MILNOR'S EXOTIC SPHERES

ABSTRACT. These are notes on Milnor's influential paper [Mil56], titled "On manifolds homeomorphic to the 7-sphere".

1. MILNOR'S λ-INVARIANT

Our presentation will follow the generalization suggested by footnote 2 of [Mil56]. (After I wrote up these notes, I found [Mat12], which does things in essentially the same way.)

Let M be a manifold that is homeomorphic to S^{4k-1}; then this is the boundary of a $4k$-manifold B. Let $\sigma(B)$ denote the signature of B: this is the signature of the quadratic form on $H^{2k}(B,M;\mathbf{Q})$ defined by $x \mapsto \langle \nu, x^2\rangle$, where ν denotes the element of $H_{4k}(B,M)$ that maps to $H_{4k-1}(M)$ under the boundary map.

Clearly B isn't closed, so we can't apply the signature theorem to conclude that

$$\sigma(B) = \langle L(p_1, \cdots, p_k), [B]\rangle.$$

Suppose B *was* closed, for now. Recall that

$$L(x) = \frac{x}{\tanh(x)} = 1 + \frac{1}{3}x^2 - \frac{1}{45}x^4 + \cdots,$$

and that

$$L(p_1, \cdots, p_k) = [L(x_1)\cdots L(x_k)]_{4k},$$

where the p_i are the elementary symmetric functions in $x_1^2, \cdots, x_k^2$. Let us write

$$L(p_1, \cdots, p_k) = f(p_1, \cdots, p_{k-1}) + a_k p_k,$$

where a_k is the coefficient of p_k in $L(p_1, \cdots, p_k)$. The integrality of $\sigma(B)$ tells us that

$$\frac{1}{a_k}(\sigma(B) - \langle f(p_1, \cdots, p_{k-1}), [B]\rangle) \in \mathbf{Z}. \tag{1}$$

But this isn't true if B isn't closed, as we remarked before; and in our case, B clearly isn't. Instead, if we *define* $\lambda(M)$ to be the quantity in Equation (1), where B is the $4k$-dimensional manifold bounding M, we find that $\lambda(M) \in \mathbf{Q}/\mathbf{Z}$. Note that we're critically utilizing the fact that M is a homology sphere (i.e. that $H_i(M)$ is zero for $i \neq 0, 4k-1$) to pull back the Pontryagin classes of B to $H^*(B,M)$.

Theorem 1.1. *[Theorem 1 of* [Mil56]*] The element $\lambda(M) \in \mathbf{Q}/\mathbf{Z}$ is independent of the bounding manifold B.*

Before we prove this, we'll state some corollaries.

Corollary 1.2. *If $\lambda(M) \neq 0$, the manifold M isn't the boundary of any $4k$-manifold with $b_{4i} = 0$ (where b_i denotes the ith Betti number) for all $i < k$.*

Date: September 25, 2017.

This is clear: otherwise, the signature and all the Pontryagin classes would just be zero.

Changing the orientation of M simply flips the sign of the signature and the function $\langle f(p_1,\cdots,p_{k-1}),[B]\rangle$, so we have:

Corollary 1.3. *If $\lambda(M)\neq 0$, there is no orienation-reversing self-diffeomorphism of M.*

Proof of Theorem 1.1. Let B_1 and B_2 denote two manifolds bounding M. Define $C = B_1 \sqcup_M B_2$; this is a compact oriented manifold *without boundary*, where we pick the opposite orientation for B_2 when gluing. Then[1] $H^*(C) \simeq H^*(B_1,M)\oplus H^*(B_2,M)$, except in the top dimension (namely, dimension $8k$), where, because of the way we picked our orientations, the fundamental class of C is given by the difference of the fundamental classes of B_1 and B_2.

We claim that $\sigma(C) = \sigma(B_1) - \sigma(B_2)$. To see this, suppose we write $x \in H^*(C)$ as $x_1 \oplus x_2$; then

$$\langle \nu_C, x^2\rangle = \langle \nu_C, x_1^2\oplus x_2^2\rangle = \langle \nu_{B_1}\oplus(-\nu_{B_2}), x_1^2\oplus x_2^2\rangle = \langle \nu_{B_1}, x_1^2\rangle - \langle \nu_{B_2}, x_2^2\rangle,$$

as desired. Similarly, the Pontryagin classes $p_1,\cdots,p_{k-1}$ of C are determined by the restrictions to $H^*(B_1)$ and $H^*(B_2)$ — but these are just the Pontryagin classes of B_1 and B_2, respectively. It follows from this discussion that

$$(\sigma(B_1)-\langle f(p_1,\cdots,p_{k-1}),[B_1]\rangle)-(\sigma(B_2)-\langle f(p_1,\cdots,p_{k-1}),[B_2]\rangle) = \sigma(C)-\langle f(p_1,\cdots,p_{k-1}),[C]\rangle.$$

But the signature formula applied to C tells us that the right hand side is divisible by a_k. Thus, the element $\lambda(M)$ is independent of the choice of bounding manifold B, as an element of $\mathbf{Q}/\mathbf{Z}$. $\square$

2. Constructing exotic spheres

To construct the example that Milnor works with, we now need to specialize to the case $n=2$.

Exercise 2.1. We have

$$L(p_1,p_2) = \frac{7p_2 - p_1^2}{45}.$$

It follows from Theorem 1.1 that

$$\frac{45}{7}\left(\sigma(B) + \frac{1}{45}p_1^2\right) \in \mathbf{Q}/\mathbf{Z}.$$

Multiplying this by 14, we get something (which we'll also denote by $\lambda(M)$) in $\mathbf{Z}$, that is an invariant mod 7:

$$\lambda(M) = 90\sigma(B) + 2p_1^2 \equiv 2p_1^2 - \sigma(B) \mod 7.$$

To proceed, we need the following result (see [Hat09, Proposition 1.14]).

Proposition 2.2. *There is a bijection between isomorphism classes of oriented real vector bundles of dimension m over S^n and homotopy classes of maps $S^{n-1}\to \mathrm{GL}_m^+(\mathbf{R})$, where $\mathrm{GL}_m^+(\mathbf{R})$ denotes the subgroup of $\mathrm{GL}_m(\mathbf{R})$ consisting of matrices of positive determinant.*

[1] This follows from the relative Mayer-Vietoris long exact sequence: if $Y\subseteq X$ is the union of $C\subseteq A$ and $D\subseteq B$, then we have a long exact sequence

$$\cdots\to H^{n-1}(A\cap B, C\cap D)\to H^n(X,Y)\to H^n(A,C)\oplus H^n(B,D)\to H^n(A\cap B,C\cap D)\to\cdots.$$

To get S^3-bundles over S^4, we therefore need to look at the unit vectors in the real vector bundles — these correspond to the orthogonal matrices of unit determinant. In conclusion:

Corollary 2.3. *There is a bijection between $\pi_3(SO(4))$ and S^3-bundles over S^4.*

We would like to compute $\pi_3(SO(4))$; to do this, we will construct a double cover of $SO(4)$ whose homotopy groups are easy to compute. Let us think of S^3 as the unit vectors inside the quaternions. Define a map $S^3 \times S^3 \to SO(4)$ via $(u,v) \mapsto \{x \mapsto uxv^{-1}\}$. This is a group homomorphism, with kernel $\{(1,1),(-1,-1)\}$. It follows that $S^3 \times S^3$ is a double cover of $SO(4)$, and hence that $\pi_3(SO(4)) \simeq \mathbf{Z} \oplus \mathbf{Z}$. This identification can be written down explicitly: for $(i,j) \in \mathbf{Z} \oplus \mathbf{Z}$, define $S^3 \to S^3 \times S^3$ by $x \mapsto (x^i, x^{-j})$; then, the map $S^3 \to SO(4)$ corresponding to (i,j) is given by $u \mapsto \{x \mapsto u^i x u^j\}$. We will denote the vector bundle associated to $(i,j) \in \mathbf{Z} \oplus \mathbf{Z} \simeq \pi_3(SO(4))$ by $\xi_{i,j}$.

Theorem 2.4. *The total space of $\xi_{i,j}$ is homeomorphic to S^7 if $i+j=1$.*

To prove this, we need some Morse theory.

Theorem 2.5 (Reeb). *Let M be a closed n-manifold. If there is a differentiable function $f : M \to \mathbf{R}$ with only two critical points, both of which are critical (so that the Hessian is nonsingular), then M is homeomorphic to S^n.*

Assuming this, let us prove Theorem 2.4. Let us define coordinate charts on S^4 given by the complements of the north and south poles. Each of these can be identified with $\mathbf{R}^4$, by stereographic projection. The transition map is given by $x' = x/|x|^2$. When $\xi_{i,j}$ is restricted to each chart, they necessarily become trivialized, so the total space can be described by taking two copies of $\mathbf{R}^4 \times S^3$ and identifying the subsets $(\mathbf{R}^4 - 0) \times S^3$ by the diffeomorphism (note: this is slightly different than what Milnor does, but it works anyway):

$$(u,v) \mapsto (u',v') = (1/u, \overline{u^i v u^j}/|u|^{i+j}).$$

To define the function f required by Reeb's theorem, it suffices to do so (compatibly) on each chart. Let

$$f(u,v) = \frac{\Re(v)}{(1+|u|^2)^{1/2}} = \frac{\Re(uv^{-1})}{(1+|uv^{-1}|^2)^{1/2}}.$$

We have to check that these agree on the intersection, so that they define a global function. Recall that $1/x = \overline{x}/|x|^2$. It follows that

$$(v')^{-1} = \left(\frac{\overline{u^i v u^j}}{|u|^{i+j}}\right)^{-1} = \frac{u^i v u^j}{|u|^{i+j}|v|^2} = \frac{u^i v u^j}{|u|^{i+j}}.$$

It follows that

$$u'(v')^{-1} = \frac{1}{u}\frac{u^i v u^j}{|u|^{i+j}} = \frac{u^{i-1} v u^j}{|u|^{i+j}}.$$

Suppose, now, that $i+j=1$; then

$$\Re\left(u'(v')^{-1}\right) = \Re\left(\frac{u^{i-1} v u^j}{|u|^{i+j}}\right) = \frac{\Re(v)}{|u|},$$

since $j = 1 - i$ and $\Re(x^{-1}yx) = \Re(y)$ (as can be checked by explicit computation with the fact that $2\Re(x) = x + \overline{x}$). Moreover,

$$\left| \frac{u^{i-1} v u^j}{|u|^{i+j}} \right| = \frac{|v|^2}{|u|^2} = \frac{1}{|u|^2}.$$

It follows that the two functions agree on the intersection (via the transition map as defined above).

We would be done with the proof of Theorem 2.4 if we knew that there were only two critical points of f, both of which were nondegenerate. This is easy: fix a u; then the critical points of $f(u,v)$ are of the form $(u, \pm 1)$. For such points, the derivative of $\pm 1/(1+|u|^2)^{1/2}$ is given by $\pm 2|u|/(1+|u|^2)^{1/2}$. If $|u| \neq 0$, this cannot be zero, so the critical points of $f(u,v)$ are of the form $(0, \pm 1)$. (There are no critical points on the other chart, where f is defined by $\Re(uv^{-1})/(1+|u|^2)^{1/2}$.)

3. Finishing the proof

Lemma 3.1. *The Pontryagin class $p_1(\xi_{i,j})$ is $\pm 2(i-j)$.*

Proof. The integer $p_1(\xi_{i,j})$ is linear as a function of i and j, since if V is given by $f : S^3 \to SO(4)$, and W is given by $g : S^3 \to SO(4)$, the direct sum $V \oplus W$ is stably isomorphic to the bundle associated to $fg : S^3 \to SO(4)$. Let's write $p_1(\xi_{i,j}) = ai + bj$. Reversing the orientation of S^3 gives an orientation-reversing isomorphism $\xi_{i,j} \cong \xi_{-j,-i}$. Since Pontryagin classes are independent of the orientation of the bundle, we know that $p_1(\xi_{i,j}) = ai + bj = -aj - bi$, so $p_1(\xi_{i,j}) = a(i-j)$. We need to determine the constant a. For this, it suffices to compute $p_1(\xi_{0,1})$.

Recall that $p_1(\xi_{0,-1}) = -c_2(\xi_{0,1} \otimes_{\mathbf{R}} \mathbf{C})$, and that $\xi_{0,1} \otimes_{\mathbf{R}} \mathbf{C} \simeq \xi_{0,1} \oplus \overline{\xi_{0,1}}$. Moreover, $c_i(\overline{\xi}) \simeq (-1)^i c_i(\xi)$, so we get that

$$1 + c_2(\xi_{0,1} \otimes_{\mathbf{R}} \mathbf{C}) = (1 + c_2(\xi_{0,1})) \cup (1 + c_2(\overline{\xi_{0,1}})).$$

Let α be a generator of $H^4(S^4)$. Milnor computes[2] that $c_2(\xi_{0,1}) = -\alpha$, from which it follows that the above formula simplifies to $(1-\alpha) \cup (1-\alpha) = 1 - 2\alpha$; this implies that $c_2(\xi_{0,1} \otimes_{\mathbf{R}} \mathbf{C}) = -2\alpha$, i.e., that $p_1(\xi_{0,1}) = 2$. Because we'd picked a generator of $H^4(S^4) \simeq \mathbf{Z}$, it follows that

$$p_1(\xi_{i,j}) = \pm 2(i-j).$$

$\square$

Let $B_{i,j}$ denote the disk bundle whose boundary is the bundle $\xi_{i,j}$. When $i + j = 1$, we saw that the total space of $\xi_{i,j}$ is homeomorphic to a sphere. We may pick an orientation for $B_{i,j}$ so that $\sigma(B_{i,j}) = 1$. It follows that

$$\lambda(E(\xi_{i,j})) \equiv 8(i-j)^2 - 1 \mod 7.$$

If $i - j \not\equiv 1 \mod 7$, it follows that the total space of $\xi_{i,j}$ cannot be diffeomorphic to S^7. But Theorem 2.4 says that if $i + j = 1$, the total space of $\xi_{i,j}$ is homeomorphic to S^7; so exotic spheres exist in dimension 7.

[2]The only way I can see this is as follows: recall that $\mathbf{HP}^1 \simeq S^4$. Moreover, if $\gamma_{\mathbf{H}}$ denotes the tautological bundle over $\mathbf{HP}^1$, then $c_2(\gamma_{\mathbf{H}}) = \alpha$ (a generator of $H^4(\mathbf{HP}^1)$), so $c_2(\gamma_{\mathbf{H}} \otimes_{\mathbf{R}} \mathbf{C}) = -2\alpha$. This means that $p_1(\gamma_{\mathbf{H}}) = 2\alpha$. If we can prove that $\gamma_{\mathbf{H}} \simeq \xi_{0,1}$, we would be done; this can be seen by explicitly identifying the transition functions.

References

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