2,2]$ (so its Levi quotient is $\mathrm{S}(\mathrm{GL}_2 \times \mathrm{GL}_2)$). Under the isomorphism $\mathrm{SL}_4/\mu_2 \cong \mathrm{SO}_6$, the quotient $P/\mu_2 \subseteq \mathrm{SO}_6$ can be identified with the parabolic subgroup whose Levi quotient is $\mathrm{SO}_4 \times \mathrm{SO}_2 = (\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2 \times \mathbf{G}_m$ (this uses the exceptional isomorphism from the previous section). One can now appeal to the well-known fact that the n-dimensional complex quadric $Q_n \subseteq \mathbf{P}^{n+1}$ can be identified with SO_{n+2}/P, where P is the parabolic subgroup whose Levi quotient is $\mathrm{SO}_n \times \mathrm{SO}_2$; this identifies SO_6/P with the Klein quadric in $\mathbf{P}^5$, as desired.

Said differently, the subgroup of PGL_6 acting on $\mathbf{P}^5$ preserving the Klein quadric is the projective orthogonal group. The connected component of the identity is PSO_6, which identifies with symmetries of $\mathrm{Gr}_2(\mathbf{C}^4)$, namely SL_4/μ_4.

3.3. Sphere geometry. One of the first things that made Lie really famous (around 1870) was what he called "sphere geometry". The basic idea was simple: Plücker and Klein described lines in $\mathbf{P}^3$, then known as "line geometry"; so, what about *spheres* (i.e., quadrics) in $\mathbf{P}^3$? There are some amusing anecdotes about this period in Lie's life. Here are two quotes.

> [Lie] was suspected of being a German spy and thrown into prison. The letters he had written encouraged the authorities in their suspicions, for when Lie wrote in German about "lines" and "spheres" they thought he was writing about "infantry" and "artillery". When Lie said it was mathematics and began to explain "let x, y and z be rectangular coordinates..", they decided he was insane! (See page 26 of Hawkins' "Emergence of the Theory of Lie Groups".)

Another one, in Klein's words:

> [O]ne morning I got up early and wanted to go out right away when Lie, who still lay in bed, called me into his room. He explained to me the relationship he had found during the night between the asymptotic curves of one surface and the lines of curvature of another, but in such a way that I could not understand a word.

Let's see if we can do any better. Consider a sphere given by the equation

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + d = 0,$$

where the radius of this sphere satisfies $r^2 = a^2 + b^2 + c^2 - d$. Therefore, a, b, c, d, and r define coordinates on the moduli of such spheres. We will only care about these variables up to scaling, so let us write

$$a = x_1/x_0,\ b = x_2/x_0,\ c = x_3/x_0,\ r = x_4/x_0,\ d = x_5/x_0.$$

Then, the equation relating a, b, c, d, and r becomes

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5 x_0 = 0. \tag{1}$$

Note that one can view *points* as being spheres of zero radius; this corresponds to setting $x_4 = 0$. Similarly, one can view *planes* as being spheres of infinite radius; this corresponds to setting $x_0 = 0$. In general, the moduli of $(n-1)$-spheres in $\mathbf{P}^n$ are parametrized by a quadric in $\mathbf{P}^{n+2}$. (As a dimension check, note that the quadric is $(n+1)$-dimensional; these dimensions parametrize the center and the radius of the sphere.)

Lie observed that if one defines

$$\begin{aligned} p_{12} &= x_1 + ix_2, \\ p_{13} &= x_3 + x_4, \\ p_{14} &= x_5, \\ p_{23} &= -x_0, \\ p_{24} &= x_4 - x_3 \\ p_{34} &= x_1 - ix_2, \end{aligned}$$

then (1) becomes

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0,$$

i.e., the Klein quadric! In other words, one can identify the space of quadrics/-spheres in $\mathbf{P}^3$ with the space of lines in $\mathbf{P}^3$. Under this identification, lines in $\mathbf{P}^3$ which intersect correspond to spheres which touch (i.e., which are tangential).

The identification is *not* true in real geometry, but only over the complex numbers. The reason for this is as follows: the action of the real group $\mathrm{SO}_{3,3}$ on $\mathbf{R}^6$ preserves the Klein quadric in $\mathbf{P}(\mathbf{R}^6) = \mathbf{R}P^5$, while the action of $\mathrm{SO}_{4,2}$ on $\mathbf{R}^6$ preserves the Lie quadric in $\mathbf{P}(\mathbf{R}^6) = \mathbf{R}P^5$. However, the groups $\mathrm{SO}_{3,3}$ and

$\mathrm{SO}_{4,2}$ are not isomorphic, and so *real* line and *real* sphere geometry are different. One can interpret Lie's transformation as describing the isomorphism beween the complexifications of $\mathrm{SO}_{3,3}$ and $\mathrm{SO}_{4,2}$ in terms of the associated moduli problems. The exceptional isomorphism $\mathrm{SL}_4/\mu_2 \cong \mathrm{SO}_6$ defines isomorphisms $\mathfrak{sl}_4(\mathbf{R}) \cong \mathfrak{so}_{3,3}$, and $\mathfrak{su}_{2,2} \cong \mathfrak{so}_{4,2}$.

4. Symplectic 2-space

4.1. Lines in quadrics. Let us end with describing the isomorphism $\mathrm{Sp}_4 \cong \mathrm{Spin}_5$.

- As usual, one can use Dynkin diagrams: one is asking for an isomorphism between B_2 and C_2, whose diagrams are both $\bullet\!\Rightarrow\!\bullet$. This gives an isomorphism of Lie algebras $\mathfrak{sp}_4 \cong \mathfrak{so}_5$.
- One can also use the isomorphism $\mathrm{SL}_4/\mu_2 \cong \mathrm{SO}_6$. Namely, fix a symplectic form $\omega_0 \in (\wedge^2\mathbf{C}^4)^*$. Then one obtains a 5-dimensional space $V = \{\omega \in (\wedge^2\mathbf{C}^4)^* | \omega \wedge \omega_0 = 0\}$, and so the action of $\mathrm{Sp}_4 \subseteq \mathrm{SL}_4$ on $\wedge^2\mathbf{C}^4$ restricts to an action on V. It is not hard to see that the symmetric bilinear form on $\wedge^2\mathbf{C}^4$ restricts to V, so we get a map $\mathrm{Sp}_4 \to \mathrm{SO}_5$, which becomes an isomorphism after killing μ_2 in the source.

Let us make one note regarding the relationship to the Klein correspondence $\mathrm{Gr}_2(\mathbf{C}^4) \cong Q_4$. Asking that a 2-plane in $\mathbf{C}^4$ be Lagrangian for ω_0 amounts to asking that the corresponding point of the Klein quadric Q_4 satisfy a degree 2 equation. Therefore, one obtains an isomorphism between the Lagrangian Grassmannian of $T^*\mathbf{C}^2$ and a quadric contained in Q_4, i.e., a quadric Q_3 in $\mathbf{P}^4$. Using the results of Klein and Plücker earlier, one finds, for instance, that:

Theorem (Klein, Plücker)**.** *The space of lines in the quadric $Q_3 \subseteq \mathbf{P}^4$ is isomorphic to $\mathbf{P}^3$ (by sending a point in $\mathbf{P}^3$ to the variety of Lagrangian subspaces containing this point).*

Just for the sake of completeness, let note that the above results of Klein and Plücker are natural continuations of the following simple fact (to prove it, note that the quadric Q_2 is the image of the Segre embedding $\mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$; so any line on $\mathbf{P}^1 \times \mathbf{P}^1$ must have bidegree $(1,0)$ or $(0,1)$).

Theorem. *The space of lines in the quadric $Q_2 \subseteq \mathbf{P}^3$ has two connected components, and each is isomorphic to $\mathbf{P}^1$.*

Just for fun, let us relate this to our preceding study of lines in $\mathbf{P}^3$. Namely, the subvariety of $\mathrm{Gr}_2(\mathbf{C}^4) \cong Q_4$ of lines contained in Q_2. Again using the isomorphism $Q_2 = \mathbf{P}^1 \times \mathbf{P}^1$, these lines have bidegree $(1,0)$ or $(0,1)$. The locus of lines of bidegree $(1,0)$ have a quadratic condition imposed on them, and so when viewed as points of the Klein quadric Q_4, they lie on a conic. In other words, the locus of $\mathrm{Gr}_2(\mathbf{C}^4)$ of lines in Q_2 corresponds to the (disjoint) union of two conics in Q_4.

In general, one can show that the space of n-planes in $Q_{2n} \subseteq \mathbf{P}^{2n+1}$ has two components, and both are isomorphic to the space of $(n-1)$-planes in $Q_{2n-1} \subseteq \mathbf{P}^{2n}$. This is related to the spinor representation, but I won't say more about this. (Maybe I'll add something here later.)

5. *Finite* geometries

There are several other interesting exceptional isomorphisms of *finite* groups. These (generally) correspond to interesting geometry over finite fields. I will not attempt to describe anything to do with orthogonal groups. Most of this is from Dieudonné's "Les Isomorphismes Exceptionnels Entre Les Groupes Classiques Finis".

We will look at $\mathrm{PSL}_n(\mathbf{F}_q)$ (note that $\mathrm{PGL}_n(F) \cong \mathrm{PSL}_n(F)$ precisely when every element of F is an nth power; in the case when $F = \mathbf{F}_q$, there are $\gcd(n, q-1)$ elements with no nth roots). This acts on $\mathbf{P}^{n-1}(\mathbf{F}_q)$, and the exceeptional isomorphisms involving $\mathrm{PSL}_n(\mathbf{F}_q)$ arise essentially from coincidences involving $\#\mathbf{P}^{n-1}(\mathbf{F}_q)$, and the fact that the order of symmetric groups on small letters is manageable. Here are some simple examples:

- $\mathrm{PSL}_2(\mathbf{F}_2)$ acts on $\mathbf{P}^1(\mathbf{F}_2)$, which has three elements. This gives an isomorphism $\mathrm{PSL}_2(\mathbf{F}_2) = \mathrm{SL}_2(\mathbf{F}_2) \cong \Sigma_3$.
- $\mathrm{PGL}_2(\mathbf{F}_3)$ acts on $\mathbf{P}^1(\mathbf{F}_3)$, which has four elements. This gives a map $\mathrm{PGL}_2(\mathbf{F}_3) \subseteq \Sigma_4$, which is an isomorphism. The determinant of the associated matrix (viewed as an element of $\mathbf{F}_3^\times$) corresponds to the sign of the permutation, hence $\mathrm{PSL}_2(\mathbf{F}_3) \cong A_4$.
- $\mathrm{PGL}_2(\mathbf{F}_5)$ acts on $\mathbf{P}^1(\mathbf{F}_5)$, which has six elements. This gives a map $\mathrm{PGL}_2(\mathbf{F}_5) \subseteq \Sigma_6$, but it is not an isomorphism. Instead, there is an isomorphism $\mathrm{PGL}_2(\mathbf{F}_5) \cong \Sigma_5$. One way to see this, I think, is as follows. The number of Sylow 5-subgroups of Σ_5 is $|\Sigma_5/N(\mathbf{Z}/5)| = 5!/(5\times 4) = 6$; in fact, one gets an isomorphism $\Sigma_5/N(\mathbf{Z}/5) \cong \mathbf{P}^1(\mathbf{F}_5)$. One can therefore label points of $\mathbf{P}^1(\mathbf{F}_5)$ by the Sylow 5-subgroups of Σ_5: namely,

$$\infty = \langle(12345)\rangle,\ 0 = \langle(12354)\rangle,\ 1 = \langle(12453)\rangle,\ 2 = \langle(12543)\rangle,\ 3 = \langle(12534)\rangle,\ 4 = \langle(12435)\rangle.$$

  These were chosen so that conjugation by (12345) acted by $x \mapsto x+1$ on $\mathbf{P}^1(\mathbf{F}_5)$, i.e., by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbf{F}_5)$. In any case, the action of Σ_5 on $\mathbf{P}^1(\mathbf{F}_5)$ defines a map $\Sigma_5 \to \mathrm{PGL}_2(\mathbf{F}_5)$, which one can verify is an isomorphism. This restricts to an isomorphism

$$A_5 \subseteq \Sigma_5 \xrightarrow{\sim} \mathrm{PGL}_2(\mathbf{F}_5) \twoheadrightarrow \mathrm{PSL}_2(\mathbf{F}_5).$$

  Note that the above embedding $\Sigma_5 \cong \mathrm{PGL}_2(\mathbf{F}_5) \subseteq \Sigma_6$ is the "exotic" embedding.
- There is an isomorphsm $\mathrm{PGL}_2(\mathbf{F}_7) \cong \mathrm{GL}_3(\mathbf{F}_2)$ (you might know that $\mathrm{PSL}_2(\mathbf{F}_7) \cong \mathrm{PGL}_3(\mathbf{F}_2)$ is the simple group of order 168). This is not entirely unreasonable, because both $\mathbf{F}_8$ and $\mathbf{P}^1(\mathbf{F}_7)$ have eight elements. This can be explained using the next bullet point as follows. Recall that $\mathrm{PSL}_2(\mathbf{F}_7)$ acts on $\mathbf{P}^1(\mathbf{F}_7)$, which has eight elements; so it is a subgroup of A_8. The image of this subgroup under the isomorphism $A_8 \cong \mathrm{PSL}_2(\mathbf{F}_4)$ then identifies with $\mathrm{PSL}_2(\mathbf{F}_3)$.
- There is an isomorphism $\mathrm{PSL}_4(\mathbf{F}_2) \cong A_8$. It turns out that this can be understood using the work of Klein and Plücker discussed above! Namely, the preceding discussion gives an action of $\mathrm{PSL}_4(\mathbf{F}_2)$ on $\mathrm{Gr}_2(\mathbf{F}_2^4) \cong Q_4 \subseteq \mathbf{P}^5(\mathbf{F}_2)$. We will now construct a natural action of A_8 on Q_4. finish
- There is also an isomorphism $\mathrm{PSL}_2(\mathbf{F}_9) \cong A_6$, but I won't prove it here.

References

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