

RELATIVE LANGLANDS DUALITY FOR LINEAR PERIODS

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ABSTRACT. We prove relative Langlands duality for $X = \mathrm{GL}_{p+q}/(\mathrm{GL}_p \times \mathrm{GL}_q)$. These symmetric spaces arise in the setting of the linear period of Friedberg–Jacquet when $p = q$, and exhibit non-tempered behavior when $q > p + 1$. Via the real-symmetric correspondence, we establish a derived Satake equivalence for the real group $G_{\mathbf{R}} = \mathrm{U}(p, q)$, and use it to obtain a constructible realization of coherent sheaves on Kato’s exotic nilpotent cones. Along the way, we describe the dual variety of X in terms of the Whittaker cotangent bundle and regular centralizers, viewed as Coulomb branches.

CONTENTS

1. Introduction	2
1.1. Relative Langlands duality	2
1.2. Linear period of Friedberg–Jacquet	2
1.3. Derived Satake equivalence for $\mathrm{U}(p, q)$	3
1.4. Whittaker cotangent bundles and regular centralizers	3
1.5. Exotic nilpotent cones	4
1.6. Hermitian symmetric spaces	5
1.7. Acknowledgements	5
2. Constructible side	5
2.1. Real groups and symmetric spaces	5
2.2. Loop spaces and affine Grassmannians	6
2.3. Real and relative Satake categories	7
2.4. Hecke actions	7
3. Spectral side	10
3.1. Hamiltonian dual varieties	10
3.2. Vector bundle structures	12
3.3. Invariant quotients and exotic nilpotent cones	13
3.4. Whittaker cotangent bundles	16
3.5. Regular centralizers	16
3.6. Dual varieties as affine closures	18
3.7. Generalized Kostant theorem	18
3.8. Lagrangian correspondence	25
3.9. Proof of Theorem 3.6.1	29
4. Main results	30
4.1. Ext algebras	30
4.2. Relative duality for linear periods and derived Satake for $\mathrm{U}(p, q)$	31
4.3. Coherent sheaves on exotic nilpotent cones	32
4.4. Abelian equivalences	33
References	34

1. INTRODUCTION

1.1. Relative Langlands duality. The relative Langlands program of Ben-Zvi, Sakellaridis and Venkatesh [BZSV] proposes geometric mechanisms for observed and conjectural relations between periods and L-functions.

A fundamental conjecture in the program predicts that given a complex reductive group G , with dual group $\check{G}$, and a homogeneous spherical G -variety X , there exists a (graded) Hamiltonian $\check{G}$ -variety $\check{M}$ with moment map $\mu : \check{M} \rightarrow \check{\mathfrak{g}}^*$ equipped with a commuting $\mathbb{G}_m$ -action satisfying $\mu(t \cdot m) = t^2 \mu(m)$, and a relative derived Satake equivalence

$$(1) \quad D_{\mathfrak{L}^+G}^b(\mathfrak{L}X) \cong \mathrm{Perf}^{\check{G}}(\check{M})^{\mathbb{Z}}$$

On the left or automorphic side, $\mathfrak{L}X$ is the formal loop space of X , $\mathfrak{L}^+G$ is the formal arc space of G , and $D_{\mathfrak{L}^+G}^b(\mathfrak{L}X)$ is the bounded constructible dg-derived category of $\mathfrak{L}^+G$ -equivariant sheaves on $\mathfrak{L}X$. On the right or spectral/Galois side, $\mathrm{Perf}^{\check{G}}(\check{M})^{\mathbb{Z}}$ is the sheared dg-category of $\check{G}$ -equivariant perfect dg-modules over the ring of functions on $\check{M}$ viewed as a dg-algebra with trivial differential and grading given by the $\mathbb{G}_m$ -action. The shearing notation and the shifted vector spaces used below are defined in Section 4.2; the parity convention is specified in Section 4.4.

Moreover, the relative derived Satake equivalence (1) should be compatible with the natural monoidal actions under the derived Satake equivalence

$$D_{\mathfrak{L}^+G}^b(\mathrm{Gr}_G) \cong \mathrm{Perf}^{\check{G}}(\check{\mathfrak{g}}^*[2])$$

where $\mathrm{Gr}_G = \mathfrak{L}G/\mathfrak{L}^+G$ is the affine Grassmannian, $D_{\mathfrak{L}^+G}^b(\mathrm{Gr}_G)$ acts on $D_{\mathfrak{L}^+G}^b(\mathfrak{L}X)$ by convolution, and $\mathrm{Perf}^{\check{G}}(\check{\mathfrak{g}}^*[2])$ acts on $\mathrm{Perf}^{\check{G}}(\check{M})^{\mathbb{Z}}$ by tensor product through μ .

1.2. Linear period of Friedberg–Jacquet. An important instance of relative duality is the linear period of Friedberg–Jacquet [FJ]: here one takes $G = \mathrm{GL}_{2p}$, so with dual group $\check{G} = \mathrm{GL}_{2p}$, and studies $X = \mathrm{GL}_{2p}/(\mathrm{GL}_p \times \mathrm{GL}_p)$ with expected dual

$$\begin{aligned} \check{M} &= \mathrm{GL}_{2p} \times^{\mathrm{Sp}_{2p}} (\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[1]) \\ \check{M}/\check{G} &= (\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[1])/\mathrm{Sp}_{2p} \end{aligned}$$

Similar phenomena can be observed more generally, when one takes $G = \mathrm{GL}_{p+q}$, $q \geq p$, so with dual group $\check{G} = \mathrm{GL}_{p+q}$, and studies $X = \mathrm{GL}_{p+q}/(\mathrm{GL}_p \times \mathrm{GL}_q)$ with expected dual

$$\begin{aligned} \check{M} &= \mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} (\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[q-p+1] \times \mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p}) \\ \check{M}/\check{G} &= (\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[q-p+1] \times \mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p})/\mathrm{Sp}_{2p} \end{aligned}$$

In the associated-bundle presentation the grading also acts on the group factor by $g \mapsto g\lambda(t)^{-1}$, with λ as in (8); after quotienting by $\check{G}$, only the displayed fiber degrees remain. When $p = q$, the final GL_{q-p} quotient factor is a point. Note this is tempered if and only if $q = p$ or $q = p + 1$.

Our main result is a relative derived Satake equivalence in this setting:

Theorem 1.2.1. (see Theorem 4.2.1) *For $G = \mathrm{GL}_{p+q}$, $q \geq p$, and $X = \mathrm{GL}_{p+q}/(\mathrm{GL}_p \times \mathrm{GL}_q)$, there is a relative derived Satake equivalence of dg-categories*

$$D_{\mathfrak{L}^+G}^b(\mathfrak{L}X) \cong \mathrm{Perf}^{\mathrm{Sp}_{2p}}(\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[q-p+1] \times \mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p})$$

compatible with the natural monoidal actions under the derived Satake equivalence

$$D_{\mathfrak{L}+G}^b(\mathrm{Gr}_G) \cong \mathrm{Perf}^{\check{G}}(\check{\mathfrak{g}}^*[2])$$

Within the above derived Satake equivalence, we also find an abelian equivalence. According to [CN, Section 8.4], the category $D_{\mathfrak{L}+G}^b(\mathfrak{L}X)$ admits a perverse t -structure. We denote the corresponding category of perverse sheaves by $\mathrm{Perv}_{\mathfrak{L}+G}(\mathfrak{L}X)$.

Theorem 1.2.2. (see Theorem 4.4.1) *There is an equivalence of abelian category*

$$\mathrm{Perv}_{\mathfrak{L}+G}(\mathfrak{L}X) \cong \begin{cases} \wedge^*(T^*\mathbb{A}^{2p})\text{-mod}(\mathrm{Rep}(\mathrm{Sp}_{2p})) & p = q, \\ \mathrm{Rep}(\mathrm{Sp}_{2p}) & p \neq q. \end{cases}$$

1.3. Derived Satake equivalence for $U(p, q)$. A key feature of the linear period of Friedberg–Jacquet, and more generally of $G = \mathrm{GL}_{p+q}$ acting on $X = \mathrm{GL}_{p+q}/(\mathrm{GL}_p \times \mathrm{GL}_q)$ is that X is a symmetric space, i.e. $K = \mathrm{GL}_p \times \mathrm{GL}_q$ is the fixed-points of an involution θ of G . Among many consequences, this allows one to reinterpret the relative derived Satake equivalence of Theorem 1.2.1 in terms of the unitary group $G_{\mathbf{R}} = U(p, q)$, which is quasi-split if and only if $q = p$ or $q = p + 1$. Following [CN] and recalled as Proposition 2.4.5 below, the automorphic side admits a natural equivalence

$$(2) \quad D_{\mathfrak{L}+G}^b(\mathfrak{L}X) \cong D_{\mathfrak{L}+U(p,q)}^b(\mathrm{Gr}_{U(p,q)})$$

where $\mathrm{Gr}_{U(p,q)} = \mathfrak{L}U(p, q)/\mathfrak{L}^+U(p, q)$ is the real affine Grassmannian of $U(p, q)$. Moreover, the equivalence (2) is compatible with natural actions of the derived Satake category.

So the relative derived Satake equivalence of Theorem 1.2.1 can be restated as a real derived Satake equivalence:

Theorem 1.3.1 (see Theorem 4.2.1). *For $G_{\mathbf{R}} = U(p, q)$ there is a real derived Satake equivalence of dg-categories*

$$D_{\mathfrak{L}+U(p,q)}^b(\mathrm{Gr}_{U(p,q)}) \cong \mathrm{Perf}^{\mathrm{Sp}_{2p}}(\wedge^2 \mathbb{A}^{2p}[2] \times T^*\mathbb{A}^{2p}[q-p+1] \times \mathfrak{g}_{q-p}^*[2] // \mathrm{GL}_{q-p})$$

compatible with the natural monoidal actions under the derived Satake equivalence

$$D_{\mathfrak{L}+G}^b(\mathrm{Gr}_G) \cong \mathrm{Perf}^{\check{G}}(\check{\mathfrak{g}}^*[2])$$

According to [N], there is natural perverse t -structure on $D_{\mathfrak{L}+U(p,q)}^b(\mathrm{Gr}_{U(p,q)})$ and we denote by $\mathrm{Perv}_{\mathfrak{L}+U(p,q)}(\mathrm{Gr}_{U(p,q)})$ the corresponding category of perverse sheaves.

Theorem 1.3.2. (see Theorem 4.4.1) *There is an equivalence of abelian category*

$$\mathrm{Perv}_{\mathfrak{L}+U(p,q)}(\mathrm{Gr}_{U(p,q)}) \cong \begin{cases} \wedge^*(T^*\mathbb{A}^{2p})\text{-mod}(\mathrm{Rep}(\mathrm{Sp}_{2p})) & p = q, \\ \mathrm{Rep}(\mathrm{Sp}_{2p}) & p \neq q. \end{cases}$$

1.4. Whittaker cotangent bundles and regular centralizers. Another key feature of the linear period of Friedberg–Jacquet, and more generally of $G = \mathrm{GL}_{p+q}$ acting on $X = \mathrm{GL}_{p+q}/(\mathrm{GL}_p \times \mathrm{GL}_q)$ is that $K = \mathrm{GL}_p \times \mathrm{GL}_q$ is a Levi subgroup of G . The work of the third author [M] suggests the alternative description of the dual variety $\check{M}$ as the affine closure of the Whittaker cotangent bundle

$$(3) \quad \check{M} \cong \overline{T_{\psi_K}^*(\check{G}/\check{U})}$$

here $T_{\psi_{\check{K}}}^*(\check{G}/\check{U})$ is the Hamiltonian reduction of $T^*\check{G}$ by a maximal unipotent subgroup $\check{U} = [\check{B}, \check{B}] \subset \check{G}$ at $-d\psi_{\check{K}}$ for the right action $g \mapsto gu^{-1}$, where $\psi_{\check{K}} : \check{U} \rightarrow \mathbb{G}_a$ is an additive character nontrivial precisely on the simple roots of the dual Levi $\check{K} = \mathrm{GL}_p \times \mathrm{GL}_q$ (see Section 3.4 for more details).

On the other hand, a recent general result of Ginzburg [Gi] suggests another description of the dual variety in terms of regular centerlizers group schemes:

$$(4) \quad \check{M} \cong \overline{\check{G} \times J_{\check{K}}/\pi^*J_{\check{G}}}$$

here the right hand side is the affine closure of the quotient $\check{G} \times J_{\check{K}}/\pi^*J_{\check{G}}$, where $J_{\check{G}} \rightarrow \mathfrak{c}_{\check{G}}$, $J_{\check{K}} \rightarrow \mathfrak{c}_{\check{K}}$ are the group schemes of regular centralizers, and $\pi : \mathfrak{c}_{\check{K}} \rightarrow \mathfrak{c}_{\check{G}}$ is the natural projection (see Section 3.5 for more details).

Indeed, we prove the isomorphisms (3) and (4), which is the key step in our approach to Relative Langlands duality for linear periods:

Theorem 1.4.1 (see Theorem 3.6.1). *There are $\check{G} \times \mathbb{G}_m$ -equivariant isomorphisms of affine varieties over $\check{\mathfrak{g}}^*$ in the following commutative diagram*

$$\begin{array}{ccc} \overline{\check{G} \times J_{\check{K}}/\pi^*J_{\check{G}}} & \xrightarrow{\cong} & \overline{T_{\psi_{\check{K}}}^*(\check{G}/\check{U})} \\ & \searrow \cong & \swarrow \cong \\ & \check{M} & \end{array}$$

In fact, the isomorphism $\overline{T_{\psi_{\check{K}}}^*(\check{G}/\check{U})} \cong \overline{(\check{G} \times J_{\check{K}})/\pi^*J_{\check{G}}}$ in Theorem 1.4.1 follows from the work in [Ga, M], which is valid when $\check{K}$ is a general Levi subgroups (not necessarily symmetric or spherical), and the main step in the proof is the isomorphism (3). The proof uses a generalization of Kostant theorem, see Section 3.7, and a Lagrangian correspondence between $T_{\psi_{\check{K}}}^*(\check{G}/\check{U})$ and $\check{M}$, see Section 27.

Since $\check{M}$ is smooth, as a corollary of the theorem we obtain the following:

Corollary 1.4.2 (see Corollary 3.6.2). *The affine closures $\overline{T_{\psi_{\check{K}}}^*(\check{G}/\check{U})} \cong \overline{(\check{G} \times J_{\check{K}})/\pi^*J_{\check{G}}}$ are smooth.*

Remark 1.4.3. It is proved in [Gi, Corollary 3.1.6] that the affine closure $\overline{\check{G} \times J_{\check{K}}/\pi^*J_{\check{G}}}$ for general connected reductive subgroup $\check{K} \subset \check{G}$ is an irreducible normal variety. However, the smoothness of the affine closures in setting of Levi symmetric subgroup $\check{K} = \mathrm{GL}_p \times \mathrm{GL}_q$ was unexpected at least by the authors.

1.5. Exotic nilpotent cones. Finally, this paper also makes contact with the exotic nilpotent cones of [Ka]. Recall the dual variety

$$\check{M}/\check{G} = (\wedge^2 \mathbb{A}^{2p}[2] \times T^*\mathbb{A}^{2p}[q-p+1] \times \mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p})/\mathrm{Sp}_{2p}$$

The null-cone for the Sp_{2p} action on $\wedge^2 \mathbb{A}^{2p}[2] \times T^*\mathbb{A}^{2p}[q-p+1]$, denoted by

$$\mathcal{N}'_{\mathrm{Sp}_{2p}} \subset \wedge^2 \mathbb{A}^{2p}[2] \times T^*\mathbb{A}^{2p}[q-p+1]$$

is known as the exotic nilpotent cones of [Ka]. Consider the dg category $D^b(\mathrm{Gr}_{U(p,q)})$ of bounded constructible complexes on $\mathrm{Gr}_{U(p,q)}$ and its full subcategory

$$D_{\{\mathfrak{L}^+ + U(p,q)\}}^b(\mathrm{Gr}_{U(p,q)}) \subset D^b(\mathrm{Gr}_{U(p,q)})$$

of bounded complexes constructible with respect to the $\mathfrak{L}^+U(p, q)$ -orbits stratification. As a corollary of derived Satake for $U(p, q)$, we obtain the following constructible description coherent sheaves on $\mathcal{N}'_{\mathrm{Sp}_{2p}}$:

Theorem 1.5.1. *(see Theorem 4.3.1) There is an equivalence of dg-categories*

$$D_{\{\mathfrak{L}^+U(p, q)\}}^b(\mathrm{Gr}_{U(p, q)}) \cong \mathrm{Perf}^{\mathrm{Sp}_{2p}}(\mathcal{N}'_{\mathrm{Sp}_{2p}})$$

1.6. Hermitian symmetric spaces. The symmetric space $X = \mathrm{GL}_{p+q}/(\mathrm{GL}_p \times \mathrm{GL}_q)$ considered in this paper is a particular example of a Hermitian symmetric space. In forthcoming work [CDMN], we expect to establish Relative Langlands duality for Hermitian symmetric spaces with no roots of type N. Modulo central factors, the Hermitian symmetric spaces under consideration are as follows:

$X = G/K$	Rank
$\mathrm{GL}_{2r+m}/(\mathrm{GL}_r \times \mathrm{GL}_{r+m})$	r
$\mathrm{PSO}_{2n}/\mathrm{GSO}_{2n-2}$	2
$\mathrm{SO}_{4r}/\mathrm{GL}_{2r}$	r
$\mathrm{SO}_{4r+2}/\mathrm{GL}_{2r+1}$	r
$\mathrm{E}_6/\mathrm{GSO}_{10}$	2
$\mathrm{E}_7/\mathrm{GE}_6$	3

A striking feature of the Hermitian symmetric case is that, after forgetting the gradings, all of the corresponding dual varieties $\check{M}$ are of the form

$$\check{G} \times^{\mathrm{Sp}(V)} (T^*(V) \times \wedge^2(V) \times \mathbf{A}^{a_X})$$

where V is a vector space of dimension $2r$ and a_X is the dimension of the remaining affine factor. For the linear periods considered here, $r = p$ and $a_X = q - p$.

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2. CONSTRUCTIBLE SIDE

Throughout, sheaves have complex coefficients, and $\mathrm{Rep}(H)$ denotes finite-dimensional algebraic representations of a complex algebraic group H . We use D^b for the bounded, finite-rank constructible category and Perv for its perverse heart. The cocomplete categories D and their compact objects are distinguished below in Section 2.3.

2.1. Real groups and symmetric spaces. Let $G = \mathrm{GL}_N$ be the complex general linear group of rank N . Let T be the diagonal maximal torus and $B = TU$ be the Borel subgroup of upper triangular matrices.

Fix integers $1 \leq p \leq q$ and put $N = p + q$. Consider the following matrix

$$I_{p,q} = \begin{pmatrix} 0 & 0 & J_p \\ 0 & I_{q-p} & 0 \\ J_p & 0 & 0 \end{pmatrix}$$

where I_{q-p} is the identity matrix of size $q-p$ and J_p is the permutation matrix with 1's on the anti diagonal and 0's elsewhere. Consider the involution $\theta : G \rightarrow G$ given by $\theta(g) = I_{p,q}g(I_{p,q})^{-1}$. The fixed point subgroup $K = G^\theta$ is a Levi symmetric subgroup of G isomorphic to $K \cong GL_p \times GL_q$. We denote by $X = G/K$ the associated symmetric space.

The block upper triangular subgroup

$$P = \begin{pmatrix} B_p & * & * \\ 0 & GL_{q-p} & * \\ 0 & 0 & B_p \end{pmatrix}$$

is a θ -split minimal parabolic subgroup, where $B_p \subset GL_p$ is the upper triangular Borel. Its Levi subgroup is

$$L = P \cap \theta(P) = \begin{pmatrix} GL_1^p & 0 & 0 \\ 0 & GL_{q-p} & 0 \\ 0 & 0 & GL_1^p \end{pmatrix}.$$

The subtorus $A = \{t \in T | \theta(t) = t^{-1}\}^\circ \subset T$ is a maximal θ -split torus consisting of diagonal matrices of the form

$$\text{diag}(a_1, \dots, a_p, 1, \dots, 1, a_p^{-1}, \dots, a_1^{-1})$$

We have $L = Z_G(A)$ and $P = LU_P = MAU_P$ where U_P is the unipotent radical of P and $M = Z_K(A)$. We see that X is quasi-split (but non-split) if and only if $p = q$ or $p + 1 = q$.

We write $\Lambda_T = X_*(T)$ for the coweight lattice and Λ_T^+ for the dominant coweights with respect to (T, B) . Let $2\rho \in X^*(T)$ be the sum of the positive roots. The real coweight lattice is $\Lambda_A = X_*(A)$, with $\Lambda_A^+ = \Lambda_A \cap \Lambda_T^+$. Under the coordinate identifications $X_*(T) \cong \mathbf{Z}^N$ and $X^*(T) \cong \mathbf{Z}^N$, we have

$$2\rho = (N-1, N-3, \dots, -N+3, -N+1).$$

The elements of Λ_A are the tuples

$$(5) \quad (\lambda_1, \dots, \lambda_p, 0, \dots, 0, -\lambda_p, \dots, -\lambda_1).$$

We have an embedding $\tau : X \rightarrow G$ sending $x = gK$ to $g\theta(g)^{-1}$ which induces an isomorphism $\tau : X \cong Q = \{g \in G | \theta(g) = g^{-1}\}^\circ$ between X and the neutral component of the θ anti-fixed points in G . We denote by $\iota_A : A \rightarrow Q \cong X$ the natural embedding.

Consider the conjugation $\eta_c : G \rightarrow G$ given by $\eta_c(g) = (\bar{g}^t)^{-1}$ with fixed points subgroup $G_c = G^{\eta_c} = U(N)$ the compact unitary group. Consider the conjugation $\eta : G \rightarrow G$ given by $\eta(g) = I_{p,q}(\bar{g}^{-1})^t I_{p,q}^{-1}$. The fixed point subgroup G^η is a real form of G , denote by $G_{\mathbf{R}} = U(p, q)$, known as the unitary group of signature (p, q) . Consider the intersections $P_{\mathbf{R}} = P \cap G_{\mathbf{R}}$, $L_{\mathbf{R}} = L \cap G_{\mathbf{R}}$, and $U_{P, \mathbf{R}} = U_P \cap G_{\mathbf{R}}$, etc. The intersection $P_{\mathbf{R}} = P \cap G_{\mathbf{R}}$ is the minimal parabolic subgroup with Iwasawa decomposition $P_{\mathbf{R}} = M_{\mathbf{R}} A_{\mathbf{R}} U_{P, \mathbf{R}}$. The intersection $K_c = K \cap G_{\mathbf{R}}$ is a maximal compact subgroup isomorphic to the product $K_c \cong U(p) \times U(q)$ of unitary groups.

2.2. Loop spaces and affine Grassmannians. Let $\mathfrak{L}G$ and $\mathfrak{L}K$ denote the loop groups of G and K , respectively. We also have the arc subgroups $\mathfrak{L}^+G \subseteq \mathfrak{L}G$ and $\mathfrak{L}^+K \subseteq \mathfrak{L}K$. Let $\text{Gr}_G = \mathfrak{L}G/\mathfrak{L}^+G$ and $\text{Gr}_K = \mathfrak{L}K/\mathfrak{L}^+K$ be the corresponding affine Grassmannians. We abbreviate $\text{Gr} = \text{Gr}_G$. There is a natural closed immersion $\iota_K : \text{Gr}_K \hookrightarrow \text{Gr}_G$. Let $\mathfrak{L}G_{\mathbf{R}}$ be the real loop group and $\mathfrak{L}^+G_{\mathbf{R}} \subseteq \mathfrak{L}G_{\mathbf{R}}$ its arc

subgroup. The quotient $\mathrm{Gr}_{\mathbf{R}} = \mathfrak{L}\mathrm{G}_{\mathbf{R}}/\mathfrak{L}^+\mathrm{G}_{\mathbf{R}}$ is the real affine Grassmannian, a real form of the ind-scheme $\mathrm{Gr}_{\mathbf{G}}$. For any $\lambda \in \Lambda_{\mathbf{T}}^+$ (resp. $\lambda \in \Lambda_{\mathbf{A}}^+$) we denote by $\mathrm{Gr}^{\lambda} \subset \mathrm{Gr}$ (resp. $\mathrm{Gr}_{\mathbf{R}}^{\lambda} \subset \mathrm{Gr}_{\mathbf{R}}$) the $\mathfrak{L}^+\mathrm{G}$ -orbit (resp. $\mathfrak{L}^+\mathrm{G}_{\mathbf{R}}$) through t^{λ} . We have $\mathrm{Gr} = \bigsqcup_{\lambda \in \Lambda_{\mathbf{T}}^+} \mathrm{Gr}^{\lambda}$ and $\mathrm{Gr}_{\mathbf{R}} = \bigsqcup_{\lambda \in \Lambda_{\mathbf{A}}^+} \mathrm{Gr}_{\mathbf{R}}^{\lambda}$.

Let $\mathfrak{L}\mathrm{X}$ be the loop space of X . The embedding $\iota_{\mathrm{A}} : \mathrm{A} \rightarrow \mathrm{X}$ of the θ -split torus gives rise to an embedding $\Lambda_{\mathrm{A}} \rightarrow \mathfrak{L}\mathrm{X}$ sending λ to $x^{\lambda} = \iota_{\mathrm{A}}(t^{\lambda})$. Let $\mathfrak{L}\mathrm{X}^{\lambda} = \mathfrak{L}^+\mathrm{G} \cdot x^{\lambda}$ be the $\mathfrak{L}^+\mathrm{G}$ -orbit through x^{λ} . It is known that $\mathfrak{L}\mathrm{X} = \bigsqcup_{\lambda \in \Lambda_{\mathbf{A}}^+} \mathfrak{L}\mathrm{X}^{\lambda}$ and $\overline{\mathfrak{L}\mathrm{X}^{\lambda}} = \bigsqcup_{\mu \in \Lambda_{\mathbf{A}}^+, \mu \leq \lambda} \mathfrak{L}\mathrm{X}^{\mu}$ (see, e.g., [CY, Proposition 17 and Theorem 18]).

2.3. Real and relative Satake categories. Let $(\mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathrm{Gr}), *)$ be the co-complete monoidal dg category of $\mathfrak{L}^+\mathrm{G}$ -equivariant constructible complexes on Gr with monoidal structure given by the convolution product: $\mathrm{F}, \mathrm{F}' \in \mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathrm{Gr})$ and $\mathrm{F} \star \mathrm{F}' = m_!(\mathrm{F} \tilde{\boxtimes} \mathrm{F}')$, where $m : \mathfrak{L}\mathrm{G} \times^{\mathfrak{L}^+\mathrm{G}} \mathrm{Gr} \rightarrow \mathrm{Gr}$ is the natural action map and $\mathrm{F} \tilde{\boxtimes} \mathrm{F}' \in \mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathfrak{L}\mathrm{G} \times^{\mathfrak{L}^+\mathrm{G}} \mathrm{Gr})$ is the twisted product of F and F' . It is known that $\mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathrm{Gr})$ is compactly generated and we denote by $\mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathrm{Gr})^c$ the subcategory of compact objects. The equivariant derived category $\mathrm{D}_{\mathfrak{L}^+\mathrm{G}}^b(\mathrm{Gr})$ coincides with the subcategories of locally compact objects¹ and we denote by $\mathrm{Ind}(\mathrm{D}_{\mathfrak{L}^+\mathrm{G}}^b(\mathrm{Gr}))$ the ind-completion of $\mathrm{D}_{\mathfrak{L}^+\mathrm{G}}^b(\mathrm{Gr})$. We denote by $(\mathrm{D}_{\mathfrak{L}^+\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathbf{R}}), *)$, etc, the corresponding monoidal categories for $\mathrm{Gr}_{\mathbf{R}}$. We will call the categories $\mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathrm{Gr})$, etc (resp. $\mathrm{D}_{\mathfrak{L}^+\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathbf{R}})$, etc) the complex Satake categories for G (resp. real Satake categories for $\mathrm{G}_{\mathbf{R}}$).

Let $\mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathfrak{L}\mathrm{X})$ be the co-complete dg category of $\mathfrak{L}^+\mathrm{G}$ -equivariant constructible complexes on $\mathfrak{L}\mathrm{X}$. Denote by $\mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathfrak{L}\mathrm{X})^c$, $\mathrm{D}_{\mathfrak{L}^+\mathrm{G}}^b(\mathfrak{L}\mathrm{X})$, and $\mathrm{Ind}(\mathrm{D}_{\mathfrak{L}^+\mathrm{G}}^b(\mathfrak{L}\mathrm{X}))$ for the corresponding categories of compact objects, locally compact objects, and the ind-completion of the latter, respectively. We have a natural action of the complex Satake category

$$* : \mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathrm{Gr}) \times \mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathfrak{L}\mathrm{X}) \rightarrow \mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathfrak{L}\mathrm{X})$$

given by $\mathrm{F} \star \mathrm{F}' = a_!(\mathrm{F} \tilde{\boxtimes} \mathrm{F}')$ where $a : \mathfrak{L}\mathrm{G} \times^{\mathfrak{L}^+\mathrm{G}} \mathfrak{L}\mathrm{X} \rightarrow \mathfrak{L}\mathrm{X}$ is the action map and $\mathrm{F} \tilde{\boxtimes} \mathrm{F}' \in \mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathfrak{L}\mathrm{G} \times^{\mathfrak{L}^+\mathrm{G}} \mathfrak{L}\mathrm{X})$ is the twisted product of F and F' . We will call the above category $\mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathfrak{L}\mathrm{X})$, etc, the relative Satake categories.

2.4. Hecke actions. Let $\mathrm{Gr}^{(2)} \rightarrow \mathbf{C}$ be the Beilinson-Drinfeld Grassmannian over the complex line classifying triples (x, E, ϕ) where $x \in \mathbf{C}$, E is a G -bundle on $\mathbf{C}$, and ϕ is a trivialization of E on $\mathbf{C} \setminus \{\pm x\}$. Consider the anti-holomorphic involution η on $\mathrm{Gr}^{(2)}$ given by $\eta(x, \mathrm{E}, \phi) = (-\bar{x}, \eta^*\mathrm{E}, \eta^*\phi)$ defines a real form $\mathrm{Gr}_{\mathbf{R}}^{(2)} \rightarrow i\mathbf{R}$ such that

$$\begin{array}{ccccc} \mathrm{Gr} \times i\mathbf{R}_{>0} & \xrightarrow{a} & \mathrm{Gr}_{\mathbf{R}}^{(2)} & \xleftarrow{b} & \mathrm{Gr}_{\mathbf{R}} \\ \downarrow & & \downarrow & & \downarrow \\ i\mathbf{R}_{>0} & \longrightarrow & i\mathbf{R} & \longleftarrow & \{0\} \end{array}$$

Consider the specialization functor

$$(6) \quad \psi : \mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathrm{Gr}) \rightarrow \mathrm{D}_{\mathfrak{L}^+\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathbf{R}}) \quad \psi(\mathrm{F}) = b^* a_*(\mathrm{F} \boxtimes \mathbf{C}_{i\mathbf{R}_{>0}}).$$

¹An object F is called locally compact if its image under the forgetful functor $\mathrm{For} : \mathrm{D}_{\mathfrak{L}^+\mathrm{G}}(\mathrm{Gr}) \rightarrow \mathrm{D}(\mathrm{Gr})$ is compact.

Let $\text{Perv}_{\mathfrak{L}^+G}(\text{Gr}) \subset D_{\mathfrak{L}^+G}^b(\text{Gr})$ be the category of $\mathfrak{L}^+G$ -equivariant perverse sheaves on Gr . The geometric Satake equivalence says that there is an isomorphism of Tannakian category

$$(7) \quad \text{Perv}_{\mathfrak{L}^+G}(\text{Gr}) \cong \text{Rep}(\check{G})$$

where $\check{G} = \text{GL}_N$ is the dual group of G such that the intersection cohomology complex IC^λ for Gr^λ maps to the irreducible representations V_λ of highest weight λ . For $V \in \text{Rep}(\check{G})$, write IC_V for its corresponding Satake sheaf. According to [N, Lemma 3.9.1], we have

$$\dim_{\mathbf{R}}(\text{Gr}_{\mathbf{R}}^\lambda) = \langle \lambda, 2\rho \rangle = 2 \sum_{i=1}^p (N+1-2i)\lambda_i \in 2\mathbb{Z}.$$

Thus there is perverse t -structure on the real Satake category $D_{\mathfrak{L}^+G_{\mathbf{R}}}^b(\text{Gr}_{\mathbf{R}})$ given by the middle perversity function

$$p_{\mathbf{R}} : \Lambda_A^+ \rightarrow \mathbb{Z}, \quad p_{\mathbf{R}}(\lambda) = -\frac{\dim_{\mathbf{R}}(\text{Gr}_{\mathbf{R}}^\lambda)}{2} = -\sum_{i=1}^p (N+1-2i)\lambda_i$$

We denote by $\text{Perv}_{\mathfrak{L}^+G_{\mathbf{R}}}(\text{Gr}_{\mathbf{R}})$ the corresponding category of $\mathfrak{L}^+G_{\mathbf{R}}$ -equivariant perverse sheaves on $\text{Gr}_{\mathbf{R}}$.

The Hecke action of $\text{Rep}(\check{G})$ on the real Satake category $D_{\mathfrak{L}^+G_{\mathbf{R}}}(\text{Gr}_{\mathbf{R}})$, denoted again by $*$, is defined as

$$\text{Rep}(\check{G}) \times D_{\mathfrak{L}^+G_{\mathbf{R}}}(\text{Gr}_{\mathbf{R}}) \rightarrow D_{\mathfrak{L}^+G_{\mathbf{R}}}(\text{Gr}_{\mathbf{R}}) \quad (V, F) \rightarrow V * F := \psi(\text{IC}_V) * F$$

Proposition 2.4.1. (1) *The specialization functor ψ , and hence the Hecke action, is monoidal. Moreover, the Hecke action is t -exact with respect to the perverse t -structure if and only if $G_{\mathbf{R}}$ is quasi-split, that is, when $G_{\mathbf{R}} = \text{U}(p, p)$ or $\text{U}(p, p+1)$.*

(2) *The Hecke action of $\text{Rep}(\check{G})$ preserves semisimplicity, that is, it maps semi-simple objects to semi-simple objects.*

Proof. Part (1) is [N, Theorem 1.2.2 and Corollary 10.6.2] and Part (2) is [CN, Corollary 1.5]. $\square$

We shall show that the real Satake category $D_{\mathfrak{L}^+G_{\mathbf{R}}}(\text{Gr}_{\mathbf{R}})$ is generated by $\text{IC}_{\mathbf{R}}^0$ under the Hecke action.

Lemma 2.4.2. *Any real dominant coweight $\lambda \in \Lambda_A^+$ is of the form $\lambda = \mu - \theta(\mu)$ for some $\mu \in \Lambda_T$*

Proof. The involution θ sends a coweight $\mu = (a_1, \dots, a_p, b_1, \dots, b_{q-p}, c_1, \dots, c_p) \in \Lambda_T \cong \mathbb{Z}^N$ to

$$\theta(\mu) = (c_p, \dots, c_1, b_1, \dots, b_{q-p}, a_p, \dots, a_1).$$

It follows that for any real dominant coweight $\lambda = (\lambda_1, \dots, \lambda_p, 0, \dots, 0, -\lambda_p, \dots, -\lambda_1) \in \Lambda_A^+$, we have $\lambda = \mu - \theta(\mu)$ where $\mu = (\lambda_1, \dots, \lambda_p, 0, \dots, 0) \in \Lambda_T$. The lemma follows. $\square$

Lemma 2.4.3. *The $\mathfrak{L}^+G_{\mathbf{R}}$ -stabilizers on $\text{Gr}_{\mathbf{R}}$ is connected.*

Proof. According to [N, Proposition 3.6.1], each $\mathfrak{L}^+G_{\mathbf{R}}$ -orbit $\text{Gr}_{\mathbf{R}}^\lambda$ is a $G_{\mathbf{R}}$ -equivariant vector bundle over a real flag manifold $G_{\mathbf{R}}/P_{\mathbf{R},\lambda}$, where $P_{\mathbf{R},\lambda}$ is a real parabolic subgroup depending on λ . Thus it suffices to check that $P_{\mathbf{R},\lambda}$ is connected. Since a real parabolic is connected if and only if its Levi subgroup

is connected, we reduce to the latter assertion. The Levi subgroups of $G_{\mathbf{R}}$ are isomorphic to

$$U(p-a, q-a) \times \prod_i \mathrm{GL}_{n_i}(\mathbb{C}), \quad a = \sum_i n_i \leq p,$$

which is connected. $\square$

Proposition 2.4.4. (1) *The irreducible objects of $\mathrm{Perv}_{\mathfrak{L}+G_{\mathbf{R}}}(\mathrm{Gr}_{\mathbf{R}})$ are given by the intersection cohomology complex $\mathrm{IC}_{\mathbf{R}}^{\lambda}$ for $\mathrm{Gr}_{\mathbf{R}}^{\lambda}$, $\lambda \in \Lambda_{\mathbf{A}}^+$.*

(2) *The real Satake category $D_{\mathfrak{L}+G_{\mathbf{R}}}(\mathrm{Gr}_{\mathbf{R}})$ is generated by $\mathrm{IC}_{\mathbf{R}}^0$ under the Hecke action of $\mathrm{Rep}(\check{G})$*

Proof. Lemma 2.4.3 implies that $D_{\mathfrak{L}+G_{\mathbf{R}}}(\mathrm{Gr}_{\mathbf{R}})$ is generated by IC-complexes $\mathrm{IC}_{\mathbf{R}}^{\lambda}$ and Lemma 2.4.2 together with [N, Corollary 10.6.2] implies, up to shifts, all the IC-complexes appear as a direct summand of $V \star \mathrm{IC}_{\mathbf{R}}^0$ for some $V \in \mathrm{Rep}(\check{G})$. The corollary follows. $\square$

It is shown in [CY, Theorem 3.6] that the collection of spherical orbits $\{\mathfrak{LX}^{\lambda}\}_{\lambda \in \Lambda_{\mathbf{A}}^+}$ provides a placid stratification of $\mathfrak{LX}$ (see [CY, Appendix A] for a review of placidness). Thus by [CN, Proposition 8.12.], there is a unique perverse t -structure on $D_{\mathfrak{L}+G}(\mathfrak{LX})$ associated to the perversity function

$$p_X : \Lambda_{\mathbf{A}}^+ \rightarrow \mathbb{Z}, \quad p_X(\lambda) = -\langle \lambda, \rho \rangle$$

characterized by the property that

$${}^p D_{\mathfrak{L}+G}^{\geq 0}(\mathfrak{LX}) = \{F \in D_{\mathfrak{L}+G}(\mathfrak{LX}) \mid i_{\lambda}^!(F) \in {}^p D_{\mathfrak{L}+G}^{\geq 0}(\mathfrak{LX}^{\lambda})[p_X(\lambda)]\}$$

here $i_{\lambda} : \mathfrak{LX}^{\lambda} \rightarrow \mathfrak{LX}$ is the embedding and ${}^p D_{\mathfrak{L}+G}^{\geq 0}(\mathfrak{LX}^{\lambda})$ is the positive part of the !-adapted perverse t -structure on the placid scheme $\mathfrak{LX}^{\lambda}$. We denote by $\mathrm{Perv}_{\mathfrak{L}+G}(\mathfrak{LX})$ the corresponding perverse heart on $D_{\mathfrak{L}+G}^b(\mathfrak{LX})$. For any $\lambda \in \Lambda_{\mathbf{A}}^+$, we denote by IC_X^{λ} the IC-complex for the orbit $\mathfrak{LX}^{\lambda}$. We have $i_{\lambda}^!(\mathrm{IC}_X^{\lambda}) \cong \omega_{\mathfrak{LX}^{\lambda}}[p_X(\lambda)]$, where $\omega_{\mathfrak{LX}^{\lambda}}$ is the dualizing complex of $\mathfrak{LX}^{\lambda}$.

Recall the following real-symmetric equivalence [CN, Theorem 1.1]:

Proposition 2.4.5. *There is a natural equivalence*

$$D_{\mathfrak{L}+G}(\mathfrak{LX}) \cong D_{\mathfrak{L}+G_{\mathbf{R}}}(\mathrm{Gr}_{\mathbf{R}})$$

which is exact with respect to the perverse t -structure and is compatible with the monoidal action of the complex Satake category $D_{\mathfrak{L}+G}(\mathrm{Gr})$. In particular, the above equivalence gives rise to an equivalence between perverse sheaves

$$\mathrm{Perv}_{\mathfrak{L}+G}(\mathfrak{LX}) \cong \mathrm{Perv}_{\mathfrak{L}+G_{\mathbf{R}}}(\mathrm{Gr}_{\mathbf{R}}).$$

Consider the Hecke action of $\mathrm{Rep}(\check{G})$ on $D_{\mathfrak{L}+G}(\mathfrak{LX})$ given by

$$\mathrm{Rep}(\check{G}) \times D_{\mathfrak{L}+G}(\mathfrak{LX}) \rightarrow D_{\mathfrak{L}+G}(\mathfrak{LX}), \quad V \star F = \mathrm{IC}_V \star F$$

The following follows directly from Proposition 2.4.1, Proposition 2.4.4 and Proposition 2.4.5:

Proposition 2.4.6. (1) *The irreducible objects of $\mathrm{Perv}_{\mathfrak{L}+G}(\mathfrak{LX})$ are given by the intersection cohomology complex IC_X^{λ} , $\lambda \in \Lambda_{\mathbf{A}}^+$.*

(2) *The Hecke action on $D_{\mathfrak{L}+G}(\mathfrak{LX})$ is t -exact if and only if X is quasi-split, that is, if $X \cong \mathrm{GL}_{2p}/(\mathrm{GL}_p \times \mathrm{GL}_p)$ or $\mathrm{GL}_{2p+1}/(\mathrm{GL}_p \times \mathrm{GL}_{p+1})$.*

(3) The relative Satake category $D_{\mathfrak{L}+\mathfrak{G}}(\mathfrak{L}X)$ is generated by $IC_X^0 \cong \omega_{\mathfrak{L}+X}$ under the Hecke action of $\text{Rep}(\check{G})$.

3. SPECTRAL SIDE

3.1. Hamiltonian dual varieties. We recall the construction in [BZSV] of the Hamiltonian dual variety of $X = \text{GL}_{p+q}/(\text{GL}_p \times \text{GL}_q)$. Put $N = p + q$, $\delta = q - p$, and $\check{G} = \text{GL}_N$. Throughout the spectral calculations, choose the lower triangular Borel $\check{B}$, with diagonal torus $\check{T}$ and unipotent radical $\check{U}$, with simple root vectors $E_{i+1,i}$. For a block of size m , set

$$\tau_m(t) = \text{diag}(t^{1-m}, t^{3-m}, \dots, t^{m-1}), \quad e_m = \sum_{i=1}^{m-1} i(m-i)E_{i+1,i}, \quad f_m = \sum_{i=1}^{m-1} E_{i,i+1}.$$

Thus $(e_m, [e_m, f_m], f_m)$ is a principal $\mathfrak{sl}_2$ -triple and τ_m is the sum of the positive coroots for the lower triangular Borel.

Let $\check{L} = \text{I}_{2p} \times \text{GL}_\delta \subset \check{G}$ be the Levi subgroup, and write

$$(8) \quad \lambda(t) = 2\rho_{\check{L}}(t) = \text{diag}(\text{I}_{2p}, \tau_\delta(t)).$$

For $\delta = 0$, put $\lambda = 1$. Choose the principal homomorphism $\iota_{\check{L}} : \text{SL}_2 \rightarrow \check{L}$ whose differentials on the upper and lower root vectors are e_δ and f_δ , respectively, and whose restriction to the diagonal torus is λ . For $\delta \leq 1$ this homomorphism is trivial. Let $\check{P}_X = \check{L}_X \check{N}_X$ be the parabolic determined by λ : its Lie algebra is the sum of the nonnegative weight spaces, $\check{L}_X = Z_{\check{G}}(\lambda)$, and $\text{Lie } \check{N}_X$ is the sum of the positive weight spaces. Explicitly,

$$\check{L}_X \cong \begin{cases} \text{GL}_{2p} \times \text{GL}_1^{q-p}, & p+q \in 2\mathbb{Z}, \\ \text{GL}_{2p+1} \times \text{GL}_1^{q-p-1}, & p+q \in 2\mathbb{Z} + 1. \end{cases}$$

We will use the distinguished commuting subgroup

$$\check{L}'_X = \text{GL}_{2p} \times \text{I}_{q-p} \cong \text{GL}_{2p}.$$

Let $V = \mathbb{C}^{2p}$, with standard basis $\mathbf{e}_1, \dots, \mathbf{e}_{2p}$ and the symplectic form characterized by $\omega(\mathbf{e}_i, \mathbf{e}_{2p+1-i}) = 1$ for $1 \leq i \leq p$, skew-symmetry, and zero on all other pairs of basis vectors. The dual group of X is

$$\check{G}_X = \text{Sp}(V) \subset \check{L}'_X \subset \check{G}.$$

The block GL_{2p} commutes with $\iota_{\check{L}}(\text{SL}_2)$, so

$$\rho_X : \check{G}_X \times \text{SL}_2 \longrightarrow \check{G}, \quad (g, h) \longmapsto g \iota_{\check{L}}(h)$$

is the Arthur homomorphism associated to X .

Denote by $\check{\mathfrak{g}}, \check{\mathfrak{g}}_X, \check{\mathfrak{n}}_X$, etc, the Lie algebra of $\check{G}, \check{G}_X, \check{N}_X$, etc. Let $\Phi_{\text{tr}} : \check{\mathfrak{g}} \cong \check{\mathfrak{g}}^*$ be the $\check{G}$ -equivariant isomorphism induced by the trace paring. Let $e_X = d\iota_{\check{L}}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$ and $y_X = d\iota_{\check{L}}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$, and $\psi_X = \Phi_{\text{tr}}(y_X) \in \check{\mathfrak{g}}^*$. Denote by $\check{\mathfrak{n}}_{X, \geq 2} \subset \check{\mathfrak{n}}_X$ be the ≥ 2 weight subspace with respect to the action of λ . Put $\check{N}_{X, \geq 2} = \exp(\check{\mathfrak{n}}_{X, \geq 2})$. Following [BZSV, Section 3.4.2], we will consider the following three Hamiltonian varieties:

- (1) the cotangent bundle $T^*\check{G}$ viewed as $\check{G} \times \check{G}$ -Hamiltonian space with moment map $T^*\check{G} \cong \check{G} \times \check{\mathfrak{g}}^* \rightarrow \check{\mathfrak{g}}^* \times \check{\mathfrak{g}}^*$ sending $(g, v) \rightarrow (\text{Ad}_g v, -v)$ (here we identify $\check{\mathfrak{g}}^*$ as left invariant differential forms on $\check{G}$).
- (2) the symplectic representation $\check{S}_X$ of $\check{G}_X = \text{Sp}_{2p}$ given by

$$\check{S}_X = \begin{cases} T^*\mathbb{A}^{2p} & p = q, \\ 0 & p \neq q. \end{cases}$$

where $T^*\mathbb{A}^{2p}$ is the cotangent bundle of the standard representation $\mathbb{A}^{2p}$ of Sp_{2p} , viewed as $\check{G}_X$ -Hamiltonian space with the standard moment map $\check{S}_X \rightarrow \check{\mathfrak{g}}_X^* = \mathfrak{sp}_{2p}^*$ sending $s \in \check{S}_X$ to $(v \rightarrow \frac{1}{2}\omega_{\check{S}_X}(vs, s)) \in \mathfrak{sp}_{2p}^*$ (here $\omega_{\check{S}_X}$ is the canonical symplectic form on $T^*\mathbb{A}^{2p}$)

- (3) the quotient space $\check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2}$ with the symplectic form $\omega_{\check{\mathfrak{n}}_X}(u, u') = \psi_X([u, u'])$ and the Hamiltonian $\check{G}_X\check{N}_X$ -action given by
 - (i) the $\check{G}_X$ -action is through the adjoint action
 - (ii) the $\check{N}_X$ -action is through the translation action $\check{N}_X/\check{N}_{X,\geq 2} \cong \check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2}$
 - (iii) the $\check{G}_X$ -moment map is given by

$$\check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2} \rightarrow \text{Lie}(\text{Sp}(\check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2}))^* \rightarrow \check{\mathfrak{g}}_X^*$$

where the first map is the canonical moment map on the symplectic vector space and the second map is induced by the inclusion $\check{G}_X \rightarrow \text{Sp}(\check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2})$

- (iv) the $\check{N}_X$ -moment map is given by ψ_X -shifted moment map

$$\check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2} \cong (\check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2})^* \rightarrow \check{\mathfrak{n}}_X^* \quad v \rightarrow v + \psi_X|_{\check{\mathfrak{n}}_X}$$

where the first isomorphism is induced by the symplectic form.

Example 3.1.1. (1) Assume $p + q$ is odd. Then we have $\check{S}_X = 0$. Moreover, it follows from (8) that the weights with respect to co-character $2\rho_{\check{L}}$ are all even numbers and it follows that $\check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2} = 0$

(2) Assume $p + q$ is even. If $p = q$, then we have $\check{S}_X \cong T^*\mathbb{A}^{2p}$ and $\check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2} = 0$. If $p \neq q$, then we have $\check{S}_X = 0$ and a direct computation show that Sp_{2p} -equivariant isomorphism

$$\check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2} \cong \check{\mathfrak{n}}_{X,1} \cong T^*\mathbb{A}^{2p}.$$

where $\check{\mathfrak{n}}_{X,1}$ is the weight one subspace.

The product space

$$T^*\check{G} \times \check{S}_X \times \check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2}$$

is naturally a Hamiltonian $\check{G}_X\check{N}_X$ -space, where $\check{N}_X$ acts trivially on $\check{S}_X$ and $\check{G}_X\check{N}_X \subset \check{G}$ acts on $T^*\check{G}$ through the right $\check{G}$ -action, with moment map

$$\mu_{\psi_X} : T^*\check{G} \times \check{S}_X \times \check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2} \rightarrow \check{\mathfrak{g}}_X^* \times \check{\mathfrak{n}}_X^*$$

and the dual variety of X is defined as the following Hamiltonian reduction

$$\check{M} = \mu_{\psi_X}^{-1}(0) // \check{G}_X\check{N}_X = T^*\check{G} \times_{\check{\mathfrak{g}}_X^* \times \check{\mathfrak{n}}_X^*}^{\check{G}_X\check{N}_X} (\check{S}_X \times \check{\mathfrak{n}}_X/\check{\mathfrak{n}}_{X,\geq 2})$$

We have the residual moment map

$$\mu : \check{M} \rightarrow \check{\mathfrak{g}}^*$$

coming from the left $\check{G}$ -action on $T^*\check{G}$. Moreover, μ is $\mathbb{G}_m$ -equivariant with respect to the square action on $\check{\mathfrak{g}}^*$ and the $\mathbb{G}_m$ on $\check{M}$ induced by

- (1) the scalar action on $\check{S}_X$ and $\check{n}_X/\check{n}_{X,\geq 2}$
- (2) the action on $T^*\check{G} \cong \check{G} \times \check{\mathfrak{g}}^*$ given by $t(g, v) = (g2\rho_L(t^{-1}), t^2\text{Ad}_{2\rho_L(t)}(v))$.

Example 3.1.2. (1) Assume $p + q$ is odd. Then it follows from Example (3.1.1) that

$$\check{M} \cong T_{\psi_X}^*(\text{GL}_{p+q}/\text{Sp}_{2p}\check{N}_X) \cong \text{GL}_{p+q} \times^{\text{Sp}_{2p}\check{N}_X} (\psi_X + (\mathfrak{sp}_{2p} \oplus \check{n}_X)^\perp)$$

is isomorphic to the ψ_X -twisted cotangent bundle of $\text{GL}_{p+q}/\text{Sp}_{2p}\check{N}_X$ with respect to the covector $\psi_X|_{\mathfrak{sp}_{2p} \oplus \check{n}_X} \in (\mathfrak{sp}_{2p}^* \oplus \check{n}_X^*)^{\text{Sp}_{2p}\check{N}_X}$, and the moment map is given by

$$\mu : \check{M} \rightarrow \check{\mathfrak{g}}^*, \quad [g, v] \mapsto \text{Ad}_g v.$$

Moreover, if $q = p + 1$, then both $\check{N}_X$ and ψ_X are trivial, and

$$\check{M} \cong T^*(\text{GL}_{2p+1}/\text{Sp}_{2p}) \cong \text{GL}_{2p+1} \times^{\text{Sp}_{2p}} \mathfrak{sp}_{2p}^\perp$$

is isomorphic to the cotangent bundle of $\text{GL}_{2p+1}/\text{Sp}_{2p}$; here $\mathfrak{sp}_{2p}^\perp$ is taken in $\mathfrak{gl}_{2p+1}$.

- (2) Assume $p = q$. Then $\check{N}_X$ is trivial, and Hamiltonian induction identifies

$$\check{M} \simeq T^*(\text{GL}_{2p}/\text{Sp}_{2p} \times V) \simeq \text{GL}_{2p} \times^{\text{Sp}_{2p}} (\mathfrak{p}_{2p} \times V \times V), \quad \mathfrak{p}_{2p} = \mathfrak{sp}_{2p}^\perp \subset \mathfrak{gl}_{2p}.$$

Use ω to identify V^* with V , and put $R_{u,v}(z) = \omega(v, z)u$. In the product cotangent coordinates the moment map is

$$(9) \quad \mu([g, T, u, v]) = \text{Ad}_g(T + R_{u,v}), \quad T \in \mathfrak{p}_{2p}.$$

For comparison with the trace-orthogonal coordinates in the Hamiltonian reduction, set

$$\beta(u, v) = \text{pr}_{\mathfrak{sp}_{2p}} R_{u,v} = \frac{1}{2}(R_{u,v} + R_{v,u}), \quad A = T + \text{pr}_{\mathfrak{p}_{2p}} R_{u,v}.$$

Then $T + R_{u,v} = A + \beta(u, v)$. Both coordinate descriptions give the grading $t[g, A, u, v] = [g, t^2 A, tu, tv]$.

3.2. Vector bundle structures. Consider the $\check{G}_X \times \mathbb{G}_m$ -representation

$$\check{V}_X = \check{S}_X \times (\check{\mathfrak{g}}_X^\perp \cap Z_{\check{\mathfrak{g}}}(e_X)).$$

The action on $\check{S}_X$ is scalar multiplication, and the action on the second factor is $x \mapsto t^2 \text{Ad}_{\lambda(t)} x$. The construction of [BZSV, Section 3.4.8, (3.18)–(3.19)] gives an isomorphism

$$(10) \quad \check{M} \simeq \check{G} \times^{\check{G}_X} \check{V}_X.$$

Here and below our associated-bundle convention is $(g, x) \sim (gh^{-1}, hx)$.

In terms of the isomorphism above the moment map is given as follows. If $\delta = 0$, then $\check{V}_X \cong \mathfrak{p}_{2p} \times V \times V$ and the moment map is described in Example 3.1.2 (2). If $\delta > 0$, then $\check{S}_X = 0$ and we set

$$\mathcal{Z}_\delta = f_\delta + (\check{\mathfrak{g}}_X^\perp \cap Z_{\check{\mathfrak{g}}}(e_\delta)),$$

where the two nilpotents occupy the final δ -block. Translation by f_δ identifies $\mathcal{Z}_\delta$ with $\check{V}_X$, and

$$(11) \quad \check{M} \simeq \check{G} \times^{\check{G}_X} \mathcal{Z}_\delta, \quad \mu([g, x]) = \text{Ad}_g x.$$

The grading is

$$(12) \quad t[g, x] = [g\lambda(t)^{-1}, t^2 \text{Ad}_{\lambda(t)} x].$$

Lemma 3.2.1. *There is a $\check{G}_X \times \mathbb{G}_m$ -equivariant isomorphism*

$$\check{V}_X \simeq \wedge^2 V \times T^*V \times \mathfrak{c}_{\mathrm{GL}_\delta}.$$

The degrees on the first two factors are 2 and $\delta + 1$, respectively. The coefficient of order i of the monic polynomial parametrized by $\mathfrak{c}_{\mathrm{GL}_\delta}$ has degree $2i$.

Proof. For $\delta = 0$, this is the decomposition $\check{V}_X = \mathfrak{p}_{2p} \times T^*V$, with $\mathfrak{p}_{2p} \simeq \wedge^2 V$. For $\delta > 0$, the nilpotent has the block form

$$e_X = \begin{pmatrix} 0_{2p} & 0 \\ 0 & e_\delta \end{pmatrix}, \quad e_\delta = \sum_{i=1}^{\delta-1} i(\delta-i)E_{i+1,i}.$$

Let $\varepsilon_1, \dots, \varepsilon_\delta$ be the standard basis of $\mathbb{C}^\delta$. The two off-diagonal centralizer spaces are

$$\begin{aligned} D &= \{d \in M_{2p,\delta} : de_\delta = 0\} = \{b\varepsilon_1^t : b \in V\}, \\ C &= \{c' \in M_{\delta,2p} : e_\delta c' = 0\} = \{\varepsilon_\delta c : c \in V^*\}. \end{aligned}$$

Here $M_{a,b}$ denotes the space of $a \times b$ matrices. Thus

$$\begin{aligned} (13) \quad \check{V}_X &= \check{\mathfrak{g}}_X^\perp \cap Z_{\check{\mathfrak{g}}}(e_X) = \begin{pmatrix} \mathfrak{p}_{2p} & D \\ C & Z_{\mathfrak{gl}_\delta}(e_\delta) \end{pmatrix} \\ &= \left\{ \begin{pmatrix} A & b\varepsilon_1^t \\ \varepsilon_\delta c & s \end{pmatrix} : A \in \mathfrak{p}_{2p}, b \in V, c \in V^*, s \in Z_{\mathfrak{gl}_\delta}(e_\delta) \right\}. \end{aligned}$$

Thus the matrix x in (11) is

$$(14) \quad x(A, b, c, s) = \begin{pmatrix} A & b\varepsilon_1^t \\ \varepsilon_\delta c & f_\delta + s \end{pmatrix}.$$

Since e_δ is regular nilpotent, its Lie algebra centralizer $\mathcal{S}_\delta$ has basis $I_\delta, e_\delta, \dots, e_\delta^{\delta-1}$. The grading gives degrees 2 on A and $\delta + 1$ on b, c . The coefficient of e_δ^{i-1} has degree $2i$. The Kostant section $f_\delta + Z_{\mathfrak{gl}_\delta}(e_\delta) \simeq \mathfrak{c}_{\mathrm{GL}_\delta}$ therefore identifies its grading with the characteristic-polynomial degrees $2, 4, \dots, 2\delta$, proving the assertion. $\square$

Proposition 3.2.2. *There is an isomorphism of varieties*

$$\check{M} \simeq \mathrm{GL}_{p+q} \times^{\mathrm{GL}_{2p}} T^*(\mathrm{GL}_{2p}/\mathrm{Sp}_{2p} \times V) \times \mathfrak{c}_{\mathrm{GL}_\delta}.$$

It is $\check{G} \times \mathbb{G}_m$ -equivariant for the action transported from (12) and the fiber degrees in Lemma 3.2.1.

Proof. Combining (10), Lemma 3.2.1 and Example 3.1.2, we have the desired isomorphism

$$\check{M} \simeq \mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} (\mathfrak{p}_{2p} \times T^*V) \times \mathfrak{c}_{\mathrm{GL}_\delta} \simeq \mathrm{GL}_{p+q} \times^{\mathrm{GL}_{2p}} T^*(\mathrm{GL}_{2p}/\mathrm{Sp}_{2p} \times V) \times \mathfrak{c}_{\mathrm{GL}_\delta}.$$

$\square$

3.3. Invariant quotients and exotic nilpotent cones. Put $\delta = q - p$ and $V = \mathbb{A}^{2p}$. Under the symplectic identification $\wedge^2 V \simeq \mathfrak{p}_{2p}$, the space $\mathfrak{p}_{2p} = \mathfrak{sp}_{2p}^\perp$ consists of the symplectic-self-adjoint endomorphisms. Explicitly,

$$\mathfrak{p}_{2p} = \left\{ \begin{pmatrix} X & Y \\ Z & r(X) \end{pmatrix} : X, Y, Z \in \mathfrak{gl}_p, \quad r(Y) = -Y, \quad r(Z) = -Z \right\}.$$

Here $r(X) = J_p X^t J_p$ is reflection across the anti-diagonal, with J_p the anti-diagonal permutation matrix. The invariant ring for the diagonal action of Sp_{2p} on $\mathfrak{p}_{2p} \times V \times V$ is polynomial, with generators $b_1, \dots, b_p, c_0, \dots, c_{p-1}$ characterized by

$$\det(z - T) = P(z)^2, \quad P(z) = z^p + b_1 z^{p-1} + \dots + b_p, \quad c_j = -\omega(T^j u, v).$$

Here $(T, u, v) \in \mathfrak{p}_{2p} \times V \times V$. The quotient map is equidimensional; see [S] and [Kn2].

Recall that $\mathfrak{c}_{\mathrm{GL}_m} = \mathfrak{gl}_m // \mathrm{GL}_m$ is the coefficient space of monic polynomials of degree m , with origin z^m . Using the trace identification, let $\chi_m : \mathfrak{gl}_m^* \rightarrow \mathfrak{c}_{\mathrm{GL}_m}$ be the characteristic-polynomial quotient map. We use the coordinates $c_0, \dots, c_{p-1}$ for a second copy of $\mathfrak{c}_{\mathrm{GL}_p}$ by associating to them the polynomial

$$C(z) = z^p + c_0 z^{p-1} + \dots + c_{p-1}.$$

The final factor $\mathfrak{c}_{\mathrm{GL}_\delta}$ has coordinates $d_1, \dots, d_\delta$, with

$$D(z) = z^\delta + d_1 z^{\delta-1} + \dots + d_\delta;$$

for $\delta = 0$ this factor is a point and $D = 1$. Lemma 3.2.1 and Proposition 3.2.2 give the natural product presentations

$$(\mathfrak{p}_{2p} \times V \times V) // \mathrm{Sp}_{2p} \simeq \mathfrak{c}_{\mathrm{GL}_p} \times \mathfrak{c}_{\mathrm{GL}_p}, \quad \check{\mathcal{M}} // \check{\mathcal{G}} \simeq \mathfrak{c}_{\mathrm{GL}_p} \times \mathfrak{c}_{\mathrm{GL}_p} \times \mathfrak{c}_{\mathrm{GL}_\delta}.$$

In particular,

$$\mathcal{O}(\check{\mathcal{M}})^{\check{\mathcal{G}}} = \mathbb{C}[b_1, \dots, b_p, c_0, \dots, c_{p-1}, d_1, \dots, d_\delta].$$

The grading transported from $\check{\mathcal{M}}$ is

$$\deg b_i = 2i, \quad \deg c_j = 2(j + \delta + 1), \quad \deg d_i = 2i.$$

Denote these quotient maps by

$$\chi_{p,p} : \mathfrak{p}_{2p} \times V \times V \rightarrow \mathfrak{c}_{\mathrm{GL}_p} \times \mathfrak{c}_{\mathrm{GL}_p}, \quad \chi_{\check{\mathcal{M}}} : \check{\mathcal{M}} \rightarrow \mathfrak{c}_{\mathrm{GL}_p} \times \mathfrak{c}_{\mathrm{GL}_p} \times \mathfrak{c}_{\mathrm{GL}_\delta}.$$

We next construct sections of $\chi_{p,p}$ and $\chi_{\check{\mathcal{M}}}$ using the companion-matrix description. For $b = (b_1, \dots, b_p)$, take the companion matrix

$$\kappa_p(b) = \begin{pmatrix} 0 & 0 & \dots & 0 & -b_p \\ 1 & 0 & \dots & 0 & -b_{p-1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -b_2 \\ 0 & \dots & 0 & 1 & -b_1 \end{pmatrix}.$$

Lemma 3.3.1. (1) *The map*

$$\kappa_{p,p}(b, c) = \left(\begin{pmatrix} \kappa_p(b) & 0 \\ 0 & r(\kappa_p(b)) \end{pmatrix}, \mathbf{e}_1, (0, \dots, 0, -c_{p-1}, \dots, -c_0)^t \right)$$

is an algebraic section of $\chi_{p,p}$.

(2) *In the product fiber coordinates above, the map $\kappa_{\check{\mathcal{M}}} = \kappa_{p,p} \times \mathrm{id}_{\mathfrak{c}_{\mathrm{GL}_\delta}}$,*

$$\begin{aligned} \mathfrak{c}_{\mathrm{GL}_p} \times \mathfrak{c}_{\mathrm{GL}_p} \times \mathfrak{c}_{\mathrm{GL}_\delta} &\xrightarrow{\kappa_{\check{\mathcal{M}}}} \check{\mathcal{V}}_X \\ &\hookrightarrow \mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} \check{\mathcal{V}}_X \simeq \check{\mathcal{M}}, \end{aligned}$$

defines an algebraic section of $\chi_{\check{\mathcal{M}}}$, with the final factor omitted when $\delta = 0$.

Proof. Write $\kappa_{p,p}(b, c) = (T, u, v)$. The characteristic polynomial of T is $(z^p + b_1 z^{p-1} + \dots + b_p)^2$. Since $T^j u = \mathbf{e}_{j+1}$ for $0 \leq j < p$, we also have $-\omega(T^j u, v) = c_j$. This proves (1); part (2) follows immediately from part (1). $\square$

The moment map $\mu : \check{M} \rightarrow \mathfrak{gl}_{p+q}^*$ is GL_{p+q} -equivariant and hence gives rise to the following commutative diagram:

$$(15) \quad \begin{array}{ccc} \check{M} & \xrightarrow{\mu} & \mathfrak{gl}_{p+q}^* \\ \chi_{\check{M}} \downarrow & & \downarrow \chi_N \\ \mathfrak{c}_{\mathrm{GL}_p} \times \mathfrak{c}_{\mathrm{GL}_p} \times \mathfrak{c}_{\mathrm{GL}_\delta} & \xrightarrow{\bar{\mu}} & \mathfrak{c}_{\mathrm{GL}_{p+q}}. \end{array}$$

Let $\mathcal{N}_{\check{\mathfrak{g}}^*} = \chi_N^{-1}(0)$ be the nilpotent cone for $\mathfrak{gl}_{p+q}^*$, and put $\mathcal{N}'_{\mathrm{Sp}_{2p}} = \chi_{p,p}^{-1}(0)$, the exotic null-cone in this representation; see [Ka]. It follows from Proposition 3.2.2 that

$$\chi_{\check{M}}^{-1}(0) \simeq \mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} \mathcal{N}'_{\mathrm{Sp}_{2p}}.$$

Consider the preimage $\mu^{-1}(\mathcal{N}_{\check{\mathfrak{g}}^*}) \subset \check{M}$ of the nilpotent cone under the moment map.

Lemma 3.3.2. *The natural closed embedding*

$$\mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} \mathcal{N}'_{\mathrm{Sp}_{2p}} \simeq \chi_{\check{M}}^{-1}(0) \hookrightarrow \mu^{-1}(\mathcal{N}_{\check{\mathfrak{g}}^*})$$

induces a bijection on $\mathbb{C}$ -points.

Proof. Denote by $\bar{\mu}$ the lower arrow in (15). In view of the commutativity of that diagram, it suffices to show that the $\mathbb{C}$ -points of $\bar{\mu}^{-1}(0)$ consist only of the origin. By Lemma 3.3.1, we have

$$\bar{\mu}(b, c, d) = \chi_N \circ \mu \circ \kappa_{\check{M}}(b, c, d),$$

and $\bar{\mu}(b, c, d) = 0$ if and only if the characteristic polynomial of $\mu(\kappa_{\check{M}}(b, c, d))$ is z^{p+q} . Thus we reduce to checking that

$$\det(z - \mu(\kappa_{\check{M}}(b, c, d))) = z^{p+q} \iff (b, c, d) = 0.$$

Write $\kappa_{\check{M}}(b, c, d) = (T, u, v, d)$, put $K_b = \kappa_p(b)$, and set

$$R(c) = \begin{pmatrix} c_0 & \cdots & c_{p-1} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Then equations (9) and (14) imply

$$\mu(\kappa_{\check{M}}(b, c, d)) = \begin{cases} \begin{pmatrix} K_b + R(c) & 0 \\ 0 & r(K_b) \end{pmatrix}, & p = q, \\ \begin{pmatrix} K_b & 0 & E_{p,\delta} \\ 0 & r(K_b) & 0 \\ F_{\delta,p}(c) & 0 & f_\delta + s(d) \end{pmatrix}, & p < q. \end{cases}$$

Here $E_{p,\delta}$ is $p \times \delta$ with its only nonzero entry 1 in position (1, 1), and $F_{\delta,p}(c)$ is $\delta \times p$ with last row $(c_0, \dots, c_{p-1})$ and all other rows zero. For $\delta > 0$, $s(d) \in Z_{\mathfrak{gl}_\delta}(e_\delta)$ is the unique element such that $\det(z - f_\delta - s(d)) = D(z)$.

First, in both cases $r(K_b)$ is a direct summand of the moment matrix. If the moment matrix is nilpotent, then $r(K_b)$ is nilpotent. Since

$$\det(z - r(K_b)) = z^p + b_1 z^{p-1} + \cdots + b_p,$$

this forces $b = 0$.

With $b = 0$, expansion of the displayed determinants gives

$$\det(z - \mu(\kappa_{\check{M}}(0, c, d))) = z^p \left(z^p D(z) - \sum_{j=0}^{p-1} c_j z^{p-1-j} \right), \quad D = 1 \text{ if } \delta = 0.$$

Thus nilpotency is equivalent to

$$z^p D(z) - \sum_{j=0}^{p-1} c_j z^{p-1-j} = z^q.$$

The term $z^p D(z)$ has no powers of degree below p , so comparison of those coefficients gives $c = 0$. We then obtain $D(z) = z^{q-p} = z^\delta$, hence $d = 0$. Conversely, the same formula gives z^{p+q} when $(b, c, d) = 0$. This proves the lemma. $\square$

3.4. Whittaker cotangent bundles. Let $\check{K} = \mathrm{GL}_p \times \mathrm{GL}_q$ and let $\check{P}$ be the lower triangular parabolic with Levi $\check{K}$. Put

$$f_{p,q} = \mathrm{diag}(f_p, f_q), \quad \tau_{\check{K}}(t) = \mathrm{diag}(\tau_p(t), \tau_q(t)), \quad \tau_{\check{G}}(t) = \tau_N(t).$$

The additive character $\psi_{\check{K}} : \check{U} \rightarrow \mathbb{G}_a$ has differential $u \mapsto \mathrm{tr}(f_{p,q}u)$: it is 1 on the pinned simple root vectors in $\check{K}$ and 0 on the simple root vectors outside $\check{K}$. For the right action $g \mapsto gu^{-1}$, the Whittaker cotangent bundle is the Hamiltonian reduction at $-d\psi_{\check{K}}$:

$$T_{\psi_{\check{K}}}^*(\check{G}/\check{U}) \simeq \check{G} \times^{\check{U}} (f_{p,q} + \check{\mathfrak{b}})$$

with moment map sending $[g, x]$ to $\mathrm{Ad}_g x$. The grading relevant to the dual variety is defined by

$$(16) \quad \gamma(t) = \tau_{\check{K}}(t)^2 \tau_{\check{G}}(t)^{-1} = \tau_{\check{K}}(t) \mathrm{diag}(t^q \mathrm{I}_p, t^{-p} \mathrm{I}_q),$$

and

$$(17) \quad t[g, x] = [g\gamma(t)^{-1}, t^2 \mathrm{Ad}_{\gamma(t)} x].$$

3.5. Regular centralizers. We recall some facts and constructions related to the group scheme of regular centralizers (see, e.g., [BF, Kn1, Ngo, Gi, L]). Let $J_{\check{G}} \rightarrow \mathfrak{c}_{\check{G}}$ (resp. $J_{\check{K}} \rightarrow \mathfrak{c}_{\check{K}}$) be the group scheme of regular centralizers of $\check{G}$ (resp. $\check{K}$). Choose the homogeneous Kostant sections $\kappa_{\check{G}} : \mathfrak{c}_{\check{G}} \rightarrow \check{\mathfrak{g}}^*$ and $\kappa_{\check{K}} : \mathfrak{c}_{\check{K}} \rightarrow \check{\mathfrak{k}}^*$, whose images on a block of size m are $f_m + Z_{\mathfrak{gl}_m}(e_m)$. They give rise to closed embeddings

$$J_{\check{G}} \longrightarrow \mathfrak{c}_{\check{G}} \times \check{G}, \quad J_{\check{K}} \longrightarrow \mathfrak{c}_{\check{K}} \times \check{K}.$$

The fiber over x is the centralizer of $\kappa_{\check{G}}(x)$, respectively of $\kappa_{\check{K}}(x)$. Moreover, we have natural isomorphisms

$$\mathrm{Lie} J_{\check{G}} \simeq T^* \mathfrak{c}_{\check{G}}, \quad \mathrm{Lie} J_{\check{K}} \simeq T^* \mathfrak{c}_{\check{K}}.$$

For $A = \check{G}, \check{K}$, the chosen sections satisfy

$$\kappa_A(t^2 x) = t^2 \mathrm{Ad}_{\tau_A(t)} \kappa_A(x),$$

where $t^2 x$ denotes square dilation on the Chevalley quotient.

The embedding $\check{K} \rightarrow \check{G}$ gives rise to the map between Chevalley quotients

$$\pi : \mathfrak{c}_{\check{K}} = \check{\mathfrak{k}}^* // \check{K} \longrightarrow \check{\mathfrak{g}}^* // \check{G} = \mathfrak{c}_{\check{G}},$$

and hence to the cotangent map

$$\pi^* \text{Lie } J_{\check{G}} \simeq \pi^* T^* \mathfrak{c}_{\check{G}} \longrightarrow T^* \mathfrak{c}_{\check{K}} \simeq \text{Lie } J_{\check{K}}.$$

As in [L] and [Gi, Proposition 5.4.1], this map is the differential of a homomorphism

$$(18) \quad \alpha : \pi^* J_{\check{G}} \longrightarrow J_{\check{K}}$$

of group schemes over $\mathfrak{c}_{\check{K}}$. Note that π is generically étale and the map (18) is generically an isomorphism.

Using the above homomorphism, we can form the quotient

$$(\check{G} \times J_{\check{K}}) / \pi^* J_{\check{G}} = ((\check{G} \times \mathfrak{c}_{\check{K}}) \times_{\mathfrak{c}_{\check{K}}} J_{\check{K}}) / \pi^* J_{\check{G}}$$

by the group scheme $\pi^* J_{\check{G}}$ over $\mathfrak{c}_{\check{K}}$. Its action on the two factors is through the Kostant-section embedding and the homomorphism (18), respectively; explicitly,

$$(g, x, y) \cdot j = (gj^{-1}, x, \alpha(j)y).$$

We have a natural $\check{G} \times \mathbb{G}_m$ -action on this quotient, given by

$$(h, t)[g, x, y] = [hg\tau_{\check{G}}(t)^{-1}, t^2x, \text{Ad}_{\tau_{\check{K}}(t)}y],$$

and a $\check{G} \times \mathbb{G}_m$ -equivariant map

$$(19) \quad (\check{G} \times J_{\check{K}}) / \pi^* J_{\check{G}} \longrightarrow \check{\mathfrak{g}}^*, \quad [g, x, y] \longmapsto \text{Ad}_g \kappa_{\check{G}}(\pi(x)).$$

The morphism (19) extends to a $\check{G} \times \mathbb{G}_m$ -equivariant map $\overline{(\check{G} \times J_{\check{K}}) / \pi^* J_{\check{G}}} \rightarrow \check{\mathfrak{g}}^*$.

Proposition 3.5.1. *There is a $\check{G} \times \mathbb{G}_m$ -equivariant isomorphism of affine varieties over $\check{\mathfrak{g}}^*$*

$$\overline{T^*_{\psi_{\check{K}}}(\check{G}/\check{U})} \simeq \overline{(\check{G} \times J_{\check{K}}) / \pi^* J_{\check{G}}},$$

with the grading (17) on the left.

Proof. According to [Ga, Theorem 1.1], there is a $\check{G} \times \check{K}$ -equivariant isomorphism over $\check{\mathfrak{g}}^*$

$$\overline{T^*(\check{G}/\check{U}_{\check{P}})} \simeq \overline{(\check{G} \times \check{K} \times \mathfrak{c}_{\check{K}}) / \pi^* J_{\check{G}}},$$

where $\check{U}_{\check{P}} \subset \check{P}$ is the unipotent radical. Taking Whittaker reduction with respect to $\check{K}$ and $\psi_{\check{K}}$, we obtain the desired isomorphism

$$\begin{aligned} \overline{T^*_{\psi_{\check{K}}}(\check{G}/\check{U})} &\simeq \overline{T^*(\check{G}/\check{U}_{\check{P}}) //_{-d\psi_{\check{K}}} \check{U}_{\check{K}}} \\ &\simeq \overline{(\check{G} \times J_{\check{K}}) / \pi^* J_{\check{G}}}. \end{aligned}$$

Here $\check{U}_{\check{K}} = \check{U} \cap \check{K}$, the differential is restricted to $\text{Lie } \check{U}_{\check{K}}$, and the sign is that of the right action $g \mapsto gh^{-1}$. The first isomorphism follows from [M, Lemma 3.1.10]; the second is the Whittaker reduction of Gannon's theorem, as in [Ga, Corollary 2.8]. $\square$

3.6. Dual varieties as affine closures. Here is the main result of the section, providing an alternative description of the dual variety $\check{M}$ as an affine closure of $T_{\psi_K}^*(\check{G}/\check{U})$.

Theorem 3.6.1. *There is a $\check{G} \times \mathbb{G}_m$ -equivariant isomorphism of affine varieties over $\check{\mathfrak{g}}^*$*

$$\check{M} \simeq \overline{T_{\psi_K}^*(\check{G}/\check{U})}.$$

The rest of the section is devoted to the proof of Theorem 3.6.1. The key ingredients include a generalization of Kostant's theorem, see Section 3.7, and a Lagrangian correspondence between $T_{\psi_K}^*(\check{G}/\check{U})$ and $\check{M}$, see Section 27.

Since the Hamiltonian dual variety $\check{M}$ is smooth, as a corollary of Theorem 3.6.1 and Proposition 3.5.1 we obtain the following smoothness of the affine closures.

Corollary 3.6.2. *The affine closures*

$$\overline{T_{\psi_K}^*(\check{G}/\check{U})} \simeq \overline{(\check{G} \times J_K)/\pi^* J_{\check{G}}}$$

are smooth.

3.7. Generalized Kostant theorem. For $m \geq 1$, let $\check{B}_m \subseteq \mathrm{GL}_m$ be the lower triangular Borel, let $\check{U}_m$ be its unipotent radical, and write $\check{\mathfrak{b}}_m$ and $\check{\mathfrak{u}}_m$ for their Lie algebras. Consider the regular nilpotent matrices

$$e_m = \begin{pmatrix} 0 & & & & \\ m-1 & & 0 & & \\ & 2(m-2) & 0 & & \\ & & \ddots & \ddots & \\ & & & m-1 & 0 \end{pmatrix}, \quad f_m = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

Put

$$\tau_m(t) = \mathrm{diag}(t^{1-m}, t^{3-m}, \dots, t^{m-1}).$$

Thus $(e_m, d\tau_m(1), f_m)$ is a principal $\mathfrak{sl}_2$ -triple. The celebrated result of Kostant [K] asserts that $\check{U}_m$ acts freely on $\check{\mathfrak{b}}_m + f_m$ and provides a transverse slice to the $\check{U}_m$ -orbits. More precisely, let $\mathcal{S}_m = Z_{\mathfrak{gl}_m}(e_m) \subseteq \check{\mathfrak{b}}_m$. Then the action map

$$\check{U}_m \times (\mathcal{S}_m + f_m) \longrightarrow \check{\mathfrak{b}}_m + f_m, \quad (u, x) \longmapsto \mathrm{Ad}_u(x),$$

is a $\check{U}_m$ -equivariant isomorphism.

The main result of this section is an analog of Kostant's theorem in which f_m is replaced by a nilpotent element with two Jordan blocks. Namely, consider the following nilpotent elements

$$e_{p,q} = \begin{pmatrix} e_p & 0 \\ 0 & e_q \end{pmatrix}, \quad f_{p,q} = \begin{pmatrix} f_p & 0 \\ 0 & f_q \end{pmatrix}.$$

The second matrix is the trace representative of $d\psi_K$ used in Section 3.4. We have a natural adjoint action of $\check{U}$ on $\check{\mathfrak{b}} + f_{p,q}$ but, unlike the setting of Kostant theorem, this action is typically not free. We shall introduce a subgroup $\check{U}_X \subseteq \check{U}$ and a $\check{U}_X$ -invariant affine subspace $\mathcal{S}_{p,q} + f_{p,q} \subseteq \check{\mathfrak{b}} + f_{p,q}$. The generalized Kostant theorem, see Theorem 3.7.5 below, says that the adjoint action map factors through an isomorphism

$$(20) \quad \check{U} \times^{\check{U}_X} (\mathcal{S}_{p,q} + f_{p,q}) \simeq \check{\mathfrak{b}} + f_{p,q}.$$

3.7.1. *Symplectic linear algebra.* We record some linear algebra results needed for the proof of (20). We focus on the case $p = q$ and write $\check{G} = \mathrm{GL}_{2p}(\mathbf{C})$. Let $V = \mathbf{C}^{2p}$, and let $\mathbf{e}_1, \dots, \mathbf{e}_{2p}$ be its standard basis. Equip V with the symplectic form $\omega(\mathbf{e}_i, \mathbf{e}_{2p+1-i}) = 1$ for $1 \leq i \leq p$, the remaining pairings being determined by alternatingness and zero. Write

$$\Omega = \begin{pmatrix} 0 & J_p \\ -J_p & 0 \end{pmatrix}, \quad \check{G}_X = \mathrm{Sp}_{2p} = \check{G}^{\theta_X}, \quad \theta_X(g) = \Omega^{-1}(g^t)^{-1}\Omega,$$

where J_p is the reversing matrix of size p . The induced involution on $\check{\mathfrak{g}}$ gives $\check{\mathfrak{g}} = \check{\mathfrak{g}}_X \oplus \mathfrak{p}_{2p}$, with $\check{\mathfrak{g}}_X = \check{\mathfrak{g}}^{\theta_X}$ and $\mathfrak{p}_{2p} = \check{\mathfrak{g}}^{-\theta_X}$. If $r(A) = J_p A^t J_p$, then

$$\theta_X \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -r(D) & r(B) \\ r(C) & -r(A) \end{pmatrix}.$$

Let $\check{B}$ be the lower triangular Borel, $\check{U}$ its unipotent radical, and $\check{T}$ the diagonal torus. These subgroups are θ_X -stable. Put $\check{U}_X = \check{U} \cap \check{G}_X = \check{U}^{\theta_X}$ and $\check{\mathfrak{u}}_X = \mathrm{Lie}(\check{U}_X)$. Let $\check{B}^-$ be the opposite Borel and $\check{U}^-$ its unipotent radical. Write $V = W_1 \oplus W_2$ for the two coordinate subspaces of dimension p . Let $\check{P} \supseteq \check{B}$ be the parabolic stabilizing W_2 , let $\check{U}_{\check{P}}$ be its unipotent radical, and let $\check{K} = \mathrm{GL}_p \times \mathrm{GL}_p$ be its Levi subgroup preserving W_1 and W_2 .

Using the weighted matrices fixed above, define

$$e = \begin{pmatrix} e_p & 0 \\ 0 & e_p \end{pmatrix}, \quad f = \begin{pmatrix} f_p & 0 \\ 0 & f_p \end{pmatrix} = f_{p,p}, \quad h = [e, f].$$

Then (e, h, f) is a principal $\mathfrak{sl}_2$ -triple in $\check{\mathfrak{k}} = \mathrm{Lie}(\check{K})$. The element h is the differential of the cocharacter $\mathrm{diag}(\tau_p, \tau_p)$, which is the sum of the positive coroots of $\check{K}$. The block formula for θ_X shows that $e, f \in \mathfrak{p}_{2p}$ and hence $h \in \check{\mathfrak{g}}_X$. Consequently, $[f, \check{\mathfrak{u}} \cap \mathfrak{p}_{2p}] \subseteq \check{\mathfrak{b}} \cap \check{\mathfrak{g}}_X = \check{\mathfrak{b}}^{\theta_X}$.

Lemma 3.7.1. *The map $\mathrm{ad}_f : \check{\mathfrak{u}} \cap \mathfrak{p}_{2p} \rightarrow \check{\mathfrak{b}}^{\theta_X}$ is injective.*

Proof. If $x \in \check{\mathfrak{u}} \cap \mathfrak{p}_{2p}$ satisfies $[f, x] = 0$, then its image under the projection $\pi : \mathrm{Lie}(\check{P}) \rightarrow \check{\mathfrak{k}}$ does the same. But $\pi(x)$ belongs to $\check{\mathfrak{u}} \cap \check{\mathfrak{k}}$ and $f \in \check{\mathfrak{u}}^- \cap \check{\mathfrak{k}}$ is a regular nilpotent in $\check{\mathfrak{k}}$, so $\pi(x) = 0$. Therefore $x \in \mathrm{Lie}(\check{U}_{\check{P}})$.

We have a $\check{P}$ -equivariant isomorphism $\mathrm{Lie}(\check{U}_{\check{P}}) \simeq \mathrm{Hom}(V/W_2, W_2)$ and therefore an $\check{K}$ -equivariant isomorphism $\mathrm{Lie}(\check{U}_{\check{P}}) \simeq \mathrm{Hom}(W_1, W_2)$. Let $\check{K}_{\Delta} \subseteq \check{K}$ be the diagonal subgroup. Then $W_1 \simeq W_2$ as $\check{K}_{\Delta}$ -modules, and $\mathrm{Lie}(\check{U}_{\check{P}})$ is isomorphic to the adjoint representation of $\check{K}_{\Delta}$. Observe that $f \in \check{\mathfrak{k}}_{\Delta}$ is principal nilpotent in this reductive subalgebra. Hence its centralizer is spanned by its powers. Thus the kernel of ad_f on $\mathrm{Lie}(\check{U}_{\check{P}})$ consists of matrices

$$x = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}, \quad f_p C = C f_p,$$

where $C \in \mathbf{C}[f_p]$ is upper triangular and constant along each diagonal. But then $r(C) = J_p C^t J_p = C$, so x is θ_X -invariant. Since $x \in \mathfrak{p}_{2p}$, we conclude that $x = 0$. $\square$

Now we will construct a complement to the image $\mathrm{ad}_f(\check{\mathfrak{u}} \cap \mathfrak{p}_{2p})$ in $\check{\mathfrak{b}}^{\theta_X}$. We have the moment map $V \times V^* \rightarrow \check{\mathfrak{g}}_X^*$. Using the trace form, we identify $\check{\mathfrak{g}}_X$ with $\check{\mathfrak{g}}_X^*$. We also use $w \mapsto \omega(w, -)$ to make the $\check{G}_X$ -equivariant identification $V \simeq V^*$. We obtain a $\check{G}_X$ -equivariant symmetric bilinear pairing

$$(21) \quad \beta(v, w) \in \check{\mathfrak{g}}_X, \quad \beta(v, w)(z) = \frac{1}{2}(\omega(w, z)v + \omega(v, z)w).$$

We define a map $i : V \rightarrow \check{\mathfrak{g}}_X$ by $i(w) = \beta(e_{2p}, w)$. Since e_{2p} is $\check{U}$ -invariant, i is $\check{U}_X$ -equivariant. The map i is injective, because β identifies $\text{Sym}^2 V$ with $\check{\mathfrak{g}}_X$. Let $\mathfrak{v} = i(V)$ be its image. It is a $\check{U}_X$ -stable subspace of $\check{\mathfrak{g}}_X$ and consists of the θ_X -fixed matrices supported in the first column and last row. In particular, $\mathfrak{v} \subseteq \check{\mathfrak{b}}^{\theta_X}$.

Lemma 3.7.2. $\mathfrak{v} \cap [f, \check{\mathfrak{u}} \cap \mathfrak{p}_{2p}] = 0$.

Proof. Write

$$x = \begin{pmatrix} A & 0 \\ C & r(A) \end{pmatrix} \in \check{\mathfrak{u}} \cap \mathfrak{p}_{2p},$$

where A is strictly lower triangular and $r(C) = -C$. Suppose $[f, x] \in \mathfrak{v}$. Its upper-left block is supported in the first column, say $[f_p, A] = d\varepsilon_1^t$, where $\varepsilon_1 = (1, 0, \dots, 0)^t \in \mathbb{C}^p$. For $0 \leq k < p$,

$$0 = \text{tr}(f_p^k [f_p, A]) = \varepsilon_1^t f_p^k d = d_{k+1}.$$

Hence $[f_p, A] = 0$ and $A = 0$. The lower-left block $[f_p, C]$ is supported in its first column and last row, so $C_{a+1,b} = C_{a,b-1}$ for $a < p$ and $b > 1$. Thus C is constant along each diagonal, which implies $r(C) = C$. Together with $r(C) = -C$ this gives $C = 0$. $\square$

Corollary 3.7.3. $\check{\mathfrak{b}}^{\theta_X} = \mathfrak{v} \oplus [f, \check{\mathfrak{u}} \cap \mathfrak{p}_{2p}]$.

Proof. By Lemma 3.7.2, it now suffices to count dimensions. On one hand,

$$\dim \check{\mathfrak{b}}^{\theta_X} = \frac{\dim \check{\mathfrak{g}}_X + \text{rank } \check{\mathfrak{g}}_X}{2} = \frac{p(2p+1) + p}{2} = p^2 + p.$$

On the other hand, by the injectivity of ad_f on $\check{\mathfrak{u}} \cap \mathfrak{p}_{2p}$ and the injectivity of i , we have $\dim \mathfrak{v} = 2p$ and $\dim [f, \check{\mathfrak{u}} \cap \mathfrak{p}_{2p}] = \dim (\check{\mathfrak{u}} \cap \mathfrak{p}_{2p})$. In the block form of Lemma 3.7.2, the strictly lower triangular block A contributes $p(p-1)/2$ parameters. The condition $r(C) = -C$ says that $J_p C$ is skew-symmetric, so C contributes another $p(p-1)/2$ parameters. Thus

$$\dim (\check{\mathfrak{u}} \cap \mathfrak{p}_{2p}) = p(p-1), \quad 2p + p(p-1) = p^2 + p. \quad \square$$

3.7.2. Construction of the slice. Let $\delta = q - p$. In what follows, we need to make use of the Levi subgroup and subalgebra given by the block embeddings

$$\text{GL}_{2p} \times \text{GL}_{\delta} = \begin{pmatrix} \text{GL}_{2p} & 0 \\ 0 & \text{GL}_{\delta} \end{pmatrix} \subseteq \check{G},$$

$$\mathfrak{gl}_{2p} \oplus \mathfrak{gl}_{\delta} = \begin{pmatrix} \mathfrak{gl}_{2p} & 0 \\ 0 & \mathfrak{gl}_{\delta} \end{pmatrix} \subseteq \mathfrak{gl}_N,$$

with the δ -block omitted when $\delta = 0$. Let $\check{G}_X = \text{Sp}_{2p} \subseteq \text{GL}_{2p}$ and $\check{\mathfrak{g}}_X = \text{Lie}(\check{G}_X)$. We use the involution θ_X of GL_{2p} defining $\check{G}_X^{\theta_X} = \text{GL}_{2p}^{\theta_X}$ and the Cartan decomposition $\mathfrak{gl}_{2p} = \check{\mathfrak{g}}_X \oplus \mathfrak{p}_{2p}$ introduced above. Here $\check{U}_{2p} = \check{U} \cap \check{L}'_X$, and $\check{\mathfrak{b}}_{2p}$ is the lower triangular subalgebra of the $2p$ -block. The subgroup $\check{U}_X \subseteq \check{U}$ is defined by

$$\check{U}_X = \check{U}_{2p}^{\theta_X} \subseteq \check{U}, \quad u \mapsto \begin{pmatrix} u & 0 \\ 0 & I_{\delta} \end{pmatrix}.$$

The construction of $\mathcal{S}_{p,q}$ is uniform for $p \leq q$. Put

$$\mathcal{S}_{p,p}^0 = \check{\mathfrak{b}}_{2p} \cap \mathfrak{p}_{2p}, \quad \mathcal{S}_{\delta} = Z_{\mathfrak{gl}_{\delta}}(e_{\delta}),$$

where $\mathcal{S}_\delta$ is defined for $\delta > 0$ and its summand is omitted when $\delta = 0$. We regard these as subspaces of $\mathfrak{gl}_N$ through the block embeddings above. Let

$$V_{p,q}^0 = \{x \in \mathfrak{gl}_N : x_{ab} = 0 \text{ if } a < N \text{ or } b > 2p\}.$$

We define

$$\mathcal{S}_{p,q} = \mathcal{S}_{p,p}^0 \oplus \mathcal{S}_\delta \oplus V_{p,q}^0, \quad \Sigma_{p,q} = f_{p,q} + \mathcal{S}_{p,q}.$$

When $\delta = 0$, the slice has the following alternative description. Put $\text{Av}_{\theta_X} = (1 + \theta_X)/2$. This projection identifies $V_{p,p}^0$ with $\mathfrak{v}$: its image lies in $\mathfrak{v}$, and it is injective on last-row matrices, because θ_X carries the last row to the first column, whose intersection with that row is spanned by the θ_X -fixed matrix $E_{2p,1}$. Since both spaces have dimension $2p$, the image is all of $\mathfrak{v}$. For $w \in V_{p,p}^0$, we have

$$w - \text{Av}_{\theta_X}(w) = \frac{1}{2}(w - \theta_X(w)) \in \mathcal{S}_{p,p}^0.$$

The same injectivity gives $\mathcal{S}_{p,p}^0 \cap V_{p,p}^0 = 0$. Consequently,

$$(22) \quad \mathcal{S}_{p,p} = \mathcal{S}_{p,p}^0 \oplus V_{p,p}^0 = \mathcal{S}_{p,p}^0 \oplus \mathfrak{v}.$$

For $\delta > 0$, let $\varepsilon_1, \dots, \varepsilon_\delta$ be the standard basis of $\mathbf{C}^\delta$ and set $E = \mathbf{e}_{2p}\varepsilon_1^t$. The matrices in $\Sigma_{p,q}$ have the form

$$(23) \quad \begin{pmatrix} A & E \\ \varepsilon_\delta c & f_\delta + s \end{pmatrix}, \quad A \in f_{p,p} + \mathcal{S}_{p,p}^0, \quad c \in (\mathbf{C}^{2p})^*, \quad s \in \mathcal{S}_\delta.$$

Since $\tilde{U}_X$ fixes $\mathbf{e}_{2p}$, this description shows that $\Sigma_{p,q}$ is $\tilde{U}_X$ -stable when $\delta > 0$. For $\delta = 0$, this follows from the stability of $\mathfrak{v}$ and $\text{Ad}_{\tilde{U}_X}(f_{p,p}) - f_{p,p} \subseteq \mathcal{S}_{p,p}^0$.

Example 3.7.4. Consider the case $p = q = 2$. We have $\delta = 0$ and

$$\mathcal{S}_{2,2}^0 = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ u & 0 & c & 0 \\ 0 & -u & b & a \end{pmatrix} : a, b, c, u \in \mathbf{C} \right\}, \quad V_{2,2}^0 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} : x_i \in \mathbf{C} \right\}.$$

Hence

$$\mathcal{S}_{2,2} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ u & 0 & c & 0 \\ v_1 & v_2 & v_3 & v_4 \end{pmatrix} : a, b, c, u, v_i \in \mathbf{C} \right\},$$

and therefore

$$\Sigma_{2,2} = f_{2,2} + \mathcal{S}_{2,2} = \left\{ \begin{pmatrix} a & 1 & 0 & 0 \\ b & c & 0 & 0 \\ u & 0 & c & 1 \\ v_1 & v_2 & v_3 & v_4 \end{pmatrix} : a, b, c, u, v_i \in \mathbf{C} \right\}.$$

We have

$$\tilde{U}_X = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ \beta & \gamma & 1 & 0 \\ \eta & \beta - \alpha\gamma & -\alpha & 1 \end{pmatrix} : \alpha, \beta, \gamma, \eta \in \mathbf{C} \right\}.$$

3.7.3. Proof of the generalized Kostant theorem. In this section we establish the generalized Kostant theorem:

Theorem 3.7.5. *For $1 \leq p \leq q$, the action map*

$$\check{U} \times \Sigma_{p,q} \longrightarrow \check{\mathfrak{b}} + f_{p,q}, \quad (u, x) \longmapsto \text{Ad}_u(x),$$

induces an isomorphism

$$\check{U} \times^{\check{U}^\times} \Sigma_{p,q} \xrightarrow{\sim} \check{\mathfrak{b}} + f_{p,q}.$$

It will be convenient to slightly reformulate Theorem 3.7.5. Put

$$\mathcal{P}_p = \exp(\check{\mathfrak{u}}_{2p} \cap \mathfrak{p}_{2p}).$$

Multiplication is an isomorphism $\mathcal{P}_p \times \check{U}_X \rightarrow \check{U}_{2p}$: for $u \in \check{U}_{2p}$, its inverse is

$$(24) \quad a = \exp\left(\frac{1}{2} \log(u\theta_X(u)^{-1})\right), \quad k = a^{-1}u.$$

Indeed, $\theta_X(a) = a^{-1}$ and $\theta_X(k) = k$. Conversely, a factorization $u = ak$ with $a \in \mathcal{P}_p$ and $k \in \check{U}_X$ forces $a^2 = u\theta_X(u)^{-1}$, so is unique. Exponential and logarithm are polynomial on these nilpotent and unipotent matrices. Let $\pi : \check{U} \rightarrow \check{U}_{2p}$ be the natural upper-left block projection, and put

$$C_{p,q} = \pi^{-1}(\mathcal{P}_p), \quad \mathfrak{c}_{p,q} = T_1 C_{p,q}.$$

Applying the factorization (24) to $\pi(u)$ gives polynomial isomorphisms

$$(25) \quad C_{p,q} \times \check{U}_X \simeq \check{U}, \quad C_{p,q} \simeq \ker \pi \times (\check{\mathfrak{u}}_{2p} \cap \mathfrak{p}_{2p}).$$

Thus $C_{p,q}$ is an affine transversal, and $\check{\mathfrak{u}} = \mathfrak{c}_{p,q} \oplus \check{\mathfrak{u}}_X$.

Example 3.7.6. Consider the case $p = q = 2$. We have

$$\mathfrak{c}_{2,2} = \check{\mathfrak{u}}_4 \cap \mathfrak{p}_4 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ r & 0 & 0 & 0 \\ s & 0 & 0 & 0 \\ 0 & -s & r & 0 \end{pmatrix} : r, s \in \mathbf{C} \right\}, \quad C_{2,2} = \mathcal{P}_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ s & 0 & 1 & 0 \\ 0 & -s & r & 1 \end{pmatrix} : r, s \in \mathbf{C} \right\}.$$

The first factorization in (25) identifies $\check{U} \times^{\check{U}^\times} \Sigma_{p,q} \cong C_{p,q} \times \Sigma_{p,q}$. Hence Theorem 3.7.5 will follow from the next proposition.

Proposition 3.7.7. *The action map*

$$a : C_{p,q} \times \Sigma_{p,q} \longrightarrow \check{\mathfrak{b}} + f_{p,q}, \quad (c, x) \longmapsto \text{Ad}_c(x),$$

is an isomorphism.

The proof will follow from the following observations:

- (i) The map a is étale at $(1, f_{p,q})$.
- (ii) In affine coordinates centered at $(1, f_{p,q})$ and $f_{p,q}$, the map a intertwines positive linear $\mathbb{G}_m$ -actions.

These assertions imply that a is an isomorphism by the polynomial inverse argument given below.

3.7.4. *Proof of (i).* The following assertion follows by differentiating the action map.

Lemma 3.7.8. *The differential of a at $(1, f_{p,q})$ is*

$$da : \mathfrak{c}_{p,q} \oplus \mathcal{S}_{p,q} \longrightarrow \check{\mathfrak{b}}, \quad (x, y) \longmapsto [x, f_{p,q}] + y.$$

Proof. A direct computation. $\square$

Therefore, (i) can be reframed as follows.

Lemma 3.7.9. *The map $[f_{p,q}, -] : \mathfrak{c}_{p,q} \rightarrow \check{\mathfrak{b}}$ is injective, and*

$$\check{\mathfrak{b}} = [f_{p,q}, \mathfrak{c}_{p,q}] \oplus \mathcal{S}_{p,q}.$$

Proof. We deduce the assertion from the next two lemmas, which give the equality of dimensions and the trivial intersection, respectively. $\square$

Lemma 3.7.10. $\dim \check{\mathfrak{b}} = \dim \mathfrak{c}_{p,q} + \dim \mathcal{S}_{p,q}$.

Proof. We have

$$\dim \mathfrak{c}_{p,q} = \frac{N(N-1)}{2} - p^2, \quad \dim \mathcal{S}_{p,q} = p^2 + \delta + 2p = p^2 + N.$$

Their sum is $N(N+1)/2$. $\square$

Lemma 3.7.11. *If $\xi \in \mathfrak{c}_{p,q}$ and $[f_{p,q}, \xi] \in \mathcal{S}_{p,q}$, then $\xi = 0$.*

Proof. We assume first that $\delta > 0$. With $E = \mathbf{e}_{2p} \varepsilon_1^t$ as above, write

$$f_{p,q} = \begin{pmatrix} f_{p,p} & E \\ 0 & f_\delta \end{pmatrix}, \quad \xi = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix},$$

where $x \in \check{\mathfrak{u}}_{2p} \cap \mathfrak{p}_{2p}$, $y \in \mathcal{M}_{\delta, 2p}$, and $z \in \check{\mathfrak{u}}_\delta$. The hypothesis gives

$$[f_{p,q}, \xi] = \begin{pmatrix} [f_{p,p}, x] + Ey & Ez - xE \\ f_\delta y - yf_{p,p} & [f_\delta, z] - yE \end{pmatrix} = \begin{pmatrix} a & 0 \\ v & b \end{pmatrix},$$

where $a \in \mathcal{S}_{p,p}^0$, $b \in \mathcal{S}_\delta$, and v is supported in its last row.

We project the first condition $[f_{p,p}, x] + Ey = a$ to the $\theta_X = 1$ and $\theta_X = -1$ eigenspaces in $\mathfrak{gl}_{2p}$. Since $a \in \mathfrak{p}_{2p}$ and $[f_{p,p}, x] \in \check{\mathfrak{g}}_X$, we obtain

$$[f_{p,p}, x] + \text{Av}_{\theta_X}(Ey) = 0, \quad \frac{1}{2}(\text{Ev} - \theta_X(\text{Ev})) = a.$$

In the first equation, $x \in \check{\mathfrak{u}}_{2p} \cap \mathfrak{p}_{2p}$ and $\text{Av}_{\theta_X}(Ey) \in \text{Av}_{\theta_X}(\mathcal{V}_{p,p}^0) = \mathfrak{v}$. Lemmas 3.7.2 and 3.7.1 therefore give $x = 0$ and $\text{Av}_{\theta_X}(Ey) = 0$. The projection Av_{θ_X} is injective on last-row matrices: θ_X carries the last row to the first column, whose intersection with that row is spanned by the θ_X -fixed matrix $E_{2p,1}$. Hence $Ey = 0$, so the first row of y vanishes.

Now we take up the equation $f_\delta y - yf_{p,p} = v$. It asserts that the first $\delta - 1$ rows of the left hand side vanish. If $\delta = 1$, the vanishing of the first row of y already gives $y = 0$. For $\delta > 1$, given a vector $t \in \mathbf{C}^\delta$, we use the notation

$$t = \begin{pmatrix} t_1 \\ \vdots \\ t_\delta \end{pmatrix}, \quad t^+ = \begin{pmatrix} t_1 \\ \vdots \\ t_{\delta-1} \end{pmatrix}, \quad t^- = \begin{pmatrix} t_2 \\ \vdots \\ t_\delta \end{pmatrix}.$$

If $y_1, \dots, y_{2p} \in \mathbf{C}^\delta$ are the columns of y , then

$$f_\delta y = \begin{pmatrix} y_1^- & y_2^- & \cdots & y_{2p}^- \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$y f_{p,p} = \begin{pmatrix} 0 & y_1^+ & \cdots & y_{p-1}^+ & 0 & y_{p+1}^+ & \cdots & y_{2p-1}^+ \\ 0 & * & \cdots & * & 0 & * & \cdots & * \end{pmatrix},$$

with empty column ranges omitted when $p = 1$. Discarding the final rows and equating the resulting matrices yields the following system:

$$y_1^- = 0, \quad y_j^- = y_{j-1}^+ \quad (2 \leq j \leq p),$$

$$y_{p+1}^- = 0, \quad y_j^- = y_{j-1}^+ \quad (p+2 \leq j \leq 2p).$$

Since the first row of y is zero, $y_1^- = 0$ implies $y_1 = 0$. Thus $y_1^+ = 0$, and repeating this argument along the first chain gives $y_1 = \cdots = y_p = 0$. The induction begins again from $y_{p+1}^- = 0$ and results in $y_{p+1} = \cdots = y_{2p} = 0$. Thus $y = 0$ in all cases. The lower-right block now gives $[f_\delta, z] = b \in \mathcal{S}_\delta$. The differential at $(1, f_\delta)$ of Kostant's isomorphism implies that $z = 0$. This completes the proof when $\delta > 0$.

When $\delta = 0$, we have $\xi \in \mathfrak{c}_{p,p} = \check{\mathfrak{u}}_{2p} \cap \mathfrak{p}_{2p}$ and $[f_{p,p}, \xi] \in \mathcal{S}_{p,p}^0 \oplus \mathfrak{v}$. Since the commutator belongs to $\check{\mathfrak{g}}_X$ and $\mathcal{S}_{p,p}^0 \subseteq \mathfrak{p}_{2p}$, it belongs to $\mathfrak{v}$. Lemma 3.7.2 gives $[f_{p,p}, \xi] = 0$, and Lemma 3.7.1 then gives $\xi = 0$. $\square$

Proof of (ii) and Proposition 3.7.7. The preceding lemmas establish (i). Let τ_N act on $C_{p,q}$ by conjugation and on the affine matrix spaces by $x \mapsto t^2 \text{Ad}_{\tau_N(t)} x$. On the first $2p$ coordinates, τ_N differs from τ_{2p} by a scalar, so it preserves $\mathcal{P}_p$ and $C_{p,q}$. In the coordinates $\ker \pi \times (\check{\mathfrak{u}}_{2p} \cap \mathfrak{p}_{2p})$ all weights are positive: the weights of conjugation on strictly lower triangular matrix entries are $2(a-b) > 0$ for $a > b$. The matrix $f_{p,q}$ is fixed, while all weights on $\check{\mathfrak{b}}$ are $2 + 2(a-b) > 0$ for $a \geq b$. The subspace $\mathcal{S}_{p,q}$ is stable: its first block and last-row summand are stable, and, when $\delta > 0$, e_δ^k has weight $2k + 2$ under the affine action. Thus the source and target are affine spaces with positive linear actions in coordinates centered at $(1, f_{p,q})$ and $f_{p,q}$, respectively. The map a is equivariant. This proves (ii).

To conclude, use these affine coordinates on $C_{p,q} \times \Sigma_{p,q}$ and $\check{\mathfrak{b}} + f_{p,q}$. The actions are positive and linear, and $da_{(1, f_{p,q})}$ is an isomorphism. After an equivariant linear change of coordinates, the pullback of a coordinate of weight d is itself plus a polynomial in coordinates of weights strictly less than d . Induction on the weights constructs a polynomial inverse to a . $\square$

For comparison with the grading on the Whittaker space, recall $\tau_{\check{K}} = \text{diag}(\tau_p, \tau_q)$ and the cocharacter from (16):

$$(26) \quad \gamma(t) = \tau_{\check{K}}(t)^2 \tau_N(t)^{-1} = \text{diag}(\tau_p(t), \tau_q(t)) \text{diag}(t^q \mathbf{I}_p, t^{-p} \mathbf{I}_q).$$

Conjugation by γ preserves $\check{\mathfrak{U}}_X$ and $C_{p,q}$, and $x \mapsto t^2 \text{Ad}_{\gamma(t)} x$ preserves $\Sigma_{p,q}$ and fixes $f_{p,q}$. Indeed, the first $2p$ entries of γ form a cocharacter of $\check{\mathfrak{G}}_X$, and its last δ entries are τ_δ . The normal-form isomorphism of Theorem 3.7.5 is therefore equivariant for this action as well. The equivariance required in Theorem 3.6.1 is for (26); τ_N is used in the contraction argument.

3.8. Lagrangian correspondence. We consider the case $p = q \geq 1$, with the notation of Section 3.7. Recall that we use the trace form to make the equivariant identifications $\check{\mathfrak{g}}^* \simeq \check{\mathfrak{g}}$ and $\check{\mathfrak{g}}_X^* \simeq \check{\mathfrak{g}}_X$. Put $f = f_{p,p}$ and $v_0 = \mathbf{e}_{2p}$. The restriction of f to $\check{\mathfrak{u}}$ is $d\psi_{\check{K}}$. We use the Whittaker space

$$T_{\psi_{\check{K}}}^*(\check{G}/\check{U}) = \check{G} \times^{\check{U}} (\check{\mathfrak{b}} + f).$$

With the convention $(g, z) \sim (gh^{-1}, hz)$, the moment map for the right action on $T^*\check{G}$ is $-\xi|_{\check{\mathfrak{u}}}$, so this is the reduction at $-d\psi_{\check{K}}$.

We now consider the Hamiltonian $\check{G}$ -variety $\check{M} = T^*\mathcal{X}$, where $\mathcal{X} = \check{G} \times^{\check{G}_X} V \simeq \check{G}/\check{G}_X \times V$. Let $x_0 = [1, v_0] \in \mathcal{X}$. The twisted conormal to the $\check{G}$ -orbit through $(1\check{U}, x_0)$ in $(\check{G}/\check{U}) \times \mathcal{X}$ gives a correspondence

$$\begin{array}{ccc} & \mathfrak{L} & \\ p_1 \swarrow & & \searrow p_2 \\ T_{\psi_{\check{K}}}^*(\check{G}/\check{U}) & & T^*\mathcal{X}. \end{array}$$

Letting $\check{G}_{X,v_0} \subseteq \check{G}_X$ denote the stabilizer of v_0 and $\mu : T^*\mathcal{X} \rightarrow \check{\mathfrak{g}}^*$ denote the moment map, we have

$$F_{x_0} = T_{x_0}^*\mathcal{X} \cap \mu^{-1}(\check{\mathfrak{b}} + f), \quad \mathfrak{L} = \check{G} \times^{\check{U} \cap \check{G}_{X,v_0}} F_{x_0},$$

with $p_1([g, \eta]) = [g, \mu(\eta)]$ and $p_2([g, \eta]) = g\eta$. We observe that $\check{U} \cap \check{G}_{X,v_0} = \check{U} \cap \check{G}_X = \check{U}_X$, since v_0 is already $\check{U}$ -invariant. Moreover, $f \in \mathfrak{p}_{2p}$ implies $d\psi_{\check{K}}|_{\check{\mathfrak{u}}_X} = 0$, and hence $\psi_{\check{K}}|_{\check{U}_X} = 0$; this is the stabilizer condition for the twisted conormal construction.

We can write $T_{x_0}^*\mathcal{X} \simeq (\check{\mathfrak{g}}/\check{\mathfrak{g}}_X)^* \times \{v_0\} \times V^*$. Using ω , we identify $V^* \simeq V$. The trace form is θ_X -invariant, and thus identifies $(\check{\mathfrak{g}}/\check{\mathfrak{g}}_X)^*$ with $\mathfrak{p}_{2p}$. In these trace-orthogonal coordinates,

$$T^*\mathcal{X} \simeq \check{G} \times^{\check{G}_X} (\mathfrak{p}_{2p} \times V \times V), \quad \mu([g, A, v, w]) = \text{Ad}_g(A + \beta(v, w)),$$

where $i(w) = \beta(v_0, w)$ is the map of the preceding subsection. Recall the affine space $\mathcal{S} = \Sigma_{p,p} = f + \mathcal{S}_{p,p}$ in (22). Since $i(V) \subset \check{\mathfrak{b}} \cap \check{\mathfrak{g}}_X$, the preimage of $\check{\mathfrak{b}} + f$ in $T_{x_0}^*\mathcal{X}$ is

$$((\check{\mathfrak{b}} + f) \cap \mathfrak{p}_{2p}) \times \{v_0\} \times V \simeq \mathcal{S}, \quad (A, v_0, w) \mapsto A + i(w).$$

Our correspondence becomes

$$(27) \quad \begin{array}{ccc} & \check{G} \times^{\check{U}_X} \mathcal{S} & \\ p_1 \swarrow & & \searrow p_2 \\ \check{G} \times^{\check{U}} (\check{\mathfrak{b}} + f) & & \check{G} \times^{\check{G}_X} (\mathfrak{p}_{2p} \times V \times V). \end{array}$$

The right arrow sends $[g, A + i(w)]$ to $[g, A, v_0, w]$. Identifying $\check{G} \times^{\check{U}_X} \mathcal{S} \simeq \check{G} \times^{\check{U}} (\check{U} \times^{\check{U}_X} \mathcal{S})$, we see that p_1 is induced from the action map $\bar{a} : \check{U} \times^{\check{U}_X} \mathcal{S} \rightarrow \check{\mathfrak{b}} + f$, which is an isomorphism by Theorem 3.7.5, applied with $q = p$. We conclude that p_1 is an isomorphism.

Lemma 3.8.1. $\dim T_{\psi_{\check{K}}}^*(\check{G}/\check{U}) = \dim T^*(\check{G} \times^{\check{G}_X} V)$.

Proof. Equivalently, we claim that $\dim \check{G}_X = \dim \check{U} + \dim V$. Since $\dim V = \dim \check{V}$, we are claiming that $\dim \check{G}_X = \dim \check{B}$. Indeed, both dimensions are $p(2p+1)$. $\square$

The correspondence is Lagrangian. Indeed, the two maps from $\check{G} \times F_{x_0}$

$$(g, \eta) \mapsto (g, \mu(\eta)) \in T^*\check{G}, \quad (g, \eta) \mapsto g\eta \in T^*\mathcal{X}$$

have equal pullbacks of the Liouville forms: on a tangent vector with $\check{G}$ -component ga , both evaluate as $\langle \mu(\eta), a \rangle$. Differentiating and descending gives $p_1^* \omega_{T^*\check{G}/\check{U}} = p_2^* \omega_{T^*\mathcal{X}}$; the dimension equality proves the assertion. Hence $j = p_2 p_1^{-1}$ preserves the symplectic forms and is a morphism of smooth varieties of the same dimension, so it is étale. It follows that p_2 is étale.

3.8.1. Big open image. We shall show that p_2 in the Lagrangian correspondence (27) is an open immersion and is surjective up to a closed subvariety of codimension at least two.

Writing $\check{G} \times^{\check{U}_X} \mathcal{S} \simeq \check{G} \times^{\check{G}_X} (\check{G}_X \times^{\check{U}_X} \mathcal{S})$, we see that p_2 is induced from a map

$$\check{G}_X \times^{\check{U}_X} \mathcal{S} \longrightarrow \mathfrak{p}_{2p} \times V \times V.$$

To simplify notation, write

$$\mathcal{T}_0 = (f + \check{\mathfrak{b}}) \cap \mathfrak{p}_{2p}, \quad \mathcal{T} = \mathcal{T}_0 \times \{v_0\}.$$

Then $\mathcal{S} \simeq \mathcal{T} \times V$, and the V factors just come along for the ride: under the isomorphism

$$\check{G}_X \times^{\check{U}_X} (\mathcal{T} \times V) \xrightarrow{\sim} (\check{G}_X \times^{\check{U}_X} \mathcal{T}) \times V, \quad [k, t, w] \longmapsto ([k, t], kw),$$

the map becomes $q \times \text{id}_V$, where

$$q : \check{G}_X \times^{\check{U}_X} \mathcal{T} \longrightarrow \mathfrak{p}_{2p} \times V, \quad [k, (A, v_0)] \longmapsto (\text{Ad}_k A, kv_0).$$

Because p_2 is étale, the same is true of q , as may be checked after the smooth surjective base change $\check{G} \rightarrow \check{G}/\check{G}_X$. Moreover, to show that p_2 is an open immersion with complement of codimension at least two, it suffices to show the same of q .

Our first step is to show that q is an open immersion.

Lemma 3.8.2. *q is an open immersion.*

Proof. Because q is étale, it suffices to show that it is injective on geometric points. Put $E_0 = 0$ and

$$E_i = \langle \mathbf{e}_{2p}, \mathbf{e}_{2p-1}, \dots, \mathbf{e}_{2p-i+1} \rangle \quad (1 \leq i \leq p).$$

For $A \in (\check{\mathfrak{b}} + f) \cap \mathfrak{p}_{2p}$, the prescribed superdiagonal entries give $A^{i-1}v_0 \equiv \mathbf{e}_{2p-i+1} \pmod{E_{i-1}}$. Suppose that $[k, (A, v_0)]$ and $[k', (A', v_0)]$ have the same image (x, v) . Both k and k' carry E_i to $\langle v, xv, \dots, x^{i-1}v \rangle$, with the same normalized generators on the successive quotients. Thus $k^{-1}k'$ preserves this isotropic flag and acts trivially on its graded pieces. Since it is symplectic, it also preserves the dual flag and acts trivially on its graded pieces. Hence $k^{-1}k' \in \check{U} \cap \check{G}_X = \check{U}_X$, so the two associated-bundle points coincide. The argument works over every algebraically closed extension of $\mathbf{C}$. Thus q is universally injective and étale, and therefore an open immersion. $\square$

Our next step is to show that q is surjective up to a closed subvariety of codimension at least two. Since $0 \in V$ has codimension $2p \geq 2$, we may replace V by $V \setminus \{0\}$. But $\check{G}_X$ acts transitively on $V \setminus \{0\}$. Letting $\check{G}_{X, v_0}$ denote the stabilizer of v_0 in $\check{G}_X$, we see that it suffices to show that the action map $\check{G}_{X, v_0} \times ((\check{\mathfrak{b}} + f) \cap \mathfrak{p}_{2p}) \rightarrow \mathfrak{p}_{2p}$ is surjective up to a codimension at least two subvariety. To this end, define

$\ell : \mathfrak{p}_{2p} \rightarrow \mathbf{Z}$ by setting $\ell(x)$ to be the dimension of the subspace V_x spanned by $v := v_0, xv, x^2v, \dots$. It is clear that the fibers of ℓ are locally closed subvarieties of $\mathfrak{p}_{2p}$.

Lemma 3.8.3. *Let $x \in \mathfrak{p}_{2p}$ and $w \in V$. The subspace spanned by $x^i w$ for $i \geq 0$ is isotropic.*

Proof. We have $\omega(x^i w, x^j w) = \omega(w, x^{i+j} w) = \omega(x^j w, x^i w)$ by the self-adjointness of x . But $\omega(x^i w, x^j w) = -\omega(x^j w, x^i w)$, since ω is alternating. So $\omega(x^i w, x^j w) = 0$. $\square$

Lemma 3.8.4. *The bound $\ell(x) \leq p$ holds for $x \in \mathfrak{p}_{2p}$. Moreover, it is attained over an open dense subvariety of $\mathfrak{p}_{2p}$.*

Proof. By Lemma 3.8.3, the subspace V_x is isotropic. So $\ell(x) = \dim V_x \leq \dim V/2 = p$. Therefore $\ell^{-1}(p)$ is open. It is nonempty because $\ell(f) = p$. The density follows from the irreducibility of $\mathfrak{p}_{2p}$. $\square$

Let $\mathcal{U} := \ell^{-1}(p)$ be the dense open subvariety of Lemma 3.8.4.

Lemma 3.8.5. *Let V be a symplectic vector space of dimension $2p$ and let $0 \leq i \leq p$. Then the variety Gr_i^ω of i -dimensional isotropic subspaces $W \subseteq V$ has dimension*

$$\dim \text{Gr}_i^\omega = i \left(2p - \frac{3i-1}{2} \right).$$

Proof. The case $i = 0$ is immediate. Let Fl_i^ω denote the variety of isotropic flags in V of height i . There is a smooth surjection $\text{Fl}_i^\omega \rightarrow \text{Gr}_i^\omega$ whose fiber over W is the variety $\text{Fl}(W)$ of complete flags in W . Hence

$$\dim \text{Gr}_i^\omega = \dim \text{Fl}_i^\omega - \dim \text{Fl}(W).$$

Recall that $\dim \text{Fl}(W) = i(i-1)/2$. The natural map $\text{Fl}_i^\omega \rightarrow \text{Fl}_{i-1}^\omega$ has fiber over $W_\bullet$ isomorphic to $\mathbf{P}(W_{i-1}^\perp/W_{i-1})$, of dimension $2p - 2(i-1) - 1$. So

$$\dim \text{Fl}_i^\omega = \sum_{j=0}^{i-1} (2p - 2j - 1) = (2p - i)i.$$

Therefore

$$\dim \text{Gr}_i^\omega = (2p - i)i - \frac{i(i-1)}{2} = i \left(2p - \frac{3i-1}{2} \right).$$

$\square$

Lemma 3.8.6. *For $1 \leq i \leq p$, we have*

$$\text{codim}_{\mathfrak{p}_{2p}} \ell^{-1}(i) = 2(p - i).$$

In particular, $\dim \ell^{-1}(i) \leq \dim \mathfrak{p}_{2p} - 2$ for $i < p$.

Proof. Let $\mathcal{Y}$ be the variety of isotropic subspaces of V of dimension i containing v . There is a map $\pi : \ell^{-1}(i) \rightarrow \mathcal{Y}$ taking x to V_x . This map is surjective. Indeed, suppose that W is any such subspace. We may choose a symplectic basis $v_1, \dots, v_{2p} = v$ such that W is spanned by $v_{2p}, \dots, v_{2p-i+1}$. Let x operate on V by $xv_j = v_{j-1}$ for $j \geq 2p - i + 2$ or $2 \leq j \leq i$, and $xv_j = 0$ otherwise. Then $x \in \mathfrak{p}_{2p}$, $x \in \ell^{-1}(i)$, and $\pi(x) = W$.

Now we compute $\dim \mathcal{Y}$. Let L be the span of v . Then isotropic i -planes W containing L correspond to isotropic $(i-1)$ -planes W/L in the symplectic vector space $L^\perp/L$, of dimension $2p-2$. By Lemma 3.8.5,

$$\dim \mathcal{Y} = (i-1) \left(2p-2 - \frac{3(i-1)-1}{2} \right) = \frac{(i-1)(4p-3i)}{2}.$$

Fix $W \in \mathcal{Y}$ and consider the fiber $\pi^{-1}(W)$. Since x preserves W and is self-adjoint, it operates on W and $W^\perp/W$. The symplectic form descends to $W^\perp/W$, and the action of x is self-adjoint with respect to it. Let $\mathfrak{p}_{W^\perp/W}$ be the space of these endomorphisms. We get a map

$$\eta : \pi^{-1}(W) \longrightarrow \mathfrak{gl}_W \times \mathfrak{p}_{W^\perp/W}.$$

Since $\dim(W^\perp/W) = 2(p-i)$, the base has dimension

$$\dim \mathfrak{gl}_W + \dim \mathfrak{p}_{W^\perp/W} = i^2 + \dim \bigwedge^2 (W^\perp/W)^* = i^2 + (p-i)(2(p-i)-1).$$

Fix $x_1 \in \mathfrak{gl}_W$ and $x_2 \in \mathfrak{p}_{W^\perp/W}$. The image of η consists of all such pairs for which v generates W under x_1 . This is an open condition on x_1 , so the image is open. Now consider a nonempty fiber $\eta^{-1}(x_1, x_2)$. Let $P_W \subseteq \check{G}$ be the stabilizer of the flag $W \subseteq W^\perp$, let U_W be its unipotent radical, and put $\mathfrak{u}_W = \text{Lie}(U_W)$. Then $\mathfrak{u}_W \cap \mathfrak{p}_{2p}$ operates on the fiber by translation, making it a $\mathfrak{u}_W \cap \mathfrak{p}_{2p}$ -torsor. In a symplectic decomposition $V = W \oplus E \oplus W^*$, its free blocks are an arbitrary $W \otimes E$ block and an alternating $W \otimes W$ block. Thus

$$d := \dim(\mathfrak{u}_W \cap \mathfrak{p}_{2p}) = 2i(p-i) + \frac{i(i-1)}{2}.$$

Putting it all together,

$$\begin{aligned} \dim \ell^{-1}(i) &= \frac{(i-1)(4p-3i)}{2} + i^2 + (p-i)(2(p-i)-1) + d \\ &= 2p^2 - 3p + 2i. \end{aligned}$$

By comparison, $\dim \mathfrak{p}_{2p} = \dim \bigwedge^2 V^* = 2p^2 - p$. So the desired codimension is

$$\text{codim}_{\mathfrak{p}_{2p}} \ell^{-1}(i) = (2p^2 - p) - (2p^2 - 3p + 2i) = 2(p-i),$$

which is at least two when $i < p$. $\square$

Corollary 3.8.7. *The complement of the open subvariety $\mathcal{U} = \ell^{-1}(p)$ in $\mathfrak{p}_{2p}$ has codimension at least two.*

Proof. The complement $\mathfrak{p}_{2p} \setminus \mathcal{U}$ is the union of the subvarieties $\ell^{-1}(i)$ for $1 \leq i < p$, so the claim follows from Lemma 3.8.6. For $p=1$ the complement is empty. $\square$

Lemma 3.8.8. *Every $x \in \mathcal{U}$ is conjugate under $\check{G}_{X, v_0}$ to an element of $(\check{\mathfrak{b}} + f) \cap \mathfrak{p}_{2p}$.*

Proof. Since $x \in \mathcal{U}$, the vectors $v_{2p} := v_0, v_{2p-1} := xv_0, \dots, v_{p+1} := x^{p-1}v_0$ span a Lagrangian subspace W . By Lemma 3.8.3 and the maximality of W , we have $x^p v_0 \in W$. Hence W is x -stable and x operates on V/W . The form ω induces a perfect pairing $W \times V/W \rightarrow \mathbf{C}$. Choose a Lagrangian complement to W and its basis $v_1, \dots, v_p$ such that $\omega(v_i, v_{2p+1-j}) = \delta_{ij}$. Then there exists a unique $k \in \check{G}_X$ such that $k\mathbf{e}_i = v_i$. Since $k\mathbf{e}_{2p} = v_0 = \mathbf{e}_{2p}$, we have $k \in \check{G}_{X, v_0}$.

In this basis the lower-right block D of $k^{-1}xk$ has ones on its superdiagonal and zeros above it. The upper-right block is zero because W is stable. Self-adjointness

makes the upper-left block $J_p D^t J_p$, with the same superdiagonal pattern. All remaining entries lie on or below the diagonal. Thus $k^{-1} x k \in \mathfrak{b} + f$. $\square$

Corollary 3.8.9. *The image of the action map $q : \check{G}_X \times^{\check{U}_X} \mathcal{T} \rightarrow \mathfrak{p}_{2p} \times V$ is*

$$\mathcal{U}_{\text{cyc}} = \{(x, v) : v, xv, \dots, x^{p-1}v \text{ are linearly independent}\}.$$

The complement of its image has codimension two.

Proof. The prescribed superdiagonal entries give $\text{im } q \subseteq \mathcal{U}_{\text{cyc}}$. The reverse inclusion follows from the preceding lemma and the transitivity of $\check{G}_X$ on $V \setminus \{0\}$. Over nonzero v , the stratum of cyclic dimension i has codimension $2(p - i)$ by Lemma 3.8.6; the locus $v = 0$ has codimension $2p$. For $p \geq 2$, the nonempty stratum $i = p - 1$ has codimension two. For $p = 1$, the complement is $\mathfrak{p}_{2p} \times \{0\}$, again of codimension two. $\square$

Putting things together, we get:

Proposition 3.8.10. *The action map $q : \check{G}_X \times^{\check{U}_X} \mathcal{T} \rightarrow \mathfrak{p}_{2p} \times V$ is an open immersion and is surjective up to a closed subvariety of codimension at least two.*

3.9. Proof of Theorem 3.6.1. When $p = q$, Proposition 3.8.10 implies that the Lagrangian correspondence gives an open embedding

$$j = p_2 p_1^{-1} : T_{\psi_{\check{K}}}^*(\check{G}/\check{U}) \longrightarrow \check{M}$$

whose complement has codimension two. The construction is $\check{G} \times \mathbb{G}_m$ -equivariant for the grading fixed in Section 3.4. Hartogs' principle therefore gives the desired isomorphism of affine closures.

Now assume $p < q$, and put $\delta = q - p > 0$. Recall that $\check{G} = \text{GL}_N$, $N = p + q$, $\check{G}_X = \text{Sp}_{2p}$, and $\check{U}_X = \check{G}_X \cap \check{U}$. The same slices are

$$\mathcal{T}_0 = (f_{p,p} + \mathfrak{b}_{2p}) \cap \mathfrak{p}_{2p}, \quad \mathcal{T} = \mathcal{T}_0 \times \{v_0\}, \quad v_0 = e_{2p},$$

and we use $\mathcal{S}_\delta = Z_{\mathfrak{gl}_\delta}(e_\delta)$. By Theorem 3.7.5,

$$T_{\psi_{\check{K}}}^*(\check{G}/\check{U}) \simeq \check{G} \times^{\check{U}_X} \Sigma_{p,q}, \quad \Sigma_{p,q} = f_{p,q} + \mathcal{S}_{p,q}.$$

On the other hand, the block description (14) gives a natural inclusion

$$\begin{aligned} \Sigma_{p,q} &\simeq \mathcal{T} \times V^* \times \mathcal{S}_\delta \hookrightarrow \mathcal{Z}_\delta, \\ \mathcal{Z}_\delta &\simeq (\mathfrak{p}_{2p} \times V) \times V^* \times \mathcal{S}_\delta. \end{aligned}$$

It induces a map

$$j : T_{\psi_{\check{K}}}^*(\check{G}/\check{U}) \simeq \check{G} \times^{\check{U}_X} \Sigma_{p,q} \simeq \check{G} \times^{\check{G}_X} (\check{G}_X \times^{\check{U}_X} \Sigma_{p,q}) \longrightarrow \check{G} \times^{\check{G}_X} \mathcal{Z}_\delta \simeq \check{M}.$$

We claim that j is a $\check{G} \times \mathbb{G}_m$ -equivariant open embedding with codimension two complement. Hartogs' principle then gives

$$\overline{T_{\psi_{\check{K}}}^*(\check{G}/\check{U})} \cong \check{M}.$$

Proof of the claim. It suffices to consider the action map

$$(28) \quad \check{G}_X \times^{\check{U}_X} \Sigma_{p,q} \longrightarrow \mathcal{Z}_\delta.$$

The $\check{G}_X$ -action on $\mathcal{Z}_\delta$ is

$$h \cdot (A, b, c, s) = (\text{Ad}_h A, hb, ch^{-1}, s).$$

Consequently, there is a product identification

$$\begin{aligned} \check{G}_X \times^{\check{U}_X} \Sigma_{p,q} &\xrightarrow{\sim} (\check{G}_X \times^{\check{U}_X} \mathcal{T}) \times V^* \times \mathcal{S}_\delta, \\ [h, (A, v_0, c, s)] &\longmapsto ([h, (A, v_0)], ch^{-1}, s). \end{aligned}$$

Under this identification, the action map (28) is

$$q \times \text{id}_{V^*} \times \text{id}_{\mathcal{S}_\delta} : (\check{G}_X \times^{\check{U}_X} \mathcal{T}) \times V^* \times \mathcal{S}_\delta \longrightarrow (\mathfrak{p}_{2p} \times V) \times V^* \times \mathcal{S}_\delta,$$

where q is the map in Proposition 3.8.10. Since q is an open embedding with codimension two complement, the same is therefore true for (28), and hence for j . This proves the claim.

4. MAIN RESULTS

4.1. Ext algebras. Let $\text{IC}_{\mathcal{O}(\check{G})} \in \text{Ind}(\mathcal{D}_{\mathfrak{L}+G}^b(\text{Gr}))$ be the regular perverse sheaf on Gr : the algebra ind-object corresponding to the regular representation $\mathcal{O}(\check{G})$ of $\check{G}$ under the geometric Satake equivalence (7). The de-equivariantized Ext algebras for the complex, real and relative Satake categories are the dg-algebras $A = \text{RHom}_{\text{Ind}(\mathcal{D}_{\mathfrak{L}+G}^b(\text{Gr}_G))}(\text{IC}^0, \text{IC}^0 \star \text{IC}_{\mathcal{O}(\check{G})})$, $A_{\mathbf{R}} = \text{RHom}_{\text{Ind}(\mathcal{D}_{\mathfrak{L}+G_{\mathbf{R}}}^b(\text{Gr}_{\mathbf{R}}))}(\text{IC}_{\mathbf{R}}^0, \text{IC}_{\mathbf{R}}^0 \star \text{IC}_{\mathcal{O}(\check{G})})$, and $A_X = \text{RHom}_{\text{Ind}(\mathcal{D}_{\mathfrak{L}+G}^b(\mathfrak{L}X))}(\text{IC}_X^0, \text{IC}_X^0 \star \text{IC}_{\mathcal{O}(\check{G})})$, respectively. Note that we have a residual $\check{G}$ -action on A (resp. $A_{\mathbf{R}}$, A_X) coming from the right $\check{G}$ -action on $\mathcal{O}(\check{G})$ and a natural $\check{G}$ -equivariant dg algebra maps $A \rightarrow A_{\mathbf{R}}$ and $A \rightarrow A_X$ induced from the Hecke action. It follows from derived Satake equivalence for G [BF] that A is formal, that is, it is isomorphic to its cohomology algebra $H^*(A)$ with trivial differentials. Moreover, there is a $\check{G} \times \mathbb{G}_m$ -equivariant isomorphism $H^*(A) \cong \mathcal{O}(\check{\mathfrak{g}}^*)$ where $\check{G} \times \mathbb{G}_m$ -acts on $\check{\mathfrak{g}}^*$ by $(g, t)(v) = t^2 \text{Ad}_g(v)$.

Theorem 4.1.1. (1) *There is a $\check{G}$ -equivariant isomorphism $A_{\mathbf{R}} \cong A_X$ of dg-algebras. Moreover, dg-algebra $A_{\mathbf{R}} \cong A_X$ is formal, that is, it is isomorphic to its cohomology algebra $H^*(A_{\mathbf{R}}) \cong H^*(A_X)$ with trivial differentials.*

(2) *There is $\check{G} \times \mathbb{G}_m$ -equivariant isomorphism $H^*(A_{\mathbf{R}}) \cong H^*(A_X) \cong \mathcal{O}(\check{M})$ of $\mathcal{O}(\check{\mathfrak{g}}^*)$ -algebras.*

Proof. Part (1) is proved in [CN, Theorem 13.4] using the pointwise purity of IC_X^λ in [CY, Theorem 81]. Proof of (2). According to [Gi, (3.1.5)] or [C, Lemma 4.2]² we have a $\check{G}$ -equivariant isomorphism of dg-algebras

$$H^*(A_X) \cong \text{Ext}_{\text{Ind}(\mathcal{D}_{\mathfrak{L}+G}^b(\mathfrak{L}X))}^*(\text{IC}_X^0, \text{IC}_X^0 \star \text{IC}_{\mathcal{O}(\check{G})}) \cong H_K^*(\text{Gr}_K, \iota_K^! \text{IC}_{\mathcal{O}(\check{G})})$$

where $\iota_K : \text{Gr}_K \rightarrow \text{Gr}_G$ is natural inclusion map. On the other hand, [M, Theorem 1.5.2] and [Gi, Corollary 3.1.6], together with Proposition 3.5.1, identify this algebra with $\mathcal{O}(T_{\psi_K}^*(\check{G}/\check{U}))$ over $\mathcal{O}(\check{\mathfrak{g}}^*)$. Now the claim follows from Theorem 3.6.1. $\square$

Introduce the non-equivariant dg extension algebra $A'_{\mathbf{R}} = \text{RHom}_{\text{Ind}(\mathcal{D}^b(\text{Gr}_{\mathbf{R}}))}(\text{IC}_{\mathbf{R}}^0, \text{IC}_{\mathbf{R}}^0 \star \text{IC}_{\mathcal{O}(\check{G})})$ for $\mathcal{D}^b(\text{Gr}_{\mathbf{R}})$.

²In *loc. cit.*, the setting is for twisted loop groups but the same argument applies to the un-twisted setting.

Theorem 4.1.2. (1) *The dg algebra $A'_{\mathbf{R}}$ is formal.*

(2) *There is a $\mathbb{G}_m$ -equivariant isomorphism $H^*(A'_{\mathbf{R}}) \cong \mathcal{O}(\chi_{\check{M}}^{-1}(0)) \cong \mathcal{O}(\mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} \mathcal{N}'_{\mathrm{Sp}_{2p}})$*

Proof. (1) Note that there are isomorphisms of dg algebras

$$\begin{aligned} A'_{\mathbf{R}} &\cong \mathrm{RHom}(\mathrm{IC}_{\mathbf{R}}^0, \psi(\mathrm{IC}_{\mathcal{O}(\check{G})})) \cong \mathrm{RHom}(\mathrm{IC}_{\mathbf{R}}^0, p_* \mathrm{IC}_{\mathcal{O}(\check{G})}) \cong \mathrm{RHom}(p^* \mathrm{IC}_{\mathbf{R}}^0, \mathrm{IC}_{\mathcal{O}(\check{G})}) \cong \\ &\stackrel{(a)}{\cong} \mathrm{RHom}(\iota_{K*} \underline{\mathbf{C}}_{\mathrm{Gr}_K}, \mathrm{IC}_{\mathcal{O}(\check{G})}) \cong \mathrm{R}\Gamma(\mathrm{Gr}_K, \iota_K^! \mathrm{IC}_{\mathcal{O}(\check{G})}). \end{aligned}$$

Here $p : \mathrm{Gr} \rightarrow \mathrm{Gr}_{\mathbf{R}}$ is the specialization map such that $p_* \cong \psi : D_c^b(\mathrm{Gr}) \rightarrow D_c^b(\mathrm{Gr}_{\mathbf{R}})$ (see, e.g., [CMNO, Lemma 4.8]) and (a) follows from the observation that $p^{-1}(t^0) = \mathrm{Gr}_K$. Since $\iota_K^! \mathrm{IC}_{\mathcal{O}(\check{G})}$ is $!$ -pure with respect to the Iwahori stratification on Gr_K , by [BY, Lemma 3.1.5], the cohomology group $H^i(\mathrm{Gr}_K, \iota_K^! \mathrm{IC}_{\mathcal{O}(\check{G})})$ is pure of weight i , and the formality of $\mathrm{R}\Gamma(\mathrm{Gr}_K, \iota_K^! \mathrm{IC}_{\mathcal{O}(\check{G})})$ follows from the general criterion in [BF].

(2) Taking $\check{G}$ -invariants in the proof of Theorem 4.1.1(2) gives $H_K^*(\mathrm{pt}) \cong \mathcal{O}(\check{M})^{\check{G}}$ as graded augmented algebras: the right regular invariants of $\mathcal{O}(\check{G})$ are the constants. By [M, Proposition 2.1.1 and Remark 2.1.2], the complex $\iota_K^! \mathrm{IC}_{\mathcal{O}(\check{G})}$ is equivariantly formal. Hence we obtain isomorphisms

$$\begin{aligned} H^*(A'_{\mathbf{R}}) &\cong H^*(\mathrm{Gr}_K, \iota_K^! \mathrm{IC}_{\mathcal{O}(\check{G})}) \cong H_K^*(\mathrm{Gr}_K, \iota_K^! \mathrm{IC}_{\mathcal{O}(\check{G})}) \otimes_{H_K^*(\mathrm{pt})} \mathbf{C} \cong \\ &\cong \mathcal{O}(\check{M}) \otimes_{\mathcal{O}(\check{M}/\check{G})} \mathbf{C} \cong \mathcal{O}(\chi_{\check{M}}^{-1}(0)) \cong \mathcal{O}(\mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} \mathcal{N}'_{\mathrm{Sp}_{2p}}). \end{aligned}$$

□

4.2. Relative duality for linear periods and derived Satake for $\mathrm{U}(p, q)$. Let $\mathrm{QCoh}^{\check{G}}(\check{M})$ be the dg category of $\check{G}$ -quasi-coherent complexes on the dual variety $\check{M}$. Let $\mathrm{QCoh}^{\check{G}}(\check{M})^{\check{G}}$ be sheared category with respect to the $\mathbb{G}_m$ -action on $\check{M}$. According to Proposition 3.2.2, we have

$$(29) \quad \mathrm{QCoh}^{\check{G}}(\check{M})^{\check{G}} \cong \mathrm{QCoh}^{\mathrm{Sp}_{2p}}(\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[q-p+1] \times \mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p}).$$

Here for a finite dimensional vector space V and an integer d , we denote by $V[d]$ the dg scheme with ring of functions the dg algebra $\mathrm{Sym}^*((\Pi^d V)^*[-d])$ with trivial differentials, where $\Pi^d V$ is the super vector space V with parity equal to $d \bmod 2$.³

Denote by $\mathrm{QCoh}^{\check{G}}(\check{M})_{\mathcal{N}_{\check{\mathfrak{g}}^*}}^{\check{G}}$ the full subcategory consisting of complexes set theoretically supported on the pre-image of the nilpotent cone $\mu^{-1}(\mathcal{N}_{\check{\mathfrak{g}}^*})$ under the moment map $\mu : \check{M} \rightarrow \check{\mathfrak{g}}^*$. It follows from Lemma 3.3.2 that the above equivalence restricts to an equivalence

$$\mathrm{QCoh}^{\check{G}}(\check{M})_{\mathcal{N}_{\check{\mathfrak{g}}^*}}^{\check{G}} \cong \mathrm{QCoh}^{\mathrm{Sp}_{2p}}(\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[q-p+1] \times \mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p})_{\mathcal{N}'_{\mathrm{Sp}_{2p}}}$$

where the right hand side is the full-subcategory consisting of complexes supported on

$$\mathcal{N}'_{\mathrm{Sp}_{2p}} \times \{0\} \subset (\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[q-p+1]) \times (\mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p})$$

where $\mathcal{N}'_{\mathrm{Sp}_{2p}}$ is the exotic nilpotent cone for Sp_{2p} in Section 3.3.

³In particular, if d is odd, then $\Pi^d V$ is an odd vector space and $\mathrm{Sym}^*((\Pi^d V)^*[-d])$ is isomorphic to the symmetric algebra with generators in V^* in degree d .

Denote by $\mathrm{Perf}^{\check{G}}(\check{M})^{\mathbb{Z}} \cong \mathrm{Perf}^{\mathrm{Sp}_{2p}}(\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[q-p+1] \times \mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p})$ and $\mathrm{Perf}^{\check{G}}(\check{M})_{\mathcal{N}_{\check{G}}^*}^{\mathbb{Z}} \cong \mathrm{Perf}^{\mathrm{Sp}_{2p}}(\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[q-p+1] \times \mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p})_{\mathcal{N}'_{\mathrm{Sp}_{2p}}}$ be full subcategory of compact objects consisting of perfect complexes.

Theorem 4.2.1. (1) *There is an equivalence of dg categories*

$$\begin{aligned} \mathrm{Ind}(\Phi^b) : \mathrm{Ind}(\mathrm{D}_{\mathfrak{L}^+ \mathrm{U}(p,q)}^b(\mathrm{Gr}_{\mathrm{U}(p,q)})) &\cong \mathrm{QCoh}^{\check{G}}(\check{M})^{\mathbb{Z}} \\ &\cong \mathrm{QCoh}^{\mathrm{Sp}_{2p}}(\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[q-p+1] \times \mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p}) \end{aligned}$$

compatible with the action of derived Satake equivalence $\mathrm{Ind}(\mathrm{D}_{\mathfrak{L}^+ \mathrm{G}}^b(\mathrm{Gr}_{\mathrm{G}})) \cong \mathrm{QCoh}^{\check{G}}(\check{\mathfrak{g}}^*[2])$ on both sides. Moreover, it restricts to an equivalence

$$\begin{aligned} \Phi^b : \mathrm{D}_{\mathfrak{L}^+ \mathrm{U}(p,q)}^b(\mathrm{Gr}_{\mathrm{U}(p,q)}) &\cong \mathrm{Perf}^{\check{G}}(\check{M})^{\mathbb{Z}} \\ &\cong \mathrm{Perf}^{\mathrm{Sp}_{2p}}(\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[q-p+1] \times \mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p}) \end{aligned}$$

on compact objects.

(2) *There is an equivalence of dg categories*

$$\begin{aligned} \Phi : \mathrm{D}_{\mathfrak{L}^+ \mathrm{U}(p,q)}(\mathrm{Gr}_{\mathrm{U}(p,q)}) &\cong \mathrm{QCoh}^{\check{G}}(\check{M})_{\mathcal{N}_{\check{G}}^*}^{\mathbb{Z}} \\ &\cong \mathrm{QCoh}^{\mathrm{Sp}_{2p}}(\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[q-p+1] \times \mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p})_{\mathcal{N}'_{\mathrm{Sp}_{2p}}} \end{aligned}$$

compatible with the action of derived Satake equivalence $\mathrm{D}_{\mathfrak{L}^+ \mathrm{G}}(\mathrm{Gr}_{\mathrm{G}}) \cong \mathrm{QCoh}^{\check{G}}(\check{\mathfrak{g}}^*[2])_{\mathcal{N}_{\check{G}}^*}$ on both sides. Moreover, it restricts to an equivalence

$$\begin{aligned} \Phi^c : \mathrm{D}_{\mathfrak{L}^+ \mathrm{U}(p,q)}(\mathrm{Gr}_{\mathrm{U}(p,q)})^c &\cong \mathrm{Perf}^{\check{G}}(\check{M})_{\mathcal{N}_{\check{G}}^*}^{\mathbb{Z}} \\ &\cong \mathrm{Perf}^{\mathrm{Sp}_{2p}}(\wedge^2 \mathbb{A}^{2p}[2] \times T^* \mathbb{A}^{2p}[q-p+1] \times \mathfrak{gl}_{q-p}^*[2] // \mathrm{GL}_{q-p})_{\mathcal{N}'_{\mathrm{Sp}_{2p}}} \end{aligned}$$

on compact objects.

The same is true when $\mathrm{Gr}_{\mathrm{U}(p,q)}$ and $\mathfrak{L}^+ \mathrm{U}(p,q)$ are replaced by $\mathfrak{L}X$ and $\mathfrak{L}^+ \mathrm{G}$.

Proof. The same proof as in [CMNO, Theorem 5.5.] or [C, Theorem 4.8] in view of Proposition 2.4.4 and the Theorem 4.1.1. $\square$

4.3. Coherent sheaves on exotic nilpotent cones. Consider the full-subcategory

$$\mathrm{D}_{\{\mathfrak{L}^+ \mathrm{U}(p,q)\}}^b(\mathrm{Gr}_{\mathrm{U}(p,q)}) \subset \mathrm{D}^b(\mathrm{Gr}_{\mathrm{U}(p,q)})$$

consisting of complexes that are constructible with respect to the $\mathfrak{L}^+ \mathrm{U}(p,q)$ -orbits stratification on $\mathrm{Gr}_{\mathrm{U}(p,q)}$. Let

$$\mathrm{QCoh}^{\mathrm{GL}_{p+q}}(\mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} \mathcal{N}'_{\mathrm{Sp}_{2p}})^{\mathbb{Z}}$$

be the sheared dg-category of equivariant quasi-coherent complexes on the null-cone $\chi_{\check{M}}^{-1}(0) \cong \mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} \mathcal{N}'_{\mathrm{Sp}_{2p}}$. It has a natural pushforward functor to $\mathrm{QCoh}^{\check{G}}(\check{M})^{\mathbb{Z}}$.

Let $\mathrm{Perf}^{\mathrm{GL}_{p+q}}(\mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} \mathcal{N}'_{\mathrm{Sp}_{2p}})^{\mathbb{Z}} \subset \mathrm{QCoh}^{\mathrm{GL}_{p+q}}(\mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} \mathcal{N}'_{\mathrm{Sp}_{2p}})^{\mathbb{Z}}$ be the subcategory of compact objects.

Theorem 4.3.1. *There is an equivalence of dg categories*

$$\mathrm{Ind}(\mathrm{D}_{\{\mathfrak{L}^+ \mathrm{U}(p,q)\}}^b(\mathrm{Gr}_{\mathrm{U}(p,q)})) \cong \mathrm{QCoh}^{\mathrm{GL}_{p+q}}(\mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} \mathcal{N}'_{\mathrm{Sp}_{2p}})^{\mathbb{Z}}$$

which restricts to an equivalence

$$\mathrm{D}_{\{\mathfrak{L}^+ \mathrm{U}(p,q)\}}^b(\mathrm{Gr}_{\mathrm{U}(p,q)}) \cong \mathrm{Perf}^{\mathrm{GL}_{p+q}}(\mathrm{GL}_{p+q} \times^{\mathrm{Sp}_{2p}} \mathcal{N}'_{\mathrm{Sp}_{2p}})^{\mathbb{Z}}$$

on compact objects.

Proof. The same proof as in [CMNO, Theorem 5.5.] or [C, Theorem 4.8] in view of Proposition 2.4.4 and the Theorem 4.1.2. $\square$

Remark 4.3.2. Unlike Theorem 4.2.1, we do not know an analogy of Theorem 4.3.1 in the setting of $\mathfrak{LX}$.

4.4. Abelian equivalences. Put $\delta = q - p$, $H = \check{G}_X = \mathrm{Sp}_{2p}$, $V = \mathbb{A}^{2p}$, and $W = T^*V = V \oplus V^*$. Consider the dg algebras with zero differential

$$\mathfrak{A}_{p,q} = \mathrm{Sym}^* \left(\wedge^2 V[1] \oplus (\Pi^{\delta+1} W)[\delta] \oplus \bigoplus_{i=1}^{\delta} \mathbb{A}^1[2i-1] \right),$$

$$\mathfrak{B}_{p,q} = \mathrm{Sym}^* \left(\wedge^2 V[-2] \oplus (\Pi^{\delta+1} W)[- \delta - 1] \oplus \bigoplus_{i=1}^{\delta} \mathbb{A}^1[-2i] \right).$$

Write $\mathfrak{A} = \mathfrak{A}_{p,q}$ and $\mathfrak{B} = \mathfrak{B}_{p,q}$. Our convention is that $E[k]$ lies in cohomological degree $-k$, and the symmetric algebras use the sign $(-1)^{ij+\epsilon\eta}$ for cohomological degrees i, j and internal parities ϵ, η . All modules are left modules. Their internal parity is prescribed by the central element $(-1)^{\delta+1} \in H$; thus an independent parity shift does not give additional simple objects.

The algebra $\mathfrak{A}$ is finite-dimensional and nonpositive, while $\mathfrak{B}$ is polynomial with generators in strictly positive degrees and $\mathfrak{B}^0 = \mathbb{C}$. Let $D_{fd}^H(\mathfrak{A}\text{-mod})$ denote the category of equivariant dg modules with finite total cohomology, and let $D_{perf}^H(\mathfrak{B}\text{-mod})$ be the thick subcategory generated by $\mathfrak{B} \otimes U$ for finite-dimensional H -representations U . Using the H -equivariant self-dualities of $\wedge^2 V$ and W , Koszul duality gives

$$\kappa : D_{perf}^H(\mathfrak{B}\text{-mod}) \xrightarrow{\sim} D_{fd}^H(\mathfrak{A}\text{-mod}), \quad M \mapsto \mathbb{C} \overset{\mathbf{L}}{\otimes}_{\mathfrak{B}} M,$$

where the derived fiber carries its Koszul $\mathfrak{A}$ -action; see [BFGT, Section 3.8]. Equivalently, κ is the inverse of $N \mapsto \mathrm{RHom}_{\mathfrak{A}}(\mathbb{C}, N)$ under $\mathfrak{B} \simeq \mathrm{REnd}_{\mathfrak{A}}(\mathbb{C})^{\mathrm{op}}$.

The standard t -structure on $D_{fd}^H(\mathfrak{A}\text{-mod})$ is bounded, with heart the finite-dimensional H -equivariant $H^0(\mathfrak{A})$ -modules. Hence

$$D_{fd}^H(\mathfrak{A}\text{-mod})^{\heartsuit} \cong \begin{cases} \wedge^* W\text{-mod}(\mathrm{Rep}(H)), & p = q, \\ \mathrm{Rep}(H), & p < q. \end{cases}$$

Recall from Theorem 4.2.1 the derived Satake equivalence

$$\Phi^b : D_{\mathfrak{L}^+U(p,q)}^b(\mathrm{Gr}_{U(p,q)}) \xrightarrow{\sim} \mathrm{Perf}^H(\wedge^2 V[2] \times W[\delta+1] \times \mathfrak{gl}_{\delta}^*[2] // \mathrm{GL}_{\delta}) \xrightarrow{\sim} D_{perf}^H(\mathfrak{B}\text{-mod})$$

Theorem 4.4.1. *The composed equivalence*

$$\kappa \circ \Phi^b : D_{\mathfrak{L}^+ \cup (p,q)}^b(\mathrm{Gr}_{\mathrm{U}(p,q)}) \xrightarrow{\sim} D_{fd}^{\mathrm{Sp}_{2p}}(\mathfrak{A}_{p,q}\text{-mod})$$

is t -exact for the perverse t -structure on the left and the standard t -structure on the right. It induces an equivalence of abelian categories

$$\mathrm{Perv}_{\mathfrak{L}^+ \cup (p,q)}(\mathrm{Gr}_{\mathrm{U}(p,q)}) \cong \begin{cases} \wedge^*(T^*\mathbb{A}^{2p})\text{-mod}(\mathrm{Rep}(\mathrm{Sp}_{2p})), & p = q, \\ \mathrm{Rep}(\mathrm{Sp}_{2p}), & p < q. \end{cases}$$

Here modules and representations are finite-dimensional. The same holds for $\mathfrak{LX}$ with its $\mathfrak{L}^+ \mathbf{G}$ -action.

Proof. Let $\lambda \in \Lambda_A^+$ be a dominant real coweight, and let $V^\lambda \in \mathrm{Rep}(\mathrm{Sp}_{2p})$ and $\mathrm{IC}_{\mathbf{R}}^\lambda$ be the corresponding irreducible representation and IC complex. It follows from the construction that

$$\Phi^b(\mathrm{IC}_{\mathbf{R}}^\lambda) \cong \mathfrak{B}_{p,q} \otimes V^\lambda, \quad \kappa(\mathfrak{B}_{p,q} \otimes V^\lambda) \cong V^\lambda.$$

Hence $\kappa \circ \Phi^b(\mathrm{IC}_{\mathbf{R}}^\lambda) \cong V^\lambda$ lies in the standard heart. According to Proposition 2.4.4(1), the simple perverse sheaves are the $\mathrm{IC}_{\mathbf{R}}^\lambda$. Since every object of the perverse heart has finite length, $\kappa \circ \Phi^b$ sends this heart into the standard heart. Both t -structures are bounded, so $\kappa \circ \Phi^b$ is t -exact and induces an equivalence

$$\mathrm{Perv}_{\mathfrak{L}^+ \cup (p,q)}(\mathrm{Gr}_{\mathrm{U}(p,q)}) \cong D_{fd}^{\mathrm{Sp}_{2p}}(\mathfrak{A}_{p,q}\text{-mod})^\heartsuit.$$

The theorem follows from the description of the heart above. The assertion for $\mathfrak{LX}$ follows from Proposition 2.4.5. $\square$

REFERENCES

- [BFGT] A. Braverman, M. Finkelberg, V. Ginzburg, and R. Travkin, “Mirabolic Satake equivalence and supergroups,” *Compositio Mathematica* 157 (2021), no. 8, 1724–1765. DOI: 10.1112/S0010437X21007387.
- [BZSV] D. Ben-Zvi, Y. Sakellaridis, and A. Venkatesh, “Relative Langlands Duality,” *arXiv:2409.04677* (2024).
- [BF] R. Bezrukavnikov, M. Finkelberg, Equivariant Satake category and Kostant-Whittaker reduction, *Moscow Mathematical Journal*, 2008, Volume 8, Number 1, Pages 39–72
- [BY] R. Bezrukavnikov and Z. Yun, “On Koszul duality for Kac–Moody groups,” *Representation Theory* 17 (2013), 1–98.
- [C] T.-H. Chen, A relative Langlands dual realization of $T^*\mathbf{G}/\mathbf{K}$ and derived Satake, *arXiv:2601.18022*
- [CDMN] T.-H. Chen, S. K. Devalapurkar, M. Macerato, D. Nadler, Relative Langlands duality for Hermitian symmetric spaces, in preparation.
- [CMNO] T.-H. Chen, M. Macerato, D. Nadler, J. O’Brien, Quaternionic Satake equivalence, *Compositio Mathematica*, Volume 161, Issue 7 (2025), 1615–1663
- [CN] T.-H. Chen, D. Nadler, Real groups, symmetric varieties and Langlands duality, *arXiv:2403.13995*
- [CY] T.-H. Chen, L. Yi, Singularities of orbit closures in loop spaces of symmetric varieties, *arXiv:2310.20006*
- [FJ] S. Friedberg, H. Jacquet, “Linear periods,” *Journal für die reine und angewandte Mathematik* 443 (1993), 91–139
- [GN] D. Gaitsgory, D. Nadler. Spherical varieties and Langlands duality, *Moscow Math. J.* 10 (2010), no. 1, (Special Issue: In honor of Pierre Deligne), 65–137.
- [Ga] T. Gannon, The cotangent bundle of $\mathbf{G}/\mathbf{U}_{\mathbf{P}}$ and Kostant-Whittaker descent, *arXiv:2407.16844*
- [Gi] V. Ginzburg, Pointwise purity, derived Satake, and symplectic duality, *arXiv:2508.15958*
- [K] B. Kostant, Lie group representations on polynomial rings, *American Journal of Mathematics* 85 (1963), no. 3, 327–404.

- [Ka] S. Kato, An exotic Deligne-Langlands correspondence for symplectic groups, *Duke Math. J.* 148, no. 2 (2009), 305–371
- [Kn2] F. Knop, [MathOverflow, answer 319822](#).
- [Kn1] F. Knop. “Automorphisms, root systems, and compactifications of homogeneous varieties”. *J. Amer. Math. Soc.* 9.1 (1996), pp. 153–174. <https://doi.org/10.1090/S0894-0347-96-00179-8>.
- [L] V. Lafforgue, Classical limits of functoriality kernels and values of L-functions, unpublished notes
- [M] M. Macerato, Levi-equivariant restriction of spherical perverse sheaves, [arXiv:2309.07279](#)
- [N] D. Nadler, Perverse sheaves on real loop Grassmannians, *Invent. Math.* 159 (2005), no. 1, 1–73.
- [Ngo] B.-C. Ngô, Le lemme fondamental pour les algèbres de Lie, *Publ. Math. Inst. Hautes Études Sci.* 111 (2010), pp. 1–169.
- [S] G.W. Schwarz, Representations of simple Lie groups with regular rings of invariants (MSN article). *Invent. Math.* 49 (1978), no. 2, 167–191

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