

# THE DYNAMICS AND STATISTICAL PHYSICS OF QUANTUM SYSTEMS WITH FRACTIONAL STATISTICS

SANATH DEVALAPURKAR

ABSTRACT. We discuss the theory of particles with fractional statistics, termed *anyons* by Wilczek. We show that anyons can be viewed as charged point particles with an infinitesimally thin flux tube, and discuss how this leads to an interpretation as bosons with a non-local interaction. The anyon system is then solved for some special potentials, and these calculations are used to write down an approximate equation of state for an ideal gas of anyons.

## 1. INTRODUCTION

In three-dimensional quantum mechanics, classical particles fit into the dichotomy of fermions and bosons, the difference being physically realized by the Pauli exclusion principle, and mathematically forced by the topological non-triviality of the real plane with the origin removed (as we discuss in §2.1). This non-triviality allows for greater flexibility in the sorts of statistics that particles in two dimensions can exhibit. In fact, the spin of a particle in two dimensions is not restricted to being an integer multiple of  $1/2$  — *any* real number is legal in this paradigm. Particles with such “fractional statistics” are (appropriately) known as anyons; special cases of anyons include bosons and fermions. The theory of anyons was originally introduced as a theoretical curiosity by Leinaas and Myrheim in [LM77], and was popularized by Wilczek in [Wil82a, Wil82b] (where the term “anyon” was also coined).

Using the classical Aharonov-Bohm effect, Wilczek showed in those papers that anyons may be realized as particles with a charge and an infinitesimally thin flux tube; we discuss this perspective in §2.1. He also showed that a system of anyons may be viewed as a system of interacting bosons or fermions, where the interaction is dictated by a *non-local* gauge field (in the sense that the gauge field at a point depends on the gauge field at all other points in space). This point of view is developed in §2.2, where we explicitly tackle the Lagrangian and Hamiltonian of a system of anyons (following [Ler92]), and discuss physical consequences. The interacting boson/fermion point of view is utilized heavily in §2.3, where the energy spectrum for a system of two anyons is calculated in a harmonic oscillator potential; these calculations follow the exposition in [Kha05]. Our eventual goal with these calculations is to study statistical properties of an ideal gas of anyons. Generalizing these results to systems of more anyons is an open problem, its hardness being owed to the non-locality of the gauge field through which the fermions/bosons interact.

We conclude this paper by applying our analysis of the dynamics of a 2-anyon system to computing the approximate equation of state of an ideal gas of  $N$  anyons. This theory was initially studied by [ASWZ85]. The challenge that sets this situation apart from the theory of ideal gases of fermions and bosons in two dimensions again stems from the non-locality of the interaction gauge field: as will be explicitly proved in §2.3, unlike in the classical setting, the energy spectrum of  $N$  anyons is not simply the sum of the energy spectrum of a single anyon. To calculate the equation of state, we review the notion of virial coefficients, before presenting a calculation of the second virial coefficient of an ideal gas of  $N$  anyons. This calculation vaguely

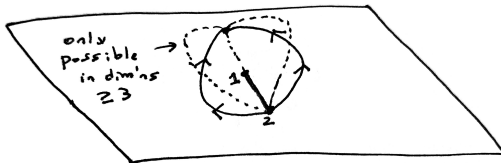


FIGURE 1. Existence of fractional statistics in two dimensions. The bold line connecting particle 1 and particle 2 shows the original orientation of the two particles relative to each other. The dotted lines that lift up above the plane can only exist in three dimensions and higher. Since particle 2 cannot pass through particle 1 in the plane, there is no way to identify the counterclockwise and clockwise paths in two dimensions.

follows the presentation in [Kha05], but is presented in what we believe to be a more elementary manner. Finally, we provide a brief qualitative discussion of how the physical consequences derived from these calculations suggest that anyons may possess superfluidity.

Before proceeding to the meat of this paper, we fix a few conventions. As mentioned above, the theory of anyons is only sensible in two dimensions, so we will restrict to working in a  $(2 + 1)$ -dimensional quantum theory unless otherwise mentioned; indices like  $\mu$  and  $\nu$  therefore run from 0 to 2, and  $i$  and  $j$  run from 1 to 2. Throughout, both  $c$  and  $\hbar$  will be set to 1 for convenience.

## 2. THE DYNAMICS OF ANYONS

**2.1. Modeling anyons via the Aharonov-Bohm effect.** It is a general fact of quantum mechanics that two physically equivalent states can differ at worst by a phase factor: if  $x_i$  denotes the position of particle  $i$ , then moving particle 2 around particle 1 by an angle of  $\phi$  changes the state by  $|x_1, x_2\rangle \rightarrow e^{i\nu\phi} |x_2, x_1\rangle$  for some parameter  $\nu$ . In dimensions at least 3, this condition forces  $|x_1, x_2\rangle = \pm |x_2, x_1\rangle$ : consider fixing particle 1, and interchanging the two particles by either moving particle 2 counterclockwise (by an angle of  $\pi$ ) or clockwise (by an angle of  $-\pi$ ) around particle 1, both in the same plane (see Figure 2.1 for an illustration). In dimensions three and above, there is no difference between these two procedures, since we can always lift the counterclockwise path from the plane into the third dimension and deform<sup>1</sup> it into the clockwise path. This implies that  $e^{i\nu\pi} = e^{-i\nu\pi}$ , which forces  $\nu \equiv 0, 1 \pmod{2}$  (when viewed as an element of the group  $\mathbf{R}/2\mathbf{Z}$ ). In particular,  $|x_1, x_2\rangle = \pm |x_2, x_1\rangle$ , as expected. In two dimensions, however, there is no way to construct a deformation between these paths, since the particles are distinct and cannot pass through each other. The parameter  $\nu$  can be any arbitrary number; the particles obeying these  $\nu$ -statistics must have physical properties distinct from those of bosons ( $\nu = 0 \pmod{2}$ ) and fermions ( $\nu = 1 \pmod{2}$ ), and therefore deserve a distinct name: such particles are called *anyons* (with  $\nu$  left implicit). Such particles necessitate a PT-violation (see [Wil90, Section II.1]), where P and T are the (discrete) parity and time-reversal symmetries, since neither reflection nor path reversal preserve the phase of the path taken by particle 2 given by  $e^{in\nu}$ , where  $n$  is the winding number of the path around  $x_1$ , i.e., the number of times the path loops around  $x_1$ .

*A priori*, particles with fractional statistics might seem like purely theoretical constructs, but, as Wilczek originally observed in [Wil82a], a physically realizable model of anyons can

<sup>1</sup>More precisely, *homotope* it.

in fact be constructed using the Aharonov-Bohm effect. Consider a particle A with charge  $q$  moving outside a solenoid with vector potential  $\mathbf{A}$ ; if particle A moves in a closed loop  $\gamma$  around the solenoid, then the Aharonov-Bohm effect says that the wavefunction acquires a phase of  $\exp\left(-iq \oint_{\gamma} \mathbf{A} \cdot d\mathbf{s}\right)$ . Letting  $\Phi$  denote the flux, this is equal to  $e^{-iq\Phi}$ . This suggests a natural model for anyons as a particle carrying both a charge  $q$  and a flux  $\Phi$  associated to a vector potential  $\mathbf{A}$ : the particles in this model for anyons are commonly called *cyons* (see [Ler92, Chapter 3]). It is important to keep in mind that although we will use the terms “flux” and “charge” in the sequel, these are merely *fictitious* fields; even though anyons can be realized physically via particles with real fluxes and charges, this is merely a specialization of the general theory developed below.

If we consider two anyons A and B having charge  $q$  and a flux  $\Phi$ , then the above discussion implies that the phase acquired by rotating them around each other is  $(e^{-iq\Phi})^2 = e^{-2iq\Phi}$ : the square arises since there are two contributions to the phase, one coming from rotating particle A around the vector potential associated to particle B, and the other coming from rotating particle B around the vector potential associated to particle A. Viewing the rotation of particles around each other as a phase associated to an angle of  $\phi = 2\pi$ , we find from the discussion at the beginning of this section that  $e^{-2iq\Phi} = e^{2\pi i\nu}$ , so the statistics is determined by  $\nu = q\Phi/\pi$ ; note that  $\nu = 0$  corresponds to a boson, and  $\nu = 1$  corresponds to a fermion. The classical relation  $\nu = 2s$  between the statistics and the spin suggests that the spin of the anyon is  $s = q\Phi/2\pi$ . For instance, an anyon with a flux  $\Phi = \pi/q$  behaves like a fermion.

**2.2. Dynamics via the Chern-Simons term.** Since the proposed model for an anyon is as a charged particle with a long and infinitesimally thin flux tube, the singular limit is the  $\delta$ -function flux tube centered at the origin; this is realized by the vector potential  $A^i(\mathbf{x}) = \frac{\Phi}{2\pi} \frac{\epsilon_{ij} x^j}{|\mathbf{x}|^2}$ . In polar coordinates,  $A^r = 0$  and  $A^\phi = \Phi/2\pi$ . Recall that the Lagrangian of a system with a vector potential  $\mathbf{A}$  of this form describing a solenoid centered at the origin, with a potential  $V(\mathbf{r})$ , is

$$\mathcal{L}_1 = \frac{m\dot{\mathbf{r}}^2}{2} + q\dot{\mathbf{r}} \cdot \mathbf{A} - V(\mathbf{r}).$$

In fact, this can be used to show (see [Eza08, Chapter 7.3]) a system of  $N$  anyons of charge  $q$  with position vectors  $\mathbf{r}_1, \dots, \mathbf{r}_N$  is determined by

$$(1) \quad \begin{aligned} \mathcal{L}_N &= \sum_{i=1}^N \left( \frac{m\dot{\mathbf{r}}_i^2}{2} + \sum_{j \neq i} q\dot{\mathbf{r}}_i \cdot \mathbf{A}(\mathbf{r}_{ij}) \right) - V(\mathbf{r}_1, \dots, \mathbf{r}_N) \\ &= \sum_{i=1}^N \frac{m\dot{\mathbf{r}}_i^2}{2} + \frac{q\Phi}{2\pi} \sum_{i < j} \dot{\Theta}_{ij} - V(\mathbf{r}_1, \dots, \mathbf{r}_N), \end{aligned}$$

where  $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$  and  $\Theta_{ij} = \arctan((y_i - y_j)/(x_i - x_j))$  is the angle between the  $x$ -axis and the vector connecting particles  $i$  and  $j$ ; the second equality in the above expression is a direct algebraic calculation using that

$$(2) \quad \sum_{l \neq k} \mathbf{A}^i(\mathbf{r}_{kl}) = \frac{\Phi}{2\pi} \sum_{l \neq k} \frac{\epsilon_{ij}(r_k^j - r_l^j)}{|\mathbf{r}_k - \mathbf{r}_l|^2} = \frac{\Phi}{2\pi} \frac{\partial}{\partial r_k^i} \sum_{l \neq k} \Theta_{kl}.$$

The Lagrangian (1) can also be rewritten in a covariant formalism, as described in [Ler92, Chapter 3] (although our presentation is simplified). Suppose there is a (fictitious) conserved 2+1-dimensional charged matter (i.e., charged anyon) current  $j^\mu$  (with charge  $q$ ) and a gauge field  $A_\mu$ ; this spatial component of this gauge field is the fictitious vector potential  $\mathbf{A}$  we considered

earlier, and we choose the time component of  $A_\mu$  to be zero. Consider adding to the classical Lagrangian ( $m\dot{\mathbf{r}}^2/2 - V(\mathbf{r})$ ) a term

$$(3) \quad \mathcal{L}_{\text{cs}} = j^\mu A_\mu - \frac{\beta}{2} \epsilon^{\mu\nu\gamma} A_\mu \partial_\nu A_\gamma$$

for some real parameter  $\beta$ ; the point of doing so, as will be clear below, is that fixing  $\beta$  is *equivalent* to giving the particle an infinitesimal flux tube. Associated to the gauge field  $A_\mu$  is the Euler-Lagrange field equation

$$j^\mu = \beta \epsilon^{\mu\nu\gamma} \partial_\nu A_\gamma.$$

This implies that  $j^i = \beta \epsilon^{ij} E_j$  and  $j^0 = \beta B$ , where  $B = \epsilon^{0ij} \partial_i A_j$  and  $E_i = -\partial_0 A_i$  are the associated “magnetic” and “electric” fields; one can regard this as a Gauss law constraint. Since  $j^0$  is the “charge density”, we can integrate the equation for  $j^0$  over a closed surface<sup>2</sup>  $S$  to find that the  $B$ -flux  $\Phi$  is equal to  $\int_S B dA = \frac{1}{\beta} \int_S j^0 dA = q/\beta$ , where  $q$  is the associated “charge”. In particular, if a particle has a charge  $q$ , then the addition of  $\mathcal{L}_{\text{cs}}$  to the Lagrangian gives the particle an additional flux  $\Phi = q/\beta$ . A few remarks are in order:

- (1) Note that  $\beta$  is determined by, and uniquely determines, the statistics parameter  $\nu$  via  $\nu = q^2\beta/\pi$ . This establishes the importance of  $\mathcal{L}_{\text{cs}}$ : upon fixing  $\beta$  (which is, by the above equation, an “intrinsic” property of a particle), we obtain what might be interpreted as a flux on the particle; this remedies the artificial requirement of having the two *a priori extrinsic* values  $\Phi$  and  $q$  determining the *intrinsic* value  $\beta$ . In light of its importance in this story, the term (3) is distinguished as the *Chern-Simons term*.
- (2) In the limit where  $j^0$  becomes a  $\delta$ -function (which is dictated by the natural requirement that anyons be point particles), one can directly check that the explicit form of the potential  $A^i$  indeed satisfies the Euler-Lagrange field equation.
- (3) We already observed that anyons necessitate a PT-violation; this can also be readily seen from the Chern-Simons term  $\mathcal{L}_{\text{cs}}$  using the fact that  $PA^i(t, \mathbf{r})P^{-1}$  is  $A^i(t, \mathbf{r}')$  if  $i = 0, 1$  and is  $-A^i(t, \mathbf{r}')$  if  $i = 2$ , where  $\mathbf{r}' = (-x, y)$  if  $\mathbf{r} = (x, y)$ , and that  $TA^i(t, \mathbf{r})T^{-1}$  is  $-A^i(-t, \mathbf{r})$  if  $i = 1, 2$  and is  $A^0(-t, \mathbf{r})$  if  $i = 0$ .

Returning to Equation (1), it is easy to calculate that the associated canonical momentum of the  $k$ th particle is  $\mathbf{p}_k = m\dot{\mathbf{r}}_k - q \sum_{j \neq k} \mathbf{A}(\mathbf{r}_{ij})$ ; taking the Legendre transform shows that the associated Hamiltonian is

$$(4) \quad \mathcal{H}_N = \frac{1}{2m} \sum_{i=1}^N \left( \mathbf{p}_i - q \sum_{j \neq i} \mathbf{A}(\mathbf{r}_{ij}) \right)^2 + V(\mathbf{r}_1, \dots, \mathbf{r}_N)$$

$$(5) \quad = \frac{1}{2m} \sum_{i=1}^N \mathbf{p}_i^2 - \frac{q\Phi}{4\pi m} \sum_{i \neq j} \frac{\mathbf{L}_{ij}}{r_{ij}^2} + \frac{q^2\Phi^2}{8\pi^2 m} \sum_{i \neq j, k} \frac{\mathbf{r}_{ij} \cdot \mathbf{r}_{ik}}{r_{ij}^2 r_{ik}^2} + V,$$

where we have defined  $\mathbf{L}_{ij} = \mathbf{r}_{ij} \times (\mathbf{p}_i - \mathbf{p}_j)$ .

The kinetic portion of the Hamiltonian (4) looks like the Hamiltonian for a charged particle interacting with an electromagnetic field, but with the usual potential term absent. In classical electrodynamics, this potential term is necessary to maintain gauge invariance; how is this issue resolved in the setting of anyons? The mystery is solved by noticing that the term  $q \sum_{j \neq i} \mathbf{A}(\mathbf{r}_{ij})$  associated to the  $i$ th particle is fundamentally *non-local*, in the sense that it depends on the position of all the other particles (see [CWWH89] for a terse discussion of this issue). In particular, the Hamiltonian equations of motion will contain terms that do not appear in the classical setting of interactions with electromagnetic fields, and these terms impose gauge invariance.

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<sup>2</sup>To obtain these results for the  $i$ th particle, one can choose a small enough closed surface containing the  $i$ th particle but not any other particle.

Consider again the kinetic term of the Hamiltonian (4). The wavefunctions  $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n)$  solving the resulting Schrödinger equation will be bosonic, in the sense that swapping  $\mathbf{r}_i$  and  $\mathbf{r}_j$  leaves the wavefunction unchanged; the fractional statistics arises entirely from the gauge field  $\mathbf{A}(\mathbf{r}_{ij})$ . Since Equation (2) shows that the terms  $\mathbf{A}(\mathbf{r}_{ij})$  are pure gauges, these terms can be purged from Equation (4) by a gauge transformation. The wavefunction and the Hamiltonian transform as (see [Kha05, Chapter 2])

$$\begin{aligned}\tilde{\psi} &= e^{\frac{iq\Phi}{2\pi} \sum_{i<j} \Theta_{ij}} \psi, \\ \tilde{\mathcal{H}}_N &= e^{\frac{iq\Phi}{2\pi} \sum_{i<j} \Theta_{ij}} \mathcal{H}_N e^{-\frac{iq\Phi}{2\pi} \sum_{i<j} \Theta_{ij}} = \sum_{i=1}^N \frac{\mathbf{P}_i^2}{2m} + V(\mathbf{r}_1, \dots, \mathbf{r}_N).\end{aligned}$$

The gauge-transformed wavefunction  $\tilde{\psi}$  is now *multi-valued*, and satisfy the property that swapping  $\mathbf{r}_i$  and  $\mathbf{r}_j$  in  $\tilde{\psi}(\mathbf{r}_1, \dots, \mathbf{r}_n)$  transforms it to  $e^{iq\Phi/2} \tilde{\psi}(\mathbf{r}_1, \dots, \mathbf{r}_n)$ . Since the gauge-transformed wavefunction more clearly displays fractional statistics, this gauge is called the *anyon gauge*; in analogy, since the untransformed Hamiltonian is dependent on the gauge field  $\mathbf{A}$ , the gauge we previously worked in is known as the *magnetic gauge*.

Returning to the magnetic gauge, we observe from the rewriting of  $\mathcal{H}_N$  in Equation (5) that obtaining exact energies for a general  $N$ -anyon system will in general be quite difficult: it describes interacting bosons (or fermions; as described above, they are interchangeable viewpoints, with different associated fluxes) with a long-range vector interaction. Nonetheless, it is possible to exactly solve a two-anyon system in simple potentials, as we will now discuss.

### 2.3. Two anyons in an oscillator potential.

*Free anyons.* In the case of a free two-anyon system, the Lagrangian  $\mathcal{L}_2$  from Equation (1) can be written in center-of-mass coordinates  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$  and  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ ; we find that  $\mathcal{L}_2$  is a sum of the center-of-mass term  $m\dot{\mathbf{R}}^2$  and the relative term  $\mathcal{L}_r = \frac{m}{4}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{q\Phi}{2\pi}\dot{\phi}$ , where  $\phi$  is the angle between the two particles. The associated center-of-mass Hamiltonian behaves like a free particle, and as such, we may ignore this term; the associated relative Hamiltonian is

$$\mathcal{H}_r = \frac{p_r^2}{m} + \frac{(p_\phi - q\Phi/2\pi)^2}{mr^2}.$$

The wavefunction  $\psi(r, \phi)$  of the free two-anyon system satisfies the Schrödinger equation associated to this Hamiltonian:

$$(6) \quad \left( -\frac{1}{mr} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{mr^2} \left( i \frac{\partial}{\partial \phi} + \frac{q\Phi}{2\pi} \right)^2 \right) \psi = E\psi.$$

Separating  $\psi(r, \phi)$  as  $\psi_r(r) \cdot \psi_\ell(\phi)$ , we find that the angular portion differs from the classical situation:

$$(7) \quad \left( i \frac{\partial}{\partial \phi} + \frac{q\Phi}{2\pi} \right)^2 \psi_\ell(\phi) = E_\ell \psi_\ell(\phi).$$

Viewing the system as one of interacting *bosons* necessitates  $\psi_\ell(\phi + \pi) = \psi_\ell(\phi)$ ; using this boundary condition, we find that the only solutions to Equation (7) are  $\psi_\ell(\phi) = e^{i\ell\phi}$ , where  $\ell$  is an even integer, with associated energies  $E_\ell = (\ell - q\Phi/2\pi)^2$ . Note that for  $\ell \neq 0$ , the energies have a two-fold degeneracy if  $q\Phi/2\pi$  is nonzero (since both  $-\ell$  and  $\ell$  give the same energy eigenvalue), i.e., if the anyon is not just a boson. Substituting this calculation into Equation (6), we obtain exactly the Bessel differential equation, with solutions  $\psi_r(r) = J_{|\ell - q\Phi/2\pi|}(kr)$  for integer  $k$  and  $E = k^2/m$ .

Before proceeding to the case of a harmonic oscillator potential, we make a few comments about the physical interpretations of these calculations.

- (1) For small values of  $r$ , the radial component  $\psi_r(r) = J_{|\ell - q\Phi/2\pi|}(kr)$  grows as  $O(r^{|\ell - q\Phi/2\pi|})$ . In the ground state  $\ell = 0$ , we find that the repulsion between two anyons grows with the statistics  $q\Phi/2\pi$ . In particular, as  $q\Phi/2\pi$  grows, so does the repulsion between two anyons; this suggests viewing anyons as bosons with a repulsive interaction, or as fermions with an attractive interaction, where this interaction is determined by the (fictitious) “charge and flux”. This quantifies the manner in which anyons satisfy a “non-integral” version of the Pauli exclusion principle.
- (2) The energy eigenvalues of noninteracting bosons (or fermions) are simply the sum of the energy eigenvalues of each individual boson (or fermion); this is *not* the case for general anyons. Indeed, the energy eigenvalues for a system of two noninteracting anyons was determined above, and one can evidently see that this is not the same as the sum of the energy eigenvalues of a free anyon (i.e., a free particle). This is not an unexpected observation: as we mentioned previously, free anyons are formally equivalent to interacting bosons with a long-range vector interaction.

*Harmonic oscillator potential.* The only other easily understood case where the system can be calculated exactly is that of a harmonic oscillator potential  $V(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2}m\omega^2(\mathbf{r}_1^2 + \mathbf{r}_2^2)$ ; essentially no other system has been calculated exactly. The calculations done in this section will be relevant in §3.2, where the energy eigenvalues will be used to compute the equation of state for an ideal gas of anyons.

Upon adding a harmonic oscillator potential, the Lagrangian (1) again breaks up as a sum of the center-of-mass term  $m\dot{\mathbf{R}}^2 - m\omega^2\mathbf{R}^2$  (which is that of a classical harmonic oscillator, and hence will be ignored in the sequel) and the relative Lagrangian  $\mathcal{L}_r = \frac{m}{4}(\dot{r}^2 + r^2\dot{\phi}^2 + \omega^2 r^2) + \frac{q\Phi}{2\pi}\dot{\phi}$ . The associated Schrödinger equation is the same as Equation (6), except with the left hand side containing an additional term  $\frac{m}{4}\omega^2 r^2 \psi(r, \phi)$ . We can again separate  $\psi(r, \phi)$  as  $\psi_r(r) \cdot \psi_\ell(\phi)$ , and note that  $\psi_\ell(\phi)$  has the same solution as in the free-anyon case. Using the discussion from the free-anyon case, we find that the radial portion satisfies

$$(8) \quad \left( -\frac{1}{mr} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{mr^2} \left( \ell - \frac{q\Phi}{2\pi} \right)^2 + \frac{m}{4} \omega^2 r^2 \right) \psi_r(r) = E \psi_r(r)$$

for  $\ell$  an even integer. This is precisely the Schrödinger equation for a harmonic oscillator with angular momentum  $\ell - q\Phi/2\pi$ ; determining the resulting energy eigenvalues arising via the relative term is classical (see also [Wil90, Chapter I.3]):

$$(9) \quad E_{n\ell} = \omega \left( 2n + \left| \ell - \frac{q\Phi}{2\pi} \right| + 1 \right),$$

for  $n$  a positive integer and  $\ell$  an even integer. Compactly, if we define  $m = n + \ell/2$ , then  $E_m = \omega(2m + 1 + q\Phi/2\pi)$  for  $\ell \leq 0$ , with degeneracy  $m + 1$ , and  $E_m = \omega(2m + 1 - q\Phi/2\pi)$  for  $\ell > 0$ , with degeneracy  $n$ . One key takeaway from this example is that as the energy spectrum varies continuously as the statistics  $q\Phi/2\pi$  varies (from 0, the bosonic case, to 1, the fermionic case). Numerical calculations have shown (see [Kha05, Chapter 3]), however, that this is no longer true in the case of multi-anyon ( $N > 2$ ) systems: the energy eigenvalues (even the ground state!) does not vary smoothly with the statistics. Another interesting observation is that the energy spectrum is not only highly degenerate, but it is also no longer equi-spaced, except when  $q\Phi/2\pi$  is 0 (the bosonic case), 1 (the fermionic case), or 1/2 (the “semion” case). Moreover, we again see the principle of interpreting anyons as interpolating between bosons and fermions in action: the ground state energy of two anyons *monotonically increases* as the statistics  $q\Phi/2\pi$  increases from 0 to 1; this makes physical sense, since in the interacting boson picture of anyons, the repulsion between two anyons should grow with  $q\Phi/2\pi$ . In the next section, we will apply

the above calculation of the energy spectrum to write down (an approximation to) the equation of state of an ideal gas of anyons.

### 3. AN IDEAL GAS OF ANYONS

The statistical physics of ideal Fermi and Boson gases is well-understood, thanks to the pure (anti)symmetry of the associated wavefunctions. This allows one to understand the Hilbert space of states of a system of  $N$  fermions (resp. bosons) as the  $N$ -fold tensor product of the Hilbert space of states of a single fermion (resp. boson). Unfortunately, as we observed in §2.3, this is no longer true for anyons with non-integer statistics. However, as emphasized in §2.2, we may view the theory of anyons as the theory of bosons that interact via a long-range vector potential. Naturally, therefore, one might approach the question of finding an equation of state for an ideal gas of anyons by performing a perturbative expansion that takes into account only interactions between two particles, then interactions between three particles, and so on.

**3.1. Virial coefficients.** In this section, we recall some classical concepts from statistical physics that allow us to perform the desired perturbative expansion; nothing we say in this section is specific to the case of anyons or mechanics in two dimensions. Recall that if  $H(\mathbf{r}_i, \mathbf{p}_i) = \sum_i \mathbf{p}_i^2/2m + U(\mathbf{r}_1, \dots, \mathbf{r}_N)$ , with  $U$  assumed to be spherically symmetric, is the Hamiltonian of a system with  $N$  identical particles and volume<sup>3</sup>  $V$ , then the associated partition function (from which all physically relevant thermodynamic quantities can be derived)  $Z_N$  is

$$Z_N = \frac{1}{N! \hbar^{3N}} \int \prod_i \frac{d^3 \mathbf{p}_i d^3 \mathbf{r}_i}{(2\pi)^{3N}} e^{-\beta H(\mathbf{r}_i, \mathbf{p}_i)} = Z_1^N \frac{1}{V^N} \int \prod_i d^3 \mathbf{r}_i e^{-\beta U},$$

where we have done the integral for the free component of the Hamiltonian, and written  $\beta$  to denote  $1/k_B T$  and  $T$  is the temperature, and  $Z_1$  to denote the partition function of a single particle. Since the free energy is  $F_N = -k_B T \ln Z_N$ , we find that  $F_N - F_1^N = -k_B T \ln(Z_N/Z_1^N)$ . If we approximate  $U(\mathbf{r})$  as a sum of two-particle potentials  $\sum_{i < j}^N U(\mathbf{r}_i - \mathbf{r}_j)$  by ignoring interactions between three or more particles, then an algebraic calculation for  $N \gg 0$  shows that by defining  $f_{ij} = e^{-\beta U(\mathbf{r}_i - \mathbf{r}_j)} - 1$ , we have:

$$(10) \quad \frac{Z_N}{Z_1^N} = \frac{1}{V^N} \int \prod_i d^3 \mathbf{r}_i e^{-\beta U} \approx 1 + \frac{1}{V^N} \int \sum_{i < j} f_{ij} d^3 \mathbf{r} \approx 1 + \frac{2\pi N^2}{V} \int r^2 (e^{-\beta U(r)} - 1) dr,$$

where we have made the approximation  $N(N-1)/2 = N \gg 0$ , used the fact that the particles are all identical, and used the volume integral of the spherically symmetric potential  $U$  is  $4\pi$  times the radial integral of  $r^2 U$ . Consequently, since the pressure is given by  $p_N = -\partial F_N / \partial V$  (with  $T$  and  $N$  held fixed), we find the equation of state:

$$(11) \quad p_N = \frac{k_B T N}{V} - \frac{2\pi k_B T N^2}{V^2} \int r^2 (e^{-\beta U(r)} - 1) dr + O(1/V^3).$$

In general, the coefficient of  $k_B T N^k / V^k$  in this equation of state is denoted by  $B_k(T)$ , and is called the  $k$ th *virial coefficient*; it describes the contribution of the  $k$ -particle interaction term to the equation of state. If the sign of  $B_k(T)$  is negative, then there is a statistical attraction in a subsystem of  $k$  particles. Comparing the definition of  $B_2(T)$  via Equation (11) to Equation (10) (without loss of generality assuming that  $N = 2$ ), we find that the second virial coefficient is given by

$$(12) \quad B_2(T) = \frac{V}{2} \left( 1 - 2 \frac{Z_2}{Z_1^2} \right).$$

<sup>3</sup>In the 2-dimensional case, this is the area.

Continuing in the manner described above, one can also calculate the third virial coefficient to be

$$B_3(T) = V^2 \left( \frac{2Z_2}{Z_1} \left( \frac{2Z_2}{Z_1} - 1 \right) - \frac{1}{3} \left( \frac{6Z_3}{Z_1^3} - 1 \right) \right).$$

We will only be concerned with  $B_2(T)$ ; its determination is sufficient to write down the equation of state up to interactions of two particles. If the particles are not confined to a box, then  $B_2(T)$  is the limit of the above expression as  $V \rightarrow \infty$ .

**3.2. Virial coefficients for anyons.** In classical statistical physics, the partition function for an ideal gas of  $N > 1$  particles will be divergent, because each energy state is infinitely degenerate; by choosing a potential (called the “regulator”), one breaks the energy degeneracy. Ideally, the calculations of the virial coefficients should be independent of the choice of regulator. We have already done the calculations for a harmonic oscillator potential in §2.3; this allows us to directly calculate the second virial coefficient via (12). Our presentation follows the skeleton of [Kha05, Chapter 4], but is a little reorganized and more elementary.

The discussion of the harmonic oscillator potential in §2.3 only considered the relative Lagrangian  $\mathcal{L}_r$ , since the center-of-mass term simply contributes the energy spectrum of the usual simple harmonic oscillator (with mass  $2m$ , but this is immaterial in the discussion of the energy spectrum). It follows that the energy spectrum of two anyons in a harmonic oscillator potential is given by a sum of the energies of a classical harmonic oscillator and the energy eigenvalues (9). Since the partition function of a quantum system with discrete energies  $\{E_n\}_{n \geq 0}$  is given by the sum  $\sum_{n \geq 0} e^{-\beta E_n}$ , we conclude that

$$Z_2 = Z_1 \sum_{n\ell} e^{-\beta E_{n\ell}} = Z_1 \sum_{n \geq 0} e^{-\beta \omega(2n+1)} \sum_{k=-\infty}^{\infty} e^{\beta \omega |2k - q\Phi/2\pi|},$$

where  $Z_1$  denotes the partition function of a single anyon (i.e., a particle) in a two-dimensional harmonic oscillator potential, which is  $Z_1 = \left( \sum_{n \geq 0} e^{-\beta \omega(n+1/2)} \right)^2 = 1/4 \sinh^2(\beta \omega/2)$ . The expression for  $Z_2$  is straightforward to evaluate: we find that

$$Z_2 = Z_1 \frac{\cosh((1 - q\Phi/2\pi)\beta \omega)}{2 \sinh^2(\beta \omega)},$$

from which Equation (12) immediately gives an expression for  $B_2(T)$ ; it remains to understand the limit as  $V \rightarrow \infty$ . In this case, we can replace the coefficient  $V/2$  by  $V$ . Computing the limit of the parenthesized expression is a little complicated, though, so we instead calculate the limit as  $\omega \rightarrow 0$ ; one should expect both limits to give physically equivalent situations, since both model the limit of weaker and weaker interactions between particles. In this limit,  $Z_1 \approx 1/\beta^2 \omega^2$ ; since both limits morally model a free particle in a volume  $V$ , we should also expect  $Z_1$  to approximately be the partition function of a free particle:

$$Z_1 = V \int \frac{d^2 \mathbf{k}}{(2\pi)^2} e^{-\beta \mathbf{k}^2/2m} = \frac{mV}{2\pi\beta}.$$

Upon accounting for the units of  $\hbar$ , the term  $m/2\pi\beta$  is seen to have units of  $\text{length}^{-2}$ , so we define  $\lambda_T = \sqrt{2\pi\beta/m}$  as the *thermal wavelength*. Comparing the two expressions for  $Z_1$  shows that  $V = (\lambda_T/\beta\omega)^2$ . Finally returning to Equation (12), a direct Taylor series expansion in the  $\omega \rightarrow 0$  limit shows that

$$(13) \quad B_2(T) \approx \left( \frac{1}{4} - \frac{1}{2} \left( 1 - \frac{q\Phi}{2\pi} \right)^2 \right) \lambda_T^2.$$



Note that we are restricting  $q\Phi/2\pi$  to lie in the interval  $[0, 1]$ ; for other values of  $q\Phi/2\pi$ , one uses Equation (13) with  $q\Phi/2\pi \pmod{2}$ ; thanks to this periodicity, one obtains cusps for even values of  $|q\Phi/2\pi|$ . Equation (13) is quite remarkable for a few reasons: first, it is not *a priori* clear that  $B_2(T)$  would even be finite, since the interactions are long-range. Second, it is known that  $B_2(T) = 1/4$  for fermions and is  $B_2(T) = -1/4$  for bosons (see [Kha05, Chapter 4.2, 4.3]), so this calculation not only recovers these known results, but also shows that  $B_2(T)$  interpolates quadratically between the fermionic and bosonic cases in the statistics  $q\Phi/2\pi$ . Third, since the sign of  $B_2(T)$  is negative (resp. positive) for  $0 \leq q\Phi/2\pi < 1/\sqrt{2}$  (resp.  $1/\sqrt{2} < q\Phi/2\pi \leq 1$ ), we find that there is a statistical attraction (resp. repulsion) between two anyons for these values of the statistics  $q\Phi/2\pi$ . Fourth, Equation (13) is independent of our choice of a harmonic oscillator potential as a regulator. We can also write down the equation of state up to two-particle interactions:

$$p_N V \approx N k_B T - \frac{2\pi N^2 k_B T}{V} \left( \frac{1}{4} - \frac{1}{2} \left( 1 - \frac{q\Phi}{2\pi} \right)^2 \right) \lambda_T^2.$$

Determining the higher virial coefficients is an extremely hard problem, since it depends on knowing the partition function for  $N \geq 3$ -anyon interactions, for which only approximate solutions are known. One might, however, expect interesting physical phenomena to occur once these multi-anyon interactions are taken into account, since (as mentioned in §2.3) the energy eigenvalues of such a system, including the ground state, does not vary smoothly with the statistics  $q\Phi/2\pi$  (unlike the two-anyon case). Unfortunately, not much seems to be known in this general situation.

**3.3. Outlook: superfluids.** We now briefly (and qualitatively) discuss, following [CWWH89], how the results obtained above suggest that an ideal gas of anyons might possess superfluidity. The Pauli exclusion principle forbids fermions from occupying the same quantum state, so it is hard for fermions to form superfluids. However, the Bardeen-Cooper-Schrieffer theory [BCS57] shows that the existence of an arbitrarily weak attractive force in an ideal gas of Fermions is sufficient to force the existence of Cooper pairs, which leads to superfluidity. In §2.3 and §3.2, we showed that one may view anyons as bosons with a repulsive interaction, or equivalently as fermions with an attractive interaction determined by the charge and flux. This argument, although quite naïve, suggests that ideal gases of anyons might exhibit superfluidity — and, in fact, in [CWWH89], it was shown that a gas of anyons with statistics that only infinitesimally differ from Fermi statistics does possess superfluidity.

#### 4. CONCLUSION

In this paper, we described a few aspects of the theory of particles with fractional statistics. These anyons, which can be modeled by charged particles with flux tubes, possess rather interesting physical properties, owing to the fact that they are equivalent to the theory of bosons (resp. fermions) with a nontrivial *non-local* repulsive (resp. attractive) interaction. The non-locality of this interaction places a great many constraints on the sort of calculations humans can perform with such systems; exact computations have only been done for systems with two anyons. Nonetheless, our analysis of this two-anyon system (in both a harmonic oscillator potential, and via the equation of state of an ideal gas of anyons) quantified the manner in which anyons interpolate between fermions and bosons.

We conclude by mentioning a major, relatively recent, part of the theory of anyons which we could not fit into this survey (of which there are a lot; the topic is quite broad!): there is a deep way to observe quasiparticles and quasiholes in the fractional quantum Hall effect as anyons; this was initially observed by Halperin and Arovas *et. al.* in [Hal84, ASW02] (see [Eza08] for a

textbook account). This led to a great surge of interest in the theory of anyons as a method to better understand the fractional quantum Hall effect. It is rather pleasing to reflect on the fact that the theoretical curiosity of the triviality of the angular momentum commutation relations in two dimensions is in fact realized by concrete physical phenomena like the Aharonov-Bohm effect and the fractional quantum Hall effect, and possesses interesting properties like superfluidity.

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