Roots of unity in $K(n)$-local E_∞-rings

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- E_∞-ring spectra are the appropriate analogues of rings in homotopy theory: these are spectra which represent cohomology theories with a good notion of cup products.
- To do any kind of algebraic geometry in homotopy theory, we need to work with affine schemes coming from E_∞-rings; homotopy commutative rings do not suffice.
- Many important examples of E_∞-rings stem from arithmetic geometry.

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- Landweber's theorem begets a homotopy commutative ring E_n such that $E_n = \mathcal{O}_{\mathrm{LT}}[u^{\pm 1}]$; Hopkins and Miller proved that E_n is an E_∞-ring. This is called Morava E-theory.

The Lubin-Tate tower

- Drinfel'd introduced a tower of finite flat extensions of LT_n:

$$\mathrm{LT}_{n,\infty} \to \cdots \to \mathrm{LT}_{n,2} \to \mathrm{LT}_{n,1} \to \mathrm{LT}_{n,0} = \mathrm{LT}_n.$$

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- Then the ℓ-adic étale cohomology $H_c^*(\mathcal{M}_{n,\infty}; \mathbb{Q}_\ell)$ has an action of $\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times \times W_{\mathbb{Q}_p}$, with W_K the Weil group.

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- Then the ℓ-adic étale cohomology $H_c^*(\mathcal{M}_{n,\infty}; \mathbf{Q}_\ell)$ has an action of $\mathrm{GL}_n(\mathbf{Q}_p) \times D^\times \times W_{\mathbf{Q}_p}$, with W_K the Weil group.
- The Jacquet-Langlands conjecture, proved by Harris-Taylor, uses this to realize (a form of) the p-adic Langlands correspondence.

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- The composite map $\mathrm{LT}_{n,k} \to \mathrm{LT}_n \to \mathcal{M}_{\mathrm{fg}}$ begets, by Landweber's theorem, a homotopy commutative ring spectrum $E_{n,k}$ — but it is not clear at all that this should be an E_∞-ring.

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- Moreover,

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Theorem (Hopkins, unpublished)

The spectrum $KU_p[\zeta_p]$ is not an E_∞-ring.

- It follows from this that the Lubin-Tate tower when $n = 1$ does not lift to the world of homotopy theory.

A generalization of Hopkins' theorem

- Let R be an E_n-module; we say that R is $K(n)$-local if $\pi_0 R$ is $(p, u_1, \cdots, u_{n-1})$-complete; for instance, $KU_p[\zeta_p]$ is $K(1)$-local.

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Theorem (D.)

There is no $K(n)$-local E_∞-ring R such that $\pi_0 R$ contains a primitive p^kth root of unity for any $n > 0$.

- This generalizes Hopkins' theorem presented above.

The method of proof

- Let R be a $K(n)$-local E_∞-ring; then $\pi_0 R$ has operations ψ^p and θ^p, where ψ^p is an additive operation such that for any $x \in \pi_0 R$, we have

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- θ^p need not be additive; for instance, since $\psi^p(x+y) = \psi^p(x) + \psi^p(y)$, we have
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- If $n = 1$, the operation ψ^p is also multiplicative.

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$$1 + \zeta_p + \cdots + \zeta_p^{p-1} = 0,$$

one can prove that, if R is a $K(1)$-local E_∞-ring such that $\pi_0 R$ contains a pth root of unity, then p is invertible in $\pi_0 R$.

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- This is a contradiction, so we get the desired result when $n = 1$.
- However, this uses the multiplicativity of ψ^p when $n = 1$, and we do not have this luxury for $n > 1$.

The general case

- Instead, we use a recent theorem of Hahn's, which states that if R is a $K(n)$-local E_∞-ring such that the "$K(1)$-localization" $L_{K(1)}R$ of R is trivial, then R itself is trivial.

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- The image of $\zeta_p \in \pi_0 R$ under this map is a primitive pth root of unity inside $\pi_0 L_{K(1)}R$. But this implies that $L_{K(1)}R$ is trivial, by the story when $n = 1$.

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- The image of $\zeta_p \in \pi_0 R$ under this map is a primitive pth root of unity inside $\pi_0 L_{K(1)}R$. But this implies that $L_{K(1)}R$ is trivial, by the story when $n = 1$.
- Hahn's theorem now implies that R is itself trivial, as desired.

Applications to the Lubin-Tate tower

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- Recall that Hopkins' theorem implies that the Lubin-Tate tower does not lift to a tower in homotopy theory.
- The general case isn't so easy, however: it is not *a priori* clear that $\mathcal{O}_{\mathrm{LT}_{n,k}}$ with $n > 1$ contains a pth root of unity, so we can't immediately utilize our main theorem.

Applications to the Lubin-Tate tower, continued

- Instead, we must resort to a recent theorem of Scholze-Weinstein:

Theorem (Scholze-Weinstein)

There is a "determinant" map $\det : \mathcal{O}_{\mathrm{LT}_{1,k}} \to \mathcal{O}_{\mathrm{LT}_{n,k}}$.

Applications to the Lubin-Tate tower, continued

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Theorem (Scholze-Weinstein)

There is a "determinant" map $\det : \mathcal{O}_{\mathrm{LT}_{1,k}} \to \mathcal{O}_{\mathrm{LT}_{n,k}}$.

- For $k > 1$, the element $\det(\zeta_p) \in \mathcal{O}_{\mathrm{LT}_{n,k}}$ is a pth root of unity in $\mathcal{O}_{\mathrm{LT}_{n,k}}$. Our main theorem gives:

Theorem (D.)

For any $n > 0$, the Lubin-Tate tower does not lift to homotopy theory.

Consequences

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Theorem

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- One interesting consequence is the following folklore result.

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- Our main result also shows that certain PEL-type Shimura varieties (see Harris-Taylor and Behrens-Lawson) do not lift to derived stacks.

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