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The main result we will discuss in this talk appears as Theorem 8.1 of the Feng-Galatius-Venkatesh paper. We begin by stating a special case of this theorem. Let X be a smooth projective variety over $\mathbf{Q}$ of dimension $2k-1$, and let $A \to X$ be a principally polarized abelian variety over X of relative dimension g; this can be understood as a morphism $X \to \mathcal{A}_g$. Then A defines a class $[X] \in \mathrm{H}_{4k-2}(\mathcal{A}_g; \mathbf{Z}_p)$, so pairing $[X]$ with the $(2k-1)$st Chern character class of the Hodge bundle over $\mathcal{A}_g$ defines a number $\mathrm{ch}_{2k-1}([X]) \in \mathbf{Q}$.

Theorem 1. *Suppose $p > 2k$ is a prime. If p divides the numerator of $\zeta(1-2k)$, then p divides the numerator of* $\mathrm{ch}_{2k-1}([X])$.

Roughly, this can be proved as follows. Taking the $(2k-1)$st Chern character class of the Hodge bundle over $\mathcal{A}_g$ defines a ($\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-equivariant) map $\mathrm{ch}_{2k-1} : \mathrm{H}_{4k-2}(\mathcal{A}_g; \mathbf{Z}_p) \to \mathbf{Q}_p(2k-1)$[1]. If $p > 2k$, then the denominator of $\mathrm{ch}_{2k-1}([X])$ is invertible in $\mathbf{Z}_p$, so we may regard $\mathrm{ch}_{2k-1}([X]) \in \mathbf{Q} \cap \mathbf{Z}_p = \mathbf{Z}_{(p)}$. If p does not divide (the numerator of) $\mathrm{ch}_{2k-1}([X])$, then the class $[X] \in \mathrm{H}_{4k-2}(\mathcal{A}_g; \mathbf{Z}_p)$ defines a splitting of ch_{2k-1}. In particular, $[X]$ defines a splitting of the extension

$$\ker(\mathrm{ch}_{2k-1}) \to \mathrm{H}_{4k-2}(\mathcal{A}_g; \mathbf{Z}_p) \xrightarrow{\mathrm{ch}_{2k-1}} \mathbf{Z}_p(2k-1). \tag{1}$$

An analogue of this argument *almost* works with $\mathrm{H}_{4k-2}(\mathcal{A}_g; \mathbf{Z}_p)$ replaced by $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p)$. Namely, we would like to say that if p does not divide the numerator of $c_{\mathrm{H}}([X])$, then the extension

$$\ker(c_{\mathrm{H}}) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p) \xrightarrow{c_{\mathrm{H}}} \mathbf{Z}_p(2k-1) \tag{2}$$

admits a splitting. It turns out that this is true since $p > 2k$. To conclude the theorem, we now apply the main result of the paper (discussed in the previous two talks): the sequence (2) does not split unless $\ker(c_{\mathrm{H}}) = 0$. In previous talks, we have identified $\ker(c_{\mathrm{H}})$ with $\mathrm{H}^2_{\mathrm{et}}(\mathrm{Spec}\,\mathbf{Z}[1/p]; \mathbf{Z}_p(2k))$, so we need this group to vanish if (2) is to split. However, it is a number-theoretic fact (which we will not discuss here) that this group is nonzero iff p divides[2] the numerator of $\zeta(1-2k)$, thereby proving Theorem 1.

Observe that, given the number-theoretic fact about $\mathrm{H}^2_{\mathrm{et}}(\mathrm{Spec}\,\mathbf{Z}[1/p]; \mathbf{Z}_p(2k))$, the key nontrivial step in the above argument is to show that the sequence (2) splits. Since we already know that the sequence (1) splits if p does not divide $\mathrm{ch}_{2k-1}([X])$, it would suffice to show that the splitting of (1) implies the splitting of (2) if $p > 2k$. This implication is in fact true, and is a special case of a general homotopy-theoretic claim which we will discuss momentarily.

Let us now begin the talk in earnest: we will first state the general version of Theorem 1 and the argument above (this generalization is essentially combinatorial), and then discuss the homotopy-theoretic claim alluded to above which will feature in the proof. Therefore, let $A \to X$ be as above, and $f : X \to \mathcal{A}_g$ the classifying map. The pullback of the Hodge bundle over $\mathcal{A}_g$ along f is the vector bundle $\omega_X := \mathrm{Lie}(A)^*$. Let $n = \dim(X)$, and let $\underline{n} = (n_1, \cdots, n_r)$ be a partition of n with each n_i odd. Define

$$s_{\underline{n}}(A/X) = \langle [X], \mathrm{ch}_{n_1}(\omega_X) \cdots \mathrm{ch}_{n_r}(\omega_X) \rangle \in \mathbf{Q}.$$

Then, Theorem 1 generalizes to:

Theorem 2. *Suppose $p \geq \max_j n_j$ is a prime such that $p | B_{n_i+1}$ for some i. Then p divides the numerator of $s_{\underline{n}}(A/X)$.*

The proof of Theorem 2 will rely on a result relating the homotopy of KSp with the homology of $\Omega^\infty \mathrm{KSp} = \mathrm{BSp}$. Let us state this result, and then describe how it implies Theorem 2.

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[1] Recall that the maps ch_{2k-1} stabilize in g, and composite $c_{\mathrm{H}} : \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p) \to \mathrm{H}_{4k-2}(\mathcal{A}_g; \mathbf{Z}_p) \xrightarrow{\mathrm{ch}_{2k-1}} \mathbf{Q}_p$ is always valued in $\mathbf{Z}_p$.

[2] This is equivalent to saying that p divides the numerator of the Bernoulli number B_{2k}, since $\zeta(1-2k) = -\frac{B_{2k}}{2k}$, and our assumption that $p > 2k$.

Definition 3. Let X be a (connected) H-space of finite type. Then the *integral decomposables* in $\mathrm{H}_*(X;\mathbf{Z}_p)$ is the ideal defined as the $\mathbf{Z}_p$-span of all monomials of the form $x\cdot y$ and $\beta_k(a\cdot b)$ for $x,y\in\mathrm{H}_*(X;\mathbf{Z}_p)$, $a,b\in\mathrm{H}_*(X;\mathbf{Z}/p^k)$ in positive degrees, and $\beta_k:\mathrm{H}_*(X;\mathbf{Z}/p^k)\to\mathrm{H}_{*-1}(X;\mathbf{Z}_p)$ is the Bockstein. Equivalently, if $I=\mathrm{H}_{*>0}(X;\mathbf{Z}_p)$ and $I_k=\mathrm{H}_{*>0}(X;\mathbf{Z}/p^k)$, then the integral indecomposables is given by the ideal $I^2+\sum_k\beta_k(I_k^2)$. Let $\mathrm{H}_*(X;\mathbf{Z}_p)_{\mathrm{ind}}$ denote the quotient of $\mathrm{H}_*(X;\mathbf{Z}_p)$ by the integral decomposables.

Theorem 4. *Let E be a p-complete connected spectrum of finite type (i.e., $\pi_iE=0$ for $i\leq 0$), and let $X=\Omega^\infty E$. Then the map*

$$\pi_i(E)\cong\pi_i(X)\xrightarrow{\text{Hurewicz}}\mathrm{H}_i(X;\mathbf{Z}_p)\to\mathrm{H}_i(X;\mathbf{Z}_p)_{\mathrm{ind}}$$

is an isomorphism for $i\leq 2p-2$.

To a seasoned topologist, the appearance of the number $2p-2$ is quite suggestive (for instance, the first Steenrod operation in mod p cohomology raises the cohomological degree by precisely $2p-2$). We will return to Theorem 4 later; let us first discuss how it implies Theorem 2.

Proof of Theorem 2. Let ω be the Hodge bundle on $\mathcal{A}_g$. Since $p\geq n_j$, the denominators of each $\mathrm{ch}_{n_j}(\omega)\in\mathrm{H}^{2n_j}(\mathcal{A}_g;\mathbf{Q}_p(n_j))$ are invertible in $\mathbf{Z}_p$, so $\mathrm{ch}_{n_j}(\omega)$ lifts to a class in $\mathrm{H}^{2n_j}(\mathcal{A}_g;\mathbf{Z}_p(n_j))$. Next, the class $[X]\in\mathrm{H}_{2n}(\mathcal{A}_g;\mathbf{Z}_p)$ defines a map $\mathbf{Z}_p(n)\to\mathrm{H}_{2n}(\mathcal{A}_g;\mathbf{Z}_p)$. Pairing with $\prod_{j\neq i}\mathrm{ch}_{n_j}(\omega)\in\mathrm{H}^{\sum_{j\neq i}2n_j}(\mathcal{A}_g;\mathbf{Z}_p)$ defines a map $\alpha_i:\mathbf{Z}_p(n_i)\to\mathrm{H}_{2n_i}(\mathcal{A}_g;\mathbf{Z}_p)$. Observe that pairing this map with $\mathrm{ch}_{n_i}(\omega)$ gives $s_{\underline{n}}(A/X)\in\mathbf{Z}_p$; so if p does not divide $s_{\underline{n}}(A/X)$, then the map α_i gives a Galois-equivariant splitting of $\mathrm{ch}_{n_i}(\omega):\mathrm{H}_{2n_i}(\mathcal{A}_g;\mathbf{Z}_p)\to\mathbf{Z}_p(n_i)$. We will show that this implies (2) splits (with $4k-2$ replaced by $2n_i$).

Using Theorem 4 and the assumption that $p\geq\max_j n_j$, we obtain a Galois-equivariant map $\mathrm{H}_{2n_i}(\mathcal{A}_g;\mathbf{Z}_p)\to\mathrm{KSp}_{2n_i}(\mathbf{Z};\mathbf{Z}_p)$ via the composite

$$\mathrm{H}_{2n_i}(\mathcal{A}_g;\mathbf{Z}_p)\to\mathrm{H}_{2n_i}(\mathrm{BSp};\mathbf{Z}_p)\to\mathrm{H}_{2n_i}(\mathrm{BSp};\mathbf{Z}_p)_{\mathrm{ind}}\xleftarrow{\cong}\mathrm{KSp}_{2n_i}(\mathbf{Z};\mathbf{Z}_p).$$

This map has the property that it makes the following diagram commute:

$$\begin{array}{ccc}\mathrm{H}_{2n_i}(\mathcal{A}_g;\mathbf{Z}_p) & & \\ \downarrow & \searrow^{\mathrm{ch}_{n_i}(\omega)} & \\ \mathrm{KSp}_{2n_i}(\mathbf{Z};\mathbf{Z}_p) & \xrightarrow[c_{\mathrm{H}}]{} & \mathbf{Z}_p(n_i).\end{array}$$

Assume for contradiction that p does not divide $s_{\underline{n}}(A/X)$; then the above discussion implies that the diagonal map admits a splitting. Therefore, c_{H} also admits a splitting. We get a contradiction exactly as before: the map c_{H} cannot split unless $\mathrm{H}^2_{\mathrm{et}}(\mathbf{Z}[1/p];\mathbf{Z}_p(n_i))=0$, but it is known (to number theorists) that this forces $p\nmid B_{n_i+1}$. ☐

Let us now turn to Theorem 4.

Example 5. To illustrate the claim, let us consider the case $E=\Sigma^n\mathrm{H}\mathbf{Z}_p$ for $n\geq 1$, i.e., $X=K(\mathbf{Z}_p,n)$. In this case, $\pi_iE=0$ for $i\neq n$, and $\pi_nE=\mathbf{Z}_p$. We therefore need to show that the same is true of $\mathrm{H}_i(K(\mathbf{Z}_p,n);\mathbf{Z}_p)$, at least when $i\leq 2p-2$. There is a canonical class in $\mathrm{H}_n(K(\mathbf{Z}_p,n);\mathbf{Z}_p)$ coming from the Hurewicz isomorphism $\pi_nK(\mathbf{Z}_p,n)\cong\mathbf{Z}_p\xrightarrow{\cong}\mathrm{H}_n(K(\mathbf{Z}_p,n);\mathbf{Z}_p)$. If $\mathbf{Z}_p[x_n]$ denotes the free commutative differential graded $\mathbf{Z}_p$-algebra on a generator in degree n, then the canonical class defines a map $\mathbf{Z}_p[x_n]\to C_*(K(\mathbf{Z}_p,n);\mathbf{Z}_p)$ of commutative differential graded $\mathbf{Z}_p$-algebras. This map is an isomorphism in dimensions $\leq 2p-1$ (so $\mathrm{H}_*(K(\mathbf{Z}_p,n);\mathbf{Z}_p)_{\mathrm{ind}}$ is generated by x_n in that range, and is therefore isomorphic to π_*E). We will not prove this here, but we can illustrate it in two examples.

(a) Suppose $n=1$, so $X=K(\mathbf{Z}_p,1)$ is a p-completed version of the circle S^1. Then $\mathrm{H}_*(X;\mathbf{Z}_p)=\mathbf{Z}_p[x_1]/x_1^2$. By graded commutativity, the class x_1 in $\mathbf{Z}_p[x_1]$ squares to zero, so $\mathbf{Z}_p[x_1]\cong\mathbf{Z}_p[x_1]/x_1^2$.

(b) Suppose $n=2$, so $X=K(\mathbf{Z}_p,2)$ is a p-completed version of $\mathbf{C}P^\infty$. Then $\mathrm{H}^*(X;\mathbf{Z}_p)\cong\mathbf{Z}_p[\beta]$, and $\mathrm{H}_*(X;\mathbf{Z}_p)$ is a divided power algebra $\Gamma_{\mathbf{Z}_p}(x_2)$. The map $\mathbf{Z}_p[x_2]\to\Gamma_{\mathbf{Z}_p}(x_2)$ is the inclusion;

the first degree where it fails to be an isomorphism is $2p$. Indeed, the divided powers $(x_2)^i/i!$ exist in $\mathbf{Z}_p[x_2]$ for $i \leq p-1$ since $i! \in (\mathbf{Z}_p)^\times$, but $(x_2)^p/p! \in \mathrm{H}_{2p}(X; \mathbf{Z}_p)$ does not exist in $\mathbf{Z}_p[x_2]$.

The general case is obtained inductively from these examples by applying the Serre spectral sequence to the fiber sequence

$$K(\mathbf{Z}_p, n) \to * \to K(\mathbf{Z}_p, n+1).$$

Example 6. For a similar example, let us consider the case $E = \Sigma^n \mathrm{H}\mathbf{Z}/p^k$ for $n, k \geq 1$, i.e., $X = K(\mathbf{Z}/p^k, n)$. In this case, $\pi_i E = 0$ for $i \neq n$, and $\pi_n E = \mathbf{Z}/p^k$. We therefore need to show that the same is true of $\mathrm{H}_i(K(\mathbf{Z}/p^k, n); \mathbf{Z}_p)$, at least when $i \leq 2p-2$. Let $A = \mathbf{Z}_p[x_n, y_{n+1} | dy = p^k x]$ denotes the commutative differential graded $\mathbf{Z}_p$-algebra on two generators equipped with the indicated differential. Then there is a map $A \to C_*(K(\mathbf{Z}/p^k, n); \mathbf{Z}_p)$ of commutative differential graded $\mathbf{Z}_p$-algebras (for instance, the image of x_n can be described as follows: by Hurewicz, we know that $\pi_n K(\mathbf{Z}/p^k, n) \cong \mathbf{Z}/p^k \xrightarrow{\cong} \mathrm{H}_n(K(\mathbf{Z}/p^k, n); \mathbf{Z}_p)$, and $x_n \in A$ is sent to a generator). As in Example 5, the map $A \to C_*(K(\mathbf{Z}/p^k, n); \mathbf{Z}_p)$ defines an isomorphism through dimension $\leq 2p-2$ (and $\mathrm{H}_*(K(\mathbf{Z}/p^k, n); \mathbf{Z}_p)_{\mathrm{ind}}$ is generated by $\mathbf{Z}_p \cdot \{x_n\}/p^k$ in that range, and is therefore isomorphic to $\pi_* E = \mathbf{Z}/p^k$). Again, we will just illustrate this in an example:

(a) Suppose $X = \mathbf{R}P^\infty = B\mathbf{Z}/2$, so that $\mathrm{H}_*(X; \mathbf{Z}/2) \cong \Gamma_{\mathbf{F}_2}(w)$ with $|w| = 1$ (one could also run this example with $B\mathbf{Z}/p$ for odd p, in which case $\mathrm{H}_*(X; \mathbf{Z}/p)$ is $\Gamma_{\mathbf{F}_p}(t) \otimes \mathbf{F}_p[w]/w^2$ with $|w| = 1$ and $|t| = 2$). Additively, $\mathrm{H}_*(X; \mathbf{Z}_2)$ is $\mathbf{Z}_2$ in degree zero, and is a copy of $\mathbf{Z}/2$ in each odd degree; moreover, the Bockstein $\mathrm{H}_*(X; \mathbf{Z}/2) \to \mathrm{H}_*(X; \mathbf{Z}_2)$ is surjective in positive degrees. The augmentation ideal I of $\mathrm{H}_*(X; \mathbf{Z}_2)$ is concentrated in odd degrees, so $I^2 = 0$ by the sign rule. Now, $\Gamma_{\mathbf{F}_2}(w) = \mathbf{F}_2[w, \gamma_2(w), \cdots]/(w^2, \gamma_2(w)^2, \cdots)$, where $\gamma_{2^i}(w)$ lives in degree 2^i. Therefore, if I_1 is the augmentation ideal of $\mathrm{H}_*(X; \mathbf{Z}/2)$, then I_1^2 is zero in degrees of the form 2^i, and is a 1-dimensional $\mathbf{F}_2$-vector space in other dimensions. Therefore, the integral indecomposables

$$I/(I^2 + \beta_1(I_1^2)) \cong \mathrm{H}_{*>0}(X; \mathbf{Z}_2)/\beta_1(I_1^2)$$

are concentrated exactly in dimensions $2^i - 1$, where it has a copy of $\mathbf{F}_2$. In particular, below dimension $2 \times 2 - 1 = 3$, this is just a copy of $\mathbf{F}_2 \cong \pi_*(\Sigma \mathrm{H}\mathbf{F}_2)$ in dimension 1.

Proof of Theorem 4. In fact, Theorem 4 will follow from the calculation in Example 6 and Example 5, and the following two claims:

(a) The space $\tau_{\leq 2p-2} X$ is homotopy equivalent (as a loop space) to a product of Eilenberg-Maclane spaces.

(b) If Y and Z are H-spaces of finite type, then $\mathrm{H}_*(Y; \mathbf{Z}_p)_{\mathrm{ind}} \oplus \mathrm{H}_*(Z; \mathbf{Z}_p)_{\mathrm{ind}} \xrightarrow{\cong} \mathrm{H}_*(Y \times Z; \mathbf{Z}_p)_{\mathrm{ind}}$.

Let us first prove (a). For this, recall that if Y is any space, then the Postnikov truncation $\tau_{\leq n} Y$ sits in a fiber sequence

$$\tau_{\leq n} Y \to \tau_{\leq n-1} Y \to K(\pi_n(Y), n+1);$$

the last map is known as a k-invariant. Therefore, $\tau_{\leq n} Y$ is built in finitely many steps from an Eilenberg-Maclane space, by iteratively taking fibers of maps to Eilenberg-Maclane spaces. Let $BX = \Omega^\infty \Sigma E$ denote the delooping of X. In order to show that $\tau_{\leq 2p-2} X$ is homotopy equivalent as a loop space to a product of Eilenberg-Maclane spaces, it suffices to show that $\tau_{\leq 2p-1} BX$ is homotopy equivalent (as an ordinary space) to a product of Eilenberg-Maclane spaces (since X is *connected*). By the above discussion, it suffices to show that all the k-invariants of $\tau_{\leq 2p-1} BX$ are nullhomotopic. Because E was assumed *connected*, we know that $\pi_i BX$ can be nonzero only for $i \geq 2$. Therefore, the k-invariants of $\tau_{\leq 2p-1} BX$ are all of the form $K(A, d) \to K(B, d+i)$ with $i \geq 1$, $2 \leq d, d+i \leq 2p-1$, and A, B are direct sums of groups of the form $\mathbf{Z}_p, \mathbf{Z}/p^k$ (by the finite type assumption on E). However, the first possible k-invariant which is not nullhomotopic in the p-complete setting is the Steenrod operation $P^1 : K(\mathbf{Z}/p, 2) \to K(\mathbf{Z}/p, 2p)$. Since $d, d+i \leq 2p-1$, we conclude that all the k-invariants of $\tau_{\leq 2p-1} BX$ are zero.

We now prove (b). The basepoints of Y and Z give maps $Y, Z \to Y \times Z$, which project onto Y and Z (respectively). Since the projections $Y \times Z \to Y, Z$ are maps of H-spaces, there is an induced map $\mathrm{H}_*(Y \times Z; \mathbf{Z}_p)_{\mathrm{ind}} \to \mathrm{H}_*(Y; \mathbf{Z}_p)_{\mathrm{ind}} \oplus \mathrm{H}_*(Z; \mathbf{Z}_p)_{\mathrm{ind}}$, and the preceding discussion implies that it admits a splitting. Therefore, $\mathrm{H}_*(Y; \mathbf{Z}_p)_{\mathrm{ind}} \oplus \mathrm{H}_*(Z; \mathbf{Z}_p)_{\mathrm{ind}} \hookrightarrow \mathrm{H}_*(Y \times Z; \mathbf{Z}_p)_{\mathrm{ind}}$ is injective. It remains to prove that it is surjective. We will in fact prove a stronger claim: the map $\mathrm{H}_*(Y; \mathbf{Z}_p) \otimes \mathrm{H}_*(Z; \mathbf{Z}_p) \to$

$\mathrm{H}_*(Y \times Z; \mathbf{Z}_p)$ is surjective upon quotienting by $\sum_k \beta_k(I_k^2)$. (Note that the integral indecomposables are obtained by a further quotient, and that the quotient of a surjective map remains surjective.) Recall that the Künneth formula tells us that there is a split exact sequence

$$0 \to \mathrm{H}_*(Y; \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathrm{H}_*(Z; \mathbf{Z}_p) \to \mathrm{H}_*(Y \times Z; \mathbf{Z}_p) \to \mathrm{Tor}^{\mathbf{Z}_p}(\mathrm{H}_*(Y; \mathbf{Z}_p), \mathrm{H}_*(Z; \mathbf{Z}_p)) \to 0.$$

If $\mathbf{Z}/p^k$ is a summand in $\mathrm{H}_*(Y; \mathbf{Z}_p)$ and $\mathbf{Z}/p^l$ is a summand in $\mathrm{H}_*(Z; \mathbf{Z}_p)$, then the Tor term contributes $\mathbf{Z}/p^d$ to $\mathrm{H}_*(Y \times Z; \mathbf{Z}_p)$, where $d = \min(k, l)$. To prove the desired claim, it suffices to observe that if $\beta_d(x)$ and $\beta_d(y)$ are generators for the p^d-torsion in these summands of $\mathrm{H}_*(Y; \mathbf{Z}_p)$ and $\mathrm{H}_*(Z; \mathbf{Z}_p)$ (respectively), then $\beta_d(xy)$ generates the aforementioned $\mathbf{Z}/p^d$-summand in $\mathrm{H}_*(Y \times Z; \mathbf{Z}_p)$. □

Email address: sdevalapurkar@math.harvard.edu