CHROMATIC HOMOTOPY THEORY

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Abstract. These are (most of the) notes from a course on chromatic homotopy theory which I taught in January 2018. I learned a lot from [Lur10] and [Hop99], and you probably will, too.

Contents

1. The Ravenel conjectures

The goal of this section is to give a high-level overview of the chromatic viewpoint on stable homotopy theory, with the Ravenel conjectures as the primary focus (we closely follow Ravenel's seminal paper [Rav84]). Essentially no precise proofs are provided, but many claims are supported by analogies with algebraic geometry.

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1.1. Introduction.

One of the goals of homotopy theory is to understand the ring π_*S. This ring is famously hard to study, thanks in part to the following theorem of Nishida's (which immediately implies that π_*S is extremely far from being Noetherian).

Theorem 1.1 (Nishida). *Any positive degree element in π_*S is nilpotent.*

This is rather surprising, and didn't really admit a conceptual explanation for a while. As an example of this nilpotency, if $\eta \in \pi_1 S$ denotes the first Hopf element, then $\eta^4 = 0$.

However, we can still attempt to find qualitative facts about π_*S. For instance, Serre showed that $\mathbf{Q}\otimes\pi_*S$ is concentrated in degree 0 (all the groups in higher degrees are finite, hence torsion). In the language of Bousfield localisation, this result can be phrased as the fact that $L_{H\mathbf{Q}}S \simeq H\mathbf{Q}$.

Toda discovered a family of elements of π_*S. In *J(X) IV*, Adams gave an explanation for the existence of these elements. He showed that, if S/p denotes the mod p Moore spectrum (given by the cofibre of the map $S \xrightarrow{p} S$), then there is a "self-map"

$$v_1 : \Sigma^{2(p-1)} S/p \to S/p,$$

at least when p was odd (if $p = 2$, the shift is 8, and this map is denoted v_1^4, for reasons to be explained later). This map is nontrivial, as Adams showed that this map is an isomorphism in K-theory: it induces multiplication by $\beta^{2(p-1)}$, where β is the Bott element.

All iterates of this map are therefore nontrivial:

$$v_1^t : \Sigma^{2(p-1)t} S/p \to S/p.$$

The spectrum S/p is a finite spectrum with two cells, since it sits inside a cofibre sequence

$$S \to S/p \to S^1;$$

the first map is the inclusion of the bottom cell, while the second is projection onto the top cell. Composing these maps with the map v_1^t, we get a nontrivial map

$$S^{2(p-1)t} \to \Sigma^{2(p-1)t} S/p \xrightarrow{v_1^t} S/p \to S^1,$$

which is a nontrivial element $\alpha_t \in \pi_{2(p-1)t-1}S$. All of these elements are in the image of J. (One can, and should, think of the "v_1-periodicity" as coming from the periodicity of the orthogonal group.) In another section, we will prove this result via explicit computations.

This is rather fabulous: we've constructed an infinite family of elements of π_*S. It's natural to ask if we can somehow iterate this procedure to procure more elements of π_*S. Namely: let $S/(p, v_1)$ denote the fibre of v_1-self map. Does $S/(p, v_1)$ admit a self map?

If $p \geq 5$, then Smith showed that the answer is yes: there is a self map

$$v_2 : \Sigma^{2(p^2-1)} S/(p, v_1) \to S/(p, v_1),$$

and composing with the inclusion of the bottom cell and projecting onto the bottom cell gives an infinite family of elements $\beta_t \in \pi_{2(p^2-1)t-1}S$. It's not obvious that these elements are nontrivial (but they are)! It would help if we knew that the map v_2 (and all of its iterates) were nontrivial, but we haven't established this yet.

1.2. **A conceptual explanation.** However, the proof of the nontriviality of these elements involved hard computations with the Adams spectral sequence (for the α family, you may remember this from Adams' paper). One might demand a more conceptual explanation for the existence of these elements.

We can try to explain the nontriviality of these elements via the Adams-Novikov spectral sequence. One can construct the *E-based Adams spectral sequence* for any (connective) ring spectrum E, whose E_2-page admits a particularly nice description when E_*E is flat over E_*. We will assume that our spectrum E is nice enough that whatever it converges to is just the Bousfield localisation, so that the spectral sequence runs

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X) \Rightarrow \pi_* L_E X.$$

As Haynes pointed out to me, this is pretty terrible notation: the E_2-page depends on the E_*E-comodule structure of E_*X. If $E = H\mathbf{F}_2$, this is the classical Adams spectral sequence. Our interest will be in the case when E is related to MU.

We computed that

$$MU_* \simeq \mathbf{Z}[x_1, x_2, \cdots],$$

with $|x_i| = 2i$. From now on, we'll localise everything at a fixed prime p; we'll be so careless that the prime will almost always be omitted. In that case, Quillen and Brown–Peterson showed that MU splits as

$$MU \simeq \bigvee \Sigma^? BP,$$

where BP is called the *Brown–Peterson spectrum*[1].

The homotopy groups of BP is also polynomial:

$$BP_* \simeq \mathbf{Z}_{(p)}[v_1, v_2, \cdots],$$

where $|v_n| = 2(p^n - 1)$. They're much sparser than that of MU, which is useful in analyzing the BP-based Adams spectral sequence. It turns out that BP_*BP is a polynomial ring over BP_* (so it's flat), which gives us the *Adams–Novikov spectral sequence* (usually shortened to ANSS)

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*X) \Rightarrow \pi_* L_{BP} X.$$

One can show that $L_{BP}X \simeq X_{(p)}$. For simplicity[2], we will usually just write $\mathrm{Ext}(M)$ for $\mathrm{Ext}_{BP_*BP}(BP_*, M)$. The ANSS is telling us that BP "sees" the p-local part of the stable homotopy category. This vague statement will be something that we'll take extremely seriously.

To begin to compute $\pi_* S_{(p)}$ using the ANSS, we'd therefore like to understand $\mathrm{Ext}(BP_*)$. This turns out to be a complicated task, but it pays off well. As homotopy theorists, our motto is: if something is hard to compute, use spectral sequences to simplify your problem!

Taking this to heart, we would like to find a spectral sequence converging to $\mathrm{Ext}(BP_*)$. Let us inductively define (BP_*, BP_*BP)-comodules[3] as follows. Let $N^0 = BP_*$, and let $v_0 = p$. Let $M^n = v_n^{-1} N^n$, where the N^n are defined inductively by the short exact sequence

$$0 \to N^n \to M^n \to N^{n+1} \to 0. \tag{1}$$

The sequence

$$BP_* \to M^0 = p^{-1}BP_* \to M^1 \to M^2 \to \cdots$$

is called the *chromatic resolution*. Standard homological algebra techniques ([Rav86, A1.3.2]) now give us a spectral sequence with

$$E_1^{n,s} = \mathrm{Ext}^s_{BP_*BP}(BP_*, M^n) = \mathrm{Ext}^s(M^n) \Rightarrow \mathrm{Ext}(BP_*).$$

This is the *chromatic spectral sequence*[4].

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[1] MU is an E_∞-ring spectrum (it is the Thom spectrum of a vector bundle over BU classified by an infinite loop map to $BGL_1(S)$, which is the classifying space for spherical fibrations; the Thom spectrum of an n-fold loop map is an E_n-ring spectrum), and it was a major conjecture for a long time as to whether BP was an E_∞-ring. *Very* recently (this year!), it was shown that BP is *not* an E_∞-ring. This has a bunch of important ramifications (which, unfortunately, we will not describe) for the global picture of homotopy theory that will be described below.

[2] Both Haynes and I dislike this notation; it should really be $H^{*,*}(M)$; but sadly, this is not the notation that is used in Ravenel's green book, which is the canonical reference for chromatic computations.

[3] Don't worry if you don't know what this means; we do not need the explicit definition for the sake of these notes.

[4] We've completely butchered the indexing above, but this is really a *tri*graded spectral sequence.

In a very vague sense, this is filtering the stable homotopy groups of spheres by "primes". Throughout, it will be helpful to borrow some intuition from commutative algebra/algebraic geometry. If R is an abelian group, there are natural maps

$$R \to R_{(p)} \to R_{\mathbf{Q}} = \mathbf{Q} \otimes R.$$

If $R = \mathbf{Z}$, this corresponds to the fact that $\mathbf{Z}$ has Krull dimension 1.

In stable homotopy theory, we also have maps $X \to X_{(p)} \to X_{\mathbf{Q}} = L_{H\mathbf{Q}}X$ for any spectrum X. But the sphere has "infinite Krull dimension", coming from the v_n's: as we said above, BP "sees" the whole p-local stable homotopy category, and BP_* has infinite Krull dimension. We therefore expect there to be an infinite number of spectra sitting in between $X_{(p)}$ and $L_{H\mathbf{Q}}X$. More precisely, we expect there to be a filtration of any p-local spectrum X:

$$X \to \cdots \to L_2X \to L_1X \to L_0X = L_{H\mathbf{Q}}X = H\mathbf{Q} \wedge X,$$

where L_nX is given[5] by "killing $v_{n+1}, v_{n+2}, \cdots$, and inverting v_n". We will make this more precise momentarily.

1.3. $E(n)$**-localisation.** This filtration is exactly what we will focus our attention on. Before this, we need to describe a geometric way to cone off elements in the homotopy of a ring spectrum. If R is a ring spectrum and $x \in \pi_*R$, we can take the cofibre of the map

$$\Sigma^{|x|}R = S^{|x|} \wedge R \xrightarrow{x \wedge 1} R \wedge R \xrightarrow{\mu} R,$$

and this is denoted R/x. Likewise, we can iterate the map to get a system $R \to \Sigma^{-|x|}R \to \Sigma^{-2|x|}R \to \cdots$, and the (homotopy) colimit is denoted $x^{-1}R$, for obvious reasons.

At this point, a warning is necessary. In general, there is no reason for the cone R/x to be a ring spectrum, in contrast to classical algebra. However, $x^{-1}R$ *is* a ring spectrum.

One way to obtain $H\mathbf{Q}$ from BP is by killing $v_1, v_2, \cdots$, and inverting p. More precisely, we can cone off the elements $v_1, v_2, \cdots$, and then invert p. In other words,

$$H\mathbf{Q} \simeq p^{-1}BP/(v_1, v_2, \cdots).$$

The bottom layer of the tower is supposed to be classical algebra, so it shouldn't see any of the higher v_n's. Motivated by this, we define $E(0) = H\mathbf{Q}$, and let L_0 denote localisation with respect to $E(0)$.

We would like the nth layer of the tower to "detect $p, v_1, \cdots, v_n$", so we define

$$E(n) = v_n^{-1}BP/(v_{n+1}, v_{n+2}, \cdots),$$

and let L_n denote localisation with respect to $E(n)$. Likewise, we can define spectra which detect "exactly" v_n:

$$K(n) = v_n^{-1}BP/(p, v_1, \cdots, v_{n-1}, v_{n+1}, \cdots).$$

For instance, $E(0) = K(0) = H\mathbf{Q}$, and $E(1)$ is one of the $(p-1)$ summands of p-local complex K-theory. The spectrum $K(1)$ is a retract of mod p complex K-theory. Ravenel proves that $E(n)$ is a ring spectrum.

However, it is not clear that there are natural transformations $L_n \to L_{n-1}$ which make the following diagram commute

$$\begin{array}{ccc} & 1 & \\ \swarrow & & \searrow \\ L_n & \longrightarrow & L_{n-1}. \end{array}$$

To construct these maps, we will resort to Bousfield classes, which provide a simpler conceptual approach to Bousfield localisation.

Let E be any spectrum. Denote by $\langle E \rangle$ the equivalence class of E under the relation on spectra defined as follows. $E \sim F$ if and only if the following condition is satisfied:

($*$) a spectrum is E_*-acyclic (i.e., $E_*X = 0$) iff it is F_*-acyclic.

[5] Of course, this isn't precise since "killing v_i" makes no sense for a general spectrum.

$\langle E\rangle \leq \langle G\rangle$ iff every G_*-acyclic spectrum is E_*-acyclic, i.e., G is "stronger" than E. It is an easy exercise to show that this implies that there is a natural transformation $L_G \to L_E$. Write $\langle E \vee F\rangle = \langle E\rangle \vee \langle F\rangle$, and $\langle E \wedge F\rangle = \langle E\rangle \wedge \langle F\rangle$.

To get a natural transformation $L_n \to L_{n-1}$, it therefore suffices to show that $\langle E(n-1)\rangle \leq \langle E(n)\rangle$. This stems from the following theorem.

Theorem 1.2 (Ravenel-Johnson-Yosimura)**.** *We have the following identification of Bousfield classes:*

$$\langle E(n)\rangle = \bigvee_{i=0}^{n} \langle K(i)\rangle = \langle v_n^{-1} BP\rangle;$$

therefore, we have an identification of functors

$$L_n X \simeq L_{K(0)\vee\cdots\vee K(n)} \simeq L_{v_n^{-1}BP}.$$

We therefore have a natural transformation

$$L_n X \to L_{n-1} X$$

compatible with the localisation maps from X, and hence a tower as above.

1.4. **Are we there yet?** This shouldn't be satisfactory: we've still been unable to make precise the idea that the functor L_n is like "killing $v_{n+1}, v_{n+2}, \cdots$ and inverting v_n". One way to do this would be to show that we can use the functors L_n to spectrally realize the exact sequences in (1).

Let X be a spectrum. Motivated by the construction of the above exact sequences, we will define $N_0 X = X$, and $M_n X = L_n N_n X$, where $N_n X$ is defined inductively by the cofibre sequence

$$N_n X \to M_n X \to N_{n+1} X. \tag{2}$$

One may therefore make the following conjecture, known as the localisation conjecture for S: if $X = S$ is the sphere, then the BP-homology of the cofibre sequences in (2) gives the exact sequences in (1). One reason for why this might be reasonable (other than the fact that the definitions are extremely similar) comes from the following theorem.

Theorem 1.3. *$N_n BP$ and $M_n BP$ are BP-modules, and the maps $N_n BP \to M_n BP$ and $M_n BP \to N_{n+1} BP$ are BP-module maps. Moreover,*

$$\pi_* N_n BP \simeq N^n,\ \ \pi_* M_n BP \simeq M^n.$$

Given the chromatic resolution of BP_*, it is also natural to suspect that the homotopy limit of the tower

$$\cdots \to L_2 X \to L_1 X \to L_0 X$$

is just $L_{BP} X$. This is the subject of the chromatic convergence theorem (which doesn't appear in this paper of Ravenel's, but is a theorem of his and Hopkins):

Theorem 1.4 (Chromatic convergence)**.** *If X is a p-local finite spectrum, then the homotopy limit of the above tower, denoted $\widehat{L}_\infty X$, is equivalent to X.*

Before proceeding, let us mention the following result about how the layers in the tower interact with the spectra $N_n X$ and $M_n X$, which is an easy consequence of the octahedral axiom in triangulated categories.

Theorem 1.5. *Let $C_n X$ be the fibre of the natural transformation $X \to L_n X$. Then*

$$N_n X \simeq \Sigma^n C_{n-1} X,$$

and

$$\mathrm{fib}(L_n X \to L_{n-1} X) \simeq \Sigma^{-n} M_n X.$$

Recall that Theorem 1.2 gives an identification $L_n X \simeq L_{v_n^{-1}BP} X$. In algebraic geometry, the localisation of a R-module M at an element $x \in R$ is just restriction to the open subset $R[x^{-1}]$; since $M[x^{-1}] \simeq M \otimes_R R[x^{-1}]$, one might expect a similar thing for $E(n)$-localisation. Namely, it is natural to conjecture:

Conjecture 1.6 (Smashing conjecture)**.** $E(n)$-localisation is *smashing*, i.e., there is an equivalence

$$L_n X \simeq X \wedge L_n S.$$

The localisation conjecture for S can be generalized:

Conjecture 1.7 (Localisation conjecture)**.** For any spectrum X, we have

$$BP \wedge L_n X \simeq X \wedge L_n BP.$$

It turns out that this is equivalent to the statement that if X is $E(n-1)_*$-acyclic, then

$$BP_* L_n X = v_n^{-1} BP_* X.$$

Clearly the localisation conjecture would follow if the smashing conjecture was true.

This is a conjecture on the "global" structure of the stable homotopy category. Namely, we are trying to think of the stable homotopy category (of p-local finite spectra, at least) of spectra as admitting a filtration, where restriction to each stratum is given by the L_n-localisation. The union of these strata should be all p-local finite spectra, by chromatic convergence. From this global point of view, one might also expect any p-local finite spectrum X to lie in some stratum; this is called the *type* of X. We will make this more precise momentarily.

It will not be relevant for the sequel, but we mention that Ravenel also explicitly computed the homotopy of $L_n BP$:

Proposition 1.8. *As expected,* $\pi_* L_0 BP = p^{-1} BP_*$*. Moreover, for* $n > 0$*, we have*

$$\pi_* L_n BP = BP_* \oplus \Sigma^{-n} N^{n+1}.$$

This comes from a cofibre sequence

$$BP \to L_n BP \to \Sigma^{-n} N_{n+1} BP$$

and the computation of Theorem 1.3. For a general spectrum, there are also spectral sequences

$$E_2^{*,*} = \mathrm{Tor}_{*,*}^{BP_*}(BP_* X, N^n) \Rightarrow \pi_*(X \wedge N_n BP),$$
$$E_2^{*,*} = \mathrm{Tor}_{*,*}^{BP_*}(BP_* X, M^n) \Rightarrow \pi_*(X \wedge M_n BP).$$

One can think of these as Künneth spectral sequences for computing the homotopy of $X \wedge N_n BP$, which one can think of as $(X \wedge BP) \wedge_{BP} N_n BP$. These two spectral sequences in turn can be stitched into a spectral sequence (the *topological chromatic spectral sequence*), exactly as for the chromatic spectral sequence:

$$E_1^{s,t} = \pi_t M_s X \Rightarrow \pi_* \widehat{L}_\infty X.$$

The localisation conjecture describes a relationship between this spectral sequence and the chromatic spectral sequence, constructed above.

1.5. **A web of conjectures.** How might we resolve the smashing conjecture? By coming up with another conjecture, of course! Before describing our plan of attack on the smashing conjecture, we need to return to a question brought up a while back which also stemmed from the global point of view à la the analogy with algebraic geometry.

Taking seriously the view that the L_n-localisation functors stratify p-local finite spectra, one might also expect any p-local finite spectrum X to lie in some stratum. Namely, we might expect there to be some integer n such that $K(n)_* X \neq 0$ and $K(i)_*(X) = 0$ for all $i < n$. It turns out that the second condition (of all the lower $K(i)$-homology groups vanishing) is equivalent to the condition that $K(n-1)_*(X) = 0$. This statement also implies that if $K(n)_* X \neq 0$, then $K(i)_*(X) \neq 0$ for all $i > n$. Since the $K(n)$'s detect homotopy theory exactly at v_n (by definition), this is a more precise way to ask for our stratification. We will call this integer n the *type* of X.

Let

$$E_p = \bigvee_{n \geq 0} K(n), \; E = \bigvee_p E_p.$$

A spectrum is said to be *harmonic* if it is E_*-local, and *dissonant* if it is E_*-acyclic. If X is a p-local finite spectrum, then $(E_\ell)_*(X) = \mathbf{Q}$ for $\ell \neq p$, so in that case we only need to consider $(E_p)_*$-locality and acyclicity. Ravenel proved the following result.

Theorem 1.9. *Every finite spectrum is harmonic.*

This is an immediate consequence of the following more general result (which in turn is proven by induction): if X is a connective spectrum of finite type, then X is harmonic if $\dim_{MU_*} MU_*X$ is finite.

It follows immediately from this result that any nontrivial finite p-local spectrum X has a type. For instance, the sphere is clearly of type 0 (note that $K(0) = H\mathbf{Q}$), while the mod p Moore spectrum S/p has type 1. The spectrum $S/(p, v_1)$, which exists for $p \geq 3$, has type 2. These spectra have BP-homology given by BP_*/p and $BP_*/(p, v_1)$, respectively.

As we saw in the introduction, these spectra admit self maps. The map $v_1 : \Sigma^{2(p-1)} S/p \to S/p$ induces an equivalence in $K(1)_*$-homology, given by multiplication by v_1. Similarly, the map $v_2 : \Sigma^{2(p^2-1)} S/(p, v_1) \to S/(p, v_1)$ induces an equivalence in $K(2)_*$-homology, given by multiplication by v_2. This naturally leads us to conjecture:

Conjecture 1.10 (Periodicity)**.** Let X be a p-local finite spectrum of type n. Then there is an equivalence $f : \Sigma^k X \to X$ which induces an equivalence in $K(n)_*$-homology, given by multiplication by some pth power of v_n.

Such a map is called a *v_n-self map*, and the same arguments as in the introduction will allow us to construct families of elements in the stable homotopy groups of spheres.

One might like to realize more homotopy types which realize the operation of quotienting out by $p, v_1, \cdots, v_i$ (in the sense that their BP-homology is given by $BP_*/(p, v_1, \cdots, v_i)$). This leads naturally to:

Conjecture 1.11 (Realizability conjecture)**.** We can realize any ideal $J \subseteq BP_*$ which is similar to an invariant regular prime ideal (essentially, any ideal of BP_* of the form $I_i \coloneqq (p, v_1, \cdots, v_i)$) via a spectrum S/I_n, which admits a v_n-self map as in the periodicity conjecture.

At $p = 3$, the spectrum $S/(p, v_1)$ does admit a v_2-self map (see [BP04])

$$v_2^9 : \Sigma^{144} S/(3, v_1) \to S/(3, v_1),$$

but it is not given by multiplication by v_2 on $K(2)$-homology; instead, it is given by multiplication by v_2^9, as the notation suggests. At $p = 2$, the spectrum $S/(2, v_1)$ admits a self map (see [BHHM08])

$$v_2^{32} : \Sigma^{192} S/(2, v_1) \to S/(2, v_1),$$

which, as the notation again suggests, is multiplication by v_2^{32}. One cannot realize $BP_*/(p, v_1, v_2)$ at $p = 2, 3$.

It is not at all clear that the spectrum $S/(p, v_1)$ admits a v_2-self map (at any prime) which is not nilpotent (i.e., any power — which is just given by iteration — of the v_n-self map is not null). This motivates us to find ring spectra E such which *detect nilpotence*. More precisely, we say that E detects nilpotence if, for any ring spectrum R, the kernel of the Hurewicz map $\pi_* R \to E_* R$ consists of nilpotent elements.

Conjecture 1.12 (Nilpotence)**.** The spectrum MU detects nilpotence.

This gives a conceptual home for Nishida's theorem (Theorem 5.2): since $MU_* S = MU_*$ is torsion free, and $\pi_* S$ is torsion in positive degrees, the Hurewicz map is zero above degree 0. Therefore every positive-degree element in $\pi_* S$ is nilpotent.

One consequence of the nilpotence conjecture is that if $W \to X \to Y$ is a cofibre sequence with connecting map $f : Y \to \Sigma W$ which is null in MU-homology, then

$$\langle X \rangle = \langle W \rangle \vee \langle Y \rangle.$$

If X is a finite spectrum, then we can construct a rational equivalence from a wedge of spheres to X (exercise). Let F denote the fibre of this map; then F has torsion homotopy (clearly) and torsion MU_*-homology. The map from F to the wedge of spheres must therefore be trivial, so we can identify

$$\langle X \rangle = \langle \text{wedge of spheres} \rangle \vee \langle F \rangle = \langle S \rangle,$$

at least if the wedge of sphere is over a nonempty indexing set.

Conjecture 1.13 (Class invariance)**.** If X is a finite spectrum, then $\langle X \rangle = \langle S \rangle$ if $\pi_* X$ is not all torsion.

This leads naturally to the telescope conjecture. Let X be a spectrum of exact type n, let $f : X \to \Sigma^{-k} X$ be a v_n-self map, and let $f^{-1} X$ denote the inverse limit of the system of iterates of f.

Conjecture 1.14 (Telescope conjecture)**.** The Bousfield class $\langle f^{-1} X \rangle$ depends only on n, and

$$\langle f^{-1} X \rangle = \langle K(n) \rangle.$$

Let Y denote the cofibre of f. As a consequence of the telescope conjecture (and Theorem 1.2), we find that

$$\langle S \rangle = \langle E(n) \rangle \vee \langle Y \rangle.$$

Since Y is $K(n)_*$-acyclic by construction, it is $E(n)_*$-acyclic. Therefore the following proposition implies the the smashing conjecture:

Proposition 1.15. *Suppose Z is a (possibly infinite) wedge of finite spectra. Then $\langle Z \rangle$ has a complement, i.e., there is a class $\langle Z \rangle^c$ such that*

$$\langle Z \rangle \vee \langle Z \rangle^c = \langle S \rangle, \ \langle Z \rangle \wedge \langle Z \rangle^c = \langle 0 \rangle.$$

If $\langle E \rangle = \langle B \rangle^c$, then E_*-localisation is smashing, i.e.,

$$L_E X \simeq X \wedge L_E S.$$

One can show that the periodicity conjecture and the telescope conjecture implies the following, also known as the periodicity conjecture.

Conjecture 1.16. If X is a finite spectrum and n is a positive integer, then there is a directed system of $K(n-1)_*$-acyclic spectra X_α such that

$$M_n X \simeq \operatorname{colim} M_n X_\alpha,$$

and for each X_α there is a homotopy equivalence for some $i = i(\alpha)$:

$$M_n X_\alpha \to \Sigma^{2p^i(p^n-1)} M_n X_\alpha.$$

Some iterate/power of these equivalences should commute with all the maps in the directed system.

Combined with the chromatic spectral sequence and the localisation conjecture, this describes a "geometric" reason for the periodicity behind the algebraic periodicity in the E_2-page of the ANSS stemming from the (algebraic) periodicity of the E_1-page of the chromatic spectral sequence.

1.6. **Outlook.** When $n = 1$, the periodicity, telescope, smashing, and localisation conjectures are true. The periodicity theorem follows from the fact that any finite spectrum whose homotopy is all p-torsion that has type 1 must be a module over the mod p^k Moore spectrum S/p^k. Constructing a v_1-self map for S/p^k therefore begets a v_1-self map for any type 1 finite spectrum. As mentioned in the introduction, this was done by Adams in *J(X) IV*.

The seminal work of Devinatz-Hopkins-Smith proved all of these conjectures, except for the telescope conjecture, which remains open to this day[6]. The proofs of these conjectures all rely on the resolution of the nilpotence theorem, which is really the heart of modern chromatic homotopy theory.

The goal of this course is to prove the Ravenel conjectures, and describe applications of these results to stable homotopy theory.

One reason the nilpotence theorem is so important (other than the fact that it implies all of these conjectures) is that it is a statement which says that topology is inherently different from algebra. Recall the ANSS, which runs

$$E_2^{s,t} = \operatorname{Ext}^{s,t}(BP_*) \Rightarrow \pi_* S_{(p)}.$$

At odd primes, one can compute that $\operatorname{Ext}^0(BP_*/p) \simeq \mathbf{F}_p[v_1]$, so the image of v_1 under the boundary map $\operatorname{Ext}^0(BP_*/p) \to \operatorname{Ext}^1(BP_*)$ gives an element $\alpha_1 \in E_2^{1,2(p-1)}$. This element survives to the E_∞-page, and begets the element $\alpha_1 \in \pi_{2(p-1)-1} S_{(p)}$. (One can similarly construct the elements β_t that way; there's a whole plethora of things which can be done with the ANSS, that are beyond the scope of this document.) The picture is a little more complicated at $p = 2$, but one can construct the element $\eta \in E_2^{1,2}$.

It is not hard to show that on the E_2-page, the elements α_1 and η are *not* nilpotent. In other words, the E_2-page of the ANSS is not sufficient to detect nilpotence. This implies the existence of differentials in the ANSS which necessarily kill some power of α_1 and η, since $\pi_* S$ consists of nilpotent elements in positive degrees.

[6]This was the subject of Ravenel's talk a few weeks ago at the topology seminar.

One equivalent way to phrase the conjectures Ravenel made is as follows. Fix an integer n, and let $f(n)$ denote the highest ANSS filtration of an element in $\pi_n S$ (i.e., the highest s such that there exists an element $x \in \pi_n S$ detected by some element in $E_2^{s,*}$). Then

$$\lim_{n\to\infty} \frac{f(n)}{n} = 0,$$

i.e., $f(n) = o(n)$ as $n \to \infty$. This is a vanishing curve for the ANSS. If $f(n) \sim n^g$, then the value g determines "how nilpotent" the elements of $\pi_* S$ are. For instance, if f is tiny, then one can show that the telescope conjecture is automatically true.

Let us close with a table of analogies between the stable homotopy category and algebraic geometry (I first found this table in [GH94]), which attempts to give evidence for the why the Ravenel conjectures should be thought of as a global perspective on homotopy theory. To this end, let X be a scheme with a unique subscheme X_n of finite codimension n, with complement $i_n : U_n \to X$.

Stable homotopy category	Quasicoherent sheaves on X
(p-local) sphere spectrum	structure sheaf
spectrum	quasicoherent sheaf
finite spectrum	coherent sheaf
smash product of spectra	tensor product
function spectrum	hom sheaf
homotopy groups	sheaf cohomology
subcategory of $E(n)$-local spectra	sheaves which are pushed forward from U_n
$E(n)_*$-localisation	the functor $(\mathbf{R}i_n)_* i_n^*$
$K(n)$-localisation	completing along $U_n \cap X_{n-1} \subseteq U_n$
chromatic convergence	the X_n give a complete filtration of X

2. The thick subcategory theorem

2.1. **Introduction.** Suppose we wanted to prove that all p-local finite spectra of type $\geq n$ were *evil*. In general, this might be extremely hard to show. The *thick subcategory theorem* allows us to reduce to finding just one spectrum of type n which is evil, if the subcategory of evil finite spectra is thick. To state the result, we need a preliminary definition.

Definition 2.1. Let $\mathscr{C}$ be a subcategory of the category of (p-local) finite spectra. We say that $\mathscr{C}$ is *thick* if it is closed under cofibrations and retracts.

An example of a thick subcategory is the subcategory $\mathscr{C}_n$ of p-local $K(n-1)_*$-acyclic finite spectra for any $n \geq 0$. The thick subcategory theorem says that these are the *only* thick subcategories:

Theorem 2.2 (Thick subcategory theorem)**.** *The only thick subcategories of the category* Sp^ω *of finite spectra are* Sp *itself, the trivial subcategory, and* $\mathscr{C}_n$ *for* $n \geq 0$.

The key reason for the utility of this theorem is that it is usually easy to check the subcategory of evil spectra is thick.

Throughout these notes, we will be assuming the nilpotence theorem, which will be discussed in section 5. We will also quietly p-localise everywhere.

2.2. **Motivating the thick subcategory theorem: an algebraic analogue.** In this subsection, we will prove an algebraic analogue of Theorem 2.2; this will be a somewhat different version of [Rav92, Theorem 3.4.2], although the proof is the same. Let $\mathscr{A}$ be the category of BP_*BP-comodules which are finitely presented as BP_*-modules. This is an abelian category. We define a thick subcategory $\mathscr{C}$ of $\mathscr{A}$ to be a subcategory satisfying the two-out-of-three property for short exact sequences: if two of A, B, and C are in $\mathscr{C}$ and there is a short exact sequence

$$0 \to A \to B \to C \to 0,$$

then the third is also in $\mathscr{C}$.

In analogy to the subcategories $\mathscr{C}_n$ of Sp^ω, we define subcategories, also denoted $\mathscr{C}_n$, of $\mathscr{A}$ by saying that $M \in \mathscr{C}_n$ iff $v_{n-1}^{-1}M = 0$, i.e., v_{n-1} is nilpotent in M. Our goal will be to show that any thick subcategory of $\mathscr{A}$ is either trivial, $\mathscr{A}$ itself, or $\mathscr{C}_n$ for some n.

Our proof will rely on the following result. Recall that $I_n = (p, v_1, \cdots, v_{n-1})$ is an ideal of BP_*; it turns out that this is an "invariant ideal", in the sense that BP_*/I_n is a BP_*BP-comodule.

Theorem 2.3 (Landweber filtration theorem)**.** *Any object M of $\mathscr{A}$ has a filtration $0 = M_k \subset \cdots \subset M_1 \subset M$ where M_j/M_{j+1} is isomorphic as a BP_*BP-comodule to (a shift of) BP_*/I_{n_j}.*

Corollary 2.4. *There is a strict inclusion $\mathscr{C}_{n+1} \subsetneq \mathscr{C}_n$.*

Proof. Concretely, this implies that if $v_n^{-1}M = 0$, then $v_{n-1}^{-1}M = 0$. Consider the filtration $0 = M_k \subset \cdots \subset M_1 \subset M$ coming from Theorem 6.4, where every quotient is isomorphic to a shift of BP_*/I_m for some m. Since $v_n^{-1}M = 0$, we know that $v_n^{-1}M_j = 0$ for every j. This means that $v_n^{-1}M_j/M_{j+1} = 0$, so $v_n^{-1}BP_*/I_{n_j}$; in other words, $n_j \geq n$. This immediately implies that $v_{n-1}^{-1}M = 0$.

Proving that the inclusion is strict involves finding an example of a BP_*BP-comodule M such that $v_{n-1}^{-1}M = 0$ but $v_n^{-1}M \neq 0$. This is easy: one example is $M = v_n^{-1}BP_*/I_n$. □

This begets a filtration

$$\cdots \subset \mathscr{C}_2 \subset \mathscr{C}_1 \subset \mathscr{A},$$

which allows us to prove the thick subcategory theorem for $\mathscr{A}$. Let $\mathscr{C}$ be a thick subcategory of $\mathscr{A}$, and let n be the largest integer for which $\mathscr{C} \subseteq \mathscr{C}_n$ but $\mathscr{C} \not\subseteq \mathscr{C}_{n+1}$. It suffices to check that $\mathscr{C}_n \subseteq \mathscr{C}$. By descending induction using Theorem 6.4, it suffices to show that all the quotients BP_*/I_{n_j} are in $\mathscr{C}$.

Let M be a comodule in $\mathscr{C} \subseteq \mathscr{C}_n$ but not in $\mathscr{C}_{n+1}$. We claim that some $n_i = n$ for some i. In Corollary 2.4 we established that $n_j \geq n$ for all j. Since $v_n^{-1}M \neq 0$, there must be some i such that $v_n^{-1}BP_*/I_{n_i} \neq 0$. This means that $n_i \leq n$, so $n_i = n$.

Considering the filtration of M coming from Theorem 6.4, we find, using the definition of thickness, that $BP_*/I_n \in \mathscr{C}$. There is a short exact sequence

$$0 \to BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0,$$

so it follows that $\mathscr{C}$ contains BP_*/I_m for every $m \geq n$. Since $n_j \geq n$, it follows that every quotient BP_*/I_{n_j} is in $\mathscr{C}$, as desired.

2.3. **The proof of the thick subcategory theorem.** Before proceeding, we need a few preliminary results.

Lemma 2.5. *Let R be a ring spectrum. If the Hurewicz image of $x \in \pi_*R$ inside $K(n)_*R$ is zero for all $0 \leq n \leq \infty$, then x is nilpotent.*

Proof. It suffices to show that the Hurewicz image of x inside MU_*R is zero, by the nilpotence theorem. In other words, it suffices to show that $MU \wedge x^{-1}R$ is contractible. Since MU splits (p-locally) as a wedge of suspensions of BP, it suffices to show that $BP \wedge x^{-1}R$ is contractible. For this, we will interpolate between BP and $H\mathbf{F}_p$. Using the idea from the previous section, we define

$$P(n) = BP/(p, v_1, \cdots, v_{n-1}).$$

In particular, $P(\infty) = H\mathbf{F}_p$. By assumption, there is some finite N for which $P(N) \wedge x^{-1}R$ is contractible. To show that $BP \wedge x^{-1}R$ is contractible, we will show by descending induction that $P(n) \wedge x^{-1}R$ is contractible.

This is true for $n = N$, so assume that it is true for $n+1$. By construction, there is a cofiber sequence relating $P(n)$ and $P(n+1)$. Smashing with $x^{-1}R$, we obtain a cofiber sequence

$$\Sigma^{2(p^n-1)}P(n) \wedge x^{-1}R \xrightarrow{v_n \wedge 1} P(n) \wedge x^{-1}R \to P(n+1) \wedge x^{-1}R \simeq *.$$

It follows that $P(n) \wedge x^{-1}R \simeq v_n^{-1}P(n) \wedge x^{-1}R$. This is a $v_n^{-1}BP$-module, so it suffices to prove that $x^{-1}R$ is $v_n^{-1}BP$-acyclic.

By Theorem 3.1 of Lecture 1, this is equivalent to proving that $x^{-1}R$ is $E(n)$-acyclic. That same result says that it suffices to show that $x^{-1}R$ is $E(n-1)$-acyclic and $K(n)$-acyclic. By (ordinary) induction, this will follow if $x^{-1}R$ is $K(m)$-acyclic for every $0 \leq m \leq n$; but this is precisely our assumption! □

Lemma 2.6. *If $f : X \to Z$ is a map from a finite spectrum which is null in $K(n)$-homology for all $0 \leq n \leq \infty$, then $f^{\wedge m} : X^{\wedge m} \to Z^{\wedge m}$ is null for some $m \gg 0$.*

Proof. Define
$$R = \bigvee_{n\geq 0} Y^{\wedge n};$$
then R is a ring spectrum. Applying Lemma 2.5 to R shows that any element x of π_*Y in the kernel of the Hurewicz map $\pi_*Y \to K(n)_*Y$ for every $0 \leq n \leq \infty$ must satisfy $x^{\wedge m} = 0 \in \pi_*Y^{\wedge m}$ for some $m \gg 0$. Applying this result to the case $Y = F(X, Z)$ proves the desired result. □

We now need an analogue of Corollary 2.4: this is [Rav84, Theorem 2.11].

Lemma 2.7. *If a finite spectrum X is $K(n)$-acyclic, then X is $K(m)$-acyclic for every $0 \leq m \leq n$.*

We will provide a proof different from that of Ravenel's; since it uses some concepts not yet developed, this may be safely skipped.

Proof. Let E_n denote Morava E-theory at height n. Then $\langle E_n\rangle = \langle E(n)\rangle$. Moreover, E_n is $K(n)$-local. If X is finite, then
$$E_n \wedge X \simeq X \wedge L_{K(n)}E_n \simeq L_{K(n)}(E_n \wedge X) \simeq L_{K(n)}(E_n \wedge L_{K(n)}X) \simeq *.$$
Therefore X is E_n-acyclic. Using Theorem 3.1 of Lecture 1, we conclude that X is $K(m)$-acyclic for every $0 \leq m \leq n$. □

To motivate the proof of the thick subcategory theorem — and, in particular, how the nilpotence theorem is applied — let us show the following result. Since $\Sigma^\infty \mathbf{C}P^2$ is an example of a type 0 complex (it has nontrivial rational homology) this is a special case of Theorem 2.2.

Proposition 2.8. *If $\mathscr{C}$ is a thick subcategory containing $\Sigma^\infty\mathbf{C}P^2$, then $\mathscr{C}$ is Sp^ω.*

Proof. It is easy to see that $\Sigma^{-2}\Sigma^\infty\mathbf{C}P^2 = C(\eta)$, where $\eta \in \pi_1(S)$. By assumption, $C(\eta) \in \mathscr{C}$, so $\mathscr{C}$ contains the thick subcategory generated by $C(\eta)$. There is a cofiber sequence
$$C(\eta) \to C(\eta^2) \to C(\eta),$$
so $C(\eta^2) \in \mathscr{C}$. Arguing similarly, we find that $C(\eta^n) \in \mathscr{C}$ for every n. Since $\eta^4 = 0$, the spectrum $C(\eta^4)$ is a wedge of spheres. Because $\mathscr{C}$ is closed under retracts, it follows that $S \in \mathscr{C}$. Every finite spectrum can be obtained from S by a sequence of cofibrations and retracts, it follows that $\mathrm{Sp}^\omega \subseteq \mathscr{C}$. □

Motivated by this, we will prove Theorem 2.2.

Proof of Theorem 2.2. Let $\mathscr{C}$ be any thick subcategory of the triangulated category of p-local finite spectra. We will show that any object X in $\mathscr{C}$ that's in $\mathscr{C}_n$ (with n minimal) gives an inclusion $\mathscr{C}_n \subseteq \mathscr{C}$. Let X be any other $K(n-1)$-acyclic spectrum that is in $\mathscr{C}$, and let DX denote its Spanier-Whitehead dual. The identity on X adjuncts to a map $S \to X \wedge DX$ (everything is p-local). For $m \geq n$, we therefore have an injection

$$K(m)_*(S) \to K(m)_*(X) \otimes_{\mathbf{F}_{p^m}[v_m^{\pm 1}]} K(m)_*(X)^\vee. \tag{3}$$

If F is the fiber of $S \to X \wedge DX$, this tells us that $K(m)_*(F) \to K(m)_*(S)$ is zero if $m \geq n$.

This shows that the composite $F \to S \to X \wedge DX$ is zero on $K(m)$-homology for $m \geq n$. Since X is $K(n-1)$-acyclic — this implies that it is $K(m)$-acyclic for $m < n$ by Lemma 2.7 — it follows that the map (3) is zero for $m < n$. But if $K(n)_*(F) \to K(n)_*(X \wedge DX)$ is null for all n, it follows from Lemma 2.6 that $F^{\wedge k} \to (X \wedge DX)^{\wedge k}$ is null for some $k \gg 0$. Composing with the evaluation map $(X \wedge DX)^{\wedge k} \to X \wedge DX$ shows that $F^{\wedge k} \to X \wedge DX$ is null, i.e., the map $F^{\wedge k} \wedge X \to X$ is null. The cofiber of this map is thus
$$X \wedge \mathrm{cofib}\,(F^{\wedge k} \to S) \simeq X \vee (X \wedge \Sigma F^{\wedge k}).$$
We know that $\mathrm{cofib}\,(F \to S) = X \wedge DX$ is in $\mathscr{C}_n$, so we use the cofiber sequence
$$\mathrm{cofib}\,(F^{\wedge k} \to S) \to \mathrm{cofib}\,(F^{\wedge (k-1)} \to S) \to \Sigma F^{\wedge (k-1)} \wedge \mathrm{cofib}\,(F \to S) \simeq \Sigma F^{\wedge (k-1)} \wedge X \wedge DX$$
to show by induction that $\mathrm{cofib}\,(F^{\wedge i} \to S) \wedge X$ is in $\mathscr{C}$ for all i. Since X is a retract of this, it follows from the fact that $\mathscr{C}$ is thick that $X \in \mathscr{C}$, as desired.

It's easy to see that $\mathscr{C} \subseteq \mathscr{C}_n$ (since otherwise, $\mathscr{C}$ would contain some spectrum that's $K(m)$-acyclic for $m < n-1$, which contradicts minimality). The thick subcategory theorem follows. □

3. The periodicity theorem

3.1. **Introduction.** In the first section, we briefly discussed the periodicity conjecture:

Conjecture 3.1 (Periodicity)**.** Let X be a p-local finite spectrum of type n. Then there is an equivalence $f : \Sigma^k X \to X$ which induces an equivalence in $K(n)_*$-homology, given by multiplication by some pth power of v_n.

The goal of this section is to describe a proof of this conjecture, using the main result of the previous section (the thick subcategory theorem).

We need to be a little more precise about the desideratum for this self-map.

Definition 3.2. Let X be a finite p-local spectrum. A v_n-self map on X is a self map $f : \Sigma^k X \to X$ such that $K(n)_* f$ is an isomorphism, and $K(m)_* f = 0$ for $m \neq n$.

If X has type n, and f is a v_n-self map of X, then the cofiber $C(f)$ has vanishing $K(n)$-homology, but nonvanishing $K(n+1)$-homology. It follows that $C(f)$ has type $n+1$. Therefore, establishing the existence of v_n-self maps for every type n spectrum allows us to construct spectra of type $n+1$.

In the terminology of the previous section, we would like to prove that the property of having a v_n-self map is evil. Then, establishing the existence of a single finite spectrum of type n with a v_n-self map will e enough to prove the periodicity theorem.

To describe more precisely how the thick subcategory theorem comes into play, let $\mathscr{C}$ denote the subcategory of p-local finite spectra which have v_n-self maps. If $X \in \mathscr{C}$ and $K(n)_* X = 0$, then X has a v_n-self map: the zero map. It follows that $\mathscr{C}_{n+1} \subseteq \mathscr{C}$, using the notation from last time.

Lemma 3.3. *There is an inclusion* $\mathscr{C} \subseteq \mathscr{C}_n$.

Proof. It will suffice to prove that if M is a BP_*BP-comodule which is a finitely presented BP_*-module such that $v_{n-1}^{-1} M \neq 0$, then M does not admit a self map $M[k] \to M$ which is an isomorphism after inverting v_n, and which is zero after inverting v_m for any $m \neq n$. □

The following two steps will therefore result in the periodicity theorem.

1. $\mathscr{C}$ is thick (so $\mathscr{C}$ is either $\mathscr{C}_n$ or $\mathscr{C}_{n+1}$ by the thick subcategory theorem).
2. There is a finite spectrum of type n with a v_n-self map.

3.2. **Step 1: thickness.** In this subsection, we will establish step 1. We need a preliminary lemma.

Lemma 3.4. *Let* $f : \Sigma^k X \to X$ *and* $g : \Sigma^d Y \to Y$ *be two* v_n*-self maps. For any map* $X \to Y$*, there are some* $a, b \gg 0$ *with* $ka = db$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \Sigma^{ka} X & \longrightarrow & \Sigma^{db} Y \\ \downarrow{\scriptstyle f^a} & & \downarrow{\scriptstyle g^b} \\ X & \longrightarrow & Y. \end{array}$$

Proof. A v_n-self map is an element x of $\pi_* R$ such that Hurewicz image of x in $K(m)_* R$ is zero for $m \neq n$, and is invertible for $m = n$, where $R = X \wedge DX$. If R is a p-local ring spectrum, we will call such an element of $\pi_* R$ a v_n-element. It will therefore suffice to prove that if R is any p-local finite ring spectrum and x, y are two v_n-elements of $\pi_* R$, then $x^a = y^b$ for some $a, b \gg 0$.

Suppose that x and y are v_n-elements and x is in the center of $\pi_* R$. If $x - y$ is torsion and nilpotent, then $x^{p^a} = y^{p^a}$ for some $a \gg 0$. Indeed, since $x = y + (x - y)$, we have

$$x^{p^a} = y^{p^a} + \sum_{k=1}^{p^a} \binom{p^a}{k} (x-y)^k y^{p^a - k}.$$

$x - y$ is nilpotent, so for some $b \gg 0$ we have $(x-y)^{p^b} = 0$. It follows that

$$x^{p^a} = y^{p^a} + \sum_{k=1}^{p^b - 1} \binom{p^a}{i} (x-y)^k y^{p^a - k}.$$

We have

$$\binom{p^a}{i} = \binom{p^a-1}{i-1}\frac{p^a}{i},$$

and p^a/i is divisible by p^{a-k}. For large enough a, it follows that this term kills $x-y$; we conclude that $x^{p^a} = y^{p^a}$.

Since X is of positive type, we know that $\pi_* R$ is torsion. If x and y are v_n-elements, it follows that $x-y$ is torsion. We claim that $x-y$ is nilpotent. To see this, let x be a v_n-element of $\pi_* R$. Since $K(n)_* R$ is a finite $K(n)_*$-algebra, the quotient $K(n)_*(R)/(v_n - 1)$ is finite with a finite group of units. The image of x inside $K(n)_*(R)/(v_n - 1)$ is a unit, so some sufficiently high power of x will map to 1 inside $K(n)_*(R)/(v_n - 1)$. This means that some high enough power of x maps to some power of v_n inside $K(n)_* R$. (This power can be chosen to be high enough so that x maps to 0 inside $K(m)_* R$ for $m \neq n$.)

It follows from this discussion that replacing x and y some sufficiently high power, we can assume that x and y both map to $v_n^k \in K(n)_* R$. Therefore $x - y \mapsto 0 \in K(m)_* R$ for every $0 \leq m \leq \infty$. Lemma 3.1 from the previous section implies that $x-y$ is nilpotent, as desired.

To conclude the proof of the lemma, it therefore suffices to show that some high enough power of x is in the center of $\pi_* R$. As proved above, we can assume that a high enough power of x maps to some power of v_n inside $K(n)_* R$. Thinking of elements of $\pi_*(R \wedge DR)$ as self maps of R, let us consider the elements f, g given by left and right multiplication by x, so f and g commute. A minor modification of the same argument as above proves that $f - g$ is torsion and nilpotent. We conclude that $f^{p^a} = g^{p^a}$ for some $a \gg 0$. This means that the elements of $\pi_*(R \wedge DR)$ associated to the element x^{p^a} are the same, i.e., x^{p^a} is in the center of $\pi_* R$. □

Proving that $\mathscr{C}$ is thick is now easy. It is clear that $\mathscr{C}$ is closed under suspension that that it contains the zero object. We need to establish that $\mathscr{C}$ is closed under retracts. Suppose $X \vee Y \in \mathscr{C}$, so $X \vee Y$ admits a v_n-self map f. There are maps $X \to X \vee Y$ and $X \vee Y \to X$. The self map of X given by

$$\Sigma^k X \to \Sigma^k X \vee Y \xrightarrow{f} X \vee Y \to X$$

is easily seen to be a v_n-self map.

It remains to prove that $\mathscr{C}$ is closed under cofibrations. The only nontrivial direction is showing that if X and Y are in $\mathscr{C}$ and $p : X \to Y$ is any map, then $C(p)$ is also in $\mathscr{C}$. For this, we use the diagram of Lemma 4, which can be extended:

$$\begin{CD} \Sigma^N X @>>> \Sigma^N Y @>>> \Sigma^N C(p) \\ @V{f}VV @V{g}VV @V{h}VV \\ X @>>> Y @>>> C(p). \end{CD}$$

We claim that h^2 is a v_n-self map of $C(p)$, so $C(p) \in \mathscr{C}$. Indeed, it is easy to see via the long exact sequence in $K(n)$-homology and the 5-lemma that h induces an isomorphism on $K(n)$-homology. We need to show that h^2 is zero on $K(m)$-homology for $m \neq n$. Again using the long exact sequence in $K(m)$-homology, we find that the image of h sits inside the image of $q : K(m)_* Y \to K(m)_* C(p)$, and that h is trivial on the image of q. It follows that $K(m)_* h^2 = 0$ for $m \neq n$, as desired.

3.3. **Step 2: finding a nontrivial type n spectrum with a v_n-self map.** It remains to establish step 2, which is to find, for every n, a nontrivial type n spectrum with a v_n-self map. This is the hardest part of the proof of the periodicity theorem. Our plan is the following:

(i) Define strongly type n spectra. Prove that every strongly type n spectrum is type n.
(ii) Construct a v_n-self map on every strongly type n spectrum.
(iii) Define partially type n spectra. For each n, construct an example.
(iv) Construct a strongly type n spectrum from every partially type n spectrum.

We will discuss every step except the last.

3.3.1. *Substep (i): strongly type n spectra.* In order to define strongly tpe n spectra, we need some prerequisites about the Steenrod algebra. Recall that

$$\mathscr{A}_* \simeq \begin{cases} P(\xi_1, \xi_2, \cdots) & p = 2, \text{ with } |\xi_i| = 2^i - 1 \\ P(\xi_1, \xi_2, \cdots) \otimes E(\tau_0, \tau_1, \cdots) & p > 2, \text{ with } |\xi_i| = 2p^i - 2, |\tau_i| = 2p^i - 1 \end{cases}$$

Let P_t^s denote the dual of $\xi_t^{p^s}$; Q_i the dual of τ_i for odd p; and Q_i the dual of ξ_{i+1}. One can prove:

Lemma 3.5. *We have*

$$(P_t^s)^p = 0, \ Q_i^2 = 0.$$

Let M be a $\mathscr{A}$-module. The map $M \xrightarrow{Q_i} M$ is a boundary operator on M because $Q_i^2 = 0$; this gives a chain complex. The homology of this chain complex is denoted $H_*(M; Q_i)$. Let $M = M_+ = M_-$; define a map $d : M_+ \oplus M_- \to M_+ \oplus M_-$ by

$$(m_+, m_-) \mapsto ((P_t^s)^{p-1} m_-, P_t^s m_+).$$

The above lemma shows that $d^2 = 0$, so d defines a boundary operator on another chain complex, whose homology is denoted $H_*(M; P_t^s)$. If $M = H^*(X)$, these are denoted $H_*(X; Q_i)$ and $H_*(X; P_t^s)$.

Definition 3.6. Let X be a p-local finite spectrum. We say that X is strongly type n if:

- Q_n acts trivially on $H^*(X)$;
- $K(n)^*(Y)$ and $H^*(Y)$ have the same rank;
- If $p = 2$, then $H_*(X; P_t^s)$ vanish for $s + t \leq n + 1$ when $(s, t) \neq (0, n + 1)$; if $p > 2$, then $H_*(X; Q_i)$ vanishes for $i < n$ and $H_*(X; P_t^s)$ vanishes for $s + t \leq n$.

We will not prove the following result, although we note that the first claim is a consequence of sparsity.

Lemma 3.7. *Let X be a connective spectrum. The first differential in the Atiyah-Hirzebruch spectral sequence for $K(n)^*(X)$ is a d_{2p^n-1}. Moreover, d_{2p^n-1} acts by Q_n.*

Using this result, we obtain:

Theorem 3.8. *Every strongly type n spectrum is type n.*

Proof. Let $m < n$, and let X be a strongly type n spectrum. The first differential in the AHSS for $K(m)^*(X)$ is a d_{2p^m-1}, which acts by Q_m by Lemma 3.7. However, since $H_*(X; Q_i)$ vanishes for $i < n$, this implies that d_{2p^m-1} is an isomorphism — hence the E_∞-page of the AHSS for $K(m)^*(X)$ is zero, so $K(m)^*(X) = 0$. In particular, $K(m)_*(X) = 0$ for $m < n$, as desired. □

3.3.2. *Substep (ii): constructing v_n-self maps.* Our goal is to now prove that every strongly type n spectrum admits a v_n-self map. Fix a strongly type n spectrum X. A v_n-self map on X adjoints to an element of $\pi_*(X \wedge DX)$. For simplicity, we will write $R = X \wedge DX$. Our goal can be restated as follows: we are looking for an element of $\pi_* R$ which, under the Hurewicz map $\pi_* R \to K(n)_* R = \mathrm{End}(K(n)_*(Y))$, maps to a power of v_n. For clarity of exposition, we will not prove most of the lemmas in this subsubsection.

The ring spectrum R is a finite spectrum, so we can attempt to construct a permanent cycle in the E_2-page of the Adams spectral sequence for $\pi_* R$:

$$E_2^{*,*} = \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(R), \mathbf{F}_p) \Rightarrow \pi_* R.$$

We first state the following lemma.

Lemma 3.9. *The E_2-page of the Adams spectral sequence for R has a vanishing line of slope $1/|v_n|$. In other words, there is some constant c such that $\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(R), \mathbf{F}_p) = 0$ for $s > (t-s)/|v_n| + c$.*

In order to construct such an element, we will consider the following commutative diagram:

(4)
$$\begin{array}{ccc} S & \longrightarrow & R \\ \downarrow & & \downarrow \\ k(n) & \longrightarrow & k(n) \wedge R, \end{array}$$

where $k(n)$ is the nth *connective* Morava K-theory. This spectrum has cohomology given by

$$H^*(k(n)) \simeq \mathscr{A}//E(Q_n) = \mathscr{A} \otimes_{E(Q_n)} \mathbf{F}_p.$$

The diagram (4) gives rise to a map of Adams spectral sequences:

$$\begin{array}{ccc} \mathrm{Ext}^{*,*}_{\mathscr{A}}(\mathbf{F}_p, \mathbf{F}_p) & \longrightarrow & \mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(R), \mathbf{F}_p) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(k(n)), \mathbf{F}_p) & \longrightarrow & \mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(k(n) \wedge R), \mathbf{F}_p). \end{array} \tag{5}$$

Note that $H^*(k(n) \wedge R) \simeq \mathscr{A}//E(Q_n) \otimes H^*(R)$. There is a change-of-rings isomorphism, which shows that

$$\mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(k(n) \wedge R), \mathbf{F}_p) \simeq \mathrm{Ext}^{*,*}_{E(Q_n)}(H^*(R), \mathbf{F}_p),$$

and that

$$\mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(k(n)), \mathbf{F}_p) \simeq \mathrm{Ext}^{*,*}_{E(Q_n)}(\mathbf{F}_p, \mathbf{F}_p).$$

The latter is an Ext over an exterior algebra, so

$$\mathrm{Ext}^{*,*}_{E(Q_n)}(\mathbf{F}_p, \mathbf{F}_p) \simeq \mathbf{F}_p[v_n],$$

where v_n is now just a *formal* variable in bidegree $(1, 2p^n - 1)$. Note that the Adams spectral sequence computing $\pi_*(k(n))$ collapses due to sparseness. The element v_n in the E_2-page survives to the element $v_n \in \pi_*(k(n)) \simeq \mathbf{F}_p[v_n]$.

Since X is a strongly type n spectrum, Q_n acts trivially on $H^*(R)$. Therefore,

$$\mathrm{Ext}^{*,*}_{E(Q_n)}(H^*(R), \mathbf{F}_p) \simeq H^*(R) \otimes \mathrm{Ext}^{*,*}_{E(Q_n)}(\mathbf{F}_p, \mathbf{F}_p) \simeq H^*(R) \otimes \mathbf{F}_p[v_n].$$

By hypothesis, $H^*(R)$ and $K(n)^*(R)$ have the same rank; therefore, the Adams spectral sequence computing $k(n)_*(R)$ also collapses.

Our discussion above implies that we can rewrite diagram (5) as:

$$\begin{array}{ccc} \mathrm{Ext}^{*,*}_{\mathscr{A}}(\mathbf{F}_p, \mathbf{F}_p) & \longrightarrow & \mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(R), \mathbf{F}_p) \\ \downarrow & & \downarrow \\ \mathbf{F}_p[v_n] & \longrightarrow & \mathbf{F}_p[v_n] \otimes H^*(R). \end{array} \tag{6}$$

We need to simplify the top line of this diagram. To do this, recall that $\mathscr{A}(N)$ is the subalgebra of $\mathscr{A}$ generated by $\mathrm{Sq}^1, \cdots, \mathrm{Sq}^{2^N}$. We need the following result.

Lemma 3.10 (Adams)**.** *Let M be an $\mathscr{A}$-module satisfying the conditions of Definition 3.6. For every $N > n$, there is a constant $c_N > 0$ such that the canonical map*

$$\phi : \mathrm{Ext}^{s,t}_{\mathscr{A}}(M, \mathbf{F}_p) \to \mathrm{Ext}^{s,t}_{\mathscr{A}(N)}(M, \mathbf{F}_p)$$

is an isomorphism for $s > (t-s)/|v_n| - c_N$. Moreover, $c_N \to \infty$ as $N \to \infty$.

As $E(Q_n) \subseteq \mathscr{A}(N) \subseteq \mathscr{A}$, the vertical maps in diagram (6) factor through $\mathrm{Ext}^{*,*}_{\mathscr{A}(N)}(H^*(R), \mathbf{F}_p)$:

$$\begin{array}{ccc} \mathrm{Ext}^{*,*}_{\mathscr{A}}(\mathbf{F}_p, \mathbf{F}_p) & \xrightarrow{i} & \mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(R), \mathbf{F}_p) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \mathrm{Ext}^{*,*}_{\mathscr{A}(N)}(\mathbf{F}_p, \mathbf{F}_p) & \xrightarrow{i} & \mathrm{Ext}^{*,*}_{\mathscr{A}(N)}(H^*(R), \mathbf{F}_p) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\ \mathbf{F}_p[v_n] & \longrightarrow & \mathbf{F}_p[v_n] \otimes H^*(R). \end{array} \tag{7}$$

Lemma 3.11. *For every $N \geq n$, there is some $k > 0$ such that*

$$v_n^k = j(x).$$

By dimension counting, the element $i(x)$ must be on a line of slope $1/|v_n|$ through the origin. Lemma 3.10 implies that there is some element $y \in \mathrm{Ext}_{\mathscr{A}}(H^*(R), \mathbf{F}_p)$ such that

$$\phi(y) = i(x).$$

Lemma 3.12. *There is some $k \geq 0$ such that y^{p^k} is a permanent cycle in the Adams spectral sequence for $\pi_*(R)$.*

Proof. As $\phi(y) = i(x)$ is in the image of i, it is a central element in $\mathrm{Ext}^{*,*}_{\mathscr{A}(N)}(H^*(R), \mathbf{F}_p)$. Therefore, Lemma 3.10 implies that y commutes with all elements of $\mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(R), \mathbf{F}_p)$ which are above the line of slope $1/|v_n|$ determined by the lemma.

Suppose y is not a permanent cycle, so $d_r(y) = z$ for some z. Since z is above the line of slope $1/|v_n|$ determined by Lemma 3.10, we learn that z commutes with y. Therefore,

$$d_r(y^p) = py^{p-1}u;$$

as $p = 0$, this is zero. Suppose, now, that there is some higher nontrivial differential on y^p. Repeating this argument again, we find that this differential kills y^{p^2}. By Lemma 3.9, there will be some k such that all differentials on y^{p^k} are zero (since their target will be above the vanishing line), as desired. □

By construction, the element y^{p^k} maps to a nontrivial element in $k(n)_*(R)$ which corresponds to multiplication by some power of v_n. Moreover, the Hurewicz map $\pi_*(R) \to K(n)_*(R)$ factors through $k(n)_*(R)$; the map $k(n)_*(R) \to K(n)_*(R)$ is just the localization map $k(n) \to k(n)[v_n^{-1}] = K(n)$. The element of $\pi_*(R)$ represented by y^{p^k} is our desired element.

3.3.3. *Substep (iii): partially type n spectra.* The condition of being a partially type n spectrum is weaker than that of being a strongly type n spectrum.

Definition 3.13. Let X be a p-local finite spectrum. We say that X is partially type n if:

- Q_n acts trivially on $H^*(X)$;
- $K(n)^*(Y)$ and $H^*(Y)$ have the same rank;
- Each Q_i, for $i < n$, and each P^0_t, for $s + t \leq n$ (when $p > 2$; for $p = 2$, we need a modification similar to that in Definition 3.6), acts nontrivially on $H^*(X)$.

It remains to construct an example of a partially type n spectrum for every n.

Proposition 3.14. *Let $X = BC_p$, and let X_k denote its k-skeleton. Let X^i_j denote X^i/X^{j-1}, so X^i_j has bottom cell in dimension j and top cell in dimension i. Then $X^{2p^n}_2$ is partially type n.*

Proof. Conditions (a) and (b) of Definition 3.13 is satisfied whenever the difference between the dimensions of the top and bottom cells of X is at most $2p^n - 2$. Indeed, under this assumption, condition (a) follows immediately. Condition (c) follows from the fact that the AHSS for $K(n)^*(X)$ collapses (again because of dimensional reasons).

It remains to prove that condition (c) of Definition 3.13 is satisfied for $X^{2p^n}_2$. First suppose $p > 2$; then,

$$H^*(X) \simeq P(x) \otimes E(y),$$

where $|x| = 2$ and $|y| = 1$. By counting dimensions, we find that there is a basis for $H^*(X^{2p^n}_2)$ which maps injectively to

$$S = \{x^k | 1 \leq k \leq p^n\} \cup \{yx^k | 1 \leq k < p^n\}.$$

Moreover, we know that

$$Q_i(x^j) = 0,\ Q_i(yx^j) = x^{j+p^i},\ P^0_t(x^j) = jx^{j+p^t-1},\ P^0_t(yx^j) = jyx^{j+p^t-1},$$

so $P^0_t(x)$ and $Q_i(y)$ are both nontrivial. If $p = 2$, then

$$H^*(X) \simeq P(x),$$

with $|x| = 1$. Again, a basis for $H^*(X^{2^{n+1}}_2)$ maps injectively to $\{x^k | 1 < k \leq 2^n\}$. We again find that $P^0_t(x)$ is nontrivial (for $t < n+1$). □

4. The smashing conjecture and chromatic convergence

In this section, we will prove the following two results, stated in the first section.

Conjecture 4.1 (Smashing conjecture)**.** Let L_n denote localization with respect to $E(n)$. Then

$$L_nX \simeq X \wedge L_nS.$$

Theorem 4.2 (Chromatic convergence)**.** *If X is a p-local finite spectrum, then $X \simeq \lim L_nX$.*

Later on, when we talk about the moduli stack of formal groups and the height filtration, we will see the algebro-geometric analogue of this result.

4.1. **The chromatic convergence theorem.** We begin with an easy lemma.

Lemma 4.3. *The category $\mathscr{C}$ of p-local finite spectra X such that $X \simeq \lim L_nX$ is a thick subcategory of all p-local finite spectra.*

Proof. It is clear that $\mathscr{C}$ is closed under weak equivalences. To show that $\mathscr{C}$ is closed under retracts, let $X \to Y$ be the inclusion of a retract. Consider:

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle\sim} & & \downarrow \\ \lim L_nX & \longrightarrow & \lim L_nY & \longrightarrow & \lim L_nX. \end{array}$$

Every functor preserves retracts (so, in particular, $\lim L_nX$ is a retract of $\lim L_nY$, as shown above). Chasing the above diagram gives the desired inverse to the map $X \to \lim L_nX$.

It remains to prove that $\mathscr{C}$ is closed under cofibrations. For this, simply note that if $f : X \to Y$ is a map of p-local finite spectra, then the five lemma applied to the following diagram gives an equivalence $\mathrm{cofib}(f) \simeq \mathrm{cofib}(\lim L_nf)$:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & \mathrm{cofib}(f) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow \\ \lim L_nX & \xrightarrow[\lim L_nf]{} & \lim L_nY & \longrightarrow & \mathrm{cofib}(\lim L_nf). \end{array}$$

This completes the proof, since

$$\mathrm{cofib}(\lim L_nf) \simeq \Sigma\,\mathrm{fib}(\lim L_nf) \simeq \lim \Sigma\,\mathrm{fib}(L_nf) \simeq \lim \mathrm{cofib}(L_nf) \simeq \lim L_n\,\mathrm{cofib}(f).$$

□

By the thick subcategory theorem, it suffices to prove Theorem 4.2 for a single type 0 spectrum. The simplest such type 0 spectrum is the sphere spectrum. We are therefore reduced to proving that $S \simeq \lim L_nS$.

Recall that

$$C_nX = \mathrm{fib}(X \to L_nX),$$

so it suffices to prove that $\lim C_nS$ is trivial. Recall the Milnor short exact sequence:

$$0 \to \lim{}^1\pi_{n+1}(\lim X_i) \to \pi_n(\lim X_i) \to \lim \pi_n(X_i) \to 0.$$

It is therefore sufficient to prove that $\lim \pi_k(C_nS) = 0$, and that the system $\{\pi_k(C_nS)\}$ is Mittag-Leffler[7].

For the moment, assume the following result.

Lemma 4.4. *The map $C_{n+1}S \to C_nS$ induces the zero map $BP_*(C_{n+1}S) \to BP_*(C_nS)$.*

We will show the following.

Lemma 4.5. *Any map $f : X \to Y$ which induces the zero map on BP_*-homology raises Adams-Novikov filtration.*

[7]Recall that a system $\{A_i\}$ of abelian groups is said to be Mittag-Leffler if the decreasing sequence of subgroups $\mathrm{Im}(A_{i+j}) \subseteq A_i$ is independent of j for $j \gg 0$. This implies that $\lim^1 A_i = 0$.

Proof. Recall that $x \in \pi_n X$ has Adams-Novikov filtration $\geq k$ if x is detected by an element of filtration $\geq k$ in the (BP-based) Adams-Novikov spectral sequence for X. Geometrically, this can be phrased as follows. Let $\overline{BP}$ denote the fiber of the map $S \to BP$. Then $x \in \pi_n X$ has Adams-Novikov filtration $\geq k$ if x is in the image of the map $\pi_n(\overline{BP}^{\wedge k} \wedge X) \to \pi_n(X)$.

We need to show that if $x \in \pi_n X$ has Adams-Novikov filtration $\geq k$, then $f(x) \in \pi_n Y$ has Adams-Novikov filtration $\geq k+1$. Let $\alpha \in \pi_n(\overline{BP}^{\wedge k} \wedge X)$ be an element mapping to x. We then obtain an element $f(\alpha) \in \pi_n(\overline{BP}^{\wedge k} \wedge Y)$ mapping to $f(x)$. To prove the result, we need to show that $f(\alpha)$ lifts to an element of $\pi_n(\overline{BP}^{\wedge k+1} \wedge Y)$, i.e., that $f(\alpha)$ maps to zero in $\pi_n(\overline{BP}^{\wedge k} \wedge Y \wedge BP)$. In other words, it suffices to show that f induces the zero map $BP_*(\overline{BP}^{\wedge k} \wedge X) \to BP_*(\overline{BP}^{\wedge k} \wedge Y)$.

We claim that $BP_*(\overline{BP})$ is free over BP_*, so the Künneth forBPla immediately implies the desired result. To see this, recall that $BP_*(BP) \simeq BP_*[b_1, b_2, \cdots]$ is a free BP_*-module, with a basis given by the monomials in the b_i. By definition, a basis for $BP_*(\Sigma\overline{BP})$ as an BP_*-module is given by monomials of positive length. In particular, $BP_*(\overline{BP})$ is a free BP_*-module, as desired. □

This lemma, combined with Lemma 4.4, implies that the image of $\pi_n C_{m+k} S \to \pi_n C_m S$ consists of elements of Adams-Novikov filtration $\geq k$. Therefore, it suffices to prove:

Proposition 4.6. *There is some $k \geq 0$ such that every element of $\pi_n C_m S$ of Adams-Novikov filtration $\geq k$ is trivial. Equivalently, the BP-based Adams-Novikov spectral sequence for $C_m S$ has a vanishing curve on the E_∞-page.*

Definition 4.7. A spectrum X is said to be *BP-convergent* if the BP-based ANSS for $C_m S$ has a vanishing curve on the E_∞-page.

The discussion in the proof of Lemma 4.5 allows us to give an equivalent condition on X. First, we need some terminology. A map $f : X \to Y$ of spectra is *phantom below dimension n* if for every finite spectrum F with top cell in dimension $\leq n$, and every map $F \to X$, the composite $F \to X \to Y$ is null.

Lemma 4.8. *A spectrum X is BP-convergent iff for every n, there exists some k such that the map $\overline{BP}^{\wedge k} \wedge X \to X$ is phantom below dimension n.*

Proposition 4.6 now follows from the claim that for any connective spectrum X, the spectrum $C_n X$ is BP-convergent.

Proposition 4.9. *The subcategory $\mathscr{C}$ of BP-convergent spectra is a thick subcategory of the category of all spectra.*

Proof. Clearly $\mathscr{C}$ is closed under weak equivalences. It is an easy exercise to prove that $\mathscr{C}$ is closed under retracts too. It therefore remains to show that $\mathscr{C}$ is closed under cofibers. For this, it suffices to show that if $X \to Y \to Z$ is a cofiber sequence, and if X and Z are BP-convergent, then Y is BP-convergent.

We begin with a digression. Let $f : X \to Y$ be a map which is phantom below dimension n, and let Z be any connective spectrum. Then $f \wedge Z$ is phantom below dimension n. To see this, let F be a finite spectrum whose top cell is in dimension $\leq n$. Write Z as a filtered colimit of finite (connective) spectra Z_α; then since F is finite, a map $F \to X \wedge Z$ factors through $X \wedge Z_\alpha$ for some α. To show that the map $F \to X \wedge Z_\alpha \to Y \wedge Z_\alpha$ is null, it suffices to show that the map $DZ_\alpha \wedge F \to X \to Y$ is null. But since Z_α is connective, $DZ_\alpha \wedge F$ is a finite spectrum with top cell in dimension $\leq n$. Since $f : X \to Y$ is phantom below dimension n, the claim follows.

Let us return to the proof of the proposition. Fix n, and suppose $k \gg 0$ is such that the maps $\overline{BP}^{\wedge k} \wedge X \to X$ and $\overline{BP}^{\wedge k} \wedge Z \to Z$ are phantom below dimension n. We claim that $\overline{BP}^{\wedge 2k} \wedge Y \to Y$ is phantom below dimension n. Indeed, let F be a finite spectrum with top cell in dimension $\leq n$. Our discussion above implies that the map $\overline{BP}^{\wedge k} \wedge (\overline{BP}^{\wedge k} \wedge Z \to Z)$ is phantom below dimension n. Therefore, the map $F \to \overline{BP}^{\wedge 2k} \wedge Y \to \overline{BP}^{\wedge k} \wedge Z$ is null. In particular, it factors through some map $F \to \overline{BP}^{\wedge k} \wedge X$. This implies that the following diagram commutes:

$$\begin{array}{ccccc} F & & \longrightarrow & & \overline{BP}^{\wedge 2k} \wedge Y \\ \downarrow & & & & \downarrow \\ \overline{BP}^{\wedge k} \wedge X & \longrightarrow & X & \longrightarrow & Y; \end{array}$$

the composite going down the left vertical arrow is null, since X is BP-convergent. ☐

Since there is a cofiber sequence

$$C_n X \to X \to L_n X,$$

Proposition 4.9 shows that it will suffice to prove that X and $L_n X$ are both BP-convergent.

Lemma 4.10. *Any connective spectrum X is BP-convergent.*

Proof. We need to show that the map $\overline{BP}^{\wedge k} \wedge X \to X$ is phantom below dimension n. It certainly suffices to show that if F is a finite spectrum with top cell in dimension $\leq n$, then any map $F \to \overline{BP}^{\wedge n+1} \wedge X$ is null. In turn, it suffices to show that $\overline{BP}^{\wedge n+1} \wedge X$ is n-connected. Since X is connective, it suffices to check that $\overline{BP}$ is connected; but this follows from the long exact sequence in homotopy associated to the fiber sequence $\overline{BP} \to S \to BP$. ☐

Lemma 4.11. *If X is any spectrum, then $L_n X$ is BP-convergent.*

Proof. Let $X^\bullet$ be the cosimplicial ring spectrum given by $E(n)^{\bullet+1} \wedge X$. Then $\mathrm{Tot}(X)$ is defined to be the homotopy inverse limit of $X^\bullet$, and it is the inverse limit of partial totalizations $\mathrm{Tot}^n(X^\bullet)$, where $\mathrm{Tot}^0(X^\bullet) = X \wedge E(n)$. The smashing conjecture (proved below) implies that, as pro-spectra, we have

$$\{L_n X\} \simeq \{\mathrm{Tot}^k X^\bullet\}.$$

In particular, $L_n X$ is a retract of $\mathrm{Tot}^k X^\bullet$ for some k. By Proposition 4.9 it suffices to show that $\mathrm{Tot}^k X^\bullet$ is BP-convergent for every k. As this is a finite limit of $E(n)$-modules, it suffices to show that every $E(n)$-module is BP-convergent. The map $BP \to E(n)$ makes every $E(n)$-module into a BP-module, so it suffices to prove that every BP-module is BP-convergent. This is easy: the map $\overline{BP} \wedge M \to M$ is null. ☐

It remains to prove Lemma 4.4.

Proof of Lemma 4.4. As L_n is a smashing localization, we know that $C_n BP \simeq BP \wedge C_n S$, so $BP_*(C_n S) \simeq \pi_* C_n BP$. Recall from Theorem 4.3 of the first section that $N_n X \simeq \Sigma^n C_{n-1} X$, so

$$BP_*(C_n S) \simeq \Sigma^{-n} \pi_* N_{n+1} BP \simeq \Sigma^{-n} N^{n+1}.$$

Recall that

$$N^{n+1} \simeq BP_*/(p^\infty, v_1^\infty, \cdots, v_n^\infty).$$

Therefore the induced map on BP-homology is a map of graded BP_*-modules (after shifting by n)

$$BP_*/(p^\infty, v_1^\infty, \cdots, v_n^\infty) \to \Sigma BP_*/(p^\infty, v_1^\infty, \cdots, v_{n-1}^\infty).$$

Any such map is zero. ☐

4.2. **The smashing conjecture.** In the above proof, we made the following claim: by the smashing conjecture, we have an equivalence of pro-spectra

$$\{L_n X\} \simeq \{\mathrm{Tot}^k E(n)^{\bullet+1} \wedge X\}.$$

Let us see how this relates to the smashing conjecture. There is a natural map $X \to \mathrm{Tot}(X^\bullet)$. As $X^\bullet$ is a cosimplicial diagram of $E(n)$-modules, we find that $\mathrm{Tot}(X^\bullet)$ is $E(n)$-local. It is therefore natural to ask if $\mathrm{Tot}(X^\bullet)$ is just $L_n X$. Equivalently, we can ask if the map $E(n) \wedge X \to E(n) \wedge \mathrm{Tot}(X^\bullet)$ is an equivalence.

There is a natural map $E(n) \wedge \mathrm{Tot}(X^\bullet) \to \mathrm{Tot}(E(n) \wedge X^\bullet)$. The cosimplicial spectrum $\mathrm{Tot}(E(n) \wedge X^\bullet)$ has an extra codegeneracy map, so general properties of Tot imply that

$$\mathrm{Tot}(E(n) \wedge X^\bullet) \simeq \mathrm{Tot}^0(E(n) \wedge X^\bullet) \simeq E(n) \wedge X.$$

The composite

$$E(n) \wedge X \to E(n) \wedge \mathrm{Tot}(X^\bullet) \to \mathrm{Tot}(E(n) \wedge X^\bullet)$$

is an equivalence. Therefore, $\mathrm{Tot}(X^\bullet)$ is an $E(n)$-localization of X iff the natural map

$$E(n) \wedge \mathrm{Tot}(X^\bullet) \to \mathrm{Tot}(E(n) \wedge X^\bullet)$$

is an equivalence. In other words, we need to show that the map

$$E(n) \wedge \lim \mathrm{Tot}^n(X^\bullet) \to \lim \mathrm{Tot}^n(E(n) \wedge X^\bullet)$$

is an equivalence. Clearly this is the case if $\{\mathrm{Tot}^n(X^\bullet)\}$ is a constant pro-spectrum.

Now, for any spectrum Y, the pro-spectrum $\{\mathrm{Tot}^n((X \wedge Y)^\bullet)\}$ is just $\{\mathrm{Tot}^n(X^\bullet \wedge Y)\}$. If $\{\mathrm{Tot}^n(X^\bullet)\}$ is a constant pro-spectrum, then it follows that the inverse limit of $\{\mathrm{Tot}^n((X \wedge Y)^\bullet)\}$ is exactly $L_n(X \wedge Y)$. Therefore, we have an equivalence

$$(L_n X) \wedge Y \to L_n(X \wedge Y).$$

We therefore need to establish that $\{\mathrm{Tot}^n(X^\bullet)\}$ is a constant pro-spectrum for every spectrum X. For this, we use the following criterion of Bousfield's:

Proposition 4.12 (Bousfield)**.** *Suppose that there is some $N > 0$ such that for every finite spectrum F, the $E(n)$-based Adams-Novikov spectral sequence for $X \wedge F$ satisfies $E_N^{s,t} = 0$ for $s \geq N$. Then $\{\mathrm{Tot}^n(X^\bullet)\}$ is constant as a pro-spectrum.*

In fact, it is sufficient to find one such spectrum:

Lemma 4.13. *If there is a type 0 finite spectrum X satisfying the hypothesis of Proposition 4.12, then L_n-localization is smashing.*

Proof. Let $\mathscr{C}$ denotes the category of p-local finite spectra X such that for every Y, the map $(L_n X) \wedge Y \to L_n(X \wedge Y)$ is an equivalence. Then $\mathscr{C}$ is a thick subcategory (exercise). Since $\mathscr{C}$ contains a spectrum of type 0 (by Proposition 4.12 and the preceding discussion), it contains the sphere spectrum by the thick subcategory theorem, as desired. $\square$

The $E(n)$-based ANSS for $X \wedge F$ is the same as the $E(n)$-based ANSS for $(L_n X) \wedge F$, which has E_2-page

$$E_2^{s,t} \simeq \mathrm{Ext}^{s,t}_{E(n)_* E(n)}(E(n)_*, E(n)_*(X \wedge F)).$$

We will show that there is some $N > 1$ such that this group vanishes for $s \geq N$. The spectrum $L_n X$ is constructed from $M_m X$ for $0 \leq m \leq n$ by a bunch of cofiber sequences. By the proof of Lemma 4.13, it therefore suffices to prove this vanishing when X is replaced by $M_m X$ for $0 \leq m \leq n$.

Moreover, by Theorem 3.1 of the first section, we know that

$$\langle L_n BP \rangle = \langle v_n^{-1} BP \rangle = \langle E(n) \rangle. \tag{8}$$

Therefore, we can consider the $L_n BP$-based ANSS for $X \wedge F$, instead of the $E(n)$-based ANSS. The $L_n BP$-based ANSS for $X \wedge F$ coincides with the BP-based ANSS for $L_n(X \wedge F)$. Note that since F is finite, $L_n(X \wedge F) \simeq L_n X \wedge F$.

We have therefore reduced to proving that

$$\mathrm{Ext}^{s,t}_{BP_* BP}(BP_*, BP_*(M_m X \wedge F)) \simeq 0$$

for $s \geq N$ and all $0 \leq m \leq n$. As F is a finite spectrum, it is built from a finite number of cofiber sequences involving the sphere spectrum. It therefore suffices to prove this vanishing for $F = S$.

Let us begin with a short digression. We can inductively build $M^m = v_m^{-1} BP_*/(p^\infty, v_1^\infty, \cdots, v_{m-1}^\infty)$ from $v_m^{-1} BP_*/(p, v_1, \cdots, v_{m-1})$ as follows. Let

$$M_i^{m-i} = v_m^{-1} BP_*/(p, v_1, \cdots, v_{i-1}, v_i^\infty, \cdots, v_{m-1}^\infty).$$

Then there are short exact sequences

$$0 \to M_{i+1}^{m-i-1} \to \Sigma^{2(p^i-1)} M_i^{m-i} \xrightarrow{\cdot v_i} M_i^{m-i} \to 0.$$

Note that

$$M_m^0 = v_m^{-1} BP_*/(p, v_1, \cdots, v_{m-1}),\ M_0^m = M^m.$$

Likewise, we can define spectra

$$M_{m-i}^i BP = v_m^{-1} BP/(p, v_1, \cdots, v_{i-1}, v_i^\infty, \cdots, v_{m-1}^\infty),$$

so that

$$M_0^m BP = v_m^{-1} BP/(p, v_1, \cdots, v_{m-1}),\ M_m^0 = M_m BP.$$

By construction, there are cofiber sequences

$$M_{m-i-1}^{i+1} BP \to \Sigma^{2(p^i-1)} M_{m-i}^i BP \xrightarrow{\cdot v_i} M_{m-i}^i BP.$$

In the next subsection, we prove the localization conjecture (Conjecture 4.5 of the first section), which, when combined with Theorem 4.3 of the first section, implies that

$$BP_*(M_m X) \simeq \pi_*(M_m BP \wedge X).$$

By our discussion above, $M_m BP$ can be built from $M_0^m BP$ by a finite collection of cofiber sequences. Suppose that X is *even*, i.e., that all cells of X are in even dimensions. Then the Atiyah-Hirzebruch spectral sequence for $BP_*(X)$ collapses, and

$$BP_*(X) \simeq H_*(X; \mathbf{Z}_{(p)}) \otimes_{\mathbf{Z}_{(p)}} BP_*.$$

In particular, the long exact sequence in homotopy associated to the cofiber sequences

$$M_{m-i-1}^{i+1} BP \wedge X \to \Sigma^{2(p^i-1)} M_{m-i}^i BP \wedge X \xrightarrow{\cdot v_i \wedge X} M_{m-i}^i BP \wedge X$$

all become short exact sequences. In particular, we can obtain

$$\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*(M_m X))$$

from

$$\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, \pi_*(v_m^{-1} BP/(p, v_1, \cdots, v_{m-1}) \wedge X))$$

by n Bockstein spectral sequences. Again using evenness, the Atiyah-Hirzebruch spectral sequence computing $\pi_*(v_m^{-1} BP/(p, v_1, \cdots, v_{m-1}) \wedge X)$ collapses, so that

$$\pi_*(v_m^{-1} BP/(p, v_1, \cdots, v_{m-1}) \wedge X) \simeq v_m^{-1} BP_*(X)/(p, v_1, \cdots, v_{m-1}).$$

Therefore, we need to show that there exists an even p-local finite spectrum X such that

$$\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, v_m^{-1} BP_*(X)/(p, v_1, \cdots, v_{m-1})) \simeq 0$$

for $s \geq N$ and all $0 \leq m \leq n$.

Work of Miller-Ravenel [MR77] shows that if M is a (BP_*, BP_*BP)-comodule such that $v_m^{-1} M = M$ and $I_m M = 0$ (where $I_m = (p, v_1, \cdots, v_{m-1})$), then

$$\mathrm{Ext}_{BP_*BP}(BP_*, M) \simeq \mathrm{Ext}_{\Sigma(m)}(K(m)_*, K(m)_* \otimes_{BP_*} M),$$

where $\Sigma(m) = K(m)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(m)_*$. Note that this is *not* $K(m)_*K(m)$. The vanishing condition is then automatic when $m = 0$, so suppose that $m \geq 1$. In this case, the algebra $\Sigma(n)$ can be identified as the group ring of a certain profinite group.

In section 6, we will study a profinite group Γ (for a fixed m), defined via

$$\Gamma = (W(\mathbf{F}_{p^m})\langle S \rangle/(S^m = p, Sx = \phi(x)S))^\times.$$

We show that this ring contains elements of order p iff $(p-1)$ divides m. Moreover, [Rav86, Theorem 6.2.3] shows that, as ungraded profinite Hopf algebras

$$\mathbf{F}_{p^m}[[\Gamma]] \simeq (\Sigma(m) \otimes_{K(m)_*} \mathbf{F}_p)^* \otimes \mathbf{F}_{p^m}.$$

Therefore,

$$\mathrm{Ext}_{\Sigma(m)}(K(m)_*, K(m)_* \otimes_{BP_*} M) \simeq H_c^*(\Gamma; \mathbf{F}_{p^m} \otimes_{BP_*} M)^{h\,\mathrm{Gal}(\mathbf{F}_{p^m}/\mathbf{F}_p)}.$$

It therefore suffices to show that the cohomology of Γ with coefficients in $\mathbf{F}_{p^m} \otimes_{BP_*} M$ vanishes above a certain range, for all $0 \leq m \leq n$.

An argument of Serre's shows that if Γ does not have finite cohomological dimension, then it contains an element of order p. If $n < p-1$, then $(p-1)$ never divides m. Therefore Γ never contains an element of order p, so it has finite cohomological dimension.

Clearly, this argument does not work for $n \geq p-1$, since in that case, there are integers $1 \leq m \leq n$ such that $(p-1)$ divides n. However, some constructions involving the representation theory of the symmetric group allow us to construct a finite spectrum X such that $H_c^*(\Gamma; \mathbf{F}_{p^m} \otimes_{BP_*} v_m^{-1} BP_*(X)/I_m)^{h\,\mathrm{Gal}(\mathbf{F}_{p^m}/\mathbf{F}_p)}$ vanishes above a certain range: X is given by a summand of $\Sigma_+^\infty \mathbf{C}P^M$ (which, recall, is an even spectrum) which is described by splitting off a certain idempotent in $\mathbf{Z}_{(p)}[\Sigma_k]$ from its natural action on $(\Sigma^\infty_+ \mathbf{C}P^M)^{\wedge k(p-1)}$, for some $k, M \gg 0$.

4.3. **The localization conjecture.** Our goal in this subsection is to show that for any spectrum X, we have

$$L_n BP \wedge X \simeq BP \wedge L_n X.$$

An enhancement of Proposition 4.6 of section 1 (see [Rav84, Theorem 6.2]) shows that there is a cofiber sequence

$$\Sigma^{-n-1} N_{n+1} BP \to BP \to L_n BP,$$

where the first map is constructed inductively using the cofiber sequences

$$\Sigma^{-1} N_{n+1} BP \to N_n BP \to M_n BP.$$

To relate $L_n BP \wedge X$ and $BP \wedge L_n X$, we note that both admit maps to $L_n BP \wedge L_n X$ such that there are cofiber sequences

$$C_n BP \wedge L_n X \to BP \wedge L_n X \to L_n BP \wedge L_n X,\ L_n BP \wedge C_n X \to L_n BP \wedge X \to L_n BP \wedge L_n X.$$

Therefore, it suffices to prove that

$$C_n BP \wedge L_n X \simeq * \simeq L_n BP \wedge C_n X.$$

We first prove that $C_n BP \wedge L_n X \simeq *$.

Lemma 4.14. *Let Y be a p-local finite spectrum of type $n+1$. Then $\langle C_n BP \rangle = \langle C_n BP \wedge Y \rangle$.*

Proof. Clearly $\langle C_n BP \wedge Y \rangle \leq \langle C_n BP \rangle$ since if $X \wedge C_n BP \simeq *$, then $X \wedge C_n BP \wedge Y$ is also contractible. Let $\mathscr{C}$ be the subcategory of finite spectra such that $\langle C_n BP \wedge Y \rangle \geq \langle C_n BP \rangle$. Then $\mathscr{C}$ is a thick subcategory, so it suffices to prove the above result for a single finite spectrum of type $n+1$. Let $Y = S/(p^\infty, v_1^\infty, \cdots, v_n^\infty)$, so Y is a finite spectrum of type $n+1$. We will show that the natural map $C_n BP \to Y \wedge C_n BP$ is an equivalence; clearly, this implies the desired result. By Theorem 4.3 of the first section, we have an equivalence

$$C_n BP \simeq \Sigma^{-n-1} N_{n+1} BP \simeq \Sigma^{-n-1} BP/(p^\infty, v_1^\infty, \cdots, v_n^\infty).$$

From this, it is easy to get the desired equivalence. □

It therefore suffices to prove that $L_n X \wedge Y \simeq *$. However, since Y is finite,

$$Y \wedge L_n X \simeq L_n X \wedge L_n Y.$$

Now, Y is $E(n)$-acyclic since it is of type $n+1$, so the identifications in (8) show that this group vanishes, as desired.

It remains to show that $L_n BP \wedge C_n X \simeq *$. However, since $C_n X$ is the cofiber of the map $X \to L_n X$ which is an $E(n)_*$-equivalence, it follows from (8) that $L_n BP \wedge C_n X$ is contractible.

5. The nilpotence theorem

5.1. **Introduction.** In previous sections, we saw the power of the nilpotence theorem. The goal of this section is to describe its proof. Recall that the result we want to prove is the following.

Theorem 5.1. *Let R be a ring spectrum. The kernel of the map $\pi_* R \to MU_* R$ consists of nilpotent elements.*

Before proceeding, let us discuss an equivalent formulation of this statement, that was discussed at the end of the first section.

Proposition 5.2. *Let $g(n)$ denote the maximal ANSS filtration of an element in $\pi_n S$. Then*

$$\lim_{n\to\infty} \frac{g(n)}{n} = 0.$$

Proof. Let $\epsilon > 0$. If we can show that there is a vanishing line of slope ϵ on the E_r-page for some r, then

$$g(n) \leq \epsilon n + O(1);$$

as ϵ was arbitrary, this implies that $g(n) = o(n)$, as desired. We will show that such a vanishing line exists if we replace S with any finite p-local spectrum X. The category of finite p-local spectra with such a vanishing line is thick, so it suffices to find one such spectrum by the thick subcategory theorem (which, recall, relies on the nilpotence theorem). But [DHS88, Proposition 4.5] shows that, for some finite spectrum Y, the ANSS for the smash product $X \wedge Y$ has such a vanishing line. It follows that the result is true for X, as desired. □

We warn the reader that we will generally localise at a prime p and keep quiet about it. Thanks to Jeremy Hahn for pointing out that our proof of Nishida's theorem falls short.

5.2. **Nishida's theorem.** In this subsection, we will give one proof of Nishida's theorem of the nilpotency of positive-degree elements in $\pi_* S$, which will be generalised by the nilpotence theorem. This will motivate certain aspects of the proof of the main result. We will work p-locally. Let $\alpha \in \pi_n S$, so α is a p-torsion element. This can be thought of as a self map

$$\alpha : S^n \to S.$$

The map α is zero on homology.

Showing that α is nilpotent is equivalent to proving that the telescope $\alpha^{-1}S = S[1/\alpha]$ is contractible, which in turn is equivalent to showing that the composite

$$S^n \xrightarrow{\alpha} S \to S[1/\alpha]$$

is null. We can *almost* reduce to the case when $p\alpha = 0$ for some prime p: if α is in odd degree, then $\alpha^2 = -\alpha^2$ by the Koszul sign rule. Therefore $2\alpha^2 = 0$. Note that the nilpotency of α^2 certainly implies the nilpotency of α. If p is odd, then $2\alpha^2 = 0$ implies that $\alpha^2 = 0$. If $p = 2$, then we need to be more careful.

If α is a general element of $\pi_* S$ such that $p\alpha = 0$, then the spectrum $S[1/\alpha]$ is an E_∞-ring with $p = 0$. We now need the following theorem, whose proof will be given at the end of this subsection.

Proposition 5.3 (Mahowald)**.** *The free (p-local) E_2-algebra with $p = 0$ is $H\mathbf{F}_p$. Moreover, $H\mathbf{F}_2$ is the*[8] *free E_2-algebra with $2 = 0$.*

It follows that (after p-localizing when $p > 2$, and integrally if $p = 2$) $S[1/\alpha]$ is a wedge of suspensions of $H\mathbf{F}_p$. Now, the mod p Hurewicz image of α is zero, so the composite $S^n \xrightarrow{\alpha} S \to S[1/\alpha]$ is null, as desired.

Proof sketch of Proposition 5.3. The space $BGL_1(S)$ classifies (p-local) spherical fibration. There is a map $S^1 \to BGL_1(S)$, which factors through $\Omega^2 S^3$ since the target is an infinite loop space (in particular, a 2-fold loop space). The Thom spectrum of the resulting spherical bundle is the free E_2-algebra with $p = 0$. Since $H\mathbf{F}_p$ is an E_2-algebra with $p = 0$, we have a map from this Thom spectrum to $H\mathbf{F}_p$. One can show that this map is an isomorphism on mod p homology, along with the Steenrod action; it follows that $H\mathbf{F}_p$ is the free E_2-algebra with $p = 0$. □

The proof of the nilpotence theorem will extract the essence of this proof: it will eventually reduce to showing that the telescope of a certain map on a certain spectrum is contractible.

Remark 5.4. This proof does not address the case $p^k\alpha = 0$ for $k > 1$ since, in that case, $S[1/\alpha]$ is an E_∞-algebra with $p^k = 0$ (and this is not $H\mathbf{F}_p$). Note that we only learn about the nilpotency of $p^{k-1}\alpha$ from this argument. It seems like an interesting (maybe undoable) exercise to use some modification of our discussion above to conclude the nilpotency of α itself.

5.3. **Attacking Theorem 5.1.** Let us outline our strategy for proving Theorem 5.1. We will first construct E_2-ring spectra, denoted $X(n)$, which interpolate between S and MU. One then has a Hurewicz map

$$\pi_* R \xrightarrow{h(n)_*} X(n)_* R.$$

It turns out that the spectra $X(n+1)$ can be written as a filtered colimit of spectra G_k, with $X(n) = G_0$.

The nilpotence theorem is then a simple consequence of the following statement: if $h(n+1)_*(x) = 0$, then $h(n)_*(x)$ is nilpotent. This in turn will then be a simple consequence of the following three steps.

[8] Note that this statement is made integrally, not 2-locally.

(1) if $h(n+1)_*(x) = 0$, then $G_k \wedge x^{-1}R \simeq 0$ for some $k \gg 0$.
(2) the Bousfield class of G_k agrees with the Bousfield class of $X(n)$ for all finite k.
(3) if $X(n) \wedge x^{-1}R \simeq 0$, then $h(n)_*(x)$ is nilpotent.

The third step is the easiest, and is left as a simple exercise; in fact, it is true that if E is any ring spectrum, then the statement in the third step is true with $X(n)$ replaced by E. In the next subsection, we will attack the first and second steps, following [DHS88].

The spectra $X(n)$ were already discussed in Ravenel's paper [Rav84]. Recall that Bott periodicity gives an equivalence (of infinite loop spaces)

$$\Omega SU \simeq BU.$$

There is a natural map $\Omega SU(n) \to \Omega SU$, so we define $X(n)$ to be the Thom spectrum of the complex vector bundle over $\Omega SU(n)$ defined by the composite $\Omega SU(n) \to \Omega SU \simeq BU$. Note that $X(1) = S$, and $X(\infty) = MU$. The map $\Omega SU(n) \to BU$ is a 2-fold loop map, so $X(n)$ is an E_2-ring spectrum.

Proposition 5.5. *$X(n)_*X(m)$ is flat over $X(n)_*$.*

Proof. Recall that

$$H_*(\Omega SU(n)) \simeq \mathbf{Z}[b_1, \cdots, b_{n-1}],$$

with $|b_i| = 2i$. Since $X(n)$ is a Thom spectrum for a vector bundle over $\Omega SU(n)$, this is the same as $H_*X(n)$ by the Thom isomorphism. An Atiyah-Hirzebruch spectral sequence computation therefore establishes that

$$X(n)_*X(m) \simeq X(n)_*[b_1, \cdots, b_{m-1}],$$

so $X(n)_*X(m)$ is flat over $X(n)_*$, as desired. □

In order to prove that if x is in the kernel of $h(n+1)_*$, then $h(n)_*(x)$ is nilpotent, we need to interpolate between $X(n)$ and $X(n+1)$. For this, we recall that there is a fiber sequence

$$SU(n) \to SU(n+1) \to S^{2n+1},$$

which gives a fiber sequence

$$\Omega SU(n) \to \Omega SU(n+1) \to \Omega S^{2n+1}.$$

The space ΩS^{2n+1} admits a filtration, called the James filtration.

Let X be a based CW-complex. The James construction JX on X is the free monoid on the set X with unit as the basepoint. There is an evident map $X^{\times k} \to JX$, whose image is denoted J_kX; this is topologized as a quotient of $X^{\times k}$. The space JX is the colimit of the spaces J_kX. James proved that

$$JX \simeq \Omega\Sigma X, \tag{9}$$

so the spaces J_kS^{2n} give a filtration of ΩS^{2n+1}. It follows that we may define spaces B_k by the pullback

$$\begin{array}{ccc} B_k & \longrightarrow & J_kS^{2n} \\ \downarrow & & \downarrow \\ \Omega SU(n+1) & \longrightarrow & \Omega S^{2n+1}. \end{array}$$

The composite $B_k \to \Omega SU(n+1) \to BU$ gives a complex vector bundle over B_k, whose Thom spectrum is denoted F_k. We define $G_k = F_{p^k-1}$. Note that $G_0 = X(n)$, and $G_\infty = X(n+1)$. Moreover, by construction, the spectrum $X(n+1)$ is a filtered colimit of the spectra G_k.

The homology of $\Omega SU(n+1)$ is isomorphic to $\mathbf{Z}[b_1, \cdots, b_n]$ with $|b_k| = 2k$. One can show, using the Eilenberg-Moore spectral sequence, that

$$H_*B_k \simeq H_*(\Omega SU(n))\langle 1, b_n, \cdots, b_n^k\rangle.$$

The Thom isomorphism, combined with our discussion in the first subsection, shows that

$$H_*G_k \simeq H_*(X(n))\langle 1, b_n, \cdots, b_n^{p^k-1}\rangle.$$

One might therefore think of $X(n+1)$ as a spectrum with "cells" given by $X(n)$, and "skeleta" G_k. We need to show that

$$\langle G_k\rangle = \langle X(n)\rangle.$$

Again using the Thom isomorphism, we find that a simple computation with the Atiyah-Hirzebruch spectral sequence gives an isomorphism

$$X(n+1)_*F_k \simeq X(n+1)_*X(n)\langle 1, b_n, \cdots, b_n^k\rangle.$$

In particular, Proposition 5.5 proves that $X(n+1)_*F_k$ is flat over $X(n+1)_*$.

5.4. **Establishing step 1.** By Proposition 5.5, $X(n+1)_*X(n+1)$ is flat over $X(n+1)_*$. It follows that the E_2-page for the $X(n+1)$-based ANSS runs

$$E_2^{*,*}(R) = \mathrm{Ext}^{*,*}_{X(n+1)_*X(n+1)}(X(n+1)_*, X(n+1)_*R) \Rightarrow \pi_*R.$$

Step 1 will be a consequence of the following result, whose proof will be given at the end of this subsection.

Theorem 5.6. *If M is a connective $X(n+1)_*X(n+1)$-comodule, then $\mathrm{Ext}_{X(n+1)_*X(n+1)}(X(n+1)_*, X(n+1)_*G_k \otimes_{X(n+1)_*} M)$ has a vanishing line of slope which goes to 0 as $k \to \infty$.*

Let $x \in \pi_*R$ be in the kernel of the Hurewicz map $h(n+1)_*$. Then x is detected by an element α in $E_2^{a,b}(R)$ of positive $X(n)$-Adams filtration (i.e., $a > 0$). As discussed at the end of the previous subsection, $X(n+1)_*G_k$ is flat over $X(n+1)_*$, so

$$X(n+1)_*(G_k \wedge R) \simeq X(n+1)_*(G_k) \otimes_{X(n+1)_*} X(n+1)_*(R).$$

Theorem 5.6 shows that, for any $M > 0$, there is some $k \gg 0$ such that the $X(n+1)$-based ANSS for $G_k \wedge R$ has a vanishing line of slope less than M. Let $y \in \pi_*(G_k \wedge R)$; this is detected by some element $\beta \in E_2^{s,t}(G_k \wedge R)$. Then the elements $\alpha\beta, \alpha^2\beta, \cdots$ lie on a line in $E_2(G_k \wedge R)$ of slope $a/(b-a)$. For a large enough k, therefore, the vanishing line in $E_2(G_k \wedge R)$ has slope less than $a/(b-a)$. This implies that, for some M, the element $\alpha^M\beta$ lives above the vanishing line; it follows that $G_k \wedge x^{-1}R$ is contractible.

Let us now return to the proof of Theorem 5.6. This is a purely algebraic argument, which brings into play some general facts about Hopf algebroids.

Proof of Theorem 5.6. Our goal will be to simplify the group $\mathrm{Ext}_{X(n+1)_*X(n+1)}(X(n+1)_*, X(n+1)_*G_k \otimes_{X(n+1)_*} M)$ as much as possible. Fix a base ring R, and let S be a commutative Hopf R-algebra. Let A be a right S-comodule algebra. Then we can define a Hopf algebroid $(A, A \otimes S)$. We need the following result, which is [Mil81, Proposition 7.6].

Theorem 5.7. *If M is an $(A, A \otimes S)$-comodule, then*

$$\mathrm{Ext}_{A\otimes S}(A, M) \simeq \mathrm{Ext}_S(R, M).$$

Using Theorem 5.7, and the fact that

$$X(n+1)_*X(n+1) \simeq X(n+1)_*[b_1, \cdots, b_n],$$

we have

$$\mathrm{Ext}_{X(n+1)_*X(n+1)}(X(n+1)_*, X(n+1)_*G_k \otimes_{X(n+1)_*} M) \simeq \mathrm{Ext}_{\mathbf{Z}_{(p)}[b_1,\cdots,b_n]}(\mathbf{Z}_{(p)}, \mathbf{Z}_{(p)}[b_1, \cdots, b_{n-1}]\langle 1, b_n, \cdots, b_n^{p^k-1}\rangle \otimes_{X(n+1)_*} X(n+1)_*R).$$

We would like to simplify this even more. Let (A, Γ) be a Hopf algebroid. If $f : A \to B$ is a faithfully flat map, then the pair $(B, B \otimes_A \Gamma \otimes_A B)$ is another Hopf algebroid, which presents the same groupoid as (A, Γ). Concretely, this implies that in our case (with the faithfully flat map $\mathbf{Z}_{(p)}[b_1, \cdots, b_n] \to \mathbf{Z}_{(p)}[b_n]$), the above Ext-group is isomorphic to

$$\mathrm{Ext}_{\mathbf{Z}_{(p)}[b_n]}(\mathbf{Z}_{(p)}, \mathbf{Z}_{(p)}\langle 1, b_n, \cdots, b_n^{p^k-1}\rangle \otimes_{X(n+1)_*} X(n+1)_*R).$$

It therefore remains to establish a vanishing line for this Ext-group.

Recall that this Ext-group can be computed by the cobar complex for $\mathbf{Z}_{(p)}\langle 1, b_n, \cdots, b_n^{p^k-1}\rangle \otimes_{X(n+1)_*} X(n+1)_*R$ over $\mathbf{Z}_{(p)}[b_n]$. We can therefore filter this cobar complex by powers of the ideal generated by p; this begets a Bockstein/May spectral sequence

$$\mathrm{Ext}_{\mathbf{F}_p[b_n]}(\mathbf{F}_p, \mathbf{F}_p\langle 1, b_n, \cdots, b_n^{p^k-1}\rangle \otimes_{X(n+1)_*} \mathrm{gr}_p X(n+1)_*R) \Rightarrow \mathrm{Ext}_{\mathbf{Z}_{(p)}[b_n]}(\mathbf{Z}_{(p)}, \mathbf{Z}_{(p)}\langle 1, b_n, \cdots, b_n^{p^k-1}\rangle \otimes_{X(n+1)_*} X(n+1)_*R) \otimes \mathbf{Z}_p.$$

Here, $\mathrm{gr}_p X(n+1)_* R$ denotes the associated graded[9] coming from the p-adic filtration on $X(n+1)_* R$. One might complain that this spectral sequence converges not to the Ext-group $\mathrm{Ext}_{\mathbf{Z}_{(p)}[b_n]}(\mathbf{Z}_{(p)}, \mathbf{Z}_{(p)}\langle 1, b_n, \cdots, b_n^{p^k-1}\rangle \otimes_{X(n+1)_*} X(n+1)_* R)$, but rather to that Ext-group tensored with $\mathbf{Z}_p$. For the sake of establishing a vanishing line, though, this is sufficient.

The general yoga of spectral sequences reduces us to proving that there is a vanishing line for

$$\mathrm{Ext}_{\mathbf{F}_p[b_n]}(\mathbf{F}_p, \mathbf{F}_p\langle 1, b_n, \cdots, b_n^{p^k-1}\rangle).$$

There is an isomorphism of coalgebras

$$\mathbf{F}_p[b_n] \simeq \bigotimes_{i\geq 0} \mathbf{F}_p[b_n^{p^i}]/(b_n^{p^{i+1}}).$$

Moreover, as $\mathbf{F}_p[b_n]$-comodules, we have

$$\mathbf{F}_p\langle 1, b_n, \cdots, b_n^{p^k-1}\rangle \simeq \bigotimes_{i\geq 0}^{k-1} \mathbf{F}_p[b_n^{p^i}]/(b_n^{p^{i+1}}).$$

It follows that

$$\mathrm{Ext}_{\mathbf{F}_p[b_n]}(\mathbf{F}_p, \mathbf{F}_p\langle 1, b_n, \cdots, b_n^{p^k-1}\rangle) \simeq \mathrm{Ext}_{\bigotimes_{i\geq k} \mathbf{F}_p[b_n^{p^i}]/(b_n^{p^{i+1}})}(\mathbf{F}_p, \mathbf{F}_p) \simeq \mathrm{Ext}_{\mathbf{F}_p[b_n^{p^k}]}(\mathbf{F}_p, \mathbf{F}_p).$$

Note that $|b_n^{p^k}| = 2np^k$. It is now an exercise with the cobar complex to show that this Ext-group has a vanishing line of slope $1/(2np^k - 1)$. Tracing through the May spectral sequence above, this concludes the proof of Theorem 5.6. □

5.5. **Finishing off the argument with step 2.**

5.5.1. *An important proposition.* It suffices to show that if E is any spectrum, then $G_k \wedge E \simeq *$ implies that $G_{k-1} \wedge E \simeq *$. We need the following result.

Proposition 5.8. *Let X be a spectrum, and let $f : \Sigma^n X \to X$ be a self-map. Then*

$$\langle X\rangle = \langle f^{-1}X\rangle \vee \langle C(f)\rangle.$$

Proof. Let E be another spectrum; we need to show that if $f^{-1}X \wedge E$ and $C(f) \wedge E$ are contractible, then $X \wedge E$ is contractible. We need to show that if $\alpha \in \pi_*(X \wedge E)$, then $\alpha = 0$. Let $\alpha : S^d \to X \wedge E$ be the map representing α. The composite

$$S^d \xrightarrow{\alpha} X \wedge E \to f^{-1}X \wedge E$$

is nullhomotopic. It factors through some $\Sigma^{-nk} X \wedge E$, so the composite

$$S^d \xrightarrow{\alpha} X \wedge E \to \Sigma^{-nk} X \wedge E$$

is null. Therefore this map factors through a map $S^d \to \Sigma^{-n(k-1)-1} X \wedge E \simeq *$. It follows by induction that $\alpha = 0$. □

According to Proposition 5.8, it therefore suffices to show that there is a self-map b of G_k whose cofiber is Bousfield equivalent to G_{k+1}, and whose telescope is contractible.

5.5.2. *Constructing the self map.* One can define an action of $\Sigma^\infty_+ \Omega^2 S^{2np^k+1}$ on G_k in (roughly) the following manner. The fiber sequence from Lemma 5.11, combined with the equivalence of (9) allows us to draw a diagram of fiber sequences

$$\begin{array}{ccc} \Omega^2 S^{2np^k+1} & = & \Omega^2 S^{2np^k+1} \\ \downarrow & & \downarrow \\ B_{p^k-1} & \longrightarrow & J_{p^k-1} S^{2n} \\ \downarrow & & \downarrow \\ \Omega SU(n+1) & \longrightarrow & \Omega S^{2n+1}. \end{array}$$

[9]This really has a *bi*grading: one coming from the actual grading on the associated graded, and the other coming from the homological grading on $X(n+1)_* R$.

If $f : X \to Y$ is a (pointed) map, then one obtains an action of ΩY on the homotopy fiber of f. In this case (modulo technical details; see [DHS88, Constructions 3.16-17], if you're so inclined), we get an action of $\Omega^2 S^{2np^k+1}$ on B_{p^k-1}. The space $\Omega^2 S^{2np^k+1}$ has the trivial bundle living over it, coming from the null vertical composite in the above diagram. Thomifying, we get an action of $\Sigma^\infty_+ \Omega^2 S^{2np^k+1}$ on G_k.

Snaith proved that there is a splitting of $\Sigma^\infty_+ \Omega^2 S^{2np^k+1}$ (see [Rav98, §2.2]):

Theorem 5.9 (Snaith)**.** *We have*

$$\Sigma^\infty_+ \Omega^2 S^{2np^k+1} \simeq \bigvee_{i\geq 0} D_i,$$

where the D_i are certain finite spectra.

Proof. See [Kuh07, §6.1] or [Dev17, Theorem 9] (with $X = S^{2np^k-1}$) for a proof. Explicitly, if $C^{(n)}(k)$ denotes the space of embeddings of k little n-cubes into a big n-cube[10], then we find that

$$D_i = \left(\Sigma^{(2np^k-1)i} \Sigma^\infty_+ C^{(2)}(i)\right)_{h\Sigma_i},$$

where Σ_i acts in the obvious way on $C^{(2)}(i)$ and by permuting the factors of S^{2np^k-1} in the suspension. □

As $\Omega^2 S^{2np^k+1}$ is a H-space, its multiplication gives rise to operations

$$D_i \wedge D_j \to D_{i+j}.$$

Note that these are the same operations that come from the operations in the little 2-disks operad. Moreover, the unit $* \to \Omega^2 S^{2np^k+1}$ realizes an equivalence $S^0 \xrightarrow{\sim} D_0$. Our explicit description of the D_i also shows that $D_1 \simeq S^{2np^k-1}$. The above multiplication therefore begets a map $\Sigma^{2np^k-1} D_i \to D_{i+1}$. Consequently, defining $L_i = \Sigma^{-(2np^k-1)i} D_i$, this gives a map $L_i \to L_{i+1}$. We define $D_\infty = \varinjlim L_i$.

Let us simplify the presentation of the splitting in Theorem 5.9. One can compute that

$$H_*(\Omega^2 S^{2n+1}; \mathbf{F}_p) \simeq \begin{cases} P(x_0, x_1, \cdots) & p = 2 \\ P(y_1, y_2, \cdots) \otimes E(x_0, x_1, \cdots) & p > 2 \end{cases},$$

where $|y_i| = 2p^i n - 2$ and $|x_i| = 2p^i n - 1$. We can assign a weight to the x_i's and y_i's by defining

$$||x_i|| = p^i = ||y_i||,$$

and letting $||a \cdot b|| = ||a|| + ||b||$. It turns out that on homology, the map $D_i \to \Omega^2 S^{2n+1}$ is the inclusion of the $\mathbf{F}_p$-vector space of monomials of weight i; this is a result of Mahowald and Brown-Peterson at the even prime, and Cohen, Hunter, and Kuhn at odd primes. The weight of every generator except for x_1 is divisible by p. It is now clear that $H_*(D_i; \mathbf{F}_p)$ is 0 if $i \neq 0, 1 \mod p$. This implies that D_∞ is contractible at odd primes — we need to find a replacement.

On mod p homology, the map $\Sigma^{2np^k-1} D_{pj} \to D_{pj+1}$ defined above, induces multiplication by x_1 (which has weight 1). The above discussion implies that this map is an isomorphism, so

$$\Sigma^{2np^k-1} D_{pj} \simeq D_{pj+1}.$$

For $i = 0$ or 1, the spectrum $\Sigma^{-(p-2)n} D_{pn+i}$ is the "nth Brown-Gitler spectrum". This establishes a (p-local) equivalence

$$\Sigma^\infty_+ \Omega^2 S^{2np^k+1} \simeq \left(\bigvee_{j\geq 0} D_{pj}\right) \wedge (S \vee S^{2np^k-1}).$$

There is an element $\alpha_j \in \pi_{(2np-2)j} D_p$ whose Hurewicz image is y_1^j. This begets a map

$$\Sigma^{(2np-2)} D_{pj} \to D_p \wedge D_{pj} \to D_{pj+p},$$

so letting $L_{pj} = \Sigma^{-(2np-2)j} D_{pj}$, we have maps $L_{pj} \to L_{p(j+1)}$. The composite

$$S^{(2np-2)j} \to D_{pj} \to \Sigma^\infty_+ \Omega^2 S^{2n+1},$$

[10]Equivalently, ordered subsets of size k of $\mathbf{R}^n$.

smashed with G_k, gives a map (when n is np^k)

$$f_j : \Sigma^{(2np^{k+1}-2)j} G_k \to D_{pj} \wedge G_k \to \Sigma^\infty_+ \Omega^2 S^{2np^k+1} \wedge G_k \xrightarrow{\mu} G_k.$$

The map $f = f_1$ is the desired self map of G_k; it follows that $f_1^k = f_k$. This begets a commuting diagram

$$\begin{array}{ccccc} G_k & \longrightarrow & L_p \wedge G_k & \longrightarrow & \cdots \\ \| & & \downarrow & & \\ G_k & \xrightarrow[f]{} & \Sigma^{-(2np^{k+1}-2)} G_k & \xrightarrow[f]{} & \cdots \end{array}$$

This means that the map $G_k \to f^{-1}G_k$ factors through $G_k \wedge \varinjlim L_{pj}$. Mahowald proved that $\varinjlim L_{pj} \simeq H\mathbf{F}_p$ (in fact, one can compute the homology of $\varinjlim L_{pj}$ — along with its Steenrod action — and this can be seen to agree with $\mathscr{A}_*$ and its action of the Steenrod algebra), so we obtain the following result.

Proposition 5.10. *The map $G_k \to f^{-1}G_k$ factors through $G_k \wedge H\mathbf{F}_p$.*

5.5.3. *Constructing another self-map.* It is not at all apparent from the construction that the cofiber of f is Bousfield equivalent to G_{k+1}. We will therefore construct another self map of G_k, which will be equivalent to f, whose cofiber will be something Bousfield equivalent to G_{k+1}. To kick off, we will need a cofiber sequence relating B_{p^k-1} and $B_{p^{k+1}-1}$. We will need the following result, which we shall not prove (see [Nei10, Proposition 5.2.2]).

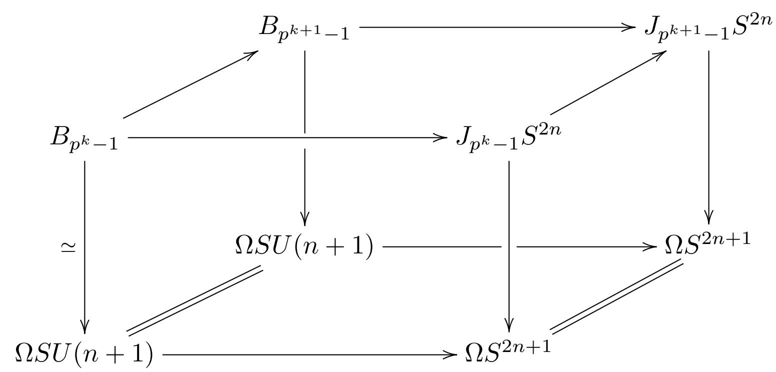
Lemma 5.11. *There is a p-local fiber sequence*

$$J_{p^k-1}S^{2n} \to JS^{2n} \to JS^{2np^k}.$$

Restricting the base of this fiber sequence gives rise to a fiber sequence, for any $m > 1$:

$$J_{p^k-1}S^{2n} \to J_{mp^k-1}S^{2n} \to J_{m-1}S^{2np^k}.$$

Our interest will be in the case $m = p$. We then have a cube:



One therefore gets a p-local fiber sequence

$$B_{p^k-1} \to B_{p^{k+1}-1} \xrightarrow{g} J_{p-1}S^{2np^k}.$$

If we invert $(p-1)!$ — which already has occurred since we are p-local — then we will use this fiber sequence to construct another self map

$$b : \Sigma^{2np^{k+1}-2} G_k \to G_k,$$

which we will prove to be equivalent to f.

Note that there is a filtration of the inclusion $B_{p^k-1} \to B_{p^{k+1}-1}$, given by the B_{rp^k-1} for $1 \leq r \leq p-1$. Devinatz-Hopkins-Smith do a lot of pretty general constructions with Thom spectra, but we will directly specialise to the case of interest.

We have a composite map

$$B_{p^{k+1}-1} \xrightarrow{g \times 1} J_{p-1}S^{2np^k} \times B_{p^{k+1}-1} \to B_{p^{k+1}-1} \to \Omega SU(n+1) \to \Omega SU \simeq BU,$$

so taking the Thom spectrum gives a map

$$G_{k+1} \to \Sigma^\infty_+ J_{p-1} S^{2np^k} \wedge G_{k+1}.$$

There is a multiplicative splitting

$$\Sigma^\infty_+ J_{p-1} S^{2np^k} \simeq \bigvee_{j=0}^{p-1} S^{2np^k j},$$

so for every $0 \le i \le p-1$, we get a map

$$\theta_i : G_{k+1} \to \Sigma^\infty_+ J_{p-1} S^{2np^k} \wedge G_{k+1} \xrightarrow{\text{collapse}} S^{2np^k i} \wedge G_{k+1}.$$

Let P_k denote the cofiber of the map θ_1, so that there is a cofiber sequence

$$G_{k+1} \xrightarrow{\theta_1} \Sigma^{2np^k} G_{k+1} \to P_k. \tag{10}$$

This implies that

$$\langle G_{k+1} \rangle \ge \langle P_k \rangle.$$

We will show that the cofiber of the as-yet-nonexistent self map b is P_k. As discussed above, the contractibility of the telescope of b implies, by Proposition 5.8, that

$$\langle P_k \rangle = \langle G_k \rangle.$$

It turns out that θ_i is "filtration preserving"[11], in the sense that there is an induced map

$$\theta_i : F_{qp^k-1}/F_{(q-h)p^k-1} \to S^{2np^k i} \wedge F_{(q-i)p^k-1}/F_{(q-h-i)p^k-1}.$$

One[12] can show that for $0 \le i \le p-1$, the filtered map θ_i gives an equivalence

$$\theta_i : F_{ip^k-1}/F_{(i-1)p^k-1} \xrightarrow{\sim} S^{2np^k i} \wedge G_k. \tag{11}$$

A trivial consequence is that there is a cofiber sequence

$$F_{(i-1)p^k-1} \to F_{ip^k-1} \to \Sigma^{2np^k i} G_k,$$

so the quotients in this filtration of G_{k+1} are just suspensions of G_k. This implies that

$$\langle G_k \rangle \ge \langle G_{k+1} \rangle.$$

Combined with the discussion above, this implies that the two Bousfield classes are indeed the same.

It remains to construct the self map b, and show that its cofiber is P_k.

Lemma 5.12. *We have*

$$\theta_j \theta_i = \binom{i+j}{i} \theta_{i+j}.$$

Proof. Again, we will only prove the "unfiltered" statement. The map

$$B_{p^{k+1}-1} \xrightarrow{g \times g \times 1} J_{p-1} S^{2np^k} \times J_{p-1} S^{2np^k} \times B_{p^{k+1}-1}$$

factors in two ways as the composite of two maps, as the following diagram illustrates:

$$\begin{array}{ccc} B_{p^{k+1}-1} & \xrightarrow{g \times 1} & J_{p-1} S^{2np^k} \times B_{p^{k+1}-1} \\ \downarrow {\scriptstyle g \times 1} & & \downarrow {\scriptstyle 1 \times g} \\ J_{p-1} S^{2np^k} \times B_{p^{k+1}-1} & \xrightarrow[\Delta \times 1]{} & J_{p-1} S^{2np^k} \times J_{p-1} S^{2np^k} \times B_{p^{k+1}-1}. \end{array} \tag{12}$$

The map $\theta_j \theta_i$ is the composite

$$G_{k+1} \to \Sigma^\infty_+ J_{p-1} S^{2np^k} \wedge \Sigma^\infty_+ J_{p-1} S^{2np^k} \wedge G_{k+1} \to S^{2np^k i} \wedge S^{2np^k j} \wedge G_{k+1},$$

[11]Devinatz-Hopkins-Smith invest a fair amount of effort to ensure that their constructions are filtration preserving. We, however, won't; unless mentioned otherwise, the results stated are also true for the filtered version of the maps θ_i.

[12]We are not that one.

where the first map is the Thomification of the composite of the top and right maps in diagram (12). To get the desired result, we need to analyse the Thomification of the composite of the left and bottom maps in that diagram.

Because the splitting of $\Sigma^\infty_+ J_{p-1}S^{2np^k}$ is multiplicative, the following diagram necessarily commutes:

$$\begin{array}{ccc} \Sigma^\infty_+ J_{p-1}S^{2np^k} & \xrightarrow{\Delta} & \Sigma^\infty_+ J_{p-1}S^{2np^k} \wedge \Sigma^\infty_+ J_{p-1}S^{2np^k} \\ \downarrow & & \downarrow \\ S^{2np^k(i+j)} & \xrightarrow[\binom{i+j}{i}]{} & S^{2np^k i} \wedge S^{2np^k j}. \end{array}$$

The desired result then follows from the commutativity of this diagram and the diagram (12). ☐

We have a diagram

$$\begin{array}{ccccc} F_{jp^k-1}/G_k & \longrightarrow & F_{(j+1)p^k-1}/G_k & \longrightarrow & F_{(j+1)p^k-1}/F_{jp^k-1} \\ \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_1} \\ \Sigma^{2np^k} F_{(j-1)p^k-1} & \longrightarrow & \Sigma^{2np^k} F_{jp^k-1} & \longrightarrow & \Sigma^{2np^k} F_{jp^k-1}/F_{(j-1)p^k-1}. \end{array}$$

Using the equivalence from (11) and Lemma 5.12, a simple induction on this diagram shows that when $(p-1)!$ is inverted, there is an equivalence

$$\theta_1 : G_{k+1}/G_k \xrightarrow{\sim} \Sigma^{2np^k} F_{(p-1)p^k-1}.$$

When working p-locally, this condition is automatically satisfied. This gives the desired self map: define b to be the composite

$$b : \Sigma^{2np^{k+1}-2} G_k \xrightarrow{\theta_p^{-1}} \Sigma^{2np^k-2} G_{k+1}/F_{(p-1)p^k-1} \to \Sigma^{2np^k-1} F_{(p-1)p^k-1} \xrightarrow{\theta_1^{-1}} \Sigma^{-1} G_{k+1}/G_k \to G_k.$$

It is an exercise to wade through these constructions and show:

Proposition 5.13. *The cofiber of b is equivalent to the spectrum P_k from the cofiber sequence* (10)*. Moreover, since the map $B_{p^k-1} \to B_{p^{k+1}-1}$ is injective on $H\mathbf{F}_p$-homology, the map b is zero on $H\mathbf{F}_p$-homology.*

This is the analogue of the first part of Section 5.2.

5.5.4. *Contractibility of the telescope.* To conclude, it therefore suffices to show that the telescope of the map b vanishes. For this, we need the following result, which is [DHS88, Proposition 3.19].

Theorem 5.14. *The map b is homotopic to f.*

It follows from Proposition 5.10 and Theorem 5.14 that the identity map $b^{-1}G_k \to b^{-1}G_k$ (which restricts to give maps $\Sigma^{-2m(np^{k+1}-1)} G_k \to b^{-1}G_k$) factors as

$$b^{-1}G_k \to b^{-1}G_k \wedge H\mathbf{F}_p \to b^{-1}G_k,$$

thanks to the diagram

$$\begin{array}{ccccc} G_k & \longrightarrow & G_k \wedge H\mathbf{F}_p & \searrow & \\ \downarrow{\scriptstyle b\simeq f} & & \downarrow & & \\ \Sigma^{-(2np^{k+1}-2)} G_k & \longrightarrow & \Sigma^{-(2np^{k+1}-2)} G_k \wedge H\mathbf{F}_p & \longrightarrow & b^{-1}G_k \\ \downarrow & & \downarrow & \nearrow & \\ \vdots & & \vdots & & \end{array}$$

But Proposition 5.13 implies that $b^{-1}G_k \wedge H\mathbf{F}_p$ is contractible. This means that $b^{-1}G_k$ is contractible, as desired.

6. Landweber exactness and heights of formal groups

6.1. **Introduction.** In the previous sections, we established the Ravenel conjectures. As discussed in the first section, these conjectures together give a global picture of the stable homotopy category. In particular, the chromatic convergence theorem tells us that we can attempt to understand $\pi_* S$ by understanding each $\pi_* L_n S$, and the induced maps $\pi_* L_n S \to \pi_* L_{n-1} S$.

Since $E(0) = H\mathbf{Q}$, the base of this tower is easy: $\pi_* L_0 S = \pi_* H\mathbf{Q} = \mathbf{Q}$. At this point in the course, however, we don't know how to compute $\pi_* L_1 S$ — so it seems like we are already stuck. So: how might we bootstrap ourselves from $\pi_* L_{n-1} S$ to $\pi_* L_n S$?

In the global picture of the stable homotopy category afforded to us by Ravenel's conjectures, we saw that L_n-localization is like restricting to an open subset, and that the natural transformation $L_n \to L_{n-1}$ is like restricting to a smaller open subset. To get from $L_{n-1}S$ to $L_n S$, we would therefore like to gain control over the locally closed subset given by the difference between these two consecutive layers.

It turns out that this difference is measured by a close friend of ours: Morava K-theory $K(n)$! In order to explain this result, let us state the following proposition.

Proposition 6.1. *Let E and F be spectra such that every E-local spectrum is F-acyclic, i.e., that $F_*(L_E X) \simeq 0$. Then there is a pullback square:*

$$\begin{array}{ccc} L_{E\vee F}X & \longrightarrow & L_E X \\ \downarrow & & \downarrow \\ L_F X & \longrightarrow & L_E L_F X. \end{array}$$

Proof. The map $L_{E\vee F}X \to L_F X$ is the unique factorization of the map $X \to L_F X$ through $L_{E\vee F}X$ since the map $X \to L_{E\vee F}X$ is an E-equivalence. The map $L_{E\vee F}X \to L_E X$ admits a similar description.

To establish the existence of the pullback square, let Y denote the pullback of the diagram. We need to show that the map $X \to Y$ is an E- and F-equivalence, and that Y is $(E \vee F)$-local. Let us first show that $X \to Y$ is an E- and F-equivalence. Since both $X \to L_E X$ and $Y \to L_E X$ are E-equivalences, we find that $X \to Y$ is an E-equivalence. Moreover, the map $X \to Y$ is an F-equivalence since, again, $X \to L_F X$ is an equivalence and $Y \to L_F X$ is an F-equivalence (since both spectra on the right hand vertical map are F-acyclic).

It remains to show that Y is $(E \vee F)$-local. For this, it suffices to show that if Z is any $(E \vee F)$-acyclic spectrum, then $[Z, Y]$ is zero. This follows from the long exact sequence

$$\cdots \to [Z, Y] \to [Z, L_E X] \oplus [Z, L_F X] \to [Z, L_F L_E X] \to \cdots$$

□

Corollary 6.2. *For any $m < n$, there is a pullback square*

$$\begin{array}{ccc} L_{K(n)\vee K(m)}X & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{K(m)}X & \longrightarrow & L_{K(m)}L_{K(n)}X. \end{array}$$

Proof. By Proposition 6.1, it suffices to prove that $K(n)_*(L_{K(m)}X)$ is zero for any spectrum X. Let Y be a $K(m)$-local spectrum. By the periodicity theorem, we can inductively construct spectra $S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m})$ for sufficiently large $(i_0, i_1, \cdots, i_m)$ which are type m finite spectra. In particular, $K(n)_*(S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m}))$ is not zero (since $n > m$); it therefore suffices to show that $Y \wedge S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m})$ is $K(n)$-acyclic.

The periodicity theorem also begets a self-map

$$v_m^N : \Sigma^{2(p^m-1)N} S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m}) \to S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m})$$

which is an isomorphism on $K(m)$-homology, but is zero on $K(n)$-homology. Therefore, the map

$$v_m^N : Y \wedge \Sigma^{2(p^m-1)N} S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m}) \to Y \wedge S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m})$$

is zero on $K(n)$-homology, but is an isomorphism on $K(m)$-homology (so, in particular, it is a homotopy equivalence since both Y and $S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m})$ are $K(m)$-local). If a homotopy equivalence is zero on $K(n)$-homology, then $K(n)_*(Y \wedge S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m}))$ must be zero, as desired. □

Recall from the first section that

$$\langle E(n)\rangle = \langle K(0)\vee K(1)\vee\cdots\vee K(n)\rangle;$$

in particular, $L_n = L_{K(0)\vee K(1)\vee\cdots\vee K(n)}$. Using Corollary 6.2, we get the chromatic fracture square:

$$\begin{array}{ccc} L_nX & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{n-1}X & \longrightarrow & L_{n-1}L_{K(n)}X. \end{array} \tag{13}$$

This tells us that in order to understand L_nX for all n, we need to understand $L_{K(n)}X$ for all n, and then understand the gluing datum. Understanding gluing data turns out to be really hard, so we will not discuss this here. Instead, we will attempt to understand $L_{K(n)}S$.

Returning to our qualms about L_1S from earlier: we only need to understand $L_{K(1)}S$. This might still seem really hard, because we haven't described any tools to understand $L_{K(n)}X$. However, in the next section, we will state a theorem of Devinatz and Hopkins which will allow us to compute $L_{K(1)}S$, and thus L_1S.

From a very broad perspective, the chromatic fracture square (13) reduces us to understanding the "local" structure of "Spec S". To attack this, we will need to build up our toolkit. The first step in this process is the Landweber exact functor theorem.

6.2. **The Landweber exact functor theorem.** We warn the reader beforehand that we will not pay attention to the grading; this is primarily my fault, so you should try to fix my writing by carefully working through the grading. Let M be a ring with a formal group law. According to Quillen's theorem, this is equivalent to a ring map $MU_*\to M$.

Question. When can this map be lifted to a map of spectra out of MU?

Landweber's theorem is an answer to this question.

The most natural way to define a homology theory, given the map $MU_*\to M$, is via the functor

$$X\mapsto MU_*(X)\otimes_{MU_*} M.$$

This satisfies all the Eilenberg-Steenrod axioms — *except* the axiom that cofiber sequences go to long exact sequences. Since MU is a homology theory, the failure of this functor to be a homology theory can be thought of as the failure of M to be a flat MU_*-module.

If M is a flat MU_*-module, this functor certainly is represented by a homology theory. In general, this condition is too strict, and there are not many interesting examples of homology theories stemming this way. To give a simpler condition, we must get into the inner workings of MU.

For simplicity, let us assume that everything is p-localized. Every formal group law over a p-local ring M is isomorphic to a so-called "p-typical formal group law" (this is known as Cartier's theorem). Just like MU_* classifies formal group laws, BP_* classifies p-typical formal group laws. Without loss of generality, let us assume that M has a p-typical formal group law defined over it, so that there is a ring map $BP_*\to M$.

Theorem 6.3. *Let M be as above. Then the functor*

$$X\mapsto BP_*(X)\otimes_{BP_*} M$$

is a homology theory iff for all n, the sequence $(p, v_1, \cdots, v_n)$ is a regular sequence in M (in other words, v_k is a non-zero-divisor in $M/(p, v_1, \cdots, v_{k-1})$).

Before we begin the proof, let us just recall the Landweber filtration theorem from the second section:

Lemma 6.4 (Landweber filtration theorem)**.** *Any BP_*BP-comodule M which is finitely presented as a BP_*-module has a filtration $0 = M_k\subset\cdots\subset M_1\subset M$ where M_j/M_{j+1} is isomorphic as a BP_*BP-comodule to (a shift of) BP_*/I_{n_j}.*

Proof. Suppose that $(p, v_1, \cdots, v_n)$ forms a regular sequence in M. Then tensoring with M preserves the short exact sequences

$$0\to BP_*/I_n\xrightarrow{\cdot v_n} BP_*/I_n\to BP_*/I_{n+1}\to 0.$$

In particular, $\mathrm{Tor}_1^{BP_*}(BP_*/I_n, M) \simeq 0$ for all n. By Lemma 6.4, we find that $\mathrm{Tor}_1^{BP_*}(N, M) \simeq 0$ for every BP_*BP-comodule N which is finitely presented as a BP_*-module. If X is a finite complex, then $BP_*(X)$ is such a comodule, so $\mathrm{Tor}_1^{BP_*}(BP_*(X), M) \simeq 0$. It follows immediately that the functor

$$X \mapsto BP_*(X) \otimes_{BP_*} M$$

defines a homology theory on finite spectra; since every spectrum is a filtered colimit of finite spectra, the result follows.

We will not prove the converse. □

There is an analogue (a corollary, in fact) of Theorem 6.3 for MU (both integrally and p-locally):

Theorem 6.5. *Let M be an MU_*-module. Then the functor*

$$X \mapsto MU_*(X) \otimes_{MU_*} M$$

is a homology theory iff for every prime p and integer n, multiplication by v_n is monic on $M/(p, v_1, \cdots, v_{n-1})$.

What if the formal group law on M comes from a complex oriented cohomology theory? In that case, the resulting functors agree:

Proposition 6.6. *Let E be a complex oriented cohomology theory such that the induced formal group law on E_* is Landweber exact. Then the map $MU \to E$ induces an isomorphism for all spectra X:*

$$MU_*(X) \otimes_{MU_*} E_* \to E_*(X).$$

Proof. Both sides are homology theories by Theorem 6.5, so it suffices to prove that the map is an isomorphism when X is the sphere spectrum. In this case, the result is obvious. □

Remark 6.7. We already knew that there is a functor

$$\{\text{Spectra}\} \to \{(BP_*, BP_*BP)\text{-comodules}\}.$$

The Landweber functor theorem is useful in taking us in the other direction.

In order to apply Theorem 6.5, it will be useful to have an interpretation of the elements v_i. Let $f(x,y)$ be a formal group law over a ring R. We define

$$[1]_f(x) = x, \ [n]_f(x) = f([n-1]_f(x), x).$$

The formal power series $[n]_f(x)$ is called the n-series.

Lemma 6.8. *Let R be a commutative ring in which $p = 0$, and let $f(x,y)$ be a formal group law over R. The p-series $[p]_f(x)$ of f has leading term ax^{p^n}.*

Proof. Recall that an endomorphism of a formal group law $f(x,y)$ is a power series $g(t) \in tR[[t]]$ such that

$$f(g(x), g(y)) = g(f(x,y)).$$

It is clear that

$$[n]_f(f(x,y)) = f([n]_f(x), [n]_f(y)),$$

so it will suffice to prove that if $h(t)$ is a nontrivial endomorphism of $f(x,y)$, then $h(t) = g(t^{p^n})$ for some n and some $g(t)$ with $g'(0) \neq 0$. This implies that $h(t)$ has leading term at^{p^n}, as desired.

We will prove this result inductively. Suppose that we have shown that $h(t) = h_k(t^{p^k})$. Without loss of generality, we can assume that $h_k'(0) = 0$ (otherwise, we are done). Define a formal power series $f_k(x,y)$ by

$$f_k(x^{p^k}, y^{p^k}) = f(x,y)^{p^k}.$$

Since R is an $\mathbf{F}_p$-algebra, $f_k(x,y)$ is a formal group law. Then it is easy to show that

$$h_k(f_k(x^{p^k}, y^{p^k})) = f(h_k(x^{p^k}), h_k(y^{p^k})),$$

so that

$$h_k(f_k(x,y)) = f(h_k(x), h_k(y)).$$

Let us differentiate with respect to y:

$$h_k'(f_k(x,y)) f_{k,2}'(x,y) = f'(h_k(x), h_k(y)) h_k'(y).$$

Setting $y = 0$, we get
$$h_k'(f_k(x,0))f_{k,2}'(x,0) = f'(h_k(x), h_k(0))h_k'(0).$$
By assumption, $h_k'(0) = 0$. Since f_k is a formal group law, $f_k(x,0) = 0$ and $f_{k,2}'(x,0) \neq 0$. Therefore,
$$h_k'(x) = 0.$$
Since $h_k'(0) = 0$, we conclude that $h_k(x)$ is of the form $h_{k+1}(x^p)$ for some power series $h_{k+1}(x)$. It is now easy to check that $h_{k+1}(x)$ is indeed an endomorphism of $f(x,y)$. Rinse and repeat until we get some $h_n(x) =: g(x)$ with $h_n'(0) \neq 0$. □

Definition 6.9. Let $f(x,y)$ denote a formal group law over a ring R, and let v_i denote the coefficient of x^{p^i} in $[p]_f(x,y)$. The formal group law has height $\geq n$ if $v_i = 0$ for $i < n$, and $f(x,y)$ has height exactly n if it has height $\geq n$ and v_n is invertible.

Example 6.10. Let $f(x,y)$ denote the formal group law $x + y - uxy$ over $R[u^{\pm 1}]$, with $|u| = 2$. Then the p-series is
$$[p]_f(x) = \frac{(1+ux)^p - 1}{u}.$$
We therefore find that $v_0 = p$, $v_1 = u^{p-1}$, and $v_i = 0$ for $i \geq 2$. In particular, $f(x,y)$ has height exactly 1 if $p = 0$ in R.

Exercise 6.11. Two formal group laws of different height cannot be isomorphic.

One might expect that the v_i from Definition 6.9 are exactly the images of the elements $v_i \in BP_*$ under the map $BP_* \to R$. This is not exactly true: the two elements coincide modulo $(p, v_1, \cdots, v_{i-1})$. To apply Theorems 6.3 and 6.5, though, we only need to know that multiplication by v_n is injective modulo $(p, v_1, \cdots, v_{n-1})$.

Example 6.12. Let $BP_* \to K(n)_*$ denote the quotient map. The formal group law over $K(n)_*$ has p-series
$$[p]_f(x) = x^{p^n}.$$

6.3. **Consequences.** There are a number of interesting consequences of our discussion above.

Example 6.13. Let $f(x,y)$ denote the (graded) formal group law $x + y - uxy$ over $\mathbf{Z}_{(p)}[u^{\pm 1}]$, with $|u| = 2$. This is not a p-typical formal group law. However, it is certainly Landweber exact (via Example 6.10). Theorem 6.5 therefore begets a cohomology theory. Since we know that $f(x,y)$ comes from (p-local) complex K-theory, we learn from Proposition 6.6 that there is an isomorphism
$$KU_*(X) \simeq MU_*(X) \otimes_{MU_*} KU_*.$$
This is the Conner-Floyd theorem.

Example 6.14. Define
$$f(x,y) = \frac{x+y}{1+xy}.$$
Then $f(x,y)$ is easily seen to be a formal group law. Recall that this is just the additional formula for the hyperbolic tangent function:
$$f(\tanh(x), \tanh(y)) = \tanh(x+y).$$
Therefore, $\tanh(t)$ defines an isomorphism between $f(x,y)$ and the additive formal group law $x + y$. Recall that
$$\tanh^{-1}(t) = \sum_{k\geq 0} \frac{x^{2k+1}}{2k+1},$$
so $f(x,y)$ and $x + y$ are in fact isomorphic over any 2-local ring. At odd primes, the two formal group laws are not isomorphic. In fact, one can compute that, p-locally (with $p > 2$), $f(x,y)$ has height 1. Exercise 6.11 proves that $f(x,y)$ is not isomorphic to the additive formal group law (which is easily seen to have infinite height).

Inspired by this example, we have:

Proposition 6.15. *Let R be a $\mathbf{Q}$-algebra. Then any formal group law is isomorphic to the additive formal group law $x + y$.*

Proof. Let $f(x,y)$ be a formal group law over R, and let $f_2(x,y)$ be the derivative with respect to y. Define

$$\log_f(x) = \int_0^x \frac{dt}{f_2(t,0)}.$$

We then have

$$\log_f(f(x,y)) = \log_f(x) + \log_f(y).$$

Indeed, let

$$g(x,y) = \log_f(f(x,y)) - (\log_f(x) + \log_f(y)).$$

Since $f(f(x,y),z) = f(x,f(y,z))$, we find that

$$f_2(f(x,y),0) = f_2(x,y)f_2(y,0).$$

Now,

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{\partial}{\partial y}\left(\log_f(f(x,y)) - (\log_f(x) + \log_f(y))\right)\\ &= \frac{f_2(x,y)}{f_2(f(x,y),0)} - \frac{1}{f_2(y,0)}\\ &= \frac{f_2(x,y)f_2(y,0) - f_2(f(x,y),0)}{f_2(f(x,y),0)f_2(y,0)} = 0.\end{aligned}$$

Since $f(x,y) = f(y,x)$, we also find that

$$\frac{\partial g}{\partial x} = 0.$$

Therefore $g(x,y)$ is constant. The constant term in $\log_f(x)$ is zero; therefore $g(x,y) = 0$, as desired. □

The inverse to $\log_f$ is denoted $\exp_f$.

Example 6.16. If $f(x,y) = x + y - uxy$, then $f_2(x,y) = 1 - ux$. Therefore,

$$\log_f(x) = \int_0^x \frac{dt}{1-ut} = \frac{\log(1+t)}{u}.$$

If $f(x,y)$ is as in Example 6.14, then

$$\log_f(x) = \int_0^x \frac{dt}{1-t^2} = \tanh^{-1}(x),$$

as we had already discovered.

Example 6.17. If $f(x,y)$ is the universal formal group law over $MU_* \otimes \mathbf{Q}$, then Mischenko's theorem states that

$$\log_f(x) = \sum_{n\geq 0} \frac{[\mathbf{C}P^n]}{n+1} x^n.$$

Recall that Hirzebruch proved that there is a bijection between ring maps $MU_* \otimes \mathbf{Q} \to R$ and power series $Q(x) \in R[[x]]$. This bijection is given by sending a formal group law $f(x,y)$ to

$$Q(x) = \frac{x}{\exp_f(x)}.$$

The genus associated to a formal power series $Q(x)$ is defined as follows: if $\mathscr{L} \to X$ is a complex line bundle, we define

$$\phi_Q(\mathscr{L}) = Q(c_1(\mathscr{L})) \in \widehat{\mathscr{H}^*}(X;R),$$

and then extend ϕ_Q to all bundles by the splitting principle. Then we define

$$\phi_Q(M) = \langle \phi_Q(\tau_M), [M]\rangle \in R.$$

For the formal group law in Example 6.14, we find that

$$\exp_f(x) = \tanh(x);$$

therefore,

$$Q(x) = \frac{x}{\tanh(x)};$$

the associated genus is exactly Hirzebruch's L-genus!

The fact that complex K-theory can be recovered from its formal group law (Example 6.13) should not be very surprising, particularly in light of the following observation.

Example 6.18. The modules $E(n)_* = v_n^{-1}BP_*/(p, v_1, \cdots, v_{n-1})$ are Landweber exact. This is clear. We have already studied the associated spectra $E(n)$ in previous lectures. Note, however, that $K(n)_* = v_n^{-1}BP_*/(p, v_1, \cdots, v_{n-1}, v_{n+1}, \cdots)$ is *not* Landweber exact.

Our discussion above leads to some natural questions: what *is* the formal group law over $E(n)$? What are some explicit examples of Landweber exact cohomology theories at height ≥ 2? In the next lecture, we will use Theorem 6.5 to construct *Morava E-theory*, denoted E_n. (This is not the same as $E(n)$, which is usually called Johnson-Wilson theory.) The formal group law over π_*E_n will have an explicit algebro-geometric interpretation. As for the second question: we will address this by constructing the spectrum of topological modular forms in the last few lectures.

7. Morava E-theory and the $K(n)$-local sphere

Our discussion in the previous section implies that the first step in answering understanding π_*S is understanding $\pi_*L_{K(n)}S$. In this section, we will describe a method to approaching $L_{K(n)}S$ and compute $\pi_*L_{K(1)}S$ as well as π_*L_1S.

7.1. **Deformations of formal group laws.**

We will begin with an analysis of the local structure of formal group laws. Fix a perfect field κ of characteristic p and a formal group law $f(x, y)$ over κ.

Definition 7.1. Let R be a local Artinian ring with residue field κ (so the maximal ideal $\mathfrak{m}$ of R is nilpotent, and each quotient $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a finite-dimensional κ-vector space). A *deformation* of $f(x, y)$ to R is a formal group law $f_R(x, y)$ over R such that

$$f_R(x, y) \equiv f(x, y) \mod \mathfrak{m}.$$

We begin with a lemma.

Lemma 7.2. *The groupoid* $\mathrm{Def}(R)$ *of deformations (up to isomorphism) of* $f(x, y)$ *to* R *is discrete.*

Proof. Concretely, this means that if $g(t)$ is an automorphism of $f_R(x, y)$ which is the identity modulo $\mathfrak{m}$, then $g(t)$ is the identity. As $\mathfrak{m}$ is nilpotent, it suffices to show that

$$g(t) \equiv t \mod \mathfrak{m}^k$$

for all k. The base case, when $k = 1$, is our assumption on $g(t)$. Let P denote the ring classifying automorphisms of $f_R(x, y)$. Then g and the identity automorphism are given by ring maps $\phi_G, \phi_1 : P \to R$. By assumption, the maps agree when composed with the quotient $R \to R/\mathfrak{m}^k$. This means that $\phi_G - \phi_1$ is an R-linear derivation $P \to \mathfrak{m}^k/\mathfrak{m}^{k+1}$. However, this map factors through $P \otimes_R \kappa$, which is étale over κ. Therefore all derivations out of $P \otimes_R \kappa$ are zero, which means that

$$\phi_G - \phi_1 \equiv 0 \mod \mathfrak{m}^{k+1},$$

as desired. ∎

Let $P = W(\kappa)[[v_1, \cdots, v_{n-1}]]$; this is a complete local ring with maximal ideal $(p, v_1, \cdots, v_{n-1})$ and residue field κ. The formal group law f is classified by a map $L \simeq \mathbf{Z}[b_1, \cdots] \to k$ sending b_{p^i-1} to 0 for $1 \leq i \leq n-1$ (recall that $f(x, y)$ has height n). Let $\widetilde{f}(x, y)$ denote the formal group law over P determined by any lift of this map to P which sends b_{p^i-1} to v_i.

Theorem 7.3 (Lubin-Tate)**.** *Let R be any local Artinian ring with residue field κ. Then the formal group law $\widetilde{f}$ defines a bijection*

$$\{\textit{local ring maps } P \to R\} \to \mathrm{Def}(R).$$

Proof. We will first prove the result when $R = \kappa[\epsilon]/\epsilon^2$. In this case, we have a map

$$\mathrm{Def}(R) \to \kappa^{n-1},$$

given by sending a deformation $L \to \kappa[\epsilon]/\epsilon^2$, which necessarily sends v_i to $c_i\epsilon$ for $0 < i < n$, to the $(n-1)$-tuple $(c_1, \cdots, c_{n-1})$. In order for this map to be well-defined, we need to show that this tuple only depends on the *isomorphism class* of the deformation of $f(x,y)$. This is left as an exercise to the reader: if $\widetilde{f}$ and $\widetilde{f}'$ are two deformations which are isomorphic via $g(t)$, then the tuples in κ^{n-1} are the same.

Now, by construction, the map

$$\{\text{local ring maps } P \to \kappa[\epsilon]/\epsilon^2\} \to \mathrm{Def}(\kappa[\epsilon]/\epsilon^2) \to \kappa^{n-1}$$

is an isomorphism. Therefore the first map in the composite is injective. Its surjectivity will follow if we can show that the second map is injective. Suppose that $\widetilde{f}$ is in its kernel, so $\widetilde{f}$ has height exactly n. We need to show that $\widetilde{f}$ is isomorphic to the trivial deformation f_{triv} (via an isomorphism which is the identity modulo (ϵ)).

However, because $\widetilde{f}$ and f_{triv} are formal group laws of height n, the ring R parametrizing isomorphisms $\widetilde{f} \simeq f_{\text{triv}}$ is an inductive limit of finite étale extensions of $\kappa[\epsilon]/\epsilon^2$. In particular, the map $R \to \kappa$ lifts uniquely (this is one way to define étaleness) to a map $R \to \kappa[\epsilon]/\epsilon^2$, as desired.

We will now reduce the general case to the case when $R = \kappa[\epsilon]/\epsilon^2$. We will argue by induction on length of R (recall that this is the supremum of the lengths of the longest chain of ideals in R). The base case is clear, since when R has length 0, we know that $R = \kappa$.

For the inductive step, recall that $\mathfrak{m}$ is nilpotent, so $\mathfrak{m}^N = 0$. Let $x \in \mathfrak{m}^{N-1}$, so $\mathrm{Ann}(x) = \mathfrak{m}$. The length of R/x is strictly less than the length of R. We claim that there is a pullback square

$$\begin{array}{ccc} \mathrm{Def}(R \times_{R/x} R) & \longrightarrow & \mathrm{Def}(R) \\ \downarrow & & \downarrow \\ \mathrm{Def}(R) & \longrightarrow & \mathrm{Def}(R/x). \end{array}$$

This follows from the fact that $\mathrm{Def}(A \times_B C) \to \mathrm{Def}(A) \times_{\mathrm{Def}(B)} \mathrm{Def}(C)$ is a bijection (it is easiest to think about this in terms of formal groups, noting that $\mathrm{Spec}(A \times_B C)$ is the gluing of $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, C$ along $\mathrm{Spec}\, B$). Now,

$$R \times_{R/x} R \simeq \kappa[x]/x^2 \times_\kappa R.$$

Indeed, we send $(x,y) \in R \times_{R/x} R$ to $(\overline{y} + cz, y) \in \kappa[x]/x^2 \times_\kappa R$, where $\overline{y} \in \kappa$ is the image of y, and $cz = x - y$ (since x and y are congruent modulo (x)).

Therefore, we find that the fiber of the map $\mathrm{Def}(R \times_{R/x} R) \to \mathrm{Def}(R)$ is $\mathrm{Def}(\kappa[x]/x^2)$. Since the square is a pullback square, we find that the fiber of the map $\mathrm{Def}(R) \to \mathrm{Def}(R/x)$ is $\mathrm{Def}(\kappa[x]/x^2)$. We can argue in exactly the same way (in fact, the argument is simpler here) to find that the fiber of the map

$$\{\text{local ring maps } P \to R\} \to \{\text{local ring maps } P \to R/x\}$$

is exactly $\{\text{local ring maps } P \to \kappa[x]/x^2\}$.

In Lemma 7.2, we established that $\pi_1 \mathrm{Def}(R) = 0$, so to prove the theorem for $\mathrm{Def}(R)$, it suffices to prove the result for $\mathrm{Def}(R/x)$ and $\mathrm{Def}(\kappa[x]/x^2)$. The result for $\mathrm{Def}(R/x)$ is the inductive hypothesis, and the result for $\mathrm{Def}(\kappa[x]/x^2)$ was proved above. ☐

7.2. **Morava E-theory.** By Theorem 7.3, the ring $W(\kappa)[[v_1, \cdots, v_{n-1}]]$ parametrizes universal deformations. The generators v_i in this Lubin-Tate ring are traditionally denoted u_i. Define a formal group law $\widetilde{f}'(x,y)$ over $W(\kappa)[[u_1, \cdots, u_{n-1}]][u^{\pm 1}]$, where now $|u| = 2$, by

$$\widetilde{f}'(x,y) = u^{-1}\widetilde{f}(ux, uy).$$

The map $L \to W(\kappa)[[u_1, \cdots, u_{n-1}]][u^{\pm 1}]$ sends

$$v_i \mapsto \begin{cases} u_i u^{p^i - 1} & 0 < i < n \\ u^{p^n - 1} & i = n \\ 0 & i > n. \end{cases}$$

By construction, the sequence $(p, v_1, \cdots, v_{n-1})$ is regular. We therefore can apply the Landweber exact functor theorem to get a cohomology theory E_n such that

$$\pi_* E_n \simeq W(\kappa)[[u_1, \cdots, u_{n-1}]][u^{\pm 1}].$$

When $\kappa = \overline{\mathbf{F}_p}$, there is a unique formal group law of every height up to isomorphism (this a theorem of Lazard's); consequently, we will only concern ourselves with the case $\kappa = \overline{\mathbf{F}_p}$. If, however, $n = 1$, this formal group law is in fact defined over $\mathbf{F}_p$ (it is the multiplicative formal group law), so in this case we will let $\kappa = \mathbf{F}_p$.

Definition 7.4. The spectrum E_n is called *Morava E-theory.*

Example 7.5. When $n = 1$, this is exactly p-adically complete K-theory.

This spectrum is extremely interesting, and it will be one of our main objects of study in forthcoming sections.

Let $f(x, y)$ be a formal group law of height n over $\kappa = \overline{\mathbf{F}_p}$. Then every automorphism of $f(x, y)$ gives rise to an automorphism of the discrete groupoid $\mathrm{Def}(R)$; by Theorem 7.3, this begets an action of $\mathrm{Aut}(f)$ on $\pi_* E_n$. This action lifts to the level of spectra: every element of $\mathrm{Aut}(f)$ gives rise to an automorphism of E_n. These automorphisms are very well-behaved; in fact, we have:

Theorem 7.6 (Goerss-Hopkins-Miller)**.** *The spectrum E_n is an E_∞-ring; moreover,*

$$\mathrm{Aut}_{E_\infty}(E_n) \simeq \mathrm{Aut}(f).$$

It still remains to find an economical description of $\mathrm{Aut}(f)$. Let us briefly indicate how one obtains the modern description of $\mathrm{Aut}(f)$.

Although we have not discussed *formal groups* in this course, one way of describing them is as "coordinateless formal group laws". One way of thinking of this description is via complex orientable cohomology theories: every complex orientable cohomology theory has an associated formal group — and we can write down the formal group law *after* picking a particular complex orientation. A precise definition is:

Definition 7.7. A formal group over a ring R is a group object in the category of smooth 1-dimensional formal schemes over R (i.e., a formal scheme which is isomorphic to $\mathrm{Spf}\, R[[t]]$).

One can define the *height* of a formal group, and show that this coincides with the definition we provided in the last section after picking a coordinate. The following theorem is one of the many big triumphs of algebraic geometry:

Theorem 7.8. *There is a contravariant equivalence of categories between formal groups over κ of finite height and* Dieudonné modules*, i.e., modules over the ring*

$$\mathrm{Cart}(\kappa) = W(\kappa)\langle F, V\rangle/(FV = VF = p, Fx = \phi(x)F, V\phi(x) = xV),$$

which are free and finite rank over $W(\kappa)$.

Under this equivalence, the Honda formal group Γ_n of height n over $\kappa = \overline{\mathbf{F}_p}$ (which is the unique formal group of height n over κ, up to isomorphism) is sent to

$$D(\Gamma_n) = \mathrm{Cart}(\kappa)/(F^n = p).$$

We therefore find, using the equivalence of categories from Theorem 7.8, that

$$\mathrm{End}(\Gamma_n) = W(\kappa)\langle F\rangle/(Fx = \phi(x)F, F^n = p);$$

in particular, we find that

$$\mathrm{Aut}(f) \simeq (W(\kappa)\langle S\rangle/(Sx = \phi(x)S, S^n = p))^\times .$$

This result was originally proven by Devinatz and Hopkins. They also wrote down formulae for the action of S on the generators u_i (which are not canonical).

Example 7.9. When $n = 1$, the description provided above shows that

$$\mathrm{Aut}(f) = \mathbf{Z}_p^\times.$$

The action of $\mathbf{Z}_p^\times$ on $E_n = K_p$ coming from Theorem 7.6 is via Adams operations. Therefore, $g \in \mathrm{Aut}(f)$ will act on u^k by multiplication by g^k. We will denote the automorphism of $\pi_* E$ induced by $g \in \Gamma$ by ψ^g.

Their crowning achievement was the following theorem:

Theorem 7.10. *There is an equivalence*

$$(E_n)^{h\,\mathrm{Aut}(f) \rtimes \mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)} \simeq L_{K(n)}S.$$

We need to be careful in the statement of this theorem, since $\mathrm{Aut}(f) \rtimes \mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)$ is actually a profinite group. Let $\mathscr{O}_n$ denote $\mathrm{End}(\Gamma_n)$; this is a maximal order in a central division algebra over $\mathbf{Q}_p$. We have normal subgroups $1 + S^N\mathscr{O}_n$ of $\mathrm{Aut}(f)$, which are of finite index. Since

$$\bigcap_{N\geq 1}(1 + S^N\mathscr{O}_n) = 1,$$

we obtain the structure of a profinite group on $\mathrm{Aut}(f)$ by letting these normal subgroups be a basis for the open neighborhoods of 1.

The action of $\mathrm{Aut}(f)$ on E_n is *continuous*. One way to interpret this is as saying that $\mathrm{Aut}(f)$ acts on π_*E_n continuously (recall that π_*E_n has the $(p, u_1, \cdots, u_{n-1})$-adic topology). The fixed points appearing in Theorem 7.10 can be defined as follows.

Let $U \leq \mathrm{Aut}(f) \rtimes \mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)$ be an open subgroup (hence of finite index). For simplicity, let us write $\Gamma = \mathrm{Aut}(f) \rtimes \mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)$; others write $\mathbf{G}_n$. Then there is a cosimplicial diagram $X^\bullet$ in spectra with

$$X^m = L_{K(n)}(E_n^{\wedge m} \wedge \mathrm{Map}(\Gamma/U, E_n)).$$

We define E_n^{hU} to be the homotopy limit of this diagram. If $V \leq \Gamma$ is a closed subgroup, we define

$$E_n^{hV} = L_{K(n)}\,\mathrm{colim}_{V\leq U\leq_o \Gamma} E_n^{hU}.$$

This allows us to define $E_n^{h\Gamma}$, as in Theorem 7.10.

Roughly speaking, the homotopy fixed point spectral sequence from the exercises generalizes to this case: we have a *continuous* homotopy fixed point spectral sequence

$$E_2^{s,t} = H_c^s(\Gamma; \pi_t E_n) \Rightarrow \pi_{t-s}L_{K(n)}S.$$

This is our key tool in approaching $L_{K(n)}S$. Of course, the complicated action of Γ on π_*E_n makes this a terribly hard computation in group cohomology. However, to illustrate the usefulness of this series of ideas, let us compute $L_{K(1)}S$ (thereby resolving the qualms we had in the previous section).

7.3. The computation of $L_{K(1)}S$ at an odd prime.

7.3.1. *Group cohomology.* When $p > 2$, there is an isomorphism

$$\Gamma \simeq \mathbf{Z}_p \times C_{p-1}.$$

Moreover, there is an isomorphism $\mathbf{Z}_p[[\mathbf{Z}_p]] \simeq \mathbf{Z}_p[[t]]$, so we have a projective resolution of $\mathbf{Z}_p$ as a $\mathbf{Z}_p[[t]]$-module:

$$0 \to P_1 = \mathbf{Z}_p[[t]] \xrightarrow{t-1} P_0 = \mathbf{Z}_p[[t]] \to \mathbf{Z}_p \to 0.$$

The projection map $\Gamma \to \mathbf{Z}_p$ allows us to view this as a complex of $\mathbf{Z}_p[[\Gamma]]$-modules; however, it is not *a priori* clear that P_1 and P_0 are projective.

As $\mathbf{Z}_p[[\Gamma]]$-modules, we have

$$P_1 = P_0 \simeq \mathbf{Z}_p[[\Gamma/C_{p-1}]] \simeq \mathbf{Z}_p[[\Gamma]] \otimes_{\mathbf{Z}_p[C_{p-1}]} \mathbf{Z}_p.$$

Because $p - 1$ is prime to p when p is odd, the $\mathbf{Z}_p[C_{p-1}]$-module $\mathbf{Z}_p$ is projective. It follows that P_0 and P_1 are projective $\mathbf{Z}_p[[\Gamma]]$-modules.

Let ℓ be a topological generator of $\Gamma/C_{p-1} \simeq \mathbf{Z}_p$. For instance, let $\ell = p+1$. Then, for any $\mathbf{Z}_p[[\Gamma]]$-module M, we have a long exact sequence

$$\cdots \to \mathrm{Ext}^*_{\mathbf{Z}_p[[\Gamma]]}(\mathbf{Z}_p, M) \to \mathrm{Ext}^*_{\mathbf{Z}_p[[\Gamma]]}(P_0, M) \xrightarrow{\psi^\ell - 1} \mathrm{Ext}^*_{\mathbf{Z}_p[[\Gamma]]}(P_1, M) \to \mathrm{Ext}^{*+1}_{\mathbf{Z}_p[[\Gamma]]}(\mathbf{Z}_p, M) \to \cdots$$

Shapiro's lemma therefore translates this into a long exact sequence

$$\cdots \to H^*(\Gamma; M) \to H^*(C_{p-1}; M) \xrightarrow{\psi^\ell - 1} H^*(C_{p-1}; M) \to H^{*+1}(\Gamma; M) \to \cdots$$

Now we will specialize to the case $M = \pi_*K_p \simeq \mathbf{Z}_p[u^{\pm 1}]$. It is then easy to see that $H^*(C_{p-1}; \mathbf{Z}_p[u^{\pm 1}])$ is concentrated in degree 0, and

$$H^0(C_{p-1}; \mathbf{Z}_p[u^{\pm 1}]) \simeq \mathbf{Z}_p[u^{\pm(p-1)}].$$

It remains to determine the action of $\psi^\ell - 1$. As $\ell = p + 1$, we have

$$(\psi^\ell - 1)_*(u^{k(p-1)}) = ((p+1)^{k(p-1)} - 1)u^{k(p-1)} = cu^{k(p-1)}p^{\nu_p(k)+1},$$

Lemma 7.11. *We have*

$$(p+1)^{k(p-1)} - 1 = cp^{\nu_p(k)+1},$$

where c is a p-adic unit.

Proof. If $k = 0$, then this is tautologically true. So suppose that $k \neq 0$. It is easy to see that

$$(p+1)^m \equiv 1 \mod p^n$$

iff $p^{n-1}|m$, since, for instance, $p+1 \in \mathbf{Z}/p^n$ generates the copy of $\mathbf{Z}/p^{n-1}$ under the isomorphism $(\mathbf{Z}/p^n)^\times \simeq C_{p-1} \times \mathbf{Z}/p^{n-1}$. It follows that

$$\nu_p((p+1)^{k(p-1)} - 1) = \nu_p(k(p-1)) + 1 = \nu_p(k) + 1,$$

as desired. □

It follows that the E_2-page of the HFPSS is given by

$$H^s(\Gamma; \pi_t K_p) \simeq \begin{cases} \mathbf{Z}_p & s = 0, 1,\ t = 0 \\ \mathbf{Z}/p^{\nu_p(k)+1} & s = 1,\ t = 2(p-1)k \\ 0 & \text{else.} \end{cases}$$

It follows that the spectral sequence collapses, so for $p > 2$ we have

$$\pi_n L_{K(1)} S \simeq \begin{cases} \mathbf{Z}_p & n = 0, -1 \\ \mathbf{Z}/p^{\nu_p(k)+1} & n = 2(p-1)k - 1 \end{cases}$$

This allows us to compute $\pi_* L_1 S$. The $E(1)$-local sphere sits inside a pullback square

$$\begin{array}{ccc} L_1 S & \longrightarrow & L_{K(1)} S \\ \downarrow & & \downarrow \\ L_{\mathbf{Q}} S = H\mathbf{Q} & \longrightarrow & L_{\mathbf{Q}} L_{K(1)} S. \end{array}$$

Note that our computation of $\pi_* L_{K(1)} S$ implies that

$$\pi_n L_{\mathbf{Q}} L_{K(1)} S \simeq \begin{cases} \mathbf{Q}_p & n = 0, -1 \\ 0 & \text{else} \end{cases}.$$

There is a long exact sequence

$$\cdots \to \pi_{*+1} L_{\mathbf{Q}} L_{K(1)} S \to \pi_* L_1 S \to \pi_* L_{K(1)} S \oplus \pi_* L_{\mathbf{Q}} S \to \pi_* L_{\mathbf{Q}} L_{K(1)} S \to \cdots$$

If $n \neq 0, -1, -2$, then

$$\pi_n L_1 S \simeq \pi_n L_{K(1)} S;$$

otherwise, we have a long exact sequence

$$0 \to \pi_0 L_1 S \to \mathbf{Z}_p \oplus \mathbf{Q} \to \mathbf{Q}_p \to \pi_{-1} L_1 S \to \mathbf{Z}_p \to \mathbf{Q}_p \to \pi_{-2} L_1 S \to 0.$$

This implies that

$$\pi_* L_1 S \simeq \begin{cases} \mathbf{Z} & n = 0 \\ \mathbf{Q}_p/\mathbf{Z}_p & n = -2 \\ \mathbf{Z}/p^{\nu_p(k)+1} & n = 2(p-1)k - 1 \\ 0 & \text{else} \end{cases}$$

7.3.2. *Chromatic spectral sequence.* Another way to understand $\pi_* L_1 S$ is via the chromatic spectral sequence. We will not be extremely precise in this subsection; instead, we will provide references to other sources. Recall that there is a fiber/cofiber sequence

$$L_1 S \to L_{\mathbf{Q}} S \to M_1 S.$$

Since $L_{\mathbf{Q}} S = H\mathbf{Q}$, it suffices to understand $M_1 S$. In general, there are spectral sequences

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{E_*^\vee E}(E_*, E_*/(p^\infty, \cdots, u_{n-1}^\infty)) \simeq H_c^*(\Gamma; E_*/(p^\infty, \cdots, u_{n-1}^\infty)) \Rightarrow \pi_* M_n S$$

$$E_2^{*,*} = \mathrm{Ext}^{s,t}_{E_*^\vee E}(E_*, E_*) \simeq H_c^*(\Gamma; E_*) \Rightarrow \pi_* L_{K(n)} S.$$

We've used the Morava change-of-rings isomorphisms above: if M is a (BP_*, BP_*BP)-comodule for which $v_n^{-1} M = M$, then

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M) \simeq \mathrm{Ext}^{*,*}_{E_*^\vee E}(E_*, E_* \otimes_{BP_*} M) \simeq H_c^*(\Gamma; E_* \otimes_{BP_*} M).$$

A related change-of-rings is one which Miller-Ravenel prove in [MR77]: if M is as above, there is an isomorphism

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M) \simeq \mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_* \otimes_{BP_*} M).$$

To understand $\pi_* M_1 S$, it therefore suffices to compute $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*/p^\infty)$. Recall that $BP_*/p^\infty = M^1$, so our spectral sequence now runs

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M^1) \Rightarrow \pi_* M_1 S.$$

This E_2-page was computed in [MRW77, §4]:

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, M^1) \simeq \begin{cases} \mathbf{Q}_p/\mathbf{Z}_p & s = 0, t = 0 \\ \mathbf{Z}/p^{\nu_p(k)+1} \left\langle \frac{v_1^k}{p^{\nu_p(k)+1}} \right\rangle & s = 0, t = 2(p-1)k \\ \mathbf{Q}_p/\mathbf{Z}_p & s = 1, t = 0, \\ 0 & \text{else.} \end{cases}$$

The chromatic spectral sequence gives an isomorphism ([Rav86, Theorem 5.2.6(b)])

$$\mathrm{Ext}^{1,t}_{BP_*BP}(BP_*, BP_*) \simeq \begin{cases} \mathrm{Ext}^{0,t}_{BP_*BP}(BP_*, M^1) & t > 0 \\ 0 & t = 0 \end{cases}$$

We will give an explicit computation of $\mathrm{Ext}^{0,2(p-1)k}_{BP_*BP}(BP_*, M^1) \simeq \mathrm{Ext}^{1,2(p-1)k}_{BP_*BP}(BP_*, BP_*)$ in Section 7.6.

According to [Rav86, Theorem 5.3.7(a)], every element in the ANSS 1-line is a permanent cycle, represented by an element in the image of the J-homomorphism[13]. Gathering together these results, we conclude:

Theorem 7.12. *The (p-local) image of the J-homomorphism maps isomorphically onto $\pi_* L_1 S$ in positive degrees under the natural map $\pi_* S \to \pi_* L_1 S$.*

At higher heights, this story becomes much more complicated: if ϕ is a lift of Frobenius to $W(\mathbf{F}_{p^n})$, then the action of $g = \sum_{k=0}^{n-1} a_k S^k$ on the (noncanonical!) polynomial generators of $\pi_* E$ is given by

$$g_*(u) \equiv a_0 u + \sum_{k=1}^{n-1} \phi(a_{n-k}) u u_k \mod (p, \mathfrak{m}^2),$$

$$g_*(u u_i) \equiv \sum_{k=1}^{i} \phi^k(a_{i-k}) u u_k \mod (p, \mathfrak{m}^2).$$

It's too much to expect a complete computation of the E_2-page. Instead, we can attempt to test the nontriviality of elements in $\pi_* L_{K(n)} S$ by pushing them forward to intermediate spectra between $L_{K(n)} S$ and E.

[13] Adams showed that if $m(2t)$ is the denominator of $B_{2t}/4t$ in lowest terms, then for odd primes we have

$$\nu_p(m(t)) = \begin{cases} \nu_p(t) + 1 & (p-1) | t \\ 0 & \text{else.} \end{cases}$$

Let G be a finite subgroup of Γ; then, the unit map $L_{K(n)}S \to E$ factors as

$$L_{K(n)}S \to E^{hG} \to E.$$

There is also a homotopy fixed point spectral sequence

$$H^*(G; \pi_* E) \Rightarrow \pi_* E^{hG},$$

which sits inside a commutative diagram of spectral sequences

$$\begin{array}{ccc} H^*(\Gamma; \pi_* E) & \Longrightarrow & \pi_* L_{K(n)}S \\ \downarrow & & \downarrow \\ H^*(G; \pi_* E) & \Longrightarrow & \pi_* E^{hG} \\ \downarrow & & \downarrow \\ H^*(*; \pi_* E) \simeq \pi_* E & \Longrightarrow & \pi_* E. \end{array}$$

Mapping an element in $H^*(\Gamma; \pi_* E)$ all the way to the bottom is probably not extremely helpful (one reason being that $\pi_* E$ is concentrated in even degrees, and, as we saw in the previous section, $\pi_* L_{K(1)}S$ has elements in odd degrees). The cohomology $H^*(G; \pi_* E)$ is a nice intermediary; for one, it's something that we can actually compute! Our main interest will be in the case when G is a *maximal* finite subgroup of Γ.

7.4. **Finite subgroups of Γ at height not divisible by $p-1$.** [Buj12] classifies all finite subgroups of the Morava stabilizer group Γ at height n and the prime p. We will not go into much detail regarding the classification. Recall that $\Gamma \simeq \mathscr{O}_n^\times$, where

$$\mathscr{O}_n := W(\mathbf{F}_{p^n})\langle S \rangle / (S^n = p, Sx = \phi(x)S).$$

Let $D = \mathscr{O}_n \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$; then $\mathscr{O}_n$ is the ring of integers in D. Moreover, D is a central division algebra over $\mathbf{Q}_p$ of dimension n^2 with Hasse invariant $1/n$. We need the following result about division algebras (see [Buj12, Theorem C.6]).

Proposition 7.13. *Let K be a local field, and let D be a central division algebra over K of dimension n^2. If L/K is a commutative extension of K in D, then $[L : K]$ divides n; moreover, any extension L/K such that $[L : K]$ divides n embeds as a commutative subfield of D.*

Since

$$[\mathbf{Q}_p(\zeta_{p^k}) : \mathbf{Q}_p] = (p-1)p^{k-1},$$

it follows that $\mathscr{O}_n^\times$ has elements of order p^k iff $(p-1)p^{k-1}$ divides n. If $(p-1)$ does not divide n, then there are no nontrivial finite p-subgroups of Γ. In fact, every finite subgroup is conjugate to C_{p^n-1} if $p > 2$ and $C_{2(p^n-1)}$ if $p = 2$. In all of these cases, any element g of the maximal finite subgroup is a Teichmüller lift of some element $\alpha \in \mathbf{F}_{p^n}^\times \subseteq W(\mathbf{F}_{p^n})^\times \subseteq \Gamma$. We then have

$$g_*(u) = \alpha u, \ g_*(u_i) = \alpha^{p^i - 1} u_i.$$

Having determined these to be the maximal finite subgroups of Γ, we might ask for a computation of $\pi_* E^{hG}$ at height not divisible by $p-1$; for this, we need to run the homotopy fixed point spectral sequence (HFPSS). Since $\gcd(|G|, p) = 1$ for any finite subgroup G of Γ, the group cohomology $H^*(G; \pi_* E)$ vanishes for $* > 0$; the HFPSS therefore collapses, and we get

$$\pi_* E^{hG} \simeq (\pi_* E)^G.$$

We've now reduced this to an algebraic problem.

7.5. **Finite subgroups at height divisible by $p-1$.** If $(p-1)$ *does* divide n, then there are nontrivial p-subgroups. Write $n=(p-1)p^{k-1}m$ with $\gcd(p,m)=1$. For $1\leq \alpha\leq k$, write

$$n_\alpha = n/\phi(p^\alpha) = n/(p-1)p^{\alpha-1}.$$

When $p>2$, there are $k+1$ conjugacy classes of maximal finite subgroups in Γ, given by

$$C_{p^n-1} \text{ and } C_{p^\alpha} \rtimes C_{(p^{n_\alpha}-1)(p-1)},$$

for $1\leq\alpha\leq k$.

When $p=2$, the situation is (as always) more complicated. Write $n=2^{k-1}m$, with $v_2(m)=0$. Then Γ has k conjugacy classes of maximal finite subgroups. If $k\neq 2$, then they are given by $C_{2^\alpha(2^{n_\alpha}-1)}$ for $1\leq\alpha\leq k$. If $k=2$, then they are given by $C_{2^\alpha(2^{n_\alpha}-1)}$ for $1\leq\alpha\neq 2\leq k$, and $Q_8\rtimes C_{3(2^m-1)}$.

The current (published) computations have been done for the case $n=p-1$, so we will focus our attention on this case for the moment. In this case, the maximal finite subgroups of Γ are conjugate to C_{p^n-1} or $C_p\rtimes C_{(p-1)^2}$. Recall that $W(\mathbf{F}_{p^n})\simeq \mathbf{Z}_p[\zeta_{p^n-1}]$, so $\zeta_{p^n-1}\subseteq W(\mathbf{F}_{p^n})^\times\subseteq\Gamma$. Define

$$X=\zeta_{p^n-1}^{(p-1)/2}S\in\Gamma.$$

It is easy to see that

$$X^n=-p.$$

The field $\mathbf{Q}_p(X)$ contains a pth root of unity ζ_p; as $\mathbf{Q}_p(X)$ and $\mathbf{Q}_p(\zeta_p)$ are both of degree $p-1$ over $\mathbf{Q}_p$, they are isomorphic. One can see that

$$X\zeta_pX^{-1}=\zeta_{p^n-1}^p.$$

Let

$$\tau=\zeta_{p^n-1}^{(p^n-1)/n^2},$$

so

$$\tau X\tau^{-1}=\tau^{-n}X.$$

Because τ^{-n} is an nth root of unity, it follows that conjugation by τ gives an automorphism of $\mathbf{Q}_p(X)$. Then the subgroup generated by ζ and τ is exactly the maximal finite subgroup $C_p\rtimes C_{(p-1)^2}$.

7.5.1. *Height* 1 *at* $p=2$. The case $n=1$, $p=2$ is the first nontrivial situation where the HFPSS does not collapse. As we saw last time, in this case,

$$\Gamma\simeq\mathbf{Z}_2^\times\simeq\mathbf{Z}_2\times C_2.$$

There is one unique maximal finite subgroup of Γ, namely C_2. Let σ be a generator of C_2; then σ acts on $(E_1)_*\simeq(K_2)_*\simeq\mathbf{Z}_2[u^{\pm1}]$ by conjugation, so

$$\sigma(u)=-u.$$

The homotopy fixed point spectral sequence for the C_2-action on K_2 runs

$$E_2^{s,t}=H^s(C_2;(K_2)_t)\Rightarrow\pi_{t-s}K_2^{hC_2}.$$

Since the cohomology of C_2 is 2-periodic, we conclude that

$$E_2^{*,*}\simeq\mathbf{Z}[u^{\pm2},\alpha]/2\alpha;$$

here, the bidegree of the elements, written as $(t-s,s)$, are $|u|=(4,0)$, and $|\alpha|=(1,1)$. The first possible differential is a d_3.

Proposition 7.14. *There is a d_3-differential on u^2:*

$$d_3(u^2)=\alpha^3,$$

which is α-linear.

We collect different proofs of this statement.

Proof 1. We know that $K_2^{hC_2}\simeq KO$. Since we know that the homotopy of KO is given by

$$KO_*\simeq\mathbf{Z}[\alpha,\tau,\delta]/(2\alpha,\alpha^3,\tau^2=4\delta),$$

the d_3-differential (as stated above) must exist. (One can also argue that $K_2^{tC_2}\simeq *$, which begets the d_3-differential.) □

This is rather silly: we're starting with our knowledge of KO_* to deduce differentials in a spectral sequence converging to KO_*!

Proof 2. In the next subsection, we will describe a map of Hopf algebroids

$$(BP_*, BP_*BP) \to ((K_2)_*, \mathrm{Map}(C_2, (K_2)_*)), \tag{14}$$

which gives a map from the Adams-Novikov spectral sequence to the homotopy fixed point spectral sequence:

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \xrightarrow{\Phi} H^s(C_2, (K_2)_t).$$

Let $v_1 \in \mathrm{Ext}^{0,2}_{BP_*BP}(BP_*, BP_*/2)$; its image $\delta(v_1)$ under the boundary map arising from the short exact sequence

$$0 \to BP_* \xrightarrow{2} BP_* \to BP_*/2 \to 0$$

is the element $\alpha_1 \in \mathrm{Ext}^{1,2}_{BP_*BP}(BP_*, BP_*)$. We claim that $\Phi(\alpha_1) = \alpha$. To see this, note that the map $BP_* \to (K_2)_*$ sends v_1 to u. It follows that the image $\delta(u)$ of u under the boundary map coming from the short exact sequence

$$0 \to (K_2)_* \xrightarrow{2} (K_2)_* \to (K_2)_*/2 \to 0$$

is the nontrivial element of $H^1(C_2; (K_2)_2) \simeq C_2 \ni \alpha$, so $\delta(u) = \alpha$. But

$$\delta(u) = \delta(\Phi(v_1)) = \Phi(\delta(v_1)) = \Phi(\alpha_1),$$

as desired. Likewise, one can prove that

$$\Phi(\alpha_3) = u^2\alpha.$$

[Rav86, Theorem 4.4.47] provides the differential

$$d_3(\alpha_3) = \alpha_1^4,$$

which pushes forward, via Φ, to the differential

$$d_3(u^2\alpha) = \alpha^4.$$

The Leibniz rule now gives the desired result. □

The next proof has a geometric slant.

Proof 3. Let C_2 act trivially on the sphere spectrum S; then, we have a C_2-equivariant map $S \to K_2$, and hence a map

$$S^{hC_2} \simeq F(BC_2, S) = D\mathbf{R}P^\infty \to K_2^{hC_2}.$$

The cell structure of $D\mathbf{R}P^\infty$ is given by

$$\cdots \underbrace{\bullet - \bullet \quad \bullet} - \bullet \quad \bullet;$$

here, each $\bullet$ denotes a cell (the righmost is in dimension 0), the $-$ denote multiplication by 2 maps, and the brace is the attaching map given by $\eta = \alpha_1 \in \pi_*S$. Since the C_2-action on S is trivial, the homotopy fixed point spectral sequence can be identified with the Atiyah-Hirzebruch spectral sequence:

$$H^*(C_2; \pi_*S) = H^*(D\mathbf{R}P^\infty; \pi_*S) \Rightarrow \pi_*D\mathbf{R}P^\infty.$$

Note that there is a map of spectral sequences

$$\Phi : H^*(D\mathbf{R}P^\infty; \pi_*S) \to H^*(C_2; (K_2)_*).$$

In the HFPSS for S^{hC_2}, the d_1-differential is given by multiplication by 2 (under our identification with the AHSS). Then, whatever survives to the E_2-page in bidegree $(t-s, s) = (-2n, 2n)$ is sent to $u^{-2n}\alpha^{2n} \in H^{2n}(C_2; \pi_0 K_2)$ (which generates the copy of $C_2 = E_2^{-2n,2n}$ in the HFPSS for $K_2^{hC_2}$).

The d_2-differential in the HFPSS/AHSS for S^{hC_2} is given by sending $H^{2n}(C_2; \pi_0 S)$ to η times $H^{2n+2}(C_2; \pi_0 S)$. One reason comes from the Steenrod action: the mod 2 reduction map

$$H^{2n}(C_2; \pi_0 S) \to H^{2n}(\mathbf{R}P^\infty; C_2)$$

is an isomorphism, and the dual Steenrod algebra acts on $x \in H^2(\mathbf{R}P^\infty; C_2)$ by

$$x \mapsto \sum_{i=0}^{\infty} \xi_i \otimes x^{2^i};$$

to get the differential, we look at the lowest degree. Since η (which maps to α, as we saw above) lies in cohomological degree 1, this leads to a jump in the filtration when we push this forward to the HFPSS for $K_2^{hC_2}$, and this gives a d_3-differential

oof, this isn't a go planation... what zhouli say?

$$d_3(u^{-2n}\alpha^{2n}) = \eta \cdot u^{-2n-2}\alpha^{2n+2}.$$

We need to identify η with α, but this can be done as in Proof 2 as above. This tells us that

$$d_3(u^{-2n}\alpha^{2n}) = u^{-2n-2}\alpha^{2n+3}.$$

We can now use the Leibniz rule to get the desired differential. □

Using any of these methods, it's easy to see that the HFPSS for $K_2^{hC_2}$ collapses on the fourth page. There are no extension problems, and we recover:

$$\pi_* K_2^{hC_2} \simeq \mathbf{Z}[\alpha, 2u^2, u^{\pm 4}]/(2\alpha) \simeq \pi_* KO.$$

7.6. **Some computations of the ANSS 1-line at an odd prime.** To avoid excessive notation, let us write

$$H^{*,*}(M) = \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M).$$

Our goal will be to construct certain elements in $H^{1,2(p-1)k}(BP_*)$ by first understanding elements of $H^{0,2(p-1)k}(BP_*/p)$. Before proceeding, we need to recall some facts about BP_* and BP_*BP, all of which can be found in [Rav86].

- There is an isomorphism
$$BP_*BP \simeq BP_*[t_1, t_2, \cdots],$$
where $|t_i| = 2(p^i - 1)$.
- Let ℓ_i be the coefficients (in $BP_* \otimes \mathbf{Q}$) of the logarithm of the universal p-typical formal group law; then $\ell_0 = v_0 = p$, and we have relations
$$p\ell_n = \sum_{i=0}^{n-1} \ell_i v_{n-i}^{p^i}.$$
- The left unit $\eta_L : BP_* \to BP_*BP$ is the inclusion, while the right unit η_R is determined by
$$\eta_R(\ell_n) = \sum_{i=0}^{n} \ell_i t_{n-i}^{p^i}.$$
- The diagonal on BP_*BP is determined by
$$\sum_{i,j} \ell_i \Delta(t_j)^{p^i} = \sum_{i,j,k} \ell_i t_j^{i} \otimes t_k^{p^{j+k}}.$$
- Denoting by I_n the invariant regular prime ideal $(p, v_1, \cdots, v_{n-1})$, we have
$$\begin{aligned} \eta_R(v_1) &= v_1 + pt_1, \\ \eta_R(v_n) &\equiv v_n \mod I_n, \\ \eta_R(v_{n+k}) &\equiv v_{n+k} + v_n t_k^{p^n} - v_n^{p^k} t_k \mod (I_n, t_1, \cdots, t_{k-1}). \end{aligned}$$

As the I_n's are invariant regular prime ideals, the quotient BP_*/I_n is a (BP_*, BP_*BP)-comodule.

Lemma 7.15. *There is an isomorphism*

$$H^{0,*}(BP_*/I_n) \simeq \mathbf{F}_p[v_n].$$

Proof. We use the cobar complex, so

$$H^{0,*}(BP_*/I_n) \simeq \ker(\eta_R - 1 : BP_*/I_n \to BP_*BP/I_n).$$

It follows from our formulae involving η_R that all powers of v_n lie in $H^{0,*}(BP_*/I_n)$. It remains to show that if $x \in H^{0,*}(BP_*/I_n)$, then x is a polynomial in v_n. Let us write

$$x = f_0 + v_{n+k} f_1 + \cdots + v_{n+k}^m f_m,$$

for some polynomials $f_i = f_i(v_n, \cdots, v_{n+k-1})$. Then

$$\eta_R(f_i) \equiv f_i \mod (I_n, t_1, \cdots, t_{k-1})$$

for $n \leq i \leq n+k-1$. Our formula for $\eta_R(v_{n+k})$ implies that

$$\eta_R(x) - x \equiv \sum_{j=0}^{m} ((v_{n+k} + v_n t_k^{p^n} - v_n^{p^k} t_k)^j f_j - v_{n+k}^j f_j) \mod (I_n, t_1, \cdots, t_{k-1}).$$

Since

$$\sum_{j=0}^{m} ((v_{n+k} + v_n t_k^{p^n} - v_n^{p^k} t_k)^j f_j - v_{n+k}^j f_j) = v_n^m t_k^{mp^n} f_m + \text{smaller powers of } t_k,$$

and $\eta_R(x) - x \equiv 0 \mod I_n$, we conclude that $f_m \equiv 0 \mod I_n$. Running this argument repeatedly shows that x must be a monomial in v_n, as desired. □

In [Rav86, Theorem 5.2.1] it is shown that

$$H^{s,t}(BP_* \otimes \mathbf{Q}) = \begin{cases} \mathbf{Q} & s = t = 0 \\ 0 & \text{else,} \end{cases}$$

and that

$$H^{0,t}(BP_*) = \begin{cases} \mathbf{Z}_{(p)} & t = 0 \\ 0 & \text{else.} \end{cases}$$

Moreover, $H^{s,*}(BP_*)$ are all p-torsion for $s > 0$. The short exact sequence

$$0 \to BP_* \xrightarrow{p} BP_* \to BP_*/p \to 0$$

gives rise to a long exact sequence

$$\cdots \to H^{*,*}(BP_*) \to H^{*,*}(BP_*/p) \xrightarrow{\delta} H^{*+1,*}(BP_*) \to \cdots$$

As $I_1 = p$, Lemma 7.15 shows that

$$H^{0,*}(BP_*/p) \simeq \mathbf{F}_p[v_1].$$

Define $\alpha_j \in H^{1,2(p-1)j}(BP_*)$ by

$$\alpha_j = \delta(v_1^j).$$

These elements are nonzero in the Ext-group. Each $H^{1,2(p-1)j}(BP_*)$ is a cyclic p-group, and the long exact sequence shows that if $* \neq 2(p-1)j$, then $H^{1,*}(BP_*) = 0$.

To determine $H^{1,*}(BP_*)$, it therefore suffices to determine the order of α_j. We have, by the definition of the boundary homomorphism:

$$\alpha_j = \delta(v_1^j) = \frac{1}{p}(\eta_R(v_1^j) - v_1^j).$$

Recall that

$$\eta_R(v_1) = v_1 + pt_1,$$

so

$$\alpha_j = \frac{1}{p}((v_1 + pt_1)^j - v_1^j) = \sum_{i=1}^{j} \binom{j}{i} p^{i-1} t_1^i v_1^{t-i} = jt_1 v_1^{j-1} + p \cdot \text{junk}.$$

It follows that α_j is divisible by $\nu_p(j)$. For $k = 1, \cdots, \nu_p(j)+1$, we define

$$\alpha_{j/k} = \alpha_j / p^{k-1}.$$

This is an element of $H^{1,2(p-1)j}(BP_*)$ of order p^k.

We claim that $\alpha_{j/\nu_p(j)+1}$ is the generator of $H^{1,2(p-1)j}(BP_*)$, so that

$$H^{1,2(p-1)j}(BP_*) \simeq \mathbf{Z}/p^{\nu_p(j)+1}.$$

Indeed, we know that $x \in H^{1,2(p-1)j}(BP_*)$ is divisible by p iff its image in $H^{1,2(p-1)j}(BP_*/p)$ is zero. But

$$\alpha_{j/\nu_p(j)+1} = t_1 v_1^{j-1} \frac{j}{\nu_p(j)} + p \cdot \text{junk},$$

so the image of $\alpha_{j/\nu_p(j)+1}$ in $H^{1,2(p-1)j}(BP_*/p)$ is represented by $t_1 v_1^{j-1} \frac{j}{\nu_p(j)}$. It remains to show that this element is nonzero in the Ext-group.

More generally, we claim that $t_1 v_1^j$ is a nonzero element of $H^{1,2(p-1)(j+1)}(BP_*/p)$. If this were not the case, there would be an element $x \in BP_*/p$ such that

$$t_1 v_1^j \equiv \eta_R(x) - x \mod p,$$

so $\eta_R(x) - x \equiv 0 \mod (p, t_1)$. Then, arguing as in Lemma 7.15, we find that $\eta_R(x) - x \equiv 0 \mod p$. This is a contradiction.

add in the notes on stacks and tmf

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