THE LUBIN-TATE STACK AND GROSS-HOPKINS DUALITY

Abstract. Morava E-theory E is an $\mathbf{E}_\infty$-ring with an action of the Morava stabilizer group Γ. We study the derived stack $\operatorname{Spf} E/\Gamma$. Descent-theoretic techniques allow us to deduce a theorem of Hopkins-Mahowald-Sadofsky on the $K(n)$-local Picard group. These techniques also allow us to rederive a few consequences of a recent result of Barthel-Beaudry-Stojanoska on the Anderson duals of higher real K-theories.

1. Introduction

Goerss-Hopkins-Miller proved that Morava E-theory E (at a fixed height n and prime p) is an $\mathbf{E}_\infty$-ring. Moreover, the profinite group Γ (also known as the Morava stabilizer group) of units in a certain division algebra of Hasse invariant $1/n$ acts continuously on E via $\mathbf{E}_\infty$-ring maps. From the perspective of derived algebraic geometry, this is saying that one can construct the object $\operatorname{Spf} E/\Gamma$ (the "Lubin-Tate stack").

Devinatz and Hopkins proved that there is an equivalence $L_{K(n)}S \simeq E^{h\Gamma}$, where the right hand side uses an appropriate notion of continuous fixed points. This result allows us to show that there is an equivalence of ∞-categories $\mathrm{QCoh}(\operatorname{Spf} E/\Gamma) \simeq L_{K(n)}\mathrm{Sp}$. In [HMS94], Hopkins-Mahowald-Sadofsky proved that the following statements are equivalent for a $K(n)$-local spectrum M.

(1) M is $K(n)$-locally invertible.
(2) $\dim_{K(n)_*} K(n)_*M = 1$.
(3) $E^\vee_* M$ is a free E_*-module of rank 1.

The above discussion suggests that one may recast this result as a descent-theoretic statement along the étale cover $\operatorname{Spf} E \to \operatorname{Spf} E/\Gamma$. This is one of the results proved in this paper. One useful computational tool in the study of the $K(n)$-local Picard group is the existence of a map $\mathrm{Pic}_n \to \mathrm{H}^1_c(\Gamma; E_0^\times)$. This descent-theoretic viewpoint allows us to think of this assignment as the monodromy action of the line bundle over $\operatorname{Spf} E/\Gamma$ corresponding to a $K(n)$-locally invertible spectrum.

As an approximation to Pic_n, one can attempt to understand the Picard group of the higher real K-theories. In the simplest case, one has an identification $\mathrm{Pic}(KO) \simeq \mathbf{Z}/8$, generated by ΣKO. This corresponds to the 8-fold periodicity of KO. Recently, Heard-Mathew-Stojanoska computed in [HMS17] that if $EO_{p-1} = E_{p-1}^{hC_p}$, then $\mathrm{Pic}(EO_{p-1}) \simeq \mathbf{Z}/(2p^2)$, again generated by ΣEO_{p-1}. This corresponds to the $2p^2$-fold periodicity of EO_{p-1}. One expects the Picard to be cyclic at any height. When $p-1$ does not divide n this is a simple computation. In [HHR], Hill-Hopkins-Ravenel describe the E_2-page for the homotopy fixed point spectral sequence for $EO_{2(p-1)}$. This suggests using tools similar to those in [HMS17] to prove that the Picard group of $EO_{2(p-1)}$ is cyclic. We will return to this computational problem in a future paper.

Barthel-Beaudry-Stojanoska used this result in [BBS17] to prove a self-duality statement. Since $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group, the functor $X \mapsto \mathrm{Hom}(\pi_{-*}X, \mathbf{Q}/\mathbf{Z})$ defines a cohomology theory. This is represented by a spectrum $I_{\mathbf{Q}/\mathbf{Z}}$, called the Brown-Comenetz dualizing spectrum. The Brown-Comenetz dual of a spectrum X is defined as $I_{\mathbf{Q}/\mathbf{Z}}X = \underline{\mathrm{Map}}(X, I_{\mathbf{Q}/\mathbf{Z}})$. There is a canonical map $H\mathbf{Q} \to I_{\mathbf{Q}/\mathbf{Z}}$, and the fiber of this map is the Anderson dualizing spectrum, $I_{\mathbf{Z}}$. One similarly defines the Anderson dual of a spectrum X to be $I_{\mathbf{Z}}X = \underline{\mathrm{Map}}(X, I_{\mathbf{Z}})$.

In [HS14], Heard-Stojanoska showed that there is an equivalence $I_{\mathbf{Z}}KO \simeq \Sigma^4 KO$. Using computational tools, Barthel-Beaudry-Stojanoska proved that, at odd primes, there is an equivalence $L_{K(n)} I_{\mathbf{Q}/\mathbf{Z}} EO_{p-1} \simeq \Sigma^{(p-1)^2} EO_{p-1}$. This implies that $L_{K(p-1)} I_{\mathbf{Z}} EO_{p-1} \simeq \Sigma^{(p-1)^2-1} EO_{p-1}$. This computational approach does not shed much light (at least to the author) on the theoretical underpinnings of Anderson self-duality. In this paper, we provide a conceptual explanation for this fact.

From an algebro-geometric point of view, $I_{\mathbf{Z}}$ can be thought of as a dualizing sheaf for $\operatorname{Spec} S$. In the first section, we recall some facts about derived stacks. We then develop methods to analyze dualizing sheaves for even periodic derived Deligne-Mumford stacks. We prove the following tool for recognizing when a spectrum is a dualizing sheaf for $\operatorname{Spec} S$, which is tangential to the discussion about Picard groups of EO_{p-1}. Let R be a coconnected p-complete spectrum such that $\pi_* R$ is a finite abelian group for $* \neq 1$ and $\pi_0 R$ is a finitely generated abelian group. Then the following statements are equivalent:

(1) $\operatorname{Map}(H\mathbf{Z}/p, R) \simeq \Sigma^{-1} H\mathbf{Z}/p$, and
(2) R is a dualizing sheaf for $\operatorname{Spec} S$.

In a later paper, we will give an application of this result to a higher Snaith theorem (see [Wes17]).

Let us return to Anderson self-duality. Let $G \subseteq \Gamma$ be a finite subgroup of the Morava stabilizer group. Consider the structure map $f : \operatorname{Spf} E/G \to \operatorname{Spec} S$; then $f^! I_{\mathbf{Z}}$ is exactly $L_{K(n)} I_{\mathbf{Z}} E^{hG}$. Using general statements about self-duality in the derived setting (see Theorem 3.15 and Proposition 3.16), we deduce that $I_{\mathbf{Z}} E^{hG}$ is an element of $\operatorname{Pic}(E^{hG})$ for any height and prime. If $G \subseteq \Gamma$ is not a finite group, then our argument does not necessarily work. However, when $G = \Gamma$, the quasicoherent sheaf $f^! I_{\mathbf{Z}}$ on $\operatorname{Spf} E/\Gamma$ is in fact $K(n)$-locally invertible, although our methods do not suffice to give a proof. As there is an equivalence $\Sigma^{-1}\widehat{I} \simeq f^! I_{\mathbf{Z}}$, where $\widehat{I}$ is the Gross-Hopkins element of Pic_n, this statement is equivalent to Gross-Hopkins duality (the classical proof is in [Str00]). Using the invertibility of this element, we deduce that — conditional on E^{hG} being Spanier-Whitehead self-dual, which is proved in Appendix A at any height divisible by $(p-1)$ for the subgroup $G = C_p$ — if the group of exotic elements of $\operatorname{Pic}(E^{hG})$, i.e., elements X such that $E_*^\vee E^{hG} \simeq E_*^\vee X$ as Morava modules, is cyclic or trivial, then $L_{K(n)} I_{\mathbf{Z}} E^{hG}$ is equivalent to a shift of E^{hG}. Thus, the result about the cyclicity of the Picard group of EO_{p-1} implies that $L_{K(p-1)} I_{\mathbf{Z}} EO_{p-1}$ is equivalent to a shift of EO_{p-1}. However, our method does not give the exact shift of $(p-1)^2 - 1$. Gross and Hopkins also describe the monodromy action on the line bundle $f^! I_{\mathbf{Z}}$, and show that it is essentially the determinant representation of Γ. The question of how one might recover this result using the methods of this paper is the subject of future work.

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2. Derived stacks

Most of the discussion in this section can be found in more detail in [Lur17].

2.1. **Generalities.** All Deligne-Mumford stacks are assumed to have affine diagonal.

Definition 2.1. A *derived (Deligne-Mumford) stack* $\mathfrak{X}$ is a Deligne-Mumford stack X along with a sheaf of $\mathbf{E}_\infty$-rings $\mathcal{O}_X^{\mathrm{der}}$ (interchangeably denoted $\mathcal{O}_{\mathfrak{X}}$) on the affine étale site of X such that $\pi_0\mathcal{O}_X^{\mathrm{der}} \simeq \mathcal{O}_X$ and $\pi_i\mathcal{O}_X^{\mathrm{der}}$ is a quasicoherent $\pi_0\mathcal{O}_X^{\mathrm{der}}$-module.

We say that a Deligne-Mumford stack X "admits a lift" if there is a derived stack with underlying stack X. This is a rather strong condition to impose on a Deligne-Mumford stack; see, for instance, [SVW99] and [Dev17] for results on non-liftability.

Let $\mathcal{F}$ be a sheaf of $\mathbf{E}_\infty$-rings on a Deligne-Mumford stack Y, and let $f : X \to Y$ denote a morphism of Deligne-Mumford stacks. Define a sheaf of $\mathbf{E}_\infty$-rings $f^{-1}\mathcal{F}$ on X as follows: for every étale map $\operatorname{Spec} R \to X$, we define $(f^{-1}\mathcal{F})(\operatorname{Spec} R)$ to be the homotopy colimit $\operatorname{colim}_{\operatorname{Spec} R\to Z\to Y, Z\to Y} \mathcal{F}(Z)$ over all such étale morphisms.

Definition 2.2. Let $\mathfrak{X}$ and $\mathfrak{Y}$ denote derived stacks. A morphism $\mathfrak{X} \to \mathfrak{Y}$ is a morphism $f : X \to Y$ along with a morphism $f^{-1}\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{Y}}$ of sheaves of $\mathbf{E}_\infty$-rings which induces the map $f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$.

If $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of derived stacks, then $f^*\mathcal{F} = \mathcal{O}_{\mathfrak{Y}} \otimes_{f^{-1}\mathcal{O}_{\mathfrak{X}}} f^{-1}\mathcal{F}$.

One can define the ∞-category of quasicoherent sheaves on derived stacks just as in the classical case: $\mathrm{QCoh}(\mathfrak{X}) = \lim_{\operatorname{Spec} R\to\mathfrak{X}} \mathrm{Mod}(\mathcal{O}^{\mathrm{der}}(\operatorname{Spec} R))$, where the homotopy limit is taken over all étale morphisms $\operatorname{Spec} R \to \mathfrak{X}$. This is a limit of presentable stable ∞-categories under colimit-preserving functors, so $\mathrm{QCoh}(\mathfrak{X})$ is also a presentable stable ∞-category. Using descent theory, we can give an equivalent presentation. Suppose $\operatorname{Spec} R \to X$ is an étale surjection. Then

$$\mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{Tot}\left(\mathrm{QCoh}(\operatorname{Spec} R) \rightrightarrows \mathrm{QCoh}(\operatorname{Spec} R \times_{\mathfrak{X}} \operatorname{Spec} R) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \right).$$

This comes from the presentation of $\mathfrak{X}$ as a semisimplicial object

$$\operatorname{Spec} R \leftleftarrows \mathrm{QCoh}(\operatorname{Spec} R \times_{\mathfrak{X}} \operatorname{Spec} R) \substack{\leftarrow\\\leftarrow\\\leftarrow} \cdots.$$

One way to obtain derived stacks is via the following theorem.

Theorem 2.3. *Let R be an $\mathbf{E}_\infty$-ring. Suppose $f : \pi_0 R \to A$ is an étale map of ordinary rings; then there is an R-algebra B with an étale map $R \to B$ such that $\pi_0 B \cong A$, and the induced map on homotopy agrees with the original map f.*

Proof. This theorem can be deduced from work of Goerss and Hopkins in [GH04], and can also be found as [Lur16, Theorem 7.5.0.6]. ☐

In what follows, we will be interested in derived formal schemes. To this end, we make the following definition.

Definition 2.4. An adic $\mathbf{E}_\infty$-ring is an $\mathbf{E}_\infty$-ring R with a topology on $\pi_0 R$, such that $\pi_0 R$ admits a finitely generated ideal I of definition.

Let M be a R-module. Pick a set of generators $x_1, \cdots, x_n$ for I. Say that M is (x_i)-complete if

$$\lim(\cdots \xrightarrow{x_i} M \xrightarrow{x_i} M \xrightarrow{x_i} M) \simeq 0,$$

where $x_i : M \to M$ is the morphism determined by $x_i \in \pi_0 R$. The R-module M is said to be I-complete if M is (x_i)-complete for $1 \le i \le n$. Let R, S, and T be adic $\mathbf{E}_\infty$-rings, such that R and T have finitely generated ideals of definition $I \subseteq \pi_0 R$ and $J \subseteq \pi_0 T$. Then we can endow $\pi_0(R \wedge_S T)$ with the K-adic topology, where K is the ideal generated by the images of I and J. The resulting adic $\mathbf{E}_\infty$-ring is denoted $R\widehat{\otimes}_S T$.

An adic $\mathbf{E}_\infty$-ring determines a derived formal scheme $\operatorname{Spf} R$, whose underlying (formal) scheme is $\operatorname{Spf} \pi_0 R$. The sheaf $\mathcal{O}^{\mathrm{der}}$ of $\mathbf{E}_\infty$-rings on the affine étale site of $\operatorname{Spf} \pi_0 R$ is defined as follows. Let $\operatorname{Spec} A \to \operatorname{Spf} \pi_0 R$ be an affine étale over $\operatorname{Spf} \pi_0 R$, given by a map $\pi_0 R \to A$;

lift A to an étale R-algebra B by Theorem 2.3. As a functor from $\mathrm{Aff}^{\text{ét}}_{/X}$ to $\mathbf{E}_\infty$-rings, we define $\mathcal{O}^{\mathrm{der}}(\operatorname{Spec} A) = B^\wedge_I$. More generally, the procedure described above allows us to construct a quasicoherent sheaf $\mathcal{F}$ on $\operatorname{Spf} R$ from any any I-complete R-module M: we send $\mathcal{F}(\operatorname{Spec} A) = (B \otimes_R M)^\wedge_I$. This begets an equivalence $\mathrm{QCoh}(\operatorname{Spf} R) \simeq \mathrm{Mod}(R)^\wedge_I$, where the right hand side denotes the ∞-category of I-complete R-modules. The smash product of adic $\mathbf{E}_\infty$-rings defined above allows us to consider the fiber product of derived (affine) formal schemes.

For the rest of this paper, any $\mathbf{E}_\infty$-ring R will be assumed to be an adic $\mathbf{E}_\infty$-ring with a fixed finitely generated ideal of definition I. There should not be any confusion as to what this ideal is; this will be clear from the context. Note that every $\mathbf{E}_\infty$-ring R can trivially be viewed as an adic $\mathbf{E}_\infty$-ring: endow $\pi_0 R$ with the discrete topology (equivalently, suppose that I is nilpotent).

Suppose G is a finite group acting on an $\mathbf{E}_\infty$-ring R by $\mathbf{E}_\infty$-maps. We can then define the quotient $\operatorname{Spf} R/G$ as the colimit of the resulting functor from BG into the ∞-category of formal derived Deligne-Mumford stacks. Using the cosimplicial model for BG (equivalently, étale descent), this can equivalently be presented via the semisimplicial diagram

$$\operatorname{Spf} R \leftleftarrows \operatorname{Spf}(R \times G) \Lleftarrow \cdots.$$

We will also need to consider special cases when G is not finite. If $\mathfrak{X}_\bullet$ is a semisimplicial object in derived stacks, we will denote by $\mathrm{QCoh}(\mathfrak{X}_\bullet)$ the totalization $\mathrm{Tot}(\mathrm{QCoh}(\mathfrak{X}_\bullet))$ of the semicosimplicial diagram $\mathrm{QCoh}(\mathfrak{X}_\bullet)$. If $\mathfrak{X}$ is a derived stack, then $\mathrm{QCoh}(\mathfrak{X}_\bullet^{\mathrm{constant}}) \simeq \mathrm{QCoh}(\mathfrak{X})$, where $\mathfrak{X}_\bullet^{\mathrm{constant}}$ is the constant semisimplicial object. We will often abuse notation by using $\mathfrak{X}$ to denote $\mathfrak{X}_\bullet^{\mathrm{constant}}$.

2.2. **Vector bundles.** Let R be an $\mathbf{E}_\infty$-ring. A projective R-module M is a retract of a free R-module. A simple consequence of this definition is that projective R-modules are flat, since direct sums and retracts of flat modules are flat. In other words, the natural map $\pi_0 M \otimes_{\pi_0 R} \pi_* R \to \pi_* M$ is an isomorphism.

Definition 2.5. Let X be a derived stack. A *vector bundle of rank n on X* is a quasicoherent sheaf $\mathcal{F}$ such that for every étale map $f : \operatorname{Spec} R \to X$, the pullback $M := f^*\mathcal{F} \in \mathrm{Mod}_R$ satisfies the following properties:

- M is a projective R-module such that $\pi_0 M$ is a finitely generated $\pi_0 R$-module.
- $\pi_0(k \otimes_R M)$ is a k-vector space of dimension n where k is a field with a map of $\mathbf{E}_\infty$-rings $R \to k$.

A line bundle is a vector bundle of rank 1. Let $\mathrm{Pic}(X)$ be the space of suspensions of line bundles on X, topologized as a subspace of the maximal subgroupoid inside $\mathrm{QCoh}(X)$. As a corollary of the discussion in [Lur17, §2.9.4-5], we find that if X is a connected derived stack, then $\mathrm{Pic}(X)$ is equivalent to the space of invertible objects of the ∞-category $\mathrm{QCoh}(X)$. Before proceeding, let us discuss how $\mathrm{Pic}(\mathrm{QCoh}(\operatorname{Spf} R))$ relates to $\mathrm{Pic}(\mathrm{QCoh}(\operatorname{Spec} R))$. Suppose $\pi_0 R$ is I-complete. It is then clear that any invertible R-module is in $\mathrm{Pic}(\mathrm{QCoh}(\operatorname{Spf} R))$. Moreover, an element of $\mathrm{Pic}(\mathrm{QCoh}(\operatorname{Spf} R))$ is in $\mathrm{Pic}(\mathrm{QCoh}(\operatorname{Spec} R))$ if and only if M is a perfect R-module.

Let R be an even periodic adic $\mathbf{E}_\infty$-ring with ideal of definition I such that:

- $\pi_0 R$ is a complete regular local Noetherian ring which is I-complete.
- An R-module is dualizable in $\mathrm{Mod}(R)$ if and only if it is perfect.

Proposition 2.6. *If R satisfies the above two conditions, then* $\mathrm{Pic}(\operatorname{Spf} R)$ *is equivalent to the space of invertible objects of* $\mathrm{QCoh}(\operatorname{Spf} R) \simeq \mathrm{Mod}(R)^\wedge_I$.

Proof. By [BR05, Theorem 8.7], any R-module in $\mathrm{Pic}(\mathrm{QCoh}(\operatorname{Spec} R))$ is equivalent to a shift of R. Clearly R and ΣR are perfect R-modules, so the second condition on R implies that any invertible object of $\mathrm{QCoh}(\operatorname{Spf} R)$ is in $\mathrm{Pic}(\operatorname{Spf} R)$. Conversely, if M is a line bundle over R, then M is dualizable. Indeed, dualizable objects are closed under retracts and wedges, so since R

is dualizable, any vector bundle over R is dualizable. In particular, M is a perfect R-module, so it suffices to show that M is an invertible object of $\mathrm{Mod}(R)$. Let $M^\vee$ denote the dual of M, so there is an evaluation map $\mathrm{ev} : M \otimes_R M^\vee \to R$. Now arguing as in [Lur17, Proposition 2.9.4.2] (which requires [Lur17, Proposition 2.9.2.3], the proof of which does not need R to be connective), we conclude that ev is an isomorphism, so M is an invertible R-module. □

It is a general fact that the functor sending a symmetric monoidal ∞-category $\mathcal{C}$ to the space of invertible objects in $\mathcal{C}$ commutes with limits and filtered colimits ([MS16, Proposition 2.2.3]). By construction, $\mathrm{QCoh}(-)$ sends colimits to limits of symmetric monoidal stable ∞-categories. As the functor from the ∞-category of symmetric monoidal stable ∞-categories to the ∞-category of symmetric monoidal ∞-categories reflects limits, it follows that $\mathrm{Pic}(-)$ takes homotopy colimits to homotopy limits. In particular, if G is a finite group acting on R by $\mathbf{E}_\infty$-maps, we have an equivalence (see also [MS16, §3.3]): $\mathrm{Pic}(\mathrm{Spf}\, R/G) \simeq \mathrm{Pic}(\mathrm{Spf}\, R)^{hG}$. Note that the G-actions on $\mathrm{Pic}(\mathrm{Spf}\, R)$ and $\mathrm{Pic}(\mathrm{Spec}\, R)$ are the same.

Let R be an even periodic $\mathbf{E}_\infty$-ring, and let M be a line bundle over R. Then $\pi_* M$ is a projective $\pi_* R$-module. Indeed, $\pi_0 M$ is a projective $\pi_0 R$-module. Since $\pi_n M \simeq \pi_n R \otimes_{\pi_0 R} \pi_0 M$, the result then follows from R being even periodic and the fact that projective modules are flat. If, moreover, $\pi_0 R$ is a local ring, then $\pi_* M$ is a free $\pi_* R$-module since projective modules over a local ring are free.

3. Dualizing sheaves

3.1. **The connective case.** If $f : X \to Y$ is a morphism of (derived) schemes, we will write $f^!$ to denote a right adjoint to $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$. This is an abuse of notation unless f is a proper morphism.

Definition 3.1. Let $\mathfrak{X}$ be a *connective* derived stack. Let $\mathcal{F}$ be a quasicoherent sheaf over $\mathfrak{X}$. We say that $\mathcal{F}$ is a *dualizing sheaf* if the following conditions are satisfied.

(1) The map $\mathcal{O}_\mathfrak{X} \to \underline{\mathrm{Map}}_{\mathcal{O}_\mathfrak{X}}(\omega_\mathfrak{X}, \omega_\mathfrak{X})$ is an equivalence.
(2) $\omega_\mathfrak{X}$ is coconnected.
(3) $\omega_\mathfrak{X}$ is coherent.
(4) $\omega_\mathfrak{X}$ has finite injective dimension.

[Lur17, Proposition 6.6.2.1] shows that if $\mathcal{F}$ and $\mathcal{G}$ are two dualizing sheaves, then there is a line bundle $\mathcal{L}$ such that $\mathcal{F} \simeq \mathcal{G} \otimes \mathcal{L}$. We give a simple tool to identify dualizing sheaves over the (p-complete) sphere spectrum.

Theorem 3.2. *Let R be a coconnected p-complete spectrum such that $\pi_* R$ is a finite abelian group for $* \neq 1$ and $\pi_0 R$ is a finitely generated abelian group. Then the following statements are equivalent:*

(1) $\underline{\mathrm{Map}}(H\mathbf{Z}/p, R) \simeq \Sigma^{n-1} H\mathbf{Z}/p$, *and*
(2) *R is a dualizing sheaf for* $\mathrm{Spec}\, S$.

Proof. It is easy to see that the Anderson dualizing spectrum $I_\mathbf{Z}$ (described in the introduction) is a dualizing S-module.

Returning to the theorem, assume (2). The above discussion implies that any dualizing sheaf is equivalent to $I_\mathbf{Z}$ up to an element of $\mathrm{Pic}(\mathrm{Sp})$, which is isomorphic to $\mathbf{Z}$ generated by S^1 (see Lemma 5.1). Without loss of generality, we may assume that $R = I_\mathbf{Z}$; then R sits in a fiber sequence $R \to I_\mathbf{Q} \to I_{\mathbf{Q}/\mathbf{Z}}$, which implies that $\underline{\mathrm{Map}}(H\mathbf{Z}/p, R) \simeq \Sigma^{-1}\underline{\mathrm{Map}}(H\mathbf{Z}/p, I_{\mathbf{Q}/\mathbf{Z}}) \simeq \Sigma^{-1} H\mathbf{Z}/p$.

For the other direction, assume R satisfies (1). Let K be a dualizing sheaf for $\mathrm{Spec}\, S$; translating the definition provided above, this means that K is a spectrum such that

(a) K is coconnected, and $\pi_n K$ is a finitely generated abelian group.
(b) K has finite injective dimension, i.e., there is an integer n such that for any n-coconnected spectrum M, we have $\pi_i \underline{\mathrm{Map}}(M, K) = 0$ for $i < 0$.
(c) The natural map $S \to \underline{\mathrm{Map}}(K, K)$ is an equivalence.

To show that R is a dualizing sheaf for $\mathrm{Spec}\, S$, we will check each of the conditions above.

(a) R is coconnected by assumption, and $\pi_n R$ is a finitely generated abelian group for all n.
(b) We have $\pi_i \underline{\mathrm{Map}}(M, R) \simeq \pi_0 \underline{\mathrm{Map}}(\Sigma^i M, R)$. Suppose R is N-coconnected for some N; then $\pi_i \underline{\mathrm{Map}}(M, R) \simeq \pi_0 \underline{\mathrm{Map}}(\tau_{\leq N} \Sigma^i M, R)$. Now, $\pi_k \Sigma^i M \simeq 0$ for $k > n + i$. If n is sufficiently large, then $\tau_{\leq N} \Sigma^i M$ is contractible, so $\underline{\mathrm{Map}}(\pi_i M, R) \simeq 0$ for some $n \gg 0$.
(c) Our proof follows [Lur17, Proposition 6.6.4.6]. It suffices to prove that for every integer k, we have an equivalence $\pi_k S \to \pi_k \underline{\mathrm{Map}}(R, R) \simeq \pi_k \underline{\mathrm{Map}}(\underline{\mathrm{Map}}(S, R), R)$. Since R is N-coconnected, we can replace S by its N-coconnected cover. In this case, $\tau_{\leq N} S$ can be written as a composite of extensions of $H\mathbf{Z}_p$ and shifts of Eilenberg-Maclane spectra annihilated by a power of p, i.e., $H\mathbf{Z}/p^k$. It therefore suffices to show that $\underline{\mathrm{Map}}(\underline{\mathrm{Map}}(H\mathbf{Z}/p^k, R), R) \simeq H\mathbf{Z}/p^k$ and that $\underline{\mathrm{Map}}(\underline{\mathrm{Map}}(H\mathbf{Z}_p, R), R) \simeq H\mathbf{Z}_p$. But

$$\underline{\mathrm{Map}}(\underline{\mathrm{Map}}(H\mathbf{Z}/p^k, R), R) \simeq \underline{\mathrm{Map}}(\underline{\mathrm{Map}}_{H\mathbf{Z}/p}(H\mathbf{Z}/p^k, R'), R') \simeq \underline{\mathrm{Map}}_{H\mathbf{Z}/p}(\underline{\mathrm{Map}}_{H\mathbf{Z}/p}(H\mathbf{Z}/p^k, R'), R'),$$

which is $H\mathbf{Z}/p^k$ since R' is a dualizing sheaf for $\mathrm{Spec}\, H\mathbf{Z}/p$, where $R' = \underline{\mathrm{Map}}(H\mathbf{Z}/p, R)$; in particular, this proves that $M \to \underline{\mathrm{Map}}(\underline{\mathrm{Map}}(M, R), R)$ is an equivalence for every spectrum M which is p-torsion. For $H\mathbf{Z}_p$, we argue as follows: there is an equivalence $\underline{\mathrm{Map}}(H\mathbf{Z}_p, R) \simeq \underline{\mathrm{Map}}(\Sigma^{-1} H\mathbf{Q}_p/\mathbf{Z}_p, R)$ since R is torsion. We are now done by the previous case.

□

The theory of dualizing sheaves over connective derived Deligne-Mumford stacks is not sufficient for our purposes; we have to extend the definition to even periodic derived stacks. Recall the following definition.

Definition 3.3. An even periodic $\mathbf{E}_\infty$-ring is an $\mathbf{E}_\infty$-ring R whose homotopy is concentrated in even dimensions such that $\pi_2 R$ is an invertible $\pi_0 R$-module, satisfying the property that $\pi_{2k} R \simeq (\pi_2 R)^{\otimes k}$ for all $k \in \mathbf{Z}$.

Definition 3.4. An even periodic derived stack $\mathfrak{X}$ is a derived stack such that for every étale morphism $\mathrm{Spec}\, R \to X$ into the underlying Deligne-Mumford stack, the $\mathbf{E}_\infty$-ring $\mathcal{O}_{\mathfrak{X}}(\mathrm{Spec}\, R)$ is a even periodic $\mathbf{E}_\infty$-ring.

Remark 3.5. Let X be a Deligne-Mumford stack with a flat map $X \to \mathcal{M}_{FG}$. An even periodic refinement of X is an even periodic derived stack $\mathfrak{X}$ lifting X such that for every étale morphism $\mathrm{Spec}\, R \to X$, the even periodic $\mathbf{E}_\infty$-ring $\mathcal{O}_{\mathfrak{X}}(\mathrm{Spec}\, R)$ has formal group given by the (flat) composite $\mathrm{Spec}\, R \to X \to \mathcal{M}_{FG}$.

We need the following result.

Proposition 3.6. *Let $\mathfrak{X}$ be an even-periodic refinement of a flat map $X \to \mathcal{M}_{FG}$ from a Noetherian Deligne-Mumford stack X which is proper and of finite type over* $\mathrm{Spec}\, \mathbf{Z}_p$. *Then $\pi_* \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a degreewise finitely generated $\mathbf{Z}_p$-module.*

Proof. It is well-known that the descent spectral sequence computing $\pi_* \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ has a finite vanishing line at the E_∞-page. Since the E_∞-page is a subquotient of the E_2-page, it will suffice to show that $E_2^{s,t}$ is finitely generated for all s, t. There is an isomorphism

$$E_2^{s,t} \cong \begin{cases} \mathrm{H}^s(X; \pi_{t/2}\mathcal{O}_{\mathfrak{X}}) & \text{if } t \equiv 0 \pmod 2 \\ 0 & \text{else,} \end{cases}$$

and $\pi_{t/2}\mathcal{O}_{\mathfrak{X}} \cong \omega^{\otimes t/2}$, where ω is the pullback of the Lie algebra line bundle on $\mathcal{M}_{FG}$. Since $\omega^{\otimes t}$ is a line bundle, it is in particular a coherent sheaf on X. Therefore, if $f : X \to \operatorname{Spec} \mathbf{Z}_p$ is the structure morphism, the higher direct image sheaves $R^s f_* \omega^{\otimes t} = \mathrm{H}^s(X; \omega^{\otimes t})$ are coherent $\mathcal{O}_{\operatorname{Spec} \mathbf{Z}_p}$-modules, i.e., are finitely generated free $\mathbf{Z}_p$-modules, as desired. □

3.2. **The even periodic case.** If R is an $\mathbf{E}_\infty$-ring, the notion of an almost perfect R-module is only well-defined when R is connective. In the nonconnective setting, we will make the following definition.

Definition 3.7. Let R be a Noetherian even periodic $\mathbf{E}_\infty$-ring. An R-module M is said to be almost perfect if it can be obtained as the geometric realization of a simplicial R-module $P_\bullet$, with each P_n a free R-module of finite rank.

If $\mathfrak{X}$ is a locally Noetherian even periodic derived stack, then a quasicoherent sheaf $\mathcal{F}$ on $\mathfrak{X}$ will be called almost perfect if, for every étale morphism $f : \operatorname{Spec} R \to \mathfrak{X}$, the pullback $f^*\mathcal{F}$ is almost perfect.

The definition of a dualizing sheaf is the following.

Definition 3.8. Let $\mathfrak{X}$ be a locally Noetherian even periodic derived stack. A quasicoherent sheaf $\omega_{\mathfrak{X}}$ on $\mathfrak{X}$ is a *dualizing sheaf* if

(1) The map $\mathcal{O}_{\mathfrak{X}} \to \underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(\omega_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ is an equivalence.
(2) The functor $\mathbf{D}(\mathcal{F}) = \underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \omega_{\mathfrak{X}})$ gives an autoequivalence of the category of almost perfect quasicoherent sheaves on $\mathfrak{X}$ with itself.
(3) For every étale map $f : \operatorname{Spec} R \to \mathfrak{X}$, the $\pi_0 R$-module $\pi_0 f^*\omega_{\mathfrak{X}}$ is a dualizing module for $\pi_0 R$.

We will need to understand when the structure sheaf (or some shift of it) of a derived stack $\mathfrak{X}$ is itself a dualizing complex. If this is the case, we say that $\mathfrak{X}$ is *self-dual* or *Gorenstein.*

We begin with a series of lemmas.

Lemma 3.9. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a étale surjection of locally Noetherian even periodic derived Deligne-Mumford stacks. Suppose that $\omega_{\mathfrak{Y}}$ is a quasicoherent sheaf on $\mathfrak{Y}$ such that $f^*\omega_{\mathfrak{Y}}$ is a dualizing sheaf on $\mathfrak{X}$. Then $\omega_{\mathfrak{Y}}$ is a dualizing sheaf on $\mathfrak{Y}$.*

Proof. Condition (1) is obvious, and condition (2) follows from the fact that f^* preserves almost perfectness. It remains to check condition (3). Let $g : \operatorname{Spec} R \to \mathfrak{Y}$ be an étale map. We need to check that $\pi_0 g^*\omega_{\mathfrak{Y}}$ is a dualizing module for $\pi_0 R$. The statement of Lemma 3.9 is true in the classical setting, so it suffices to check that $p_0^*\pi_0 g^*\omega_{\mathfrak{Y}}$ is a dualizing sheaf on T, for some étale surjection $p_0 : T \to \operatorname{Spec} \pi_0 R$. Let $\mathfrak{Z}$ denote the even periodic Deligne-Mumford stack $\mathfrak{X} \times_{\mathfrak{Y}} \operatorname{Spec} R$, and let $\operatorname{Spec} A \to \mathfrak{Z}$ be an étale surjection. Let $q : \operatorname{Spec} A \to \mathfrak{X}$ denote the induced étale morphism. The map $\mathfrak{Z} \to \operatorname{Spec} R$ is also an étale surjection, so the composite $p : \operatorname{Spec} A \to \operatorname{Spec} R$ is an étale surjection. Since p is étale, we have an isomorphism $p_0^*\pi_0 g^*\omega_{\mathfrak{Y}} \simeq \pi_0 p^* g^*\omega_{\mathfrak{Y}}$. The equivalence $p^*g^* = q^*f^*$ shows that $p_0^*\pi_0 g^*\omega_{\mathfrak{Y}}$ is equivalent to $\pi_0 q^* f^*\omega_{\mathfrak{Y}}$ as a $\pi_0 A$-module. Since $f^*\omega_{\mathfrak{Y}}$ is a dualizing sheaf for $\mathfrak{X}$, and q is an étale morphism, it follows that $\pi_0 q^* f^*\omega_{\mathfrak{Y}}$ is a dualizing module for $\pi_0 A$, as desired. □

Lemma 3.10. *Suppose $\mathfrak{X}$ is a locally Noetherian separated derived Deligne-Mumford stack which arises as an even-periodic refinement of a tame and flat map $X \to \mathcal{M}_{FG}$. Assume that $\mathfrak{X}$ is perfect and X is proper. Let $f : \mathfrak{X} \to \operatorname{Spec} S$ be the structure morphism. Then $f^! I_{\mathbf{Z}}$ is a dualizing sheaf on $\mathfrak{X}$.*

Proof. In this case, $f^!$ is (defined as) a right adjoint to f_*. We will check the conditions of Definition 3.8.

(1) We need to show that the map $\mathcal{O}_{\mathfrak{X}} \to \underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(f^! I_{\mathbf{Z}}, f^! I_{\mathbf{Z}})$ is an equivalence, i.e., that for each $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, the map $\theta : \mathrm{Map}_{\mathfrak{X}}(\mathcal{F}, \mathcal{O}_{\mathfrak{X}}) \to \mathrm{Map}(f_*(\mathcal{F} \otimes f^! I_{\mathbf{Z}}), I_{\mathbf{Z}})$ is an equivalence. The same proof as [Lur17, Proposition 6.6.3.1] can be used here, but we will recall the details for the sake of completeness. As $\mathfrak{X}$ is a perfect stack, the sheaf $\mathcal{F}$ is a filtered colimit of perfect objects. In particular, we may assume that $\mathcal{F}$ is perfect, so that $f_*(\mathcal{F} \otimes f^! I_{\mathbf{Z}}) \simeq \underline{\mathrm{Map}}(f_*\mathcal{F}^\vee, I_{\mathbf{Z}})$. Since $I_{\mathbf{Z}}$ is a dualizing complex for $\mathrm{Spec}\, S$, it suffices to show that $\underline{\mathrm{Map}}(\underline{\mathrm{Map}}(f_*\mathcal{F}^\vee, I_{\mathbf{Z}}), I_{\mathbf{Z}}) \simeq f_*\mathcal{F}^\vee$. Since $f_*\mathcal{F}^\vee = \Gamma(\mathfrak{X}, \mathcal{F}^\vee)$, the desired result follows from the observation that $\pi_*\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is degreewise finitely generated by Proposition 3.6, and that the global sections functor (to $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$-modules) preserves filtered colimits, which follows from [MM15, Theorem 4.14].
(2) We need to show that there is an equivalence $\mathcal{F} \xrightarrow{\simeq} \underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(\underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, f^! I_{\mathbf{Z}}), f^! I_{\mathbf{Z}})$ for every almost perfect quasicoherent sheaf $\mathcal{F}$. The assertion is local, so we may assume that $\mathfrak{X} = \mathrm{Spec}\, R$ is affine. In this case, the result follows from the fact that for any almost perfect R-module M, the map $M \to \underline{\mathrm{Map}}_R(\underline{\mathrm{Map}}_R(M, I_{\mathbf{Z}}R), I_{\mathbf{Z}}R)$ is an equivalence.
(3) Let $u : \mathrm{Spec}\, R \to \mathfrak{X}$ be an étale morphism. We need to show that $\pi_0 u^* f^! I_{\mathbf{Z}}$ is a dualizing sheaf for $\pi_0 R$. The R-module $u^* f^! I_{\mathbf{Z}}$ is the function spectrum $\underline{\mathrm{Map}}(R, I_{\mathbf{Z}}) = I_{\mathbf{Z}}R$. There is a short exact sequence
$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(\pi_{-n-1}R, \mathbf{Z}) \to \pi_n I_{\mathbf{Z}}R \to \mathrm{Hom}_{\mathbf{Z}}(\pi_{-n}R, \mathbf{Z}) \to 0.$$
Since R is an even periodic cohomology theory, it follows that $\pi_n I_{\mathbf{Z}}R \simeq \mathrm{Hom}_{\mathbf{Z}}(\pi_{-n}R, \mathbf{Z})$, so $\pi_0 I_{\mathbf{Z}}R$ is indeed a dualizing sheaf for $\pi_0 R$, as desired.

□

Remark 3.11. Suppose $\mathfrak{Y}$ is a Deligne-Mumford stack for which the structure morphism $\mathfrak{Y} \to \mathrm{Spec}\, S$ factors as $\mathfrak{Y} \hookrightarrow \mathfrak{X} \to \mathrm{Spec}\, S$, where $u : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ is an open immersion, and $f : \mathfrak{X} \to \mathrm{Spec}\, S$ is a Deligne-Mumford stack satisfying the conditions of Lemma 3.10. Then $u^* f^! I_{\mathbf{Z}}$ is a dualizing sheaf on $\mathfrak{Y}$. In the classical setting, Nagata's compactification theorem gives such a factoring when $\mathfrak{Y}$ is a scheme which is separated and of finite type. We do not know of an analogue of this result in the derived setting.

Remark 3.12. Lemma 3.10 is also true if $\mathfrak{X}$ is replaced by $\mathrm{Spf}\, E$, where E is a Morava E-theory (see Section 4).

We say that a locally Noetherian Deligne-Mumford stack X has finite global dimension if there is a finite étale cover $\mathrm{Spec}\, R \to X$ with R a Noetherian ring of finite global dimension.

Lemma 3.13. *Let $\mathfrak{X}$ be a locally Noetherian even periodic derived Deligne-Mumford stack of finite global dimension*[1]*, and let ω be a dualizing sheaf on $\mathfrak{X}$. A quasicoherent sheaf ω' on $\mathfrak{X}$ is a dualizing sheaf if and only if there is an equivalence $\omega' \simeq \omega \otimes \mathcal{L}$ for $\mathcal{L}$ a line bundle on $\mathfrak{X}$.*

Proof. Suppose $\mathcal{L}$ is a line bundle on $\mathfrak{X}$, and let $\omega' = \omega \otimes \mathcal{L}$. Conditions (1) and (2) of Definition 3.8 are immediate. Suppose $f : \mathrm{Spec}\, R \to \mathfrak{X}$ is an étale morphism. There is an isomorphism $\pi_0 f^*\omega' \simeq \pi_0 f^*\omega \otimes_{\pi_0 R} \pi_0\mathcal{L}$. Since $\mathcal{L}$ is a line bundle, $\pi_0\mathcal{L}$ is a line bundle over $\pi_0 R$, so $\pi_0 f^*\omega'$ is a dualizing sheaf on $\pi_0 R$, establishing condition (3).

The proof of the converse follows [Lur17, Proposition 6.6.2.1]. Suppose ω and ω' are dualizing sheaves. Let $\mathcal{L} = \underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(\omega, \omega')$. We will show that $\mathcal{L}$ is a line bundle, and that $\omega \otimes \mathcal{L} \simeq \omega'$. Suppose $\mathcal{F}$ is an almost perfect quasicoherent sheaf on $\mathfrak{X}$. We will first show that $\mathcal{F} \otimes \mathcal{L} \to \underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(\underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \omega), \omega')$ is an equivalence. To show this, it in turn suffices to show that,

[1]In general, this is stronger than having finite Krull dimension.

if R is an even periodic Noetherian $\mathbf{E}_\infty$-ring of finite global dimension, and K and K' are dualizing complexes, then for every almost perfect R-module M, the map $M \otimes_R \underline{\mathrm{Map}}_R(K, K') \to \underline{\mathrm{Map}}_R(\underline{\mathrm{Map}}_R(M, K), K')$ is an equivalence. Since the statement is true for perfect R-modules, it suffices to reduce the result to this case. If M is almost perfect, then $\pi_0 M$ and $\pi_1 M$ are both finitely generated $\pi_0 R$-modules. The statement for $\pi_0 M$ is clear from the definition. Choose a finitely generated free module $P \to M$ inducing the surjection $\pi_0 P \to \pi_0 M$. The fiber P' of $P \to M$ is almost perfect, so $\pi_0 P'$ is also finitely generated. The long exact sequence in homotopy gives a short exact sequence

$$0 \to \mathrm{coker}(\pi_1 P' \to \pi_1 P) \to \pi_1 M \to \ker(\pi_0 P' \to \pi_0 P) \to 0.$$

The $\pi_0 R$-modules $\mathrm{coker}(\pi_1 P' \to \pi_1 P)$ and $\ker(\pi_0 P' \to \pi_0 P)$ are finitely generated, so $\pi_1 M$ is also finitely generated. By [Mat15, Proposition 2.1], the R-module M is perfect. Having established that $\mathcal{F} \otimes \mathcal{L} \to \underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(\underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \omega), \omega')$ is an equivalence for any almost perfect sheaf $\mathcal{F}$, it follows that, setting $\mathcal{F} = \underline{\mathrm{Map}}_{\mathcal{O}_{\mathfrak{X}}}(\omega', \omega)$, there is an equivalence $\mathcal{F} \otimes \mathcal{L} \simeq \mathcal{O}_{\mathfrak{X}}$, so $\mathcal{L}$ is a line bundle. Moreover, the same result, when applied to $\mathcal{F} = \omega$, shows that $\omega \otimes \mathcal{L} \simeq \omega'$, as desired. □

Remark 3.14. Suppose $\mathfrak{X}$ satisfies the conditions of Lemma 3.13. Let X denote the underlying stack of $\mathfrak{X}$. Then $\mathfrak{X}$ is self-dual if and only if X is Gorenstein. Indeed, suppose $\mathfrak{X}$ is self-dual. Let $f_0 : \mathrm{Spec}\, R \to X$ be an étale map. This refines to an étale map $f : \mathrm{Spec}\, \widetilde{R} \to \mathfrak{X}$. By construction, $\pi_0 \widetilde{R} = R$. It follows from the definition that the R-module $\pi_0 f^* \mathcal{O}_{\mathfrak{X}} = \pi_0 \widetilde{R} = R$ is a dualizing module, as desired. For the converse, suppose X is Gorenstein, and let $\omega_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}}$. Condition (1) of Definition 3.8 is immediate. To prove condition (2), note that the proof of Lemma 3.13 shows that an almost perfect quasicoherent sheaf $\mathcal{F}$ on $\mathfrak{X}$ is perfect, in which case the condition is easy to establish. Finally, condition (3) follows from the assumption that X is self-dual.

The above discussion yields the following result.

Theorem 3.15. *Let $\mathfrak{X}$ be a perfect locally Noetherian separated derived Deligne-Mumford stack which arises as the even-periodic refinement of a tame and flat map $X \to \mathcal{M}_{FG}$, where X has proper and finite global dimension. If $\mathfrak{X}$ is self-dual, then $f^! I_{\mathbf{Z}}$ is invertible, where $f : \mathfrak{X} \to \mathrm{Spec}\, S$ is the structure map.*

We will also prove the following result, which we learnt from Jacob Lurie.

Proposition 3.16. *Let $f_0 : X \to \mathrm{Spec}\, \mathbf{Z}_p$ be a smooth and proper scheme of relative dimension d. Suppose $f : \mathfrak{X} \to \mathrm{Spec}\, S$ is an even-periodic refinement of X. Then $f^! I_{\mathbf{Z}}$ is in* $\mathrm{Pic}(\mathfrak{X})$.

Proof. Denote by $g : \tau_{\geq 0}\mathfrak{X} \to \mathrm{Spec}\, S$ the connective cover of $\mathfrak{X}$; there are morphisms $i : X \to \tau_{\geq 0}\mathfrak{X}$ and $j : \mathfrak{X} \to \tau_{\geq 0}\mathfrak{X}$. By Serre duality, if $f_0 : X \to \mathrm{Spec}\, \mathbf{Z}_p$ is of relative dimension d, then $f_0^! \mathbf{Z}_p$ is isomorphic to the line bundle ω_X shifted up to degree d. Therefore $f_0^! \mathbf{Z}_p \in \mathrm{Pic}(X)$. There is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f_0"] & \tau_{\geq 0}\mathfrak{X} \arrow[d, "g"] \\ \mathrm{Spec}\, \mathbf{Z}_p \arrow[r, "q"'] & \mathrm{Spec}\, S. \end{tikzcd}$$

It follows that $f_0^! \mathbf{Z} = f_0^! q^! I_{\mathbf{Z}} = i^! g^! I_{\mathbf{Z}}$, so $i^! g^! I_{\mathbf{Z}} \in \mathrm{Pic}(X)$. It is easy to see that this implies that the sheaf $g^! I_{\mathbf{Z}}$ on $\tau_{\geq 0}\mathfrak{X}$ looks like $\omega_X[d] \oplus \omega_X[d-2] \oplus \cdots$, so that $f^! I_{\mathbf{Z}} = \mathrm{Hom}_{\mathcal{O}_{\tau_{\geq 0}\mathfrak{X}}}(\mathcal{O}_{\mathfrak{X}}, g^! I_{\mathbf{Z}})$ looks like a 2-periodic version of ω_X, concentrated in degrees of the same parity as d. In particular, $f^! I_{\mathbf{Z}}$ is in $\mathrm{Pic}(\mathfrak{X})$, as desired. □

4. E-theory

Let $\mathbf{G}$ be a formal group of height n over a perfect field k of characteristic $p > 0$.

Definition 4.1. An $\mathbf{E}_\infty$-ring E is said to be a *Morava E-theory* if the following conditions are satisfied:

(1) E is even periodic, with a(n invertible) periodicity generator $\beta \in \pi_2 E$.
(2) $\pi_0 E$ is a complete local Noetherian ring with residue field k.
(3) The formal group $\operatorname{Spf} \pi_0 \underline{\operatorname{Map}}_S(\Sigma^\infty_+ \mathbf{C}P^\infty, E)$ over $\pi_0 E$ is a universal deformation of $\mathbf{G}$.

A priori, it is not clear that Morava E-theory exists; however, it is a theorem of Goerss-Hopkins-Miller that every pair $(k, \mathbf{G})$ of an perfect field k along with a finite height formal group begets a Morava E-theory E. The choice of k and $\mathbf{G}$ will be remain implicit.

A theorem of Lazard's says that all formal groups of the same height are isomorphic over an algebraically closed field k of characteristic p. A particular choice for a formal group law of height n is the Honda formal group law H_n over k, whose p-series is given by $[p]_{\Gamma_n}(x) = x^{p^n}$. By Dieudonné theory, one can show that the profinite group $\mathbf{S}_n$ of automorphisms of H_n over k is given by the units in the maximal order $\mathcal{O}_n$ of the central division algebra of Hasse invariant $1/n$ over $\mathbf{Q}_p$. Explicitly, $\mathbf{S}_n \cong (W(k)\langle S \rangle / (Sx = \phi(x)S, S^n = p))^\times$, where ϕ is a lift of Frobenius to $W(k)$ and $x \in W(k)$. As H_n is defined over $\mathbf{F}_{p^n}$, we can construct the semidirect product $\mathbf{S}_n \rtimes \operatorname{Gal}(k/\mathbf{F}_{p^n})$; we will call this the Morava stabilizer group, and denote it by Γ. For $N \geq 1$, we have normal subgroups $1 + S^N \mathcal{O}_n$ of Γ, which are of finite index. Moreover, we have $\bigcap_{N\geq 1}(1 + S^N \mathcal{O}_n) = 1$, so letting these be a basis for the open neighborhoods of 1 provides Γ the structure of a profinite group.

Goerss-Hopkins-Miller showed that the action of Γ on $\pi_0 E$ lifts to an action of Γ on the $\mathbf{E}_\infty$-ring by $\mathbf{E}_\infty$-maps. Choosing $\mathbf{G} = \mathrm{H}_n$, Lubin-Tate theory allows us to noncanonically identify $\pi_0 E \simeq W(k)[[u_1, \cdots, u_{n-1}]]$. This is a complete local ring, with maximal ideal $\mathfrak{m} = (p, u_1, \cdots, u_{n-1})$. We remark that there are explicit, but inhumanly complicated, formulas for the action of Γ on the generators u_i. The $\mathbf{E}_\infty$-ring E is therefore an adic $\mathbf{E}_\infty$-ring, complete with respect to the finitely generated ideal $(p, u_1, \cdots, u_{n-1})$. The action of the Morava stabilizer group on E is continuous in the sense that it acts via maps of *adic* $\mathbf{E}_\infty$-rings.

Theorem 4.2 (Devinatz-Hopkins). *The* continuous *homotopy fixed points $E^{h\Gamma}$ is equivalent to the $K(n)$-local sphere $L_{K(n)}S$.*

Working through the definition of the homotopy fixed points, this is saying that

$$L_{K(n)}S \simeq \operatorname{Tot}\left(E \rightrightarrows E \widehat{\wedge} E \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots \right)$$

As Γ acts continuously on E, we can form the quotient stack $\operatorname{Spf} E/G$ for any finite subgroup $G \subsetneq \Gamma$. However, we cannot immediately define the quotient stack $\operatorname{Spf} E/\Gamma$ in the same manner as above; instead, inspired by the above result of Devinatz-Hopkins, we make the following definition.

Definition 4.3. The derived Lubin-Tate stack $\mathfrak{X}$ is defined to be the semisimplicial stack $\operatorname{Spf} E/\Gamma$, described via the semisimplicial diagram

$$\operatorname{Spf} E \leftleftarrows \operatorname{Spf}(E \widehat{\wedge} E) \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} \cdots$$

The following result is the analogue of the identifications $\operatorname{QCoh}(\operatorname{Spf} E/G) \simeq \operatorname{Mod}(E)^{\wedge, G}_{\mathfrak{m}} \simeq (L_{K(n)}\operatorname{Mod}(E))^{hG}$, where $\mathfrak{m} = (p, u_1, \cdots, u_{n-1})$ and G is a finite subgroup of Γ.

Lemma 4.4. *There are symmetric monoidal equivalences* $\operatorname{QCoh}(\operatorname{Spf} E) \simeq L_{K(n)}\operatorname{Mod}(E)$ *and* $\operatorname{QCoh}(\mathfrak{X}) \simeq L_{K(n)}\operatorname{Sp}$.

Proof. To prove the first statement, it suffices to prove that an E-module is $\mathfrak{m}$-complete if and only if it is $K(n)$-local. This follows from [DFHH14, Chapter 6, Proposition 4.1]. As $v_n = u^{(p^n-1)}$, we can invert u in a $K(n)$-local E-module; the statement that $K(n)$-local is equivalent to $\mathfrak{m}$-complete then follows from [Lur17, Corollary 7.3.3.3].

The second equivalence is a formal consequence of descent. Indeed, we have an equivalence:

$$L_{K(n)}\mathrm{Sp} \simeq \mathrm{Tot}\Big(\mathrm{QCoh}(\mathrm{Spf}\, E) \rightrightarrows \mathrm{QCoh}(\mathrm{Spf}\, E \widehat{\wedge} E) \Rrightarrow \cdots \Big)$$

Since $\mathrm{Spf}\, E \to \mathfrak{X}$ is a Γ-Galois étale cover and $\mathrm{Spf}(E\widehat{\wedge}E) \simeq \mathrm{Spf}\, E \times_{\mathfrak{X}} \mathrm{Spf}\, E$, it follows that the cosimplicial diagram is the cobar construction for homotopy fixed points. Altogether, this means that $\mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{QCoh}(\mathrm{Spf}\, E)^{h\Gamma}$, giving the desired equivalence $\mathrm{QCoh}(\mathfrak{X}) \simeq L_{K(n)}\mathrm{Sp}$. □

Note that the map $f^* : \mathrm{QCoh}(\mathrm{Spec}\, S) \simeq \mathrm{Sp} \to \mathrm{QCoh}(\mathfrak{X})$ induced by the structure map $f : \mathfrak{X} \to \mathrm{Spec}\, S$ is exactly $K(n)$-localization. It is important to remark here that the naïve guess that $\mathfrak{X}$ is $\mathrm{Spf}\, L_{K(n)}S$ is not correct. For instance, let $L_{K(1)}S$ denote the $K(1)$-local sphere, with the p-adic topology. By [Lur17, Corollary 8.2.4.15], we know that $\mathrm{QCoh}(\mathrm{Spf}\, L_{K(1)}S) \simeq \mathrm{Mod}(L_{K(1)}S)^\wedge_p$; but this is not equivalent to $L_{K(1)}\mathrm{Sp} \simeq \mathrm{QCoh}(\mathfrak{X})$.

Vector bundles on $\mathrm{Spf}\, E/G$ when G is a finite subgroup of Γ are "easy". Suppose $X = \mathrm{Spf}\, E$; then every vector bundle is a perfect E-module. Our goal in this section is to study vector bundles over the quotient stack $\mathrm{Spf}\, E/G$ for $G \subseteq \Gamma$ a finite subgroup. This is equivalent to studying the ∞-category of perfect E-modules with a G-action.

Proposition 4.5. *The ∞-category of vector bundles on* $\mathrm{Spf}\, E/G$ *is generated by* $E[G] = E \wedge \Sigma^\infty_+ G$ *as a thick subcategory.*

Proof. Let M be a perfect E-module with a G-action. Since $\pi_0 E$ is a local ring, $\pi_* M$ is a (finitely generated) free $\pi_* E[G]$-module. Let $x_1, \cdots, x_m$ be a basis for $\pi_* M$ over $\pi_* E[G]$; this begets a map $f : E[G]^{\vee k} \vee \Sigma E[G]^{\vee n} \to M$, which is a surjection on homotopy. The fiber of f is also a free $E[G]$-module $E[G]^{\vee i} \vee \Sigma E[G]^{\vee j}$. Therefore, if K is the cofiber of $E[G]^{\vee i} \to \Sigma E[G]^{\vee k}$ and L is the cofiber of $\Sigma E[G]^{\vee j} \to \Sigma E[G]^{\vee n}$, we have a splitting of M as $K \vee \Sigma L$.

We provide an alternative proof in the case that $p \nmid \#G$. Let M be any perfect E-module with a G-action. We claim that M is a retract of $M \widehat{\wedge}_E E[G]$. Indeed, we have maps $\pi_1 : E[G] \to E$ (coming from $G \to *$) and $\pi_2 : E \to E[G]$ (coming from the basepoint). Moreover, our assumptions imply that $\#G$ is invertible in $(\pi_0 E)^\times \supseteq \mathbf{Z}_p^\times$, so $\frac{1}{\#G}\pi_2\pi_1$ gives an idempotent map from $E[G]$ to itself. The image is E, which establishes that E is a retract of $E[G]$, and hence the claim. To finish the proof of the proposition, we note that $M \widehat{\wedge}_E E[G]$ is in the thick subcategory generated by $E[G]$; since M is a retract, the desired result follows. □

One can ask for more satisfying descriptions along the lines of the following result of Bousfield's.

Theorem 4.6 (Bousfield)**.** *Every vector bundle over* $\mathrm{Spf}\, K_2/C_2$ *is a direct sum of suspensions of KO_2, K_2, and $KT = K_2^{h\mathbf{Z}}$.*

Remark 4.7. There are two avenues for generalization.

(1) One can attempt to describe all vector bundles over $\mathrm{Spf}\, E_{p-1}/C_p$. At odd primes, there are a lot more indecomposable representations. Nonetheless, a partial generalization of Bousfield's result is the subject of ongoing work by Hood Chatham.
(2) One can attempt to prove Bousfield's result in the equivariant setting. In [Dev18], we describe a genuine G-equivariant generalization of this result for finite abelian groups G.

5. Picard groups and Anderson duality

We now turn our attention to understanding Picard groups.

5.1. The $K(n)$-local Picard group.

Lemma 5.1. *There is an isomorphism* $\pi_0 \operatorname{Pic} \mathrm{Sp} \simeq \mathbf{Z} \simeq \langle S^1 \rangle$.

Proof. Let $X \in \operatorname{Pic} \mathrm{Sp}$. Then X is a finite spectrum (i.e., is compact), since the sphere is. We might assume that X is connective with $\pi_0 X \neq 0$. The Künneth formula tells us that $Hk_* X$ is concentrated in degree 0 for every field k. It follows from the universal coefficients theorem that $H\mathbf{Z}_* X$ is torsion-free and concentrated in degree zero. Using the Hurewicz theorem, we can conclude that $X \simeq S$. □

We could now attempt to understand the Picard space of $L_{K(n)}\mathrm{Sp}$ – or, perhaps a simpler task, the Picard group of $L_{K(n)}\mathrm{Sp}$. This category is not symmetric monoidal under the ordinary smash product; rather, one has to consider a completed smash product. For this, we have the following calculation due to Hopkins-Mahowald-Sadofsky ([HMS94]).

Theorem 5.2. *At an odd prime*[2]*, there is an isomorphism* $\pi_0 \operatorname{Pic}(L_{K(1)}\mathrm{Sp}) \simeq \mathbf{Z}_p \times \mathbf{Z}/|v_1|$.

As this is really the only computation that is known in general, we will sketch the proof. This relies on the following incredible theorem, again by Hopkins-Mahowald-Sadofsky, a geometric proof of which is the goal of this section.

Theorem 5.3 (Hopkins-Mahowald-Sadofsky [HMS94])**.** *The following conditions are equivalent.*

1. *A $K(n)$-local spectrum M is in* $\operatorname{Pic} L_{K(n)}\mathrm{Sp}$.
2. $\dim_{K(n)_*} K(n)_* M = 1$.
3. *$E_*^\vee M$ is a free E_*-module of rank* 1.

It is worthwhile to remark that since E_* is a complete local ring, the last condition is equivalent to $E_*^\vee M$ being an invertible E_*-module.

Let $M(p^k)$ denote the spectrum obtained by taking the cofiber of $S^{-1} \xrightarrow{p^k} S^{-1}$. There are maps $M(p^k) \to M(p^{k+1})$, which, in the limit, give a spectrum $M(p^\infty)$. This is an invertible spectrum: it sits in a cofiber sequence $S^{-1} \to p^{-1}S^{-1} \to M(p^\infty)$, and multiplication by p annihilates $K(n)$-homology for $n > 0$, so that $L_{K(n)}M(p^\infty) \simeq L_{K(n)}S$ – this certainly has $K(n)$-homology of dimension 1. Since $M(p^k)$ is a finite spectrum, it is of type k for some integer k. A theorem of Adams says that $k = 1$. By the periodicity theorem, we therefore obtain a v_1-self map $v_1^{p^{k-1}} : \Sigma^{2p^{k-1}(p-1)} M(p^k) \to M(p^k)$. We can use this map to construct other $K(1)$-locally invertible spectra; in fact, we will be able to define an injection $\mathbf{Z}_p \to \operatorname{Pic}(L_{K(1)}\mathrm{Sp})$.

Let $a \in \mathbf{Z}_p$, so that $a = \sum_{k=0}^\infty \lambda_k p^k$. Let a_m denote the truncation $\sum_{k=0}^m \lambda_k p^k$. Define a spectrum $S^{-|v_1|a}$ by the homotopy colimit of the diagram

$$\cdots \to \Sigma^{-|v_1|a_{k-1}} M(p^k) \to \Sigma^{-|v_1|a_{k-1}} M(p^{k+1}) \xrightarrow{v_1^{p^k \lambda_k}} \Sigma^{-|v_1|a_k} M(p^{k+1}) \to \Sigma^{-|v_1|a_k} M(p^{k+2}) \to \cdots$$

If $a \in \mathbf{Z} \subset \mathbf{Z}_p$, then $L_{K(1)}S^{-|v_1|a} \simeq L_{K(1)}M(p^\infty)$, as $\lambda_k = 0$ for $k \gg 0$. Since $K(n)$-homology plays nicely with homotopy colimits, we compute that $\dim_{K(1)_*} K(1)_*(S^{-|v_1|a}) = 1$ for every $a \in \mathbf{Z}_p$. This provides us with a continuous homomorphism $\mathbf{Z}_p \to \pi_0 \operatorname{Pic} L_{K(1)}\mathrm{Sp}$. Hopkins-Mahowald-Sadofsky show that this is an injective homomorphism (we will not, as this will take us too far afield), and the cosets of its image are the ordinary spheres $S^1, \cdots, S^{|v_1|}$. In particular, they construct a short exact sequence $0 \to \mathbf{Z}_p^\times \to \pi_0 \operatorname{Pic} L_{K(1)}\mathrm{Sp} \to \mathbf{Z}/2 \to 0$ and show that this

[2] An analogous result is true at $p = 2$; there, we have $\pi_0 \operatorname{Pic}(L_{K(1)}\mathrm{Sp}) \simeq \mathbf{Z}_2 \times \mathbf{Z}/2 \times \mathbf{Z}/2$, generated by the elements described below and the "dual question mark complex".

does not split. Since $\mathbf{Z}_p^\times \simeq \mathbf{Z}_p \times \mathbf{Z}/(p-1)$, this implies that $\pi_0 \operatorname{Pic} L_{K(1)}\mathrm{Sp} \simeq \mathbf{Z}_p \times \mathbf{Z}/(2p-2)$. We know that $|v_1| = 2(p-1)$, so the result follows.

Proof of Theorem 5.3. Since $K(n)$ is a field spectrum, the implication (1) $\Rightarrow$ (2) is easy: if M is $K(n)$-locally invertible, then there exists M' such that $M\widehat{\wedge}M' \simeq L_{K(n)}S$; the result follows by applying $K(n)$-homology and using the Künneth isomorphism.

For the other direction, suppose $\dim_{K(n)_*} K(n)_*M = 1$. Let $Z = \underline{\mathrm{Map}}(M, L_{K(n)}S)$; there is an evaluation map $M \wedge \underline{\mathrm{Map}}(M, L_{K(n)}S) \to L_{K(n)}S$. It suffices to show that this is an equivalence on $K(n)$-homology. Let $\mathcal{C}$ be the subcategory of Sp spanned by all spectra X for which the map $M \wedge \underline{\mathrm{Map}}(M, L_{K(n)}X) \xrightarrow{e_X} L_{K(n)}X$ is an equivalence on $K(n)$-homology. Any finite type n spectrum X admits a finite filtration on $L_{K(n)}X$ with each cofiber a wedge of $K(n)$s. The category $\mathcal{C}$ is closed under cofibrations and wedges, so to show that e_X is an equivalence for any finite type n spectrum, it suffices to observe that $e_{K(n)}$ is an equivalence on $K(n)$-homology. Using the finiteness of X, we deduce that e_X is an $K(n)$-equivalence if and only if $M \wedge \underline{\mathrm{Map}}(M, L_{K(n)}S) \wedge X \to X \wedge L_{K(n)}S$ (which is the same map as e_X) is an equivalence. In turn, this happens if and only if e_S is a $K(n)$-equivalence, as desired.

Hopkins-Mahowald-Sadofsky prove that (2) is equivalent to (3). We will instead show that (1) is equivalent to (3) using the tools from derived algebraic geometry developed in the previous sections. Lemma 4.4 shows that $\operatorname{Pic} L_{K(n)}\mathrm{Sp} = \operatorname{Pic}(\mathfrak{X})$. The Picard space satisfies descent[3], and hence $\operatorname{Pic}(\mathfrak{X}) \simeq \operatorname{Pic}(\operatorname{Spf} E)^{h\Gamma}$. Let $\tau : \operatorname{Spf} E \to \mathfrak{X}$ denote the étale cover. Assume statement (1) of Theorem 5.3, i.e., suppose M is in $\operatorname{Pic} L_{K(n)}\mathrm{Sp}$. Since E satisfies the conditions appearing before Proposition 2.6, every invertible object of $\mathrm{QCoh}(\operatorname{Spf} E)$ is of the form $\Sigma^k\mathcal{L}$ where $\mathcal{L}$ is a line bundle on $\operatorname{Spf} E$ and $k \in \mathbf{Z}$. This means that we can assume that τ^*M is a line bundle. It is not hard to prove that $\tau^*M \simeq E \mathbin{\widehat{\wedge}} M$. Since Γ acts on the first factor, it follows that $E_*^\vee(M)$ is a free E_*-module of rank 1. Conversely, assume (3). As a consequence of [BR05], we know that $E \mathbin{\widehat{\wedge}} M$ is in $\operatorname{Pic}(\operatorname{Spf} E)$, where $M \in \mathrm{QCoh}(\mathfrak{X})$. It suffices to prove that this has a Γ-linearization. But by Goerss-Hopkins-Miller Γ acts continuously on $E \mathbin{\widehat{\wedge}} M$ via the first factor, and E descends to the structure sheaf $L_{K(n)}S$ on $\mathfrak{X}$, so $E \mathbin{\widehat{\wedge}} M$ has a Γ-linearization, as desired. □

Remark 5.4. The same argument proves that the following statements are equivalent, for G a finite subgroup of Γ.

- An E^{hG}-module M is in $\operatorname{Pic}(E^{hG})$.
- $E_*^\vee M$ is a free $E_*^\vee E^{hG}$-module of rank 1.

Remark 5.5. A direct proof of the equivalence between (2) and (3) is also possible. By replacing M be ΣM if necessary, we may assume that $E_*^\vee M$ (resp. $K(n)_*M$) is concentrated in even degrees. Using [HS99, Proposition 8.4], we see that in this case, the rank of $E_*^\vee M$ as an E_*-module agrees with the dimension of $K(n)_*M$ as a $K(n)_*$-module. This is a version of Nakayama's lemma in the case of spectra with *even* completed E-homology.

Lemma 5.6. *There is an equivalence* $\operatorname{Pic}(\operatorname{Spf} E) \simeq \operatorname{Pic}(E)$ *that respects the* Γ*-action.*

Proof. This follows from [Lur17, Theorem 8.5.0.3]. Here is another, more topological, proof of this claim: clearly, any element of $\operatorname{Pic}(E)$ is contained in $\operatorname{Pic}(\operatorname{Spf} E)$. Conversely, an element of $\operatorname{Pic}(\operatorname{Spf} E)$ is contained in $\operatorname{Pic}(E)$ if and only if it is a perfect E-module. This follows from [Mat16, Proposition 10.11]. □

This tells us that $\operatorname{Pic}(E)^{hG} \simeq \operatorname{Pic}(\operatorname{Spf} E/G) \simeq \operatorname{Pic}(E^{hG})$ for any finite subgroup $G \subseteq \Gamma$.

Some results follow directly from our proof of Theorem 5.3.

[3]The Picard *group*, however, generally does not satisfy any form of descent.

Remark 5.7. There are isomorphisms $\pi_0 \operatorname{Pic}(E) \simeq \mathbf{Z}/2$ and $\pi_1 \operatorname{Pic}(E) \simeq (W(\mathbf{F}_{p^n})[[u_1, \cdots, u_{n-1}]])^\times$. In fact, there is a fiber sequence $\mathrm{bgl}_1(E) \to \mathfrak{pic}(E) \to H\mathbf{F}_2$.

We remark that one can construct a map

$$\epsilon : \pi_0 \operatorname{Pic} L_{K(n)}\mathrm{Sp} \to \mathrm{H}^1_c(\Gamma; \pi_1 \operatorname{Pic} E) \simeq \mathrm{H}^1_c(\Gamma; E_0^\times)$$

as follows. Let $\mathcal{L}$ be an element of $\pi_0 \operatorname{Pic} L_{K(n)}\mathrm{Sp}$, thought of as (an equivalence class of) a line bundle on $\operatorname{Spf} E/\Gamma$. This is a Γ-equivariant line bundle on $\operatorname{Spf} E$. The underlying line bundle gives rise to a Γ-equivariant line bundle on $\operatorname{Spf} \pi_0 E$. The monodromy action ($\Gamma$ is the "étale fundamental group" of the quotient stack $\operatorname{Spf} E/\Gamma$; see [Mat16]) gives rise to a (continuous) representation $\Gamma \to \mathrm{GL}_1(\pi_0 E) = E_0^\times$, which gives the desired map ϵ.

Likewise, the equivalence $\operatorname{Pic}(E)^{hG} \simeq \operatorname{Pic}(E^{hG})$ begets a spectral sequence $E_2^{s,t} = \mathrm{H}^s(G; \pi_t \operatorname{Pic}(E)) \Rightarrow \pi_{t-s} \operatorname{Pic}(E^{hG})$.

5.2. **Anderson self-duality.** In this section, we will abuse notation by writing $I_{\mathbf{Z}}$ for $L_{K(n)} I_{\mathbf{Z}_p} L_{K(n)} S$. If $G \subseteq \Gamma$ is a finite subgroup of the Morava stabilizer group (and if $G = \Gamma$), the pushforward q_* coming from the quotient map $q : \operatorname{Spf} E \to \operatorname{Spf} E/G$ admits a right adjoint $q^!$. Explicitly, one has $q^!(M) = L_{K(n)}\underline{\mathrm{Map}}_{E^{hG}}(E, M)$. We begin with the trivial observation that $\operatorname{Spf} E$ is self-dual.

Theorem 5.8. *Let G be a finite subgroup of Γ. Then $I_{\mathbf{Z}} E^{hG}$ is in the Picard group of E^{hG}.*

Proof. Let G be a finite subgroup of Γ. Then there is an equivalence $\mathrm{QCoh}(\operatorname{Spec} E/G) \simeq \mathrm{Mod}(E)^{hG}$. Since the extension $E^{hG} \to E$ is G-Galois ([MM15, Example 6.2]), there is an equivalence $\mathrm{Mod}(E)^{hG} \simeq \mathrm{Mod}(E^{hG})$. Utilizing Lemma 3.9, we learn that $\operatorname{Spec} E/G$ is self-dual, so that Theorem 3.15 (and Remark 3.12) shows that $I_{\mathbf{Z}} E^{hG}$ is in $\operatorname{Pic} \operatorname{Spec} E/G \simeq \operatorname{Pic}(E^{hG}) \simeq \operatorname{Pic}(E)^{hG}$. □

In future work, we will generalize this (using Theorem 3.15 again) to "global" cases like Tmf with level structure, and PEL Shimura varieties as considered in [BL10], as well as to genuine K-equivariant versions, where K is a finite abelian group.

As a corollary, we obtain a reproof of a consequence of a recent result of Barthel-Beaudry-Stojanoska ([BBS17]).

Corollary 5.9. *Let G be a finite subgroup of Γ at height $p-1$. Then $L_{K(n)} I_{\mathbf{Z}} E^{hG}$ is equivalent to a shift of E^{hG}.*

Proof. At height $p-1$, since $\pi_0 \operatorname{Pic}(E^{hG})$ is cyclic ([HMS17]), we conclude from Theorem 5.8 that E^{hG} is Anderson self-dual. □

Remark 5.10. We can deduce the $K(n)$-local Spanier-Whitehead self-duality of E^{hG} at height $p-1$ from the above example. (This self-duality is true more generally, as we will prove below, but this example illustrates an application of Theorem 5.8.) Since $I_{\mathbf{Z}}$ is invertible by Gross-Hopkins duality (see Remark 5.19), we know that $DE^{hG} \simeq I_{\mathbf{Z}}^{-1} \widehat{\wedge} I_{\mathbf{Z}} E^{hG}$. From the above example, we know that $L_{K(n)} I_{\mathbf{Z}} E^{hG}$ is equivalent to a shift of E^{hG} at $n = p-1$. We will be done if $I_{\mathbf{Z}} \widehat{\wedge} E^{hG}$ is equivalent to a shift of E^{hG}. As $(I_{\mathbf{Z}} \widehat{\wedge} E^{hG}) \widehat{\wedge}_{E^{hG}} M \simeq I_{\mathbf{Z}} \widehat{\wedge} M$, we can use Gross-Hopkins duality to deduce that $M = E^{hG} \widehat{\wedge} I_{\mathbf{Z}}^{-1}$ is an inverse to $I_{\mathbf{Z}} \widehat{\wedge} E^{hG}$ in $L_{K(n)}\mathrm{Mod}(E^{hG})$. It follows from $\pi_0 \operatorname{Pic}(E^{hG})$ being cyclic that $I_{\mathbf{Z}} \widehat{\wedge} E^{hG}$ is a shift of E^{hG}, as desired.

Remark 5.11. For instance, we recover the well-known result that $KO_2^\wedge$ is $K(1)$-locally Spanier-Whitehead self-dual. At the prime 3, there is an equivalence $L_{K(2)} TMF \simeq EO_2$; therefore, we also recover the $K(2)$-local Spanier-Whitehead self-duality of $L_{K(2)} TMF$. This result is originally due to Behrens ([Beh06, Proposition 2.6.1]).

This motivates a natural conjecture, which is widely believed to be true:

Conjecture 5.12. Let $G \subseteq \Gamma$ be a finite subgroup of the Morava stabilizer group at height n. Then $D(E^{hG}) \simeq (DE)^{hG}$ is a shift of E^{hG}, i.e., E^{hG} is Spanier-Whitehead self-dual.

Remark 5.13. Conjecture 5.12 is true if $(p-1)$ does not divide n. In Appendix A, we prove Conjecture 5.12 when $(p-1)$ divides n in the case when G has Sylow p-subgroup C_p (hinging on unpublished work of Hill-Hopkins-Ravenel in [HHR] and [Hil]). This property is satisfied by all finite subgroups with nontrivial p-torsion of the Morava stabilizer group whenever p does not divide $n/(p-1)$.

Definition 5.14. Let $\kappa(G)$ be the group of "exotic" invertible E^{hG}-modules, i.e., the group of invertible E^{hG}-modules M such that, as $E_*[[\Gamma]]$-modules, $E_*^\vee(M) \simeq E_*^\vee(E^{hG})$.

Conditional on Conjecture 5.12, we obtain the following result (whose proof is just Remark 5.10 run backwards), which is a generalization of Corollary 5.9:

Theorem 5.15. *Assume Conjecture 5.12. Suppose $G \subset \Gamma$ is a finite subgroup. If $\kappa(G)$ is cyclic or trivial, then $L_{K(n)}I_{\mathbf{Z}}E^{hG}$ is equivalent to a shift of E^{hG}.*

Proof. Since $I_{\mathbf{Z}}$ is $K(n)$-locally invertible (see Remark 5.19), there is an equivalence $L_{K(n)}I_{\mathbf{Z}}E^{hG} \simeq I_{\mathbf{Z}}^{-1}\widehat{\wedge}DE^{hG}$. The Tate spectrum E^{tG} is contractible, so

$$DE^{hG} \simeq \underline{\mathrm{Map}}(E^{hG}, L_{K(n)}S) \simeq \underline{\mathrm{Map}}(E_{hG}, L_{K(n)}S) \simeq \underline{\mathrm{Map}}(E, L_{K(n)}S)^{hG} \simeq (DE)^{hG}.$$

By Conjecture 5.12, DE^{hG} is equivalent to a shift of E^{hG}. We are reduced to proving that $I_{\mathbf{Z}}^{-1}\widehat{\wedge}E^{hG}$ is equivalent to a shift of E^{hG}. To prove that $I_{\mathbf{Z}}\widehat{\wedge}E^{hG}$ is equivalent to a shift of E^{hG}, we need to understand the image of Pic_n inside $\mathrm{Pic}(E^{hG})$, under the map $\mathrm{Pic}_n \to \mathrm{Pic}(E^{hG})$ given by $X \mapsto X\widehat{\wedge}E^{hG}$. Our hypotheses on $\kappa(G)$ are enough to guarantee that the image of Pic_n inside $\mathrm{Pic}(E^{hG})$ is cyclic; this shows that $I_{\mathbf{Z}}^{-1}\widehat{\wedge}E^{hG}$ is equivalent to a shift of E^{hG}, as desired. $\square$

We illustrate some examples of Theorem 5.15.

Remark 5.16. Suppose G has order coprime to p. We claim that $L_{K(n)}I_{\mathbf{Z}}E^{hG} \simeq \Sigma^? E^{hG}$. This is the easiest case of Theorem 5.15, so we will provide two proofs.

(1) It follows from the homotopy fixed point spectral sequence for $\mathrm{Pic}(E^{hG})$ that $\pi_0 \mathrm{Pic}(E^{hG})$ is cyclic if $\gcd(|G|, p) = 1$. Since $I_{\mathbf{Z}}E^{hG}$ is an invertible E^{hG}-module, it follows that E^{hG} is Anderson self-dual.

(2) We claim that $\kappa(G) = 0$; the desired result follows from Theorem 5.15. Let $X \in \kappa(G)$, and pick an isomorphism $f : E_*^\vee(X) \xrightarrow{\sim} E_*^\vee E^{hG}$. Shapiro's lemma provides an isomorphism $\widetilde{f} : \mathrm{H}^*(G; \pi_*E) \xrightarrow{\sim} \mathrm{H}_c^*(\Gamma; E_*^\vee(X))$. Since $\gcd(|G|, p) = 1$, the group cohomology $\mathrm{H}^s(G; \pi_*E)$ is trivial for $s > 0$. Any differential $d_k^X : \mathrm{H}_c^0(\Gamma; E_0^\vee(X)) \to \mathrm{H}_c^k(\Gamma; E_{k+1}^\vee(X))$ is therefore trivial, so the identity class in $\mathrm{H}_c^0(\Gamma; E_0^\vee(X))$ survives to the E_∞-page; this begets a map $L_{K(n)}S \to X$, which extends to an equivalence $X \simeq E^{hG}$. Since X was arbitrary, we conclude that $\kappa(G) = 0$. If n is not divisible by $p-1$, it is known that all maximal finite subgroups $G \subseteq \Gamma$ have order coprime to p (this is proved, for instance, in [Hew99, Theorem 1.3] and [Buj12, Proposition 1.7]). The above discussion now implies that $L_{K(n)}I_{\mathbf{Z}}E^{hG}$ is equivalent to a shift of E^{hG}.

Example 5.17. At height 2 and the prime 2, it is known that if G contains all the p-torsion in Γ, the group $\kappa(G)$ is isomorphic to $\mathbf{Z}/8$ ([Bea16, Page 18]). Theorem 5.15 proves that at $p = 2$, the spectrum $L_{K(2)}I_{\mathbf{Z}}E^{hG}$ is equivalent to a shift of E^{hG}.

Remark 5.18. One does not need $\kappa(G)$ to vanish in order to get self-duality: if $F \subseteq \Gamma$ (at any height and prime) is in the kernel of the determinant map, then $\mathrm{Spf}\, E/F$ is self-dual; indeed, the proof of Proposition 5.15 shows that, in order to prove the Anderson self-duality of E^{hF}, we only need to know that $L_{K(n)}I_{\mathbf{Z}}^{-1} \wedge E^{hF}$ is equivalent to a shift of E^{hF}. This follows (e.g., from analyzing the homotopy fixed points spectral sequence) from the fact that $F \subseteq \ker\det$.

In [GH94], Gross and Hopkins prove the following result.

Remark 5.19 (Gross-Hopkins duality)**.** Let MS denote the fiber of the map $L_nS \to L_{n-1}S$. Gross-Hopkins duality asserts that the spectrum $I_{\mathbf{Q}/\mathbf{Z}}MS$ is invertible. There is an equivalence $I_{\mathbf{Q}/\mathbf{Z}}MS \simeq \Sigma L_{K(n)}I_{\mathbf{Z}}L_{K(n)}S$. This follows immediately from the fact that $L_{K(n)}I_{\mathbf{Q}}X \simeq 0$. It therefore suffices to prove that $f^!I_{\mathbf{Z}}$ (whose underlying $K(n)$-local spectrum is $L_{K(n)}I_{\mathbf{Z}}L_{K(n)}S$) is invertible, where $f : \operatorname{Spf} E/\Gamma \to \operatorname{Spec} S$ is the structure map.

We have an étale cover $\tau : \operatorname{Spf} E \to \operatorname{Spf} E/\Gamma$, but it is not a finite morphism. This map therefore does not satisfy the hypotheses of Lemma 3.9. However, one can use the equivalence (see [Str00]) $\Sigma^{n^2}DE \simeq E$ to show that $\operatorname{Spf} E/\Gamma$ is self-dual. In order to establish that $f^!I_{\mathbf{Z}}$ is invertible, it suffices to establish an analogue of Theorem 3.15. However, the finiteness assumptions there do not apply to τ, so we do not know how to prove this.

Remark 5.20. As $f^!I_{\mathbf{Z}}$ is invertible, the dualizing spectrum $\tau^*f^!I_{\mathbf{Z}}$ defines a line bundle on $\operatorname{Spf} E$. The action of Γ defines a map $\Gamma \to \mathrm{GL}_1(E)$. Gross and Hopkins show that composing with the map $\mathrm{GL}_1(E) \to \mathrm{GL}_1(\pi_0 E)$ defines the determinant representation of Γ. We will return to the problem of proving this result via derived algebro-geometric methods in a later paper.

Appendix A. Spanier-Whitehead self-duality of $E_{n(p-1)}^{hG}$

In this section, we will work at height $n(p-1)$ for some integer n. Fix the notation G for a finite subgroup of Γ whose Sylow p-subgroup is C_p. In this section, we will prove the following two results:

Proposition A.1. *Under the assumptions in the beginning of this section, Conjecture 5.12 is true for E^{hG}.*

Remark A.2. Note that every finite subgroup of Γ with nontrivial p-torsion has Sylow p-subgroup C_p whenever p does not divide n, so Conjecture 5.12 is true for *every* finite subgroup G in this case.

Our proofs are computational. We will prove Proposition A.1 by following the argument in [BBS17, Corollary 4.11]. We will rely on the following unpublished result of Hill-Hopkins-Ravenel from [Hil, Propositions 1 and 2] (see also [HHR] for a more detailed exposition in the case $n = 2$):

Theorem A.3 (Hill-Hopkins-Ravenel)**.** *Modulo the image of the transfer (all such elements are permanent cycles), the E_2-term of the HFPSS for $E_{n(p-1)}^{hC_p}$ is given by*

$$\Lambda(\alpha_1, \cdots, \alpha_n) \otimes P(\beta, \delta_1, \cdots, \delta_n^{\pm 1}),$$

where the bidegrees of the elements, written in Adams indexing, are $|\alpha_i| = (-3, 1)$, $|\beta| = (-2, 0)$, and $|\delta_i| = (-2p, 0)$. Moreover, all of the differentials are determined by

- *For $1 \le i \le n$, there are differentials*

$$d_{2p^i-1}(\delta_n^{p^{i-1}}) = a_i \delta_n^{p^{i-1}} h_{i,0} \beta^{p^i-1}; \tag{1}$$

  here, $h_{i,0}$ are certain elements obtained by translating the elements α_i by powers of δ_n, and the elements a_i are units in $\mathbf{F}_{p^n}$.
- *For $1 \le i \le n$, there are "Toda-style" differentials on the $E_{2(p^i-1)(p-1)-1}$-page which truncate the β-towers on δ_i.*
- *The classes $\delta_i\delta_n^{-1}$ and $\delta_n^{p^n}$ are permanent cycles.*

Proof of Proposition A.1. Let $E = E_{n(p-1)}$, and let $G = C_p$. According to [Str00, Proposition 16], the $\pi_* E^{hG}$-module $\pi_* DE^{hG}$ is free of rank one as a C_p-$\pi_* E$-module on a generator that we shall denote by γ, and the HFPSS for DE^{hG} is a module over that of E^{hG}. These spectral sequences collapse at a finite stage, so by [BBS17, Lemma 4.7], it suffices to prove that $\delta_n^N \gamma$ is a permanent cycle for some integer N. Before proceeding with the proof, let us show how this proves the result for finite subgroups $G \subsetneq \Gamma$ with Sylow p-subgroup C_p. As the Leray-Hochschild-Serre spectral sequence degenerates, there is an isomorphism of E_2-pages $\mathrm{H}^*(G, \pi_* DE) \simeq \mathrm{H}^*(C_p, \pi_* DE)^{G/C_p}$. The norm of $\delta_n^N \gamma$ under the action of G/C_p is a permanent cycle in the HFPSS for DE^{hG}, so we are done.

To prove the result when $G = C_p$, we argue inductively. It follows from Theorem A.3 that γ is a $(2p-2)$-cycle, and that

$$d_{2p-1}(\gamma) = b_1 h_{1,0} \beta^{p-1} \gamma,$$

for some unit $b_1 \in \mathbf{F}_{p^n}^\times$. It follows that

$$d_{2p-1}(\delta_n^{k_1} \gamma) = (k_1 a_1 + b_1) h_{1,0} \beta^{p-1} \delta_n^{k_1 - 1} \gamma$$

is zero if k_1 is chosen to be congruent to $-b_1/a_1$ modulo p. Therefore, $\delta_n^{k_1} \gamma$ is a $(2p-1)$-cycle (and hence a $(2p^2-2)$-cycle, by sparsity). For the inductive step, suppose $\delta_n^{k_i} \gamma$ is a $(2p^{i+1}-2)$-cycle; we need to show that there is some N such that $\delta_n^N \gamma$ is a $(2p^{i+1}-1)$-cycle. We have

$$d_{2p^{i+1}-1}(\delta_n^{k_i} \gamma) = b_{i+1} h_{i+1,0} \beta^{p^{i+1}-1} \gamma$$

for some $b_{i+1} \in \mathbf{F}_{p^n}^\times$. Arguing as above, we have

$$d_{2p^{i+1}-1}(\delta_n^{\ell_{i+1} p^i + k_i} \gamma) = (\ell_{i+1} a_{i+1} + b_{i+1}) \delta_n^{k_{i+1} p^i} h_{i+1,0} \beta^{p^{i+1}-1} \gamma,$$

so choosing ℓ_{i+1} congruent to $-b_{i+1}/a_{i+1}$ modulo p, we find that $\delta_n^{\ell_{i+1} p^i + k_i} \gamma$ is a $(2p^{i+1}-1)$-cycle, as desired. Having completed the inductive step, we find that DE^{hC_p} is a shift of E^{hC_p} by $2pN = 2p \sum_{i=0}^n \ell_{i+1} p^i$. This finishes the proof of Proposition A.1. $\square$

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