BOTT PERIODICITY AT THE PRIME 2 IN THE UNSTABLE HOMOTOPY OF SPHERES

M. MAHOWALD AND A. UNELL

These are lecture notes of Mahowald's and Unell's, for which I found only three references to the published literature (one of which was pointed out by Charles Rezk):

- J. P. May. Applications and generalizations of the approximation theorem. *Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978)*, pp. 3869, Lecture Notes in Math., 763, *Springer, Berlin*, 1979.
- M. Mahowald. Some homotopy classes generated by η_j. *Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), pp. 2337,* Lecture Notes in Math., 763, *Springer, Berlin-New York*, 1979.
- N. Kuhn. The geometry of the James-Hopf maps. *Pacific J. Math.* 102 (1982), no. 2, 397412.

I asked about these notes on MathOverflow[1], and Peter May said that he had a copy of them. There are a few typos, and some results are incorrect. Note that the two pages before Chapter 5 were printed in the wrong order.

The notes are extremely interesting, and contain some results which I couldn't find in the published literature, so I thought it would be a good idea to have an online copy of it. Thanks to Peter May for lending me a copy of the notes, and for letting me post it online. Thanks also to Catherine Ray for letting me use the scanner at Northwestern.

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[1] https://mathoverflow.net/questions/334095/lecture-notes-by-mahowald-and-unell

Bott Periodicity at the Prime 2
in the Unstable Homotopy of Spheres

by

Mark Mahowald and Alan Unell

Based on lectures by Mark Mahowald
and notes by Alan Unell

✓ 6.1.2 false
Milgram [illegible]
(6.2.7, above 6.2.10)
✓ 6,2,12 proof?
✓ above 6.4.3:
Triviality to
✓ obtain [illegible]
✓ include [22]

251-0648

✓ Cohen and
homology op's

adams-Priddy
bo ~ bso

These notes are the outcome of a series of lectures given by the first author originally in the fall of 1969 during which the results announced in [20] were obtained. During the years after this seminar the unstable setting was studied and results related to this were described in [22]. Finally the results about ring spectra which are Thom complexes were found. A second series of lectures going through this material was held during the spring of 1977, and the second named author took the notes. This is a report of that seminar. The new mathematics described here represents primarily the work of the seminar author. Between 1969 and the present other workers have become interested in this material. See Milgram's article in [27] for example. The presentation here has profited from that interest.

We would also like to express our thanks to Vicki Davis for an excellent job of typing this manuscript.

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Table of Contents

Chapter 1

Introduction

1.1. v_1-periodicity.

These notes will study the 2-primary homotopy of S^n for all n. All homotopy groups will be 2 primary homotopy groups, unless otherwise stated, and all cohomology groups will be with $\mathbb{Z}_2$ for coefficients. The primary emphasis will be on the stable image of the J homomorphism and elements, stable and unstable, which are related to them. Much of the material here represents new work of the first author and some of it has been announced in various places [20], [21], and [22]. In particular Chapter 9 contains details of the results of [22], among other things. The central result there can be summarized by the following key theorem which needs some notation to state. In Chapter 8 we will define "v_1-periodic" elements. Heuristically they are a sequence of elements $\{\alpha\}$, $\alpha_i \epsilon \pi_{i\cdot 2^k+j+n}(S^n)$ for some $k \geq 3$, $\alpha_i \neq 0$, and α_i and α_{i-1} are related by a particular Toda bracket. Elements in the image of J are "v_1-periodic". We will define a spectrum J such that under $S^0 \to J$ there is an isomorphism of v_1-periodic elements.

Theorem 1.1.1. The "v_1-periodic" elements in $\pi_*(S^{2n+1})$ are mapped isomorphically to the "v_1-periodic" of $\pi_*(P^{2n} \wedge J)$ under the composite map $\Omega^{2n}S^{2n+1} \to Q(\Sigma P^{2n}) \to Q(\Sigma P^{2n} \wedge J)$ where the first is the Snaith map [32] and the second is the Hurewicz homomorphism.

1.2. EHP sequences.

In this section we will introduce several spectral sequences which are useful for understanding the point of view which lead to the results discussed here. Very little use will be made of this material directly.

First we will be interested in studying several spectral sequences which are given by the following.

Theorem 1.2.1. There is a mapping between towers of fibration

$$\begin{tikzcd}
\mathcal{P} & Q(P^1) \arrow[r] & QP^2 \arrow[r] & QP^3 \arrow[r] & \cdots \arrow[r] & QP^n \arrow[r] & QP^{n+1} \arrow[r] & \cdots \\
\mathcal{S} & \Omega^2S^2 \arrow[r] \arrow[u] & \Omega^3S^3 \arrow[r] \arrow[u] & \Omega^4S^4 \arrow[r] \arrow[u] & \cdots \arrow[r] & \Omega^{n+1}S^{n+1} \arrow[r] \arrow[u] & \Omega^{n+2}S^{n+2} \arrow[r] \arrow[u] & \cdots \\
\mathcal{O} & SO(2) \arrow[r] \arrow[u] & SO(3) \arrow[r] \arrow[u] & SO(4) \arrow[r] \arrow[u] & \cdots \arrow[r] & SO(n+1) \arrow[r] \arrow[u] & SO(n+2) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

<u>Proof</u>. The top diagram follows from Snaith's theorem [32] and the bottom is the Whitehead J-homomorphism.

There are a variety of functors which can be applied to these towers.

1) Ordinary homology. The Serre spectral sequence for each fibration

$$QP^{n-1} \to QP^n \to QS^n$$

$$\Omega^nS^n \to \Omega^{n+1}S^{n+1} \to \Omega^{n+1}S^{2n+1}$$

$$SO(n) \to SO(n+1) \to S^n$$

collapses. Thus the homology of each is easily described.

Theorem 1.2.2

$$E^0H_*(QP^n) \cong \bigotimes_{j\le n} H_*(QS^j)$$

$$E^0H_*(\Omega^{n+1}S^{n+1}) \cong \bigotimes_{j\le n} H_*(\Omega^{j+1}S^{2j+1})$$

$$E^0H_*(SO(n-1) \cong \bigotimes_{j\le n} H_*(S^j).$$

and the maps between the left hand sides are induced by the standard maps $S^j \to \Omega^{j+1}S^{2j+1} \to QS^j$.

Proof. The parts dealing with each sequence separately is standard. The only possible new thing is the observation that the Snaith map in homology induces the usual map from $\Omega^{j+1}S^{2j+1} \to QS^j$, i.e., is a $j+1$ loop map. To see this note that the composite $\Omega^{n+1}S^{n+1} \to QP^n \to QS^n$ is the loops n times of the composite $\Omega S^{n+1} \to \Omega S^{2n+1} \to QS^{2n}$. Thus

$$\begin{tikzcd}
QS^{2n-1} \arrow[r] & Q\Sigma^{n-1}P^n \arrow[r] & Q(S^{2n-1}) \\
\Omega S^n \arrow[r] \arrow[u] & \Omega^2 S^{n+1} \arrow[r] \arrow[u, "g"'] & \Omega^2 S^{2n+1} \arrow[u, "f"']
\end{tikzcd}$$

Commutes with f being a double loop map. Notice that at most g is a loop map. In fact it probably is not a loop map at all. But we now can continue by induction to conclude that $\Omega^{n-1}f$ is a $n+1$ fold loop map.

2) Homotopy functor. This gives the three standard "EHP" type spectral sequences.

$$E_1^{s,t}(\mathcal{O}) = \pi_t(S^s)$$

$$E_1^{s,t}(\mathcal{S}) = \pi_t(\Omega^{s+1}S^{2s+1})$$

$$E_1^{s,t}(\mathcal{P}) = \pi_t(Q(S^s))$$

The maps between the E_1 terms are again the stablization maps. Note one important property. The E_1 term for each is itself a result of the calculations of $E(\mathcal{S})$. To calculate $\pi_\ell(S^{2n+1})$ we start with $E_1^{*,*} = \underset{s\leq n}{\oplus}\,\pi_*(\Omega^{s+1}S^{2s+1})$. The point is that we need information about $\pi_j(S^{2s+1})$ for $j < \ell$. This spectral sequence is a bootstrap operation. This is, in part, the approach taken by Toda [34] and his school.

3) Adams spectral sequence type functors.

In Chapter 3 we will describe an Adams' type spectral sequence for S^n with the property: there is a map of spectral sequences $f_n\colon E_r^{s,t}(S^n) \to E_r^{s,t}(S^0)$ where $E_r^{s,t}(S^0)$ is the stable Adams spectral sequence and at E_2 level f_n is an isomorphism for $t - s < n - 1$.

For many spaces a similar unstable spectral sequence exists. In particular if $SO = \underset{n}{\cup} SO(n)$ then $E_2^{s,t}(SO)$ is the E_2 term for such a spectral sequence. Details are in Chapter 3.

Theorem 1.2.2. For each sequence $\mathcal{O}$, $\mathcal{S}$, and $\mathcal{P}$ there is a spectral sequence whose E_1 term is

$$E_1^{\sigma,s,t}(\mathcal{O}) = E_2^{s,t-\sigma}(S^{\sigma})$$

$$E_1^{\sigma,s,t}(\mathcal{S}) = E_2^{s-1,t-\sigma}(S^{2\sigma+1})$$

$$E_1^{\sigma,s,t}(\mathcal{P}) = \mathrm{Ext}_A^{s,t-\sigma}(Z_2,Z_2)$$

and

$$\bigoplus_{\sigma} E_\infty^{\sigma,s,t}(\mathcal{O}) \cong E_2^{s,t}(SO)$$

$$\bigoplus_{\sigma} E_\infty^{\sigma,s,t}(\mathcal{S}) \simeq E_2^{s,t}(\mathcal{S}) = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2,Z_2)$$

$$\bigoplus_{\sigma} E_\infty^{\sigma,s,t}(\mathcal{P}) \simeq \mathrm{Ext}_A^{s,t}(\tilde{H}^*(P),Z_2).$$

This will take a little work to set up the machinery. Note that no claim is made about maps between these sequences. There exist ways of doing things so there are maps but then one can hardly identify the objects.

1.3. bo resolutions

Let bo be the Ω-spectrum given by Bott periodicity. This spectrum is a ring spectrum with a unit and $H^*(bo) = A/A(Sq^1,Sq^2)$. (Unless otherwise noted coefficient groups are always $\mathbb{Z}_2$.) We will assume that the reader is familiar with the standard properties of bo.

Associated to a spectrum with unit, like bo, we have a tower of spaces

$$\begin{tikzcd}
S^0 \arrow[d] & S_1 \arrow[l] \arrow[d, "id\wedge i"] & S_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & S_s \arrow[l] \arrow[d] & S_{s+1} \arrow[l] & \cdots \arrow[l] \\
bo & S_1 \wedge bo & S_2 \wedge bo & & S_s \wedge bo & &
\end{tikzcd}$$

where $S_s \wedge bo \xleftarrow{id \wedge i} S_s \leftarrow S_{s+1}$ is a fibration and $i: S^0 \to bo$ is the unit. If we use the homotopy functor we get an exact couple with $E_1^{s,t} = \pi_{t-s}(S_s \wedge bo)$. Under reasonable hypothesis $E_\infty^{*,*}$ is an associated graded group of $\pi_*(S^0)$. This is true for bo since $\pi_j(S_s) = 0$ $j < 3s$ and so for $t - s < 3s$ $E_r^{s,t} = E_\infty^{s,t}$ for large enough r. It is also true if bo is replaced by K(Z). This spectral sequence will be written $E_r(S^0, bo, \pi)$.

Clearly $\pi_*(bo)$ acts on E_r and each (E_r, d_r) is a $\pi_*(bo)$ module. A $\pi_*(bo)$ module M is said to be $\mathbb{Z}_2$-vector space if the $\pi_* bo$ action factors through the map $\pi_* bo \to \mathbb{Z}_2$ given by i_* where $i: bo \to K(\mathbb{Z}_2, 0)$ is the obvious degree one map. Under the action of $\pi_*(bo)$ the class which generates $\pi_8(bo)$ plays the role of v_1^4 and classes which have iterates of this class non-zero are v_1-periodic. Precise definitions are given in Chapter 8.

Chapters 7 and 8 investigate this spectral sequence in some detail. The principle result is

Thoerem 1.3.1. a) $E_\infty^{s,t}(S^0, bo, \pi) = \mathbb{Z}$ $t = 0$

$= \mathbb{Z}_2$ $t = 1,2 \bmod 8$

$= 0$ all other t.

b) $E_\infty^{1,t}(S^0, bo, \pi) = \mathbb{Z}_2\rho(k)$ $t = 4k$

$= \mathbb{Z}_2$ $t = 1,2 \bmod 8$

$= 0$ otherwise.

where $\rho(k)$ is defined by $4k \equiv 2^{\rho(k)-1} \bmod 2^{\rho(k)}$.

c) $E^{s,t}(S^0, bo, \pi) = 0$ for $6s \geq t + 6$ and is a $\mathbb{Z}_2$ vector space as a $\pi_*(bo)$ module for all $s > 1$ and all t.

The proof of this result uses much of the theory developed in these notes. The final steps are in §8.3. The vanishing line asserted in part c is an immediate consequence of 4.4.12. Note that this vanishing line prevents any v_1 periodicity from arrising anonymously. The only v_1 periodicity possible is what occurs from part a and b.

Chapter 2

The Λ-algebra

2.1. Statement of the results

In this chapter we will develop the Λ-algebra [8] to facilitate calculations as well as to prove Theorem 1.2.4. The development given here is a modification of the approach of Priddy [29]. In Chapter 3 we will discuss unstable resolutions. The main result of these two chapters can by summarized by

Theorem 2.1.1 [14]. For every $n > 0$ there exists a graded differential chain complex $(\Lambda(n),d)$ such that

2.1.2a) $\Lambda(n)$ is the Z_2 vector space generated by symbols $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_\ell}$ for $I = (i_0,\ldots,i_\ell)$ such that $2i_j \geq i_{j+1}$ for $j \leq \ell - 1$ and $i_0 < n$

2.1.3b) $d(\lambda_n) = \sum_{j+k=n} \binom{j}{k} \lambda_j \lambda_{k-1}$ and d is a derivation with respect to the product. The product satisfies

$$\lambda_i \lambda_{2i+m} = \sum_{j\geq 0} \binom{m-j-1}{j} \lambda_{i+m-j} \lambda_{2i+1+j}.$$

2.1.4c) $H_{**}(\Lambda(n),d) \cong E_2^{**}S^n$ where $E_2^{**}S^n$ is the E_2-term for the unstable Adams' spectral sequence for S^n.

The two gradings arise by assigning λ_i bidegree $(1,i+1)$. Then the first grading represents the length of an element and the second represents the internal degree.

2.2 Some auxiliary algebras

As a first step towards proving Theorem 2.1.1 consider the algebra with unit over $\mathbb{Z}_2$, $\bar{\bar{A}}$, generated by symbols Sq^a, $a > 0$ an integer. These symbols are subject to the relation $Sq^a Sq^b = 0$ if $a < 2b$. Note that as $\mathbb{Z}_2$-vector spaces, $\bar{\bar{A}}$ is isomorphic to A, the mod-2 Steenrod algebra.

Recall the following definitions and lemmas.

Definition 2.2.1. Let B be a graded connected algebra over R a commutative ring with unit, for example $\bar{\bar{A}}$ over $\mathbb{Z}_2$. Let M and N be modules over B and $f\colon M \to N$ be a B-map. Then f is minimal if $\ker f \subseteq IB\cdot M$. Here IB is the $\ker \epsilon$, the augmentation, $\epsilon\colon B \to R$. A B resolution $\{\mathcal{C}_s, d_s\}$ of a B-module is minimal if each d_s is a minimal B-module homomorphism.

Lemma 2.2.2 [28]. Suppose $IB\cdot R = 0$, B and R as above, and

$$0 \longleftarrow M \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \cdots$$ is a B-minimal resolution of M, a B-module. Then $d_s^*\colon \operatorname{Hom}_B^t(C_{s-1}, R) \to \operatorname{Hom}_B^t(C_s, R)$ are zero homomorphisms. The super script t denotes those maps which decrease filtration by t.

The proof is an easy exercise. Details may be found in [28].

We now obtain

Corollary 2.2.3. $\operatorname{Ext}_B^{s,t}(M,R) \cong \operatorname{Hom}_B^t(C_s, R)$ for $B, R, M, \{C_s\}$ as above.

Proposition 2.2.4. $\operatorname{Ext}^{s,t}_{{}_\ell\bar{A}}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \Lambda^{s,t}$ as $\mathbb{Z}_2$ vector spaces where λ_I has filtration $(\ell, \sum_{j=1} (i_j + 1))$ for $I = (i_0, \ldots, i_\ell)$.

Proof: Let $L_n = \{Sq^n, Sq^{2n+j}Sq^n,\ j \geq 0;\ Sq^{2n+2j+k}Sq^{2n+j}Sq^n, j,k \geq 0$ etc.$\}$. Let $\epsilon: \overline{\overline{A}} \to \mathbb{Z}_2$ be the augmentation.

$I\overline{\overline{A}} = \bigoplus_{n>0} L_n$. Thus we can exhibit an explicit minimal $\overline{\overline{A}}$ resolution of $\mathbb{Z}_2$ as follows:

$$\mathbb{Z}_2 \xleftarrow{\epsilon} \overline{\overline{A}} \xleftarrow{d_0} \bigoplus_{i_0>0} A\,\sigma_{i_0} \xleftarrow{d_1} \bigoplus_{\substack{i_0>0\\ i_1<2i_0}} \overline{\overline{A}}\,\sigma_{i_1}\sigma_{i_0} \xleftarrow{d_2} \underset{\substack{i_0>0\\ i_1<2i_0\\ i_2<2i_1}}{} \overline{\overline{A}}\,\sigma_{i_2}\sigma_{i_1}\sigma_{\epsilon} \leftarrow \cdots$$

where $d_j(\sigma_{i_j} \cdots \sigma_{i_0}) = Sq^{i_j}\sigma_{i_{j-1}} \cdots \sigma_{i_0} \epsilon L_{i_j}\sigma_{i_{j-1}} \cdots \sigma_{i_0}$.

This sequence is clearly acyclic and minimal. Applying $\mathrm{Hom}_{\overline{\overline{A}}}(\ ,\mathbb{Z}_2)$ to this sequence we see that by Corollary 2.2.3 $\mathrm{Ext}^{s,t}_{\overline{\overline{A}}}(\mathbb{Z}_2,\mathbb{Z}_2) \cong \mathrm{Hom}^t_{\overline{\overline{A}}}(C_s,\mathbb{Z}_2)$ where C_s is the s^{th} term in the resolution. There is an anti isomorphism of $\mathbb{Z}_2$ vector spaces $\psi: \mathrm{Ext}^{s,t}_{\overline{\overline{A}}}(\mathbb{Z}_2,\mathbb{Z}_2) \to \Lambda^{s,t}$ given by $\psi(\sigma_j) = \lambda_{j-1}$. Here σ_j represents its own image in $\mathrm{Ext}_{\overline{\overline{A}}}(\mathbb{Z}_2,\mathbb{Z}_2)$.

Now define $\overline{A}$ to be the algebra with identity over $\mathbb{Z}_2$ generated by the symbols Sq^a, $a > 0$, an integer, subject to the relation

2.2.5 $$Sq^a Sq^b = \sum_{j=1}^{[a/2]} \binom{b-j-1}{a-2j} Sq^{a+b-j}Sq^j \text{ for } a < 2b.$$

Note that as $\mathbb{Z}_2$ vector $\overline{\overline{A}}$ is isomorphic to $\overline{A}$

Proposition 2.2.6. There is an anti isomorphism of $\mathrm{Ext}_{\overline{A}}(\mathbb{Z}_2,\mathbb{Z}_2)$ with Λ as algebras over $\mathbb{Z}_2$.

Proof: Consider the following sequence of $\overline{A}$ modules.

$$\mathbb{Z}_2 \xleftarrow{\varepsilon} \bar{A} \xleftarrow{d_0} \bigoplus_{i_0>0} \bar{A}\,\sigma_{i_0} \xleftarrow{d_1} \bigoplus_{\substack{i_0>0\\ i_0<2i_1}} \bar{A}\,\sigma_{i_1}\sigma_{i_0} \leftarrow \cdots$$

Here ε is the augmentation

$$d_r\sigma_{i_r}\cdots\sigma_{i_0} = Sq^{i_r}\sigma_{i_{r-1}}$$

$$+\sum_{j=1}^{[\ell_r/2]} \genfrac{}{}{0pt}{}{i_{r-1}-j-1}{i_r-2j}\, Sq^{i_r+i_{r-1}-j}\,\sigma_j(\sigma_{i_{r-1}}\cdots\sigma_{i_0})$$

Note that $j < 2i_{r-1}$. The chain complex 2.2.5 is acyclic and minimal which can easily be checked by the reader. Applying $\mathrm{Hom}_{\bar{A}}(-;\mathbb{Z}_2)$ to 2.2.5 and taking homology we obtain an algebra $\bar{\Sigma}^{s,t} = \mathrm{Ext}_{\bar{A}}^{s,t}(\mathbb{Z}_2,\mathbb{Z}_2)$ where the product is the Yoneeda composition.

σ_a will denote the element in $\mathrm{Ext}_{\bar{A}}^{1,a}(\mathbb{Z}_2,\mathbb{Z}_2)$ dual to σ_a. $\bar{\Sigma}$, as a module, has a homogeneous basis σ_I where $I = (i_0,\ldots,i_\ell)$ and $i_j < 2i_{j+1}$.

The anti-homomorphism $\varphi(\sigma_j) = \lambda_{j-1}$ is clearly an isomorphism of $\mathbb{Z}_2$-vector spaces. We need only check that the relations are carried isomorphically and to do this we calculate the Yoneeda product $\sigma_a\sigma_b$.

Since σ_a is a cocycle there is a unique map $f_a\colon \mathrm{im}\, d_0 \to \mathbb{Z}_2$ such that $f_a d_0 = \sigma_a$. We can define maps $(\sigma_a)_0\colon \bigoplus_{i_0>0} \bar{A}\,\sigma_{i_0} \to \bar{A}$ and

$$(\sigma_a)_1\colon \bigoplus_{\substack{i_0>0\\ i_1<2i_0}} \bar{A}\,\sigma_{i_1}\sigma_{i_0} \to \bigoplus_{i_0>0} \bar{A}\,\sigma_{i_0}$$

by

$$(\sigma_a)_0 Sq^I \sigma_{i_0} = \begin{cases} 0 & i_0 \neq a \\ Sq^I & i_0 = a \end{cases}$$

$$(\sigma_a)_1(\sigma_{i_1}\sigma_{i_0}) = \begin{cases} \binom{i_0-a-1}{i_1-2a}\sigma_{i_0+i_1-a} & \\ 0 & \text{otherwise} \end{cases}$$

Clearly the following diagram commutes

$$\begin{tikzcd}
\operatorname{im} d_0 \arrow[d, "f_a"] & \bigoplus_{i_0>0} \bar{A}\sigma_{i_0} \arrow[l, "d_0"'] \arrow[d, "(\sigma_a)_0"] & \bigoplus_{\substack{i_0>0 \\ i_1<2i_0}} \bar{A}\sigma_{i_1}\sigma_{i_0} \arrow[l, "d_1"'] \arrow[d, "(\sigma_a)_1"] \\
Z_2 & \bar{A} \arrow[l, "\epsilon"'] & \bigoplus_{i_0>0} \bar{A}\sigma_{i_0} \arrow[l]
\end{tikzcd}$$

and $(\sigma_a)_0$ and $(\sigma_a)_1$ are unique up to chain homotopy.

Let σ_b be a cocycle with $b < 2a$. Then the composition $\sigma_b(\sigma_a)_1$ represents the Yoneeda composition of σ_a with σ_b.

Since $\{\sigma_{i_1}\sigma_{i_0} \mid i_0 > 0,\ i_1 < 2i_0\}$ form a basis for the vector space of elements of length 2 we can compute $\sigma_b\sigma_a$ for $b > 2a$. Now

$$\sigma_b\sigma_a(\sigma_{i_1}\sigma_{i_0}) = \begin{cases} \binom{i_0-a-1}{i_1-2a} & \text{if } i_0 + i_1 = a + b \\ 0 & \text{otherwise.} \end{cases}$$

2.2.7. Thus $\sigma_b \sigma_a = \sum_{i_0+i_1=a+b} \binom{i_0-a-1}{i_1-2a} \sigma_{i_1} \sigma_{i_0}$.

Now, consider the anti homomorphism $\sigma_k \xrightarrow{\varphi} \lambda_{k-1}$. We will show that these relations are carried to the relations in Λ.

Letting $i_1 = j$ we obtain

$$\sigma_b \sigma_a = \sum_{j=2a}^{2[\frac{a+b}{3}]} \binom{b-j-1}{j-2a} \sigma_j \sigma_{a+b-j}$$

Applying the anti homomorphism φ and letting

$$a = i + 1; \; b = 2i + 1 + n, \; n \geq 1; \quad k = j - 2a$$

we obtain

$$\lambda_i \lambda_{2i+1+n} = \sum_{j>0} \binom{n-j-1}{j} \lambda_{i+n-j} \lambda_{2i+1+j}$$

which are precisely the relations in the Λ-algebra. Thus $\mathrm{Ext}_{\bar{A}}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \Lambda$ as algebras.

2.3. The resolution for A.

The mod-2 Steenrod algebra is generated by symbols Sq^a, $a \geq 0$, subject to the relations

$$Sq^a Sq^b = \binom{b-1}{a} Sq^{a+b} + \sum_{j=1}^{[a/2]} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j .$$

In order to construct an A resolution for $\mathbb{Z}_2$, as above, we will add this relation to the $\bar{A}$ resolution for $\mathbb{Z}_2$ to obtain

Theorem 2.3.1. $\mathrm{Ext}_A^{**}(\mathbb{Z}_2,\mathbb{Z}_2) \cong H_{**}(\Lambda,d)$ where $d(\lambda_i) = \sum_{j+k=i} \binom{j}{k}\lambda_j\lambda_{k-1}$ and d is a derivation with respect to products.

Proof: Consider the resolution of Proposition 2.2.6. We convert it to an A-resolution of $\mathbb{Z}_2$ as follows.

$$\mathcal{C}: \quad \mathbb{Z}_2 \xleftarrow{\epsilon} A \xleftarrow{d_0} \bigoplus_{i_0>0} A\,\sigma_{i_0} \xleftarrow{d_1} \bigoplus_{\substack{i_0>0 \\ i_1<2i_0}} A\,\sigma_{i_1}\sigma_{i_0} \leftarrow \cdots$$

where ϵ is the usual augmentation

$$d_0(\sigma_{i_0}) = Sq^{i_0}$$

$$d_1(\sigma_{i_1}\sigma_{i_0}) = \binom{i_0-1}{i_1}\sigma_{i_1+i_0} + Sq^{i_1}\sigma_{i_0} + \sum_{j=1}^{[i_1/2]}\binom{i_0-1-j}{i_1-2j} Sq^{i_0+i-j}\sigma_j$$

Applying $\mathrm{Hom}_A(-,\mathbb{Z}_2)$ we see that the algebra generated by σ_i's is isomorphic to Σ. Taking homology we have that the nonzero part of d are the first terms of $d_i(\sigma_k\sigma_I)$. This induces a differential d' on the algebra Σ, namely $d'(\sigma_{k+n}) = \sum_{j+\ell=k+n} \binom{j}{\ell}(\sigma_j\sigma_{\ell-1})$. Thus, the desired differential in Λ is obtained by applying φ to (Σ,d'). Therefore taking homology and applying the anti homomorphism $\sigma_k \to \lambda_{k-1}$ we obtain the isomorphism of the theorem.

2.4. Brown-Gitler approach.

In this section we will describe a second approach to resolutions over the Steenrod algebra which is based on a conversation with

Ed Brown. It is related to the Brown-Gitler spectrum [9]. We will filter the Steenrod algebra by $F_n(A) = \{\chi(Sq^I) \mid I \text{ admissible and } i_1 \geq n\}$. Then $A \otimes F_n(A) \supset F_n(A)$ and $F_n(A) \supset F_{n+1}(A)$. Also $F_n(A)/F_{n+1}(A) = M(n) = A/A\{\chi Sq^i \mid i > n\}$. Let $\overline{\overline{B}}$ be the associated graded algebra; $\overline{\overline{B}} = \bigoplus_{n>0} M_n$. Then $\overline{\overline{B}}$ can be thought of as the algebra generated by symbols χSq^a, a an integer > 0, subject to the relation $\chi Sq^a \chi Sq^b = 0$ if $2a > b$. This algebra is related to $\overline{\overline{A}}$ but one should note that $M(n)$ is finite for each n. As before we can write down a minimal $\overline{\overline{B}}$ resolution

$$\mathfrak{B}: \quad \overline{\overline{B}} \leftarrow \bigoplus_{n>0} \overline{\overline{B}}\,\tau_n \quad \bigoplus_{\substack{n>0\\ 2k>n}} \overline{\overline{B}}\,\tau_k\tau_n \quad \bigoplus_{\substack{n>0\\ 2k\geq n\\ 2j\geq k}} \overline{\overline{B}}\,\tau_j\tau_k\tau_n \leftarrow \cdots$$

where $\tau_n \to \chi Sq^n \in M_n$; $\tau_k\tau_n \to \chi Sq^k \in M_k\tau_n$; etc.

Proposition 2.4.1. $\mathfrak{B}$ is a minimal free acylic $\overline{\overline{B}}$ resolution.

Proof. We can write $I\overline{\overline{B}} = \bigoplus_{n>0} M_n$. The map $\overline{\overline{B}}\,\tau_k\tau_n \to M_k\tau_n$ has kernel $\bigoplus_{2j\geq k} M_j\tau_k\tau_n$ because of the relation. Since each map of the resolution is a similar map the proposition is clear.

Following closely the ideas of §2.2 and 2.3 we can pass from this associated graded resolution to a free A-resolution. The relation we end up with is $\tau_a\tau_b = \sum_{n=2a}^{[2(a+b)/3]} \binom{a-n-1}{n-2b}\tau_{a+b-n}\tau_n$. The balance of the identification of this resolution with the Λ-algebra is straight forward. Note that the result is directly isomorphic to Λ as opposed to the anti-isomorphism of the other approach. The resolution described here is exploited in some fashion in the papers of Brown and Gitler [9] and the recent paper of Brown and Peterson [36]. Understanding this approach helps to see the motivation behind the calculations in §5 of [23].

2.5 The Λ-algebra for a space X.

In this section we show how to modify the results of 2.2 and 2.3 to obtain

Theorem 2.5.1 [14]. $\mathrm{Ext}_A^{**}(\tilde{H}^*(X); \mathbb{Z}_2)$ is isomorphic to $H_{**}(\tilde{H}_*(X) \otimes \Lambda, d)$. The differential in $\tilde{H}_*(X) \otimes \Lambda$ is given by $d'(y \otimes \lambda_I) = \Sigma\, y\, Sq^i \otimes \lambda_{i-1}\lambda_I + y \otimes d\lambda_I$. Here d is the usual Λ-algebra differential and $y\, Sq^i$ represents the right action of the Steenrod algebra on $\tilde{H}_*(X)$.

We will outline the proof since many of the details are similar to those presented in 2.2 and 2.3.

Consider the resolution of 2.3.1 tensored on the right with $\tilde{H}^*(X)$. $\mathcal{C}'$:

2.5.2 $$\tilde{H}^*(X) \xleftarrow{\varepsilon\otimes 1} A \otimes \tilde{H}^*(X) \xleftarrow{d_1\otimes 1} \bigoplus_{i_0>0} A_{\sigma_{i_0}} \otimes \tilde{H}^*(X) \leftarrow \cdots$$

where $(\varepsilon \otimes 1)(1 \otimes x) = x$ $(d_1 \otimes 1)(\sigma_{i_0} \otimes x) = (Sq^{i_0} \otimes x)$ etc.

Now a basis, over A, of $\mathcal{C}'_s$, the s^{th} term of 2.5.2 is given by $(\sigma_I \otimes x)$, $x \in \tilde{H}^*(X)$, $I = (i_0, i_1, \ldots, i_{s-1})$ and $i_0 > 0$, $i_1 < 2i_0, \ldots$ $i_1 < 2i_0, \ldots, i_{s-1} < 2i_{s-2}$. The A-module structure of $\mathcal{C}'_s$ is the diagonal one. This implies, for example,

$$(Sq^i \otimes x) = \sum_j Sq^{i-j}(1 \otimes \chi Sq^j x)$$

Thus, the maps in 2.5.2 can be rewritten as follows.

Let $I = (i_0, i_1, \ldots, i_s)$, and $I'' = (i_2, \ldots, i_{s+1})$

$$(d_s \otimes 1)(\sigma_I \otimes x) = (\binom{i_0-1}{i_1}\sigma_{i_1+i_0,I''} + Sq^{i_1}\sigma_{i_0,I''}$$

$$+ \sum_{j=1}^{[i_1/2]} \binom{i_0-i-j}{i_1-2j} Sq^{i_0+i_1-j}\sigma_j\sigma_{I''} \otimes x)$$

$$= \binom{i_0-1}{i_1}\sigma_{i_1+i_0}\sigma_{I''} \otimes x + Sq^{i_1}\sigma_{i_0}\sigma_{I''} \otimes x$$

$$+ \sum_{j=1}^{[i_1/2]} \binom{i_0-1-j}{i_1-2j} Sq^{i_0+i_1-j}\sigma_j\sigma_{I''} \otimes x$$

$$= \binom{i_0-1}{i_1}\sigma_{i_1+i_0}\sigma_{I''} \otimes x + \sum_{k_1} Sq^{i_1-k_1}(1 \otimes \chi Sq^{k_1}x)$$

$$+ \sum_{j=1}^{[i_1/2]} \binom{i_0-1-j}{i_1-2j}(\sum_{k_j} Sq^{i_0+i_1-j-k_j}(1 \otimes \chi Sq^{k_j}x).$$

Using this differential one can check that 2.5.2 is a resolution.

Applying $\mathrm{Hom}_A(-; \mathbb{Z}_2)$ to 2.5.2 we see that, using the adjointness of $\otimes$ and Hom, we are left with $\Sigma \otimes \tilde{H}_*(X)$. (Recall that $\chi Sq^i x = x^* Sq^i$ for $x \in H^*(X)$.)

Using the methods of 2.3 and the anti isomorphism $\varphi \circ T\colon \Sigma \otimes H_*(X) \to \tilde{H}_*(X) \otimes \Lambda$, T is the map which exchanges factors, one can show the above differential corresponds to $d'(a \otimes \lambda_I) = \Sigma a Sq^i \otimes \lambda_{i-1}\lambda_I + a \otimes d\lambda_I$ for $a \in \tilde{H}_*(X)$.

Chapter 3

Unstable Resolutions

3.1 Massey-Peterson Theory

This section is an attempt to summarize some of the work of Massey and Peterson [25] [26].

Definition 3.1.1. A graded module M over A, the mod-2 Steenrod algebra is called unstable if for any $m \in M$ $Sq^i(m) = 0$ for $i > |m|$. For a graded module M, $|m|$ is the dimension of $m \in M$.

Definition 3.1.2. Let M be an unstable A-module. Let $\lambda: M \to M$ be defined by $\lambda(m) = Sq^{|m|}(m)$ for all $m \in M$. Thus M can be considered as a $\mathbb{Z}_2[\lambda]$-module with $\lambda^i(m) = \lambda(\lambda^{i-1}(m))$ for $m \in M$. M is then called a λ-module. More generally, if N is a graded module over $\mathbb{Z}_2$ and λ is a $\mathbb{Z}_2$ vector space homomorphism from N to N with $\lambda(N)^j \subset (N)^{2j}$ then N inherits, as above, a $\mathbb{Z}_2[\lambda]$-module structure. As usual a λ-module will be called free if it has a basis.

Definition 3.1.3. Let M be a λ-module. Then $U(M)$ is the free symmetric algebra on M modulo the ideal generated by all elements of the form $m^2 - \lambda(m)$.

Proposition 3.1.4 (10.4 of [26]). Let M be as in 3.1.3 and suppose also that M is locally finite. Then $U(M)$ is a polynomial algebra if and only if M is a free λ-module.

Definition 3.1.5. Let M be a graded module over $\mathbb{Z}_2$. Define σM to be the free λ-module generated by $\overline{M}$, where $(\overline{M})^i$ is equal to $(M)^{i-1}$.

Definition 3.1.6. Let M be a graded module over $\mathbb{Z}_2$ and N be a λ-module. A boundary-type map $f: M \to N$ is given by the composite $M \xrightarrow{i} \sigma M \xrightarrow{\bar{f}} N$ for some λ-module map $\bar{f}$ where i is the obvious degree one inclusion.

Recall that a graded ring R over $\mathbb{Z}_2$ has a simple system of generators $\{x_\alpha\}$ if the monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $i_1 < i_2 < \cdots < i_r$, form a $\mathbb{Z}_2$ basis for R. For example, the polynomial algebra $\mathbb{Z}_2[x]$, with $|x| = 1$, has a simple system of generators $\{x^{2^k}\}$, $k \geq 0$.

Proposition 3.1.7 [25]. Let M be an unstable A-module with base point $\eta: \mathbb{Z}_2 \to M$. Let $b_0 = \eta(1), b_1, b_2, \ldots$ be a set of homogeneous generators for M as a $\mathbb{Z}_2$ vector space. Then $\{b_i\}_{i \geq 0}$ is a simple system of generators for $U(M)$ as an algebra over $\mathbb{Z}_2$.

We also recall

Theorem 3.1.8 (A. Borel). Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibre space with E acyclic over $\mathbb{Z}_2$. Suppose that $H^*(F)$ has a simple system of transgressive generators $\{x_\alpha\}$. Then $H^*(B)$ is the polynomial algebra on (τx_α) where τ is the transgression.

In terms of λ-modules the Serre-Cartan basis theorem has a particularly simple formulation. Let $\mathbb{Z}_2$ be the $\mathbb{Z}_2$ vector space with one generator in dimension zero.

Proposition 3.1.9. The $\mathbb{Z}_2$ cohomology of $K(\mathbb{Z}_2,n)$ is $U(\sigma^n \mathbb{Z}_2)$.

Proof: Note that $(\sigma \mathbb{Z}_2)^{2^i} = \mathbb{Z}_2$ for $i \geq 0$, generated by $\lambda^i(1)$. (1 represents the generator of $\bar{\mathbb{Z}}_2$, the $\mathbb{Z}_2$ vector space with only one generator in dimension one). $\lambda^i(1)$ is equal to

$Sq^{2^i} Sq^{2^{i-1}} \cdots Sq^2 Sq^1(1)$. Let $I = \{i_1, \ldots, i_\ell\}$ be admissable, that is, $i_j \geq 2i_{j+1}$ and let $e(I) = 2i_1 - \sum_{j=1}^{\ell} i_j$. Then a $\mathbb{Z}_2$ basis for $\sigma^n \mathbb{Z}_2$ is $\{Sq^I(u) | e(I) \leq n\}$ where $u = \sigma^n(1)$.

The proposition is clearly true for $n = 1$. Suppose that $H^* K(\mathbb{Z}_2, q-1)) \cong U(\sigma^{q-1}\mathbb{Z}_2)$. Consider the path fibration $K(\mathbb{Z}_2, q-1) \xrightarrow{i} P \to K(\mathbb{Z}_2, q)$. A simple system of generators for $H^*K(\mathbb{Z}_2, q-1)$ is given by $(\iota_{q-1} = b_0, b_1, b_2, \ldots)$ where ι_{q-1} is the fundamental class and the b_i's are a homogeneous system of generators $\sigma^{n-1}\mathbb{Z}_2$. Clearly these generators are transgressive and so Borel's theorem implies that $H^*K(\mathbb{Z}_2, q) \cong \mathbb{Z}_2[\tau_{bo}, \tau b_1, \ldots]$. A short admissable sequence argument shows that τb_i is admissable with $e(\tau b_i) < q$ and that these are the only such sequences. Thus τb_i generate $\sigma^q \mathbb{Z}_2$. $U(\sigma^q \mathbb{Z}_2)$ is a polynomial algebra by Proposition 3.1.4 and has a homogeneous basis for $\sigma^q \mathbb{Z}_2$.

We will prove the proposition by induction. The proposition is clearly true for $n = 1$. Suppose $H^*K(\mathbb{Z}_2, q) \cong U(\sigma^q \mathbb{Z}_2)$ for all $q < n - 1$. Consider the path fibration

$$K(\mathbb{Z}_2, n-1) \to E \to K(\mathbb{Z}_2, n).$$

A simple system of generators for $H^*K(\mathbb{Z}_2, n-1)$ is given by $\{\iota_{n-1} = b_0, b_1, b_2, \ldots\}$, a homogeneous basis for $\sigma^{n-1}\mathbb{Z}_2$ as a $\mathbb{Z}_2$ vector space. Clearly the b_i's are transgressive and so Borel's theorem implies that $H^* K(\mathbb{Z}_2, n)$ is $\mathbb{Z}_2[\tau b_i]_{i \geq 0}$.

Consider the following diagram

$$\begin{tikzcd}
\sigma^{n-1}\mathbb{Z}_2 \arrow[r, "\tilde{\sigma}"] \arrow[d, "i"'] & \mathbb{Z}_2[\tilde{\sigma} b_i] \cong H^*K(\mathbb{Z}_2,n) \arrow[dl, shift right, "f"'] \\
U\sigma^n\mathbb{Z}_2 \arrow[ur, shift right, "g"'] &
\end{tikzcd}$$

The maps f and g exist since $H^*K(\mathbb{Z}_2,n)$ and $U(\sigma^n\mathbb{Z}_2)$ are polynomial algebras. Since $f\tilde{\sigma}(\iota_{n-1}) = g\iota(\iota_{n-1})$ and $\tilde{\sigma}$ commutes with squaring operations we have $f\circ g = g\circ f$ and $H^*K(\mathbb{Z}_2,n) \cong U\sigma^n\mathbb{Z}_2$.

We will find it useful to decompose $\sigma^n\mathbb{Z}_2$ as a sum of free λ-modules and $\mathbb{Z}_2$. Let $L_0(1) = (\sigma(\mathbb{Z}_2))^1$ and $L_1(1) = \bigoplus_{i>1} (\sigma(\mathbb{Z}_2))^1$. Then $\sigma(\mathbb{Z}_2) = L_0(1) \oplus L_1(1)$. Applying σ to both sides we obtain

$$\sigma^2(\mathbb{Z}_2) = \sigma(L_0(1)) \oplus \sigma(L_1(1)).$$

Let $\sigma(L_0(1)) = (\sigma(L_0(1))^2 \oplus \bigoplus_{i>2} (\sigma L_0(1))^i$.

The first factor is called $L_0(2)$. The second $L_1(2)$ and $\sigma(L_1(1))$ is $L_2(2)$. Inductively proceeding in this fashion $\sigma^n(\mathbb{Z}_2) \quad \bigoplus_{n\geq j\geq 0} L_j(n)$. Each $L_i(n)$, $i > 0$, is a free λ module.

Definition 3.1.7. A chain complex of free λ-modules is a collection of free λ-modules C_i and boundary type maps $d_i\colon C_i \to C_{i-1}$ so that $d_{i-1}d_i = 0$. $H_n(C_i,d_i) = \ker d_i/\mathrm{in}\ \overline{d_{i+1}}$ where $\overline{d_{i+1}}$ is given by the following diagram

$$\begin{tikzcd}
C_{i+1} \arrow[r, "i"] \arrow[d, "d_{i+1}"] & \sigma C_{i+1} \arrow[dl, dashed, "\overline{d_{i+1}}"] \\
C_i &
\end{tikzcd}$$

Note that $\overline{d_{i+1}}$ exists since σC_{i+1} is free.

The key result of [25] is the following

Theorem 3.1.8 (7.4 of [25]). If C_1 and C_0 are λ-free, unstable A-modules and $H^*(X\ ) = U(C_i)$, $i = 0,1$, and $X_1 \xrightarrow{i} E \xrightarrow{p} X_0$ is a fibre space with $C_1 \subset H^*(X_1)$ transgressive then $\tau(C_1) \subset C_0$ and $H^*(E) \cong U(\ker \tau) \otimes \mathrm{im}\, p^*$ and $\ker p^* = U\, \mathrm{im}(\overline{\tau})$.

3.2. A particular unstable resolution

In a purely formal fashion we can construct a chain complex of λ-free unstable A-modules whose homology will be $\tilde{H}^*(S^{i_0})$.

$$L_0(i_0) \xleftarrow{\varepsilon} \sigma^{i_0}\mathbb{Z}_2 \xleftarrow{d_1} \bigoplus_{0<i_1\le i_0} (\sigma^{i_0+i_1-1}\mathbb{Z}_2)\sigma_{i_1}$$

$$\xleftarrow{d_2} \bigoplus_{\substack{i_1<i_0\\ 0<i_2<2i_1}} (\sigma^{i_2+i_1+i_0-2}\mathbb{Z}_2)\sigma_{i_2}\sigma_{i_1} \leftarrow \cdots$$

where d_i is essentially as in 2.3. That is,

$$d_1\sigma_{i_1} = Sq^{i_1}$$

$$d_2(\sigma_{i_2}\sigma_{i_1}) = Sq^{i_2}\sigma_{i_1} + \sum_j \binom{i_1-j-1}{i_2-2j} Sq^{i_2+i_1-j}\sigma_j$$

and in general

$$d_r(\sigma_{i_r}\sigma_{i_{r-1}} \cdots \sigma_{i_1})$$

$$= [Sq^{i_r}\sigma_{i_{r-1}} + \sum_j \binom{i_{r-1}-j-1}{i_r-2j} Sq^{i_r+i_{r-1}-j}\sigma_j](\sigma_{i_{r-2}} \cdots \sigma_{i_1})$$

Proposition 3.2.2. With the augmentation ε this chain complex of free λ-modules is acylic (as λ-modules).

We will outline the proof since it is very similar to 2.2 and 2.3.

Proof: It will suffice to prove the proposition under the hypothesis that $Sq^a Sq^b = 0$ if $a < 2b$. This is equivalent to writing $\sigma^n \mathbb{Z}_2 = \bigoplus_{j\geq 0} L_j(n)$. Notice that if $\sigma^{\ell}\mathbb{Z}_2$ is a summand of C_i, the i^{th} term in the chain complex, then $\overline{d_i}$ is defined on $\sigma^{\ell+1}\mathbb{Z}_2$. Ker $\varepsilon = \bigoplus_{j>0} L_j(i_0)$ and $\overline{d_1}$ restricted

to $\bigoplus_{0< \ 1\leq L_0} [L_0(i_0 \oplus i_1) \oplus \bigoplus_{2i_1<\ell< \ 0+ \ 1} L_{\ell}(\ell_0 + \ell_1)\sigma_{i_1}]$ is an isomorphism onto ker ε. The ker $d_1 = \bigoplus_{\substack{0<j<2i_1\\ 0<i_1<i_0}} L_j(i_1 + i_0 - 1)$.

Again it is easy to verify that $\overline{d_2}$ restricted to $\oplus L_0(i_0 + i_1 + i_2 - 1)\sigma_{i_2}\sigma_{i_1} \oplus \bigoplus_{2i_1< \ \leq i_1+i_0} L_r(i_0 + i \ + i_2 - 1)$ is an isomorphism onto ker d_1. A similar argument proves the case for d_s.

We next wish to show that 3.2.1 is related to a geometrically constructed resolution of S^{i_0}.

Theorem 3.2.3. There is a sequence of spaces X_i and maps P_i such that the following diagram commutes

$$\begin{tikzcd}
& S^{i_0} \arrow[dl, "f_0"'] \arrow[d, "f_1"] \arrow[dr, "f_2"] \arrow[drrr, "f_s"] & & & & \\
X_0 & X_1 \arrow[l, "P_1"'] & X_2 \arrow[l, "P_2"'] & \cdots \arrow[l, "P_3"'] & X_s \arrow[l, "P_s"'] & {} \arrow[l, "P_{s+1}"'] \\
K(\mathbb{Z}_2,i_0) \arrow[u, "i_0", "\text{h.e.}"'] & K(V_1) \arrow[u, "i_1"'] & K(V_2) \arrow[u, "i_2"'] & \cdots & K(V_s) \arrow[u, "i_s"'] &
\end{tikzcd}$$

and

1) P_i is a fibration with $K(V_i)$ as fibre and $\ker P_i^* = \ker f_{i-1}^*$ in $\mathbb{Z}_2$ cohomology.

2) f_i^* is an epimorphism

3) V_s is a graded $\mathbb{Z}_2$ vector space generated by σ_J where $J = (j_1,\ldots,j_s)$ and $j_k < 2j_{k+1}$. The dimension of σ_J is $\sum_{j=1}^{s} j_i - s$.

4) Let M_{V_s} be the free unstable A-module such that $U(M_{V_s}) \cong H^*(K(V_s))$. Then $M_{V_s} = C_s$ of 3.2.1 and the composite

$$C_s \xrightarrow{d_s} C_{s-1} \longrightarrow U(C_{s-1})$$

Proof: Let $X_0 = K(\mathbb{Z}_2,i_0)$. Let X_i be the fibre of $g_1\colon X_0 \to \prod_{j=1}^{i_0} K(\mathbb{Z}_2,i_0+j)$ where g_1 is defined by the cohomology class $(Sq^1,Sq^2,\ldots,Sq^{i_0})$. Let V_1 be generated by $\{\tau_j\}$ $j = 1,\ldots,i_0$. This gives a fibration $K(V_1) \to X_1 \to X_0$. Theorem 3.1.8 asserts that $H^*(X_1) \cong L_0(i_0) \otimes U \ker \tau$. However, τ is just

$d_1: \bigoplus_{0<i_1\leq i_0} \sigma^{n+k-1}\mathbb{Z}_2\tau_{i_1} \to \sigma^{i_0}\mathbb{Z}_2$. Thus $H^*(X_1) \cong L_0(i_0) \otimes U(\ker d_1)$.

Let $f_0: S^{i_0} \to K(\mathbb{Z}_2, i_0)$ be the generator. Let f_1 be the unique lift of f_0 to X_1. Now suppose we have defined X_s and f_s with $H^*(X_s) \cong L_0(i_0) \otimes U \ker d_s$, then we define X_{s+1} as the fibre of $X_s \xrightarrow{g_s} B(KV_{s+1})$.

Where g_s is induced by $\overline{d_{s+1}}: C_{s+1} \to C_s$. Note that this is well defined since im $\overline{d_{s+1}}$ is isomorphic to ker d_s. Note also that $U(\sigma C_{s+1})$ is isomorphic to $H^*(BKV_{s+1})$. This yields the fibration

$$K(V_{s+1}) \xrightarrow{i_{s+1}} X_{s+i} \xrightarrow{p_{s+1}} X_s.$$

Thus ker p_{s+1} is generated by im d_{s+1} which clearly equals ker f^*_{s+1} by 3.1.8. Again by 3.1.8 we have that $H^*X_{s+1} \cong L_0(K) \otimes U(\ker d_{s+1})$ and the induction is complete.

3.3. Spectral sequences from a resolution

In this section we generalize the resolution of 3.2.1 to locally finite C-W complexes. This leads us to a proof of Theorem 2.1.1.

Definition 3.3.1. Let Y be a locally finite C-W complex. Then $\chi = \{X_s, p_s, f_s, i_s, V_s\}$ is a resolution of Y if

1. The following diagram commutes

$$\begin{tikzcd}
 & & Y \arrow[dll, "f_0"'] \arrow[dl, "f_1"] \arrow[d, "f_2"] \arrow[drr, "f_s"] & & & \\
X_0 & X_1 \arrow[l, "P_1"'] & X_2 \arrow[l, "P_2"'] & \cdots \arrow[l] & X_s \arrow[l, "P_s"'] & \cdots \arrow[l] \\
K(V_0) \arrow[u] & K(V_3) \arrow[u] & K(V_2) \arrow[u] & & K(V_s) \arrow[u] &
\end{tikzcd}$$

where $K(V_s)$ is the Eilenberg-Maclane space associated to the graded Z_2 vector space V_s. In addition $K(V_s) \xrightarrow{i_s} X_s \xrightarrow{P_s} X_{s-1}$ is a fibration and

2. $M_{V_s} \xrightarrow{\tau} H^*X_{s-1} \xrightarrow{i^*_{s-1}} H^*KV_{s-1}$ factors through $M_{V_{s-1}}$.

(Notation as in 3.2.3,4)

Note that no assumptions about ker P^*_s and ker f^*_s are made. The resolutions which satisfy 3.2.3,1,2 are often called Adams' resolutions.

Associated to any resolution is a spectral sequence obtained from its homotopy exact couple. This spectral sequence has the property that $E_1^{s,t+s} = (V_s)^t$ and $E_\infty^{s,t+s}$ is an associated graded group to $\underleftarrow{\lim}\, \pi_t X_s$ (The limit is taken with respect to $(P_i)_*$, the induced maps in homotopy.) Under the additional hypothesis that for each n there exists an s such that $f^*_s\colon H^n X_s \to H^n Y$ is onto, $E_\infty^{s,t+s}$ is an associated graded group to $\pi_t Y$.

Associated to the resolution χ of a space Y is a chain complex of λ-free unstable A modules $\{C_s, d_s\}$ with $C_i = M_{V_i}$. Note that $E_1^{s,t+s} = \mathrm{Hom}^t_A(C_s, Z_2)$ where the superscript t denotes maps which decrease filtration by t. Likewise, $E_2^{s,t-s} = H_s(\mathrm{Hom}_A^t(C_s, Z_2), d^*)$, where d^* indicates $\mathrm{Hom}_A(d, \mathbb{Z}_2)$ and d is the differential of the chain complex.

Another interesting spectral sequence results from the homology of the above chain complex $\{C_i, d_i\}$. The Massey-Peterson theory asserts that $UH_s(C_i, d_i)$ is the E_2-term of a spectral sequence whose E_∞-term is $H^*(Y)$ as a $\mathbb{Z}_2$ vector space.

Now to complete the proof of Theorem 2.1.1 we apply the homotopy spectral sequence to the resolution given by 3.2.1. Thus $E_1^{s,t} = \Sigma^{s,t}(k) \subset \Sigma^{s,t}$ consists of those σ_J where $J = (j_1, \ldots, j_s)$ and $j_s \leq k$. (Σ is defined in Chapter 2.) The anti-isomorphism $\varphi\colon \Sigma(k) \to \Lambda(k)$ given by $\varphi(\sigma_i) = \lambda_{i-1}$ preserves the differential (and relations as in Chapter 2). The proofs are almost identical and are left to the reader. This completes the proof of 2.1.1.

3.4. The loop functor applied to resolutions.

In the last section we discussed resolutions which were not necessarily Adams' resolutions. A simple way to obtain such a resolution is to use the functor Ω. That is given a resolution χ of a space X we apply Ω to every object and map in χ.

Recall that for an associative H-space Y, $H^*(X) \cong U(P(H_*(X))$ as $\mathbb{Z}_2$ vector spaces, where $P(H_*X)$ denotes the set of primitive elements in $H_*(X)$ [37]. Using this result and Borel's theorem we have

Proposition 3.4.1. If $H^*(X) \cong U(\sigma M)$ for some unstable A-module M then $H^*(\Omega X) \cong U(M)$ as vector spaces.

If we take the resolution of 3.2.3 and apply Ω to it we obtain

$$\begin{tikzcd}
\Omega X_0 & \Omega X_1 \arrow[l, "\Omega P_1"'] & \cdots \arrow[l] & \Omega X_s \arrow[l, "\Omega P_s"'] & \cdots \arrow[l] \\
& \Omega K(V_1) \arrow[u] & & \Omega K(V_s) \arrow[u] &
\end{tikzcd}$$

This corresponds to a chain complex analagous to 3.2.1.

3.4.2. $$\sigma^{k-1}\mathbb{Z}_2 \xleftarrow{d_1} \bigoplus_{0<n\leq k} (\sigma^{n+k-2}\mathbb{Z}_2)\sigma_n \xleftarrow{d_2} \bigoplus_{\substack{0<n\leq k \\ 0<i\leq 2n}} (\sigma^{n+i+k-3}\mathbb{Z}_2)\sigma_i\sigma_n$$

$$\xleftarrow{d_s} \bigoplus_{J_s} (\sigma^{k-s+\Sigma j_i}\mathbb{Z}_2)\sigma_J \longleftarrow \cdots$$

where $J_s = \{J \mid J = \{j_1,\ldots,j_s\},\ 0 < j_i < 2j_{i+1}, j_s \leq k\}$. The map d_s is defined as before. Recalling definition 3.1.7 we can prove

Proposition 3.4.3. The homology of 3.4.2 is given by

$$H_{s,t}(C(3.4.2),d) = \mathbb{Z}_2 \text{ for each } s > 0,\ t = 2^s k$$

and is generated by $\sigma_{2^{s-1}k}\sigma_{2^s k} \cdots \sigma_k$ for $s > 0$ and is $L_0(k-1)$ if $s = 0$.

Proof: Let k be a fixed integer greater than zero. As before it is sufficient to work in the setting where $Sq^a Sq^b = 0$ if $a < 2b$. In this setting the chain complex 3.2.1 for $k - 1$ is a subcomplex of 3.4.2. The quotient chain complex is easily seen to be 3.4.2 for $2k - 2$ and starting in dimension 1 instead of zero. That is,

$$\{\mathbf{C}_s\ (3.2.1 \text{ for } k-1),\ _2\} \subset \{\mathbf{C}_s\ (3.4.2 \text{ for } k)\}$$

$$\to \{\mathbf{C}_{s-1}\ (3.4.2) \text{ for } 2k-2)\}\sigma_k.$$

The long exact homology sequence associated with these short exact sequences completes the proof.

This is analagous to the EHP sequence map which we will discuss later. Also note that $H^*(\Omega S^k) = U(H_*(C\ (3.4.2),d))$. This does represent an independent proof of this result if $k > 2$ since classes once produced in this resolution cannot be annilated for dimensional reasons. We can get analagous results for iterated loops. The results of Dyer and Lashof [16] imply that the homology of the complex which results from 3.2.1 after applying iterated loops, $\Omega^i, i < k$, satisfies $U(H_*(\Omega^i C\ (3.2.1))) = H^*(\Omega^i S^k)$. We won't use this but it seems worth noting because it, in principle at least, describes the cohomology operation represented by each Dyer-Lashof homology operation.

3.5. A mapping theorem for resolutions

Because of the results of the preceeding section we would like to look at resolution which are not acyclic.

Definition 3.5.1. A regular resolution of a space X is

i) a tower of fiber spaces and maps $f_s: X \to X_s$

$$\begin{tikzcd}
X_0 & X_1 \arrow[l, "p_1"'] & X_2 \arrow[l, "p_2"'] & \cdots \arrow[l] & X_s \arrow[l, "p_s"'] & \cdots \arrow[l] \\
& X \arrow[ul, "f_0"] \arrow[u, "f_1"] \arrow[ur, "f_2"'] \arrow[urrr, "f_s"'] & & & &
\end{tikzcd}$$

ii) The fiber at the s stage in $K(V_s)$ where V_s is a graded group.

iii) The k-invariants at each stage are stable in the sense of [26].

iv) $\ker p_s^* = \ker f_{s-1}^*$.

Note that we do not require f_0^* to be an epimorphism. If f_0^* is an epimorphism in a regular resolution we have the usual idea of an unstable Adams spectral sequence. If we have such a resolution for a sphere and take its loop resolution we get a regular resolution with f_0^* no longer an epimorphism.

Theorem 3.5.2. Suppose we have $k: X \to Y$ and we have a regular resolution of X and some resolution of Y. Let $F_i(X)(F_i(Y))$ be $\operatorname{im} f_i^*(\operatorname{im} \bar{f}_i^*)$. Suppose $k^*F_i(Y) \subset F_i(X)$ for all i. Then there is a mapping $k_i: X_i \to Y_i$ of the resolution covering k.

Proof. Since $k^*F_0(Y) \subset F_0(X)$ we can define

$$X_0 \xrightarrow{k_0} Y_0.$$

Suppose now we have

$$\begin{tikzcd}
 & X_0 \arrow[r, "k_0"] & Y_0 & \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \\
 & X_i \arrow[r, "k_i"] & Y_i \arrow[r, "g_i"] & BK(V_i) \\
 & X_{i+1} \arrow[u, "P_{i+1}"] \arrow[r] & Y_{i+1} \arrow[u] & \\
X \arrow[uur, "f_i"] \arrow[ur, "f_{i+1}"] \arrow[rrr, "k"'] & & & Y \arrow[ul] \arrow[uul, "\bar{f}_i"']
\end{tikzcd}$$

Since k exists $f_i^* k_i^* g_i^* = 0$. But $\ker P_{i+1}^* = \ker f_i^*$. Thus $P_{i+1}^* k_i^* g_i^* = 0$ and the lifting $X_{i+1} \xrightarrow{k_{i+1}} Y_{i+1}$ exists. Since $k^*F_{i+1}(Y) \subset F_{i+1}(X)$ we can choose a lifting to make the diagram

$$\begin{tikzcd}
X_{i+1} \arrow[r, "k_{i+1}"] & Y_{i+1} \\
X \arrow[u, "f_{i+1}"] \arrow[r] & Y \arrow[u, "\bar{f}_{i+1}"']
\end{tikzcd}$$

commute.

3.6. The cone construction

In this section we discuss the geometric analogue for the cone construction for chain complexes. We first do a stable version and then do it unstably.

Construction 3.6.1. Let X and Y be spectra and $f\colon X \to Y$ a map with cofibre $Y \cup_f CX$. Suppose we have a minimal resolution for X

$$X = X_0 \xleftarrow{P_1} X_1 \xleftarrow{P_2} X_2 \leftarrow \cdots \xleftarrow{P_s} X_s \leftarrow$$

Let $c\colon Y \cup_f CX \to \Sigma X$ be the usual collapse map. Then we have the following diagram

3.6.2
$$\begin{tikzcd}
X_1 \arrow[dd, "P_1"'] \arrow[ddr, "f\,P_1"] & & & & \\
& H_1 \arrow[r] & Z_1 \arrow[r, "\pi'"] \arrow[d] & \Sigma X_1 \arrow[d] & \\
X \arrow[r, "f"] & Y \arrow[r, "\pi"] \arrow[ur] & Y \cup_f CX \arrow[r, "c"] & \Sigma CX \arrow[r, "x"] & K\left(\bigoplus_{i=n}^{2n} H^i(\Sigma X), \mathbb{Z}_2\right)
\end{tikzcd}$$

Using the null homotopy of $c\circ\pi\colon Y \to \Sigma$ we obtain a map $q\colon Y \to Z_1$ where Z_1 is induced from the path fibration over $K\left(\bigoplus_{k=n+1}^{2n} H^i(\Sigma X), \mathbb{Z}_2\right)$ by kc. H_1 is the fibre of the map $Z_1 \to Y \cup_f CX$. By an easy homology argument Z_1 is homotopy equivalent to $Y \cup_{fP_1} CX_1$. With the usual Adams' resolution for this space we obtain a resolution which we call a resolution of the map f. Applying homotopy, π_*, we obtain an exact couple

$$\begin{tikzcd}
\sum_i \pi_* Z_i \arrow[rr] & & \sum_i \pi_* Z_i \arrow[dl] \\
& \Sigma\, \pi_* A_i \arrow[ul] &
\end{tikzcd}$$

for the map, denoted $E_r(f)$, from the fibrations $A_i \to X_i \to X_{i-1}$.

The following lemma is standard. ([1] 2.6.1).

Lemma 3.6.3. Let A and B be finite C-W complexes and $f\colon A \to B$

a map with cofibre $B \cup_f CA$. If $f^*: \tilde{H}^*B \to \tilde{H}^*A$ is onto then there is along exact sequence.

$$\to E_2^{s,t}A \xrightarrow{f_*} E_2^{s,t}B \to E_2^{s,t}B \cup_f CA \to E_2^{s+1,t}A \to$$

Lemma 3.6.4. Let X,Y,Z be as defined in 3.6.2. Then $q^*: \tilde{H}^*Z_1 \to \tilde{H}^*Y$ is onto and $Y \xrightarrow{q} Z_1 \to \Sigma X_1$ is a cofibration.

Proof. The first part of the lemma is clear by the construction. The second part follows from an easy homology argument.

Thus applying Lemma 3.4.3 to the cofibration $Y \xrightarrow{q} Z_1 \to \Sigma X$ we obtain a long exact sequence

$$E_2^{s,t}(f) \to E_2^{s,t}X \xrightarrow{f} E_2^{s,t}Y \to E_2^{s+1,t}(f) \to$$

Proposition 3.6.5. There exists a map $b: E_r(YU_fCX) \to E_r(f)$ where $E_r(YU_fCX)$ denotes the usual Adams spectral sequence for YU_fCX.

Proof. Observe that a map exists on the level of resolutions.

Proposition 3.6.6.

$$\mathrm{Ext}_A^{s,t}(H^*(X \cup_f CY); \mathbb{Z}_2) \cong E_2^{s,t}(f) \text{ if } f^* \text{ is zero in } Z_2 \text{ cohomology.}$$

Proof. This is implied by Lemma 3.6.3 and 3.6.4.

The following example will be useful later.

Let $f: X \to Y$ be as above and suppose $H^m(X,Z)$ and $H^m(Y,Z)$ have a free generator. We would like to calculate the effect of these classes and the degree of f in $\{E_r(f)\}$.

Consider

3.6.6'.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[r] \arrow[d, "j"] & Y \cup_f CX \arrow[d, dashed, "\pi"] \\
K(Z,m) \arrow[r, "k"] & K(Z,m) \arrow[r] & K(Z_k,m),
\end{tikzcd}$$

where i and j represent the integer classes, π is the induced map on cofibrations, and k is the degree of f on the integer classes.

If $k \equiv 0 (\mathrm{mod}\ 2)$, then in $E_2(f)$ we have adjacent infinite towers representing arbitrary non zero h_0 multiplication.

Lemma 3.6.7. Let $k = 2^i(2a + b)$ and let $a \in E_2(f)$ be the class arrising from the ho tower from $\mathrm{Ext}_A^{s,t}$ $(\tilde{H}^*(X), \mathbb{Z}_2)$. Then there is a j such that for all $j' > j$ $d_{i+1}h_0^{j'} a \neq 0$ in $E_{i+1}(f)$.

Proof. This is exactly the case in the sequence $K(Z,m) \xrightarrow{k} K(\mathbb{Z},m) \rightarrow K(\mathbb{Z}_k,m)$. Note that if k is odd the differential is a d_1 and if $k = 0$ there is no differential. Naturality completes the argument.

A construction similar to 3.6.1 can be carried out to yield, under certain hypothesis results similar to 3.6.5 in the unstable case.

Construction 3.6.8. Let F be the fibre of a map $f: X \rightarrow Y$ between C-W complexes. Let $\{X_i\}$ and $\{Y_i\}$ be regular Adams' resolutions (unstable) for X and Y respectively. Let F_1 be the fibre space over F induced by the path fibration over $K(\tilde{H}^*(X))$ via $F \rightarrow X \rightarrow K(H^*X)$. Using the null homotopy of $\Omega Y \rightarrow K(\tilde{H}^*(X))$ we obtain a lifting $h: \Omega Y \rightarrow F_1$. Let i_1 be the induced map from $F_1 \rightarrow X_1$

$$\begin{tikzcd}
\Omega Y_1 \arrow[d] & F_1 \arrow[r, "i_1"] \arrow[d] & X_1 \arrow[r] \arrow[d] & K(\tilde{H}^*X_1) \\
\Omega Y \arrow[r] \arrow[ur, dashed, "h"] & F \arrow[r] & X \arrow[r] & K(\tilde{H}^*X)
\end{tikzcd}$$

Note that H^*F_1 maps onto $H^*\Omega Y$. Thus there is a map $F_1 \to K(H^*\Omega Y.)$ Let $F_{1\frac{1}{2}}$ be the fibre space over F_1 induced by the path fibration over $K(H^*(\Omega Y)$. Let F_2 be the fibre space over $F_{1\frac{1}{2}}$ induced by the path fibration over $K(\tilde{H}^*X_1)$ via the composite

$$F_{1\frac{1}{2}} \to F_1 \to X_1 \to K(H^*(X_1)$$

Continuing in this fashion we obtain the resolution $\{F_i\}_{i\in\mathbb{Z}^+}$ of the map f. Applying homotopy, π_*, to this resolution we obtain an exact couple for the map f denoted $\{E_r^u(f)\}$. Note that $E_1^u(f) \cong E_1\Omega Y \oplus E_1X$ and we have a short exact sequence of chain complexes

$$0 \to E_1^{s,t}\,\Omega Y \to E_1^{us,t}(f) \to E_1^{s,t}\,X \to 0.$$

Taking homology we obtain

Proposition 3.6.9. For X, Y and f as above there exists a long exact sequence

$$\to {}^{u}E_2^{s,t}(f) \to E_2^{s,t}\,X \xrightarrow{f} E_2^{s,t}\,Y \to {}^{u}E_2^{s+1,t}f \to$$

Proposition 3.6.10. The usual Adams' spectral sequence for F, $E_r(F)$ maps to $E_r^u(f)$.

Chapter 4

Some Stable Calculations

4.1 A spectral sequence.

In this section we will introduce a spectral sequence which we will use extensively in the rest of this chapter and in Chapter 7. As before let A be the mod 2 Steenrod algebra and let A_i be the sup-Hopf algebra generated by $\{Sq^1, \ldots, Sq^{2^i}\}$. Let $\mathcal{C} = \{C_s, d_s\}$ be a chain complex of A_i modules and A_i maps d_s with an augmentation $\epsilon\colon C_0 \to M$. That is, $\epsilon\colon (C_0)^0 \to M$ is an isomorphism. If $\tilde{H}_*(\mathcal{C}) = 0$ the chain complex is called acyclic with augmentation M. The chain complex is called convergent if $\lim\limits_{s\to\infty}(\text{connectivity of } C_s) = \infty$.

Proposition 4.1.1. Associated to a convergent acylic chain complex over A_i with augmentation M is a spectral sequence with $E_1^{\sigma,s,t} = \mathrm{Ext}_{A_i}^{s-\sigma,t}(C_\sigma, \mathbb{Z}_2)$ and $E_\infty^{\sigma,s,t} = E_0^\sigma \mathrm{Ext}_{A_i}^{s,t}(M, \mathbb{Z}_2)$.

(Recall that $E_0^\sigma \mathcal{G}$ for a filtered group $\mathcal{G}$ is the graded group of successive quotients.)

The spectral sequence which arises from a complex $\mathcal{C}$ in this fashion we will designate $E_r(\mathcal{C})$.

Proof: The acylic requirement gives short exact sequences $0 \leftarrow \ker d_s \xleftarrow{d_{s+1}} \mathcal{C}_{s+1} \leftarrow \ker d_{s+1} \leftarrow 0$. Thus $\mathrm{Ext}_{A_i}^{s,t}(\ ,\mathbb{Z}_2)$ gives a long exact sequence for each t and produces an exact couple in the standard way. The spectral sequence is a separate exact couple for each t and this accounts for the trigrading. The convergence hypothesis guarantees the convergence of the spectral sequence in a strong sense. Indeed, for t fixed the chain complex is finite.

Some examples of the above which we will use later include the following. Let $I_j(A_i)$ be the kernel of $A_i \otimes_{A_j} \mathbb{Z}_2 \xrightarrow{\varepsilon} \mathbb{Z}_2$ where ε is the obvious augmentation. Let $I_j^s(A_i)$ be defined inductively as the kernel of $A_i \otimes_{A_j} \mathbb{Z}_2 \otimes I_j^{s-1}(A_i) \to I_j^{s-1}(A_i)$. Let $\mathcal{C}_s = A_i \otimes_{A_j} \mathbb{Z}_2 \otimes I_j^s(A_i)$ and let d_s be the composite map

$$A_i \otimes_{A_j} \mathbb{Z}_2 \otimes I_j^s(A_i) \to I_j^s(A_i) \subset A_i \otimes_{A_j} \mathbb{Z}_2 \otimes I_j^{s-1}(A_i).$$

Proposition 4.1.2. The above chain complex $(\mathcal{C}_s, d_s)$ is a convergent acylic chain complex of A_i modules with augmentation $\mathbb{Z}_2$.

The proof is immediate. The E_1 term is $\mathrm{Ext}_{A_i}^{s-\sigma,t}(\mathcal{C}_\sigma, \mathbb{Z}_2)$.
By the standard change of rings theorem
$\mathrm{Ext}_{A_i}^{s,t}(A_i \otimes_{A_j} \mathbb{Z}_2 \otimes M, \mathbb{Z}_2) \cong \mathrm{Ext}_{A_j}^{s,t}(M, \mathbb{Z}_2)$. This gives us for any A_i module M

Theorem 4.1.3. There is a spectral sequence such that

$$E_1^{\sigma,s,t} = \mathrm{Ext}_{A_j}^{s-\sigma,t}(I_j^\sigma(A_i) \otimes M; \mathbb{Z}_2)$$

which converges to $\mathrm{Ext}_{A_i}^{s,t}(M, Z_2)$.

4.1.4. We will have frequent recourse to this particular example and so the above chain complex will be written $\mathcal{C}(i,j)$.

It is worth noting that if in 4.1.1 $\mathcal{C}_s$ is a free A_i module then $E_1^{\sigma,s,t} = \mathrm{Hom}_{A_i}^t(\mathcal{C}_\sigma, \mathbb{Z}_2)$ if $s = \sigma$ and $E_1^{\sigma,s,t} = 0$ for $\sigma \neq s$. Thus $E_2(\mathcal{C}) = E_\infty$ and this is the standard way to calculate Ext_{A_i}.

4.2. A_0-free modules.

In this section we will prove an important result of Adams [2]. The proof is intended primarily to illustrate, in a simple example, the methods which we will use later.

Theorem 4.2.1 (Adams). If M is a connected free A_0 module such that $M^j = 0$ if $j < 0$, then $\mathrm{Ext}_{A_i}^{s,t}(M, Z_2) = 0$ for $t < 3s - 2$ for all $i \geq 0$.

The first step is

Lemma 4.2.2. If the conclusion of 4.2.1 holds for A_0, then it holds for any connected free A_0 module.

Proof: Let $\mathcal{C}$ be a minimal resolution over A_i for A_0. Let $V = Z_2 \otimes_{A_0} M$. Then $\mathcal{C} \otimes V$ is a resolution for M and $\mathrm{Hom}_{A_i}^t(C_s \otimes V, \mathbb{Z}_2) = 0$ for $t < 3s - 2$ if $\mathrm{Hom}_{A_i}^t(\mathcal{C}_s, \mathbb{Z}_2) = \mathrm{Ext}_{A_i}^{s,t}(A_0, \mathbb{Z}_2)$ does.

The next lemma is important in its own right.

Lemma 4.2.3. $\mathrm{Ext}_{A_1}^{s,t}(A_0, \mathbb{Z}_2)$ is given by $\mathbb{Z}_2[h_1, a, P]/\{h_1^3 = 0, a^2 = 0\}$. The filtrations of the generator are

h_1, (1,2)
a , (1,3)
p , (4,12)

Proof. It is easy to check that the following sequence is exact.

$$A_0 \xleftarrow{f_0} A_1 \xleftarrow{f_1} A_1h_1 \oplus A_1a \xleftarrow{f_2} A_1h_1^2 \oplus A_1ah_1 \xleftarrow{f_3} A_1ah_1^3 \xleftarrow{f_4} A_0P.$$

f_0 is the augmentation

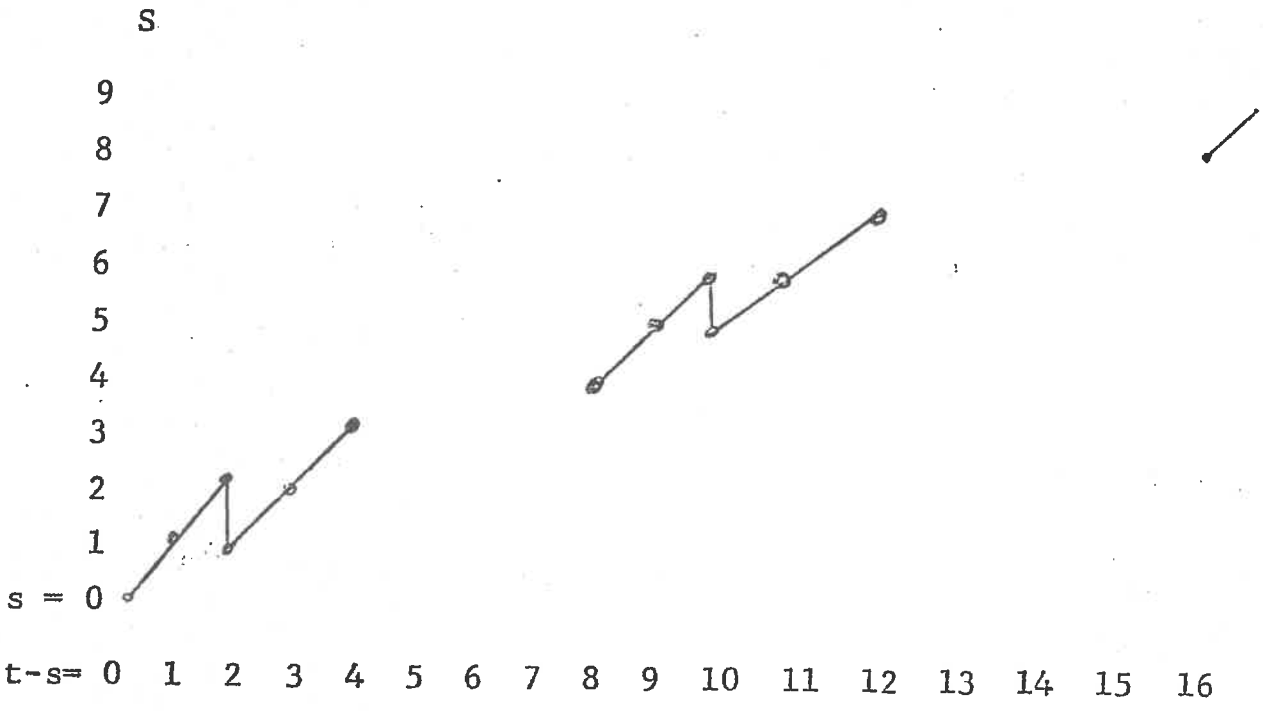
$$f_1h_1 = Sq^2 \qquad f_1a = Sq^2Sq^1$$

$$f_2h_1^2 = Sq^2h_1 + Sq^1a \qquad f_2ah_1 = Sq^2a$$

$$f_3ah_1^3 = Sq^2ah_1 + Sq^3h_1^2$$

$$f_4P = Sq^2Sq^3ah_1^2.$$

Iterating this sequence we obtain a long exact sequence of free A_1 modules resolving A_0. It is easily seen that this sequence yields $\mathrm{Ext}_{A_1}^{s,t}(A_0,\mathbb{Z}_2)$ as stated in the lemma. Chart 4.2.4 illustrates the result.



$\mathrm{Ext}_{A_1}^{s,t}(A_0,Z_2)$

Chart 4.2.4

Vertical lines represent h_0 multiplication. Diagonal lines

We will complete the proof of 4.2.1 by using $E_r(\mathcal{C}(i,1))$. Since $(C_s)^t = 0$ for $t < 4s$ and $E_1^{\sigma,s,t}(\mathcal{C}(i,1) \otimes A_0) = \mathrm{Ext}_{A_1}^{s-\sigma,t}(I_1^{\sigma}(A_i) \otimes A_0, \mathbb{Z}_2)$ we see that this group is zero by 4.2.2 and 4.2.3 for $2(s-\sigma) - 2 > t - s - \sigma$ or $3s - 2 > t$ when $\sigma = 0$ which is the extreme case. This completes the proof of 4.2.1.

4.3. Some A_1 modules.

In this section we will calculate $\mathrm{Ext}_{A_1}^{s,t}(M, Z_2)$ for various A_1-modules which we will have occasion to use latter. First we will prove a standard result in two ways. Both will use the spectral sequence of 4.1. One way is simple and easy. The other way, which appears labored, is intended to illustrate a technique which will be crucial later on. It should be viewed as pedagogically interesting. The result is probably originally due to Adams. The reader should compare also the result of Toda [34].

Theorem 4.3.1. $\mathrm{Ext}_{A_1}^{**}(Z_2, Z_2) = Z_2[h_0, h_1, \bar{a}, P]/R$ where R is generated by $h_1^3, h_0h_1, \bar{a}h_1$ and $\bar{a}^2 = h_0^2P$.

First proof: Consider the following exact sequence

$$\begin{tikzcd} Z_2 & A_1 \otimes_{A_0} Z_2 \arrow[l, "\varepsilon"'] & \Sigma^2 A_1 \arrow[l, "d_1"'] & \Sigma^4 A_1 \arrow[l, "d_2"'] & \Sigma^7 A_1 \otimes_{A_0} \mathbb{Z}_2 \arrow[l, "d_3"'] & \Sigma^{12}\mathbb{Z}_2 \arrow[l, "\eta"'] \end{tikzcd}$$

where $\mathbb{Z}_2$ is the A_1 module which is $\mathbb{Z}_2$ in dimension zero and zero elsewhere and Σ is the usual module suspension. The map ε is clear. $d_i(\text{generator}) = Sq^2$ $i = 1,2$, $d_3(\text{generator}) = Sq^3$. Then $\ker d_3 = Z_2$ and η is the inclusion of the $\ker d_3$ into $\Sigma^7 A_1 \otimes_{A_0} \mathbb{Z}_2$. That this sequence is exact is an easy calculation in A_1. We can apply 4.1.1 to this chain complex giving

$$E_1^{\sigma,s,t} = \mathrm{Ext}_{A_0}^{**}(Z_2,Z_2) = P(h_0) \text{ with bidegree of } h_0 = (1,1).$$

$$E_1^{1,s,t} = \mathrm{Ext}_{A_1}^{s-1,t}(A,Z_2) = \mathbb{Z}_2 \quad s = 1,\ t = 2 \text{ (call this class } h_1)$$

$$= 0 \quad \text{otherwise.}$$

$$E_1^{2,s,t} = \mathbb{Z}_2 \quad s = 2,\ t = 4$$

$$= 0 \quad \text{otherwise}$$

$$E_1^{3,*,*} = \bar{a}P(h_0) \text{ with bidegree } \bar{a} = (3,7)$$

$$E_1^{4,*,*} = \mathrm{Ext}_{A_1}^{s-4,t-12}(\mathbb{Z}_2,\mathbb{Z}_2).$$

This defines a polynomial generator P of bidegree (4,12). There can be no differential in this spectral sequence and thus the theorem is proved.

Second Proof of 4.3.1. Let the generators of A_0 be a_0 and a_1. Then $\overset{s}{\underset{i=1}{\otimes}} A_0$ can be viewed as the $\mathbb{Z}_2$ module generated by all words in A_0 of length s. Let B_s be the sub module of $\overset{s}{\underset{i=1}{\otimes}} A_0$ generated by linear combinations of words which are symmetric. By using the Cartan formula we have an action of A on $\overset{s}{\underset{i=1}{\otimes}} A_0$ and on B_s. The Cartan formula guarantees that B_s is a submodule over A. Also observe the following sequence

4.3.2. $$0 \to B_{s+1} \xrightarrow{g_{s+1}} B_1 \otimes B_s \xrightarrow{f_s} \Sigma' B_{s-1} \to 0,$$

where $g_{s+1}(a_{i_1} \otimes \cdots \otimes a_{i_{s+1}}) = \sum_{j=1}^{s+1} a_{i_j} \otimes a_{i_1} \otimes \cdots \otimes a_{i_j} \otimes \cdots \otimes a_{i_{s+1}}$, is exact as A modules. (This is an easy special case of the Koszul comples; see [10], Chapter VIII, §4).

Recall that $A_1 \otimes_{A_0} \mathbb{Z}_2$ as an A_0 module is $\mathbb{Z}_2 \oplus \Sigma^2 A_0 \oplus \Sigma^5 \mathbb{Z}_2$ where Σ is usual module suspension and $\mathbb{Z}_2$ is the module which is $\mathbb{Z}_2$ in dimension zero and zero elsewhere. Let $d_s\colon A_1 \otimes_{A_0} \mathbb{Z}_2 \otimes \Sigma^{2s} B_s \to A_1 \otimes_{A_0} \mathbb{Z}_2 \otimes \Sigma^{2s-2} B_{s-1}$ be defined by d_s on $\Sigma^{2s} B_s \to \Sigma^2 A_0 \otimes \Sigma^{2s-2} B_{s-1}$ being the map g_s and d_2 on $\Sigma^{2s+2} A_0 \otimes B_s \to \Sigma^{2s+3} B_s$ being f_s. Let $C_s = A_1 \otimes_{A_0} \mathbb{Z}_2 \otimes \Sigma^{2s} B_s$; then $\mathcal{C} = \{C_s, d_s\}$ is a chain complex of A_0 modules with augmentation $\mathbb{Z}_2$.

Lemma 4.3.3. The chain complex $\mathcal{C}$ is a convergent acylic chain complex with augmentation $\mathbb{Z}_2$.

Proof: Using the exact sequence described above the chain complex can be expanded to look like

$$\begin{tikzcd}
 & \Sigma^{2s} B_s & \Sigma^{2s+2} B_{s+1} \arrow[dl, "g_{s+1}"'] & \Sigma^{2s+4} B_{s+2} \arrow[dl, "g_{s+2}"'] & \\
\cdots & \Sigma^{2s+2} A_0 \otimes B_s \arrow[l] & \Sigma^{2s+4} A_0 \otimes B_{s+1} \arrow[dl, "f_{s+1}"'] & \Sigma^{2s+6} A_0 \otimes B_{s+2} \arrow[dl, "f_{s+2}"'] & \arrow[l] \\
 & \Sigma^{2s+5} B_s & \Sigma^{2s+7} B_{s+1} & \Sigma^{2s+9} B_{s+2} &
\end{tikzcd}$$

The slanting lines are just examples of 4.3.3. The convergence property is immediate. This completes the lemma.

This chain complex can be used to calculate $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, \mathbb{Z}_2)$. We have $E_1^{\sigma,s,t} = \mathrm{Ext}_{A_1}^{s-\sigma,t-\sigma}(A_1 \otimes_{A_0} \mathbb{Z}_2 \otimes B_\sigma, \mathbb{Z}_2)$ which is equal to $\mathrm{Ext}_{A_0}^{s-\sigma,t-\sigma}(B_\sigma, \mathbb{Z}_2)$. One easily sees that

$$\mathrm{Ext}_{A_0}^{**}(\mathbb{Z}_2,\mathbb{Z}_2) = P(h_0) \text{ and } h_0 \text{ has bidegree } (1,1)$$

$$\begin{aligned} \mathrm{Ext}_{A_0}^{s,t}(A_0,\mathbb{Z}_2) &= \mathbb{Z}_2 \text{ if } s = t = 0 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

4.3.4 $$\begin{aligned} \mathrm{Ext}_{A_0}^{s,t}(B_2,\mathbb{Z}_2) &= \mathbb{Z}_2 \text{ if } s = t = 0 \\ &= P(h_0)(a) \text{ with } a \text{ having bidegree } (0,2) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

$$\begin{aligned} \mathrm{Ext}_{A_0}^{s,t}(B_3,\mathbb{Z}_2) &= \mathbb{Z}_2 \text{ if } s = t = 0; \text{ or if } s = 0,\ t = 2 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Since as A_0 modules $B_{4k+i} = B_{4k-1} \oplus \Sigma^{4k} B_i$ these calculations completely determine the E_1 term.

We can summarize these calculations by Chart 4.3.5

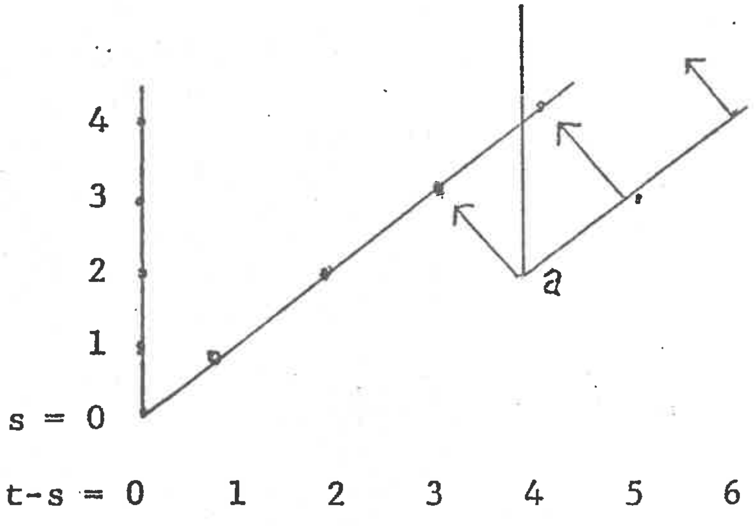
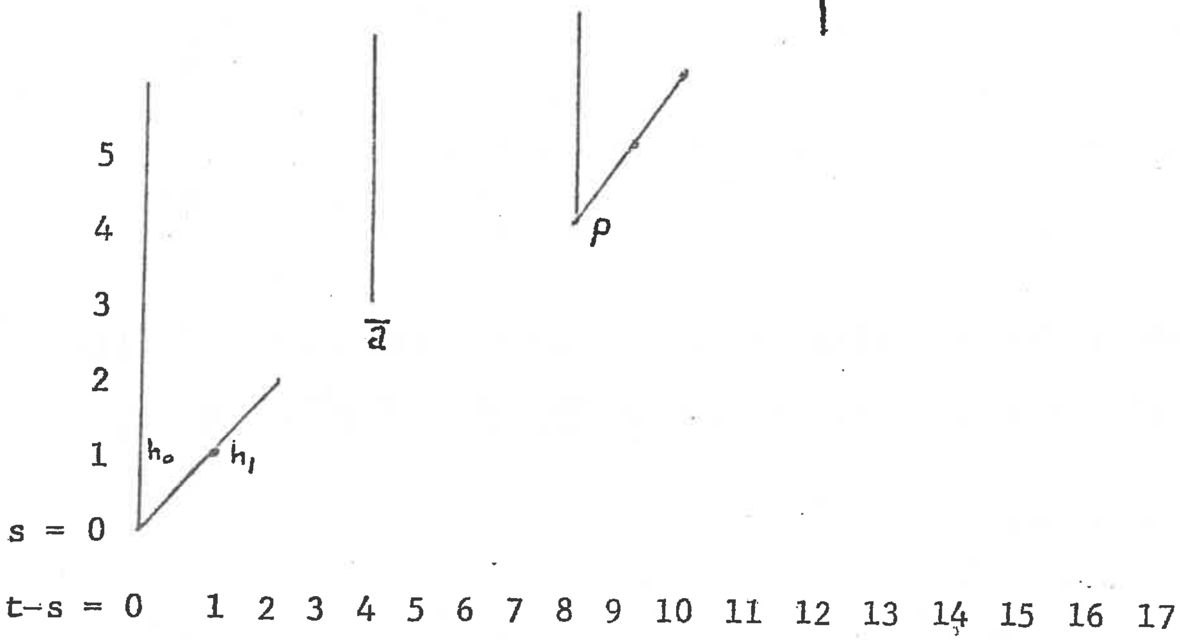


Chart 4.3.5

The element in (2,6) comes from $\mathrm{Ext}_{A_0}^{0,2}(B_2,\mathbb{Z}_2)$ and is represented by $\{Sq^2 1\}$ where 1 generates B_2 as an A-module. This determines the differential $d_1\colon \mathrm{Ext}_{A_0}^{0,2}(B_2,\mathbb{Z}_2) \to \mathrm{Ext}_{A_0}^{0,2}(B_3,Z_2)$ which is non zero. This differential implies the remaining ones indicated. Since the non zero class in $(B_4)^4$ is not in the image of an A_1 operation there are no further differentials. This gives the theorem.

Chart 4.3.6 is the result of this calculation.



$\mathrm{Ext}_{A_1}^{s,t}(\mathbb{Z}_2,\mathbb{Z}_2)$.

Chart 4.3.6.

4.3.7. Consider P^k as the set of lines through the origin in $\mathbb{R}^{k+1}$. To each point of P^k we can assign a linear transformation of $\mathbb{R}^{k+1}$ by reflection in the hyperplane, in $\mathbb{R}^{k+1}$, perpendicular to the line determined by $X \in P^k$. Composition with a fixed orientation reversing transformation provides an element of $SO(k+1)$. One can check that the map $\lambda_k\colon P^k \to SO(k+1)$ defined in this fashion is continuous. Let $J_k\colon SO(k+1) \to \Omega^{k+1}S^{k+1}$ be defined by $J_k(T)\colon S^{k+1}_{,*} \to S^{k+1}_{,*}$, $T \in SO(k+1)$, is the extension of T to the one point compactification of $\mathbb{R}^{k+1}$ and demanding that $J_k(T)$ fix the base point. J_* in homotopy is the usual Whitehead J-homomorphism.

Let $R(k)$ be the cofibre of $J_k = \bar{\lambda}_k$ and denote by $\bar{R}(k)$ its cohomology with Z_2 for coefficients.

Similarly one can define $J\circ\lambda\colon\ P^\infty \to \Omega^\infty\Sigma^\infty$ with cofibre R and $H^*(R;\ \mathbb{Z}_2) = \bar{R}$.

Proposition 4.3.8. $\mathrm{Ext}^{s,t}_{A_1}(\tilde{R};\ \mathbb{Z}_2) = \mathbb{Z}_2$ if $t - s = 4k$

0 otherwise

Proof. Filter $\bar{R}$ by requiring $\mathfrak{F}_n \subset \bar{R}$ to be the image of $A_1 \otimes \bar{R}(4n)$ in $\bar{R}$ under the standard map. Note that $\mathfrak{F}_n/\mathfrak{F}_{n-1} \cong \Sigma^{4n}(A_1 \otimes_{A_0} \mathbb{Z}_2)$.

We consider the sequence

$$\mathfrak{F}_0 \subset \mathfrak{F}_1 \to \cdots \to \mathfrak{F}_n \to \cdots .$$

We can apply $\mathrm{Ext}_{A_1}(\ ,Z_2)$ and obtain an exact couple

$$E_1^{n,s,t} = \mathrm{Ext}^{s,t}_{A_1}(\mathfrak{F}_n/\mathfrak{F}_{n-1}, \mathbb{Z}_2) \cong \mathrm{Ext}^{s,t-4n}_{A_0}(\mathbb{Z}_2, \mathbb{Z}_2).$$

There are no possible differentials thus $\mathrm{Ext}^{s,t}_{A_1}(\bar{R}, \mathbb{Z}_2) \cong \bigoplus_k \mathrm{Ext}^{s,t-4n}_{A_0}(\mathbb{Z}_2, \mathbb{Z}_2)$ and this is the proposition.

Note that the above proof also yields

Proposition 4.3.9. If $k = 4n + i$, $1 \leq i \leq 3$ then

$$\mathrm{Ext}_{A_1}^{s,t}(\bar{R}(k);\ \mathbb{Z}_2) = \bigoplus_{j=0}^{n} \mathrm{Ext}_{A_0}^{s,t-4j}(\mathbb{Z}_2,\mathbb{Z}_2) \oplus \bigoplus_{i=1}^{3} \mathrm{Ext}_{A_1}^{s,t-4n}(C_i;\ \mathbb{Z}_2) \text{ where}$$

$$(C_1)^t = \begin{cases} \mathbb{Z}_2 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(C_2)^t = \begin{cases} \mathbb{Z}_2 & \text{if } t = 0,2 \text{ and } Sq^2 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(C_3)^t = \begin{cases} \mathbb{Z}_2 & \text{if } t = 0,2,3 \text{ and } Sq^3 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark. Proposition 4.3.9 does not indicate the action of $\mathrm{Ext}_{A_1}^{**}(\mathbb{Z}_2,\mathbb{Z}_2)$ on $\mathrm{Ext}_{A_1}^{**}(\bar{R}(k),\mathbb{Z}_2)$, however, we note that P^1, the Bott periodicity operator, acts monomorphically. (P^1 is the class in $(4,12)$ in $\mathrm{Ext}_{A_1}(\mathbb{Z}_2,\mathbb{Z}_2)$; see 4.3.2.)

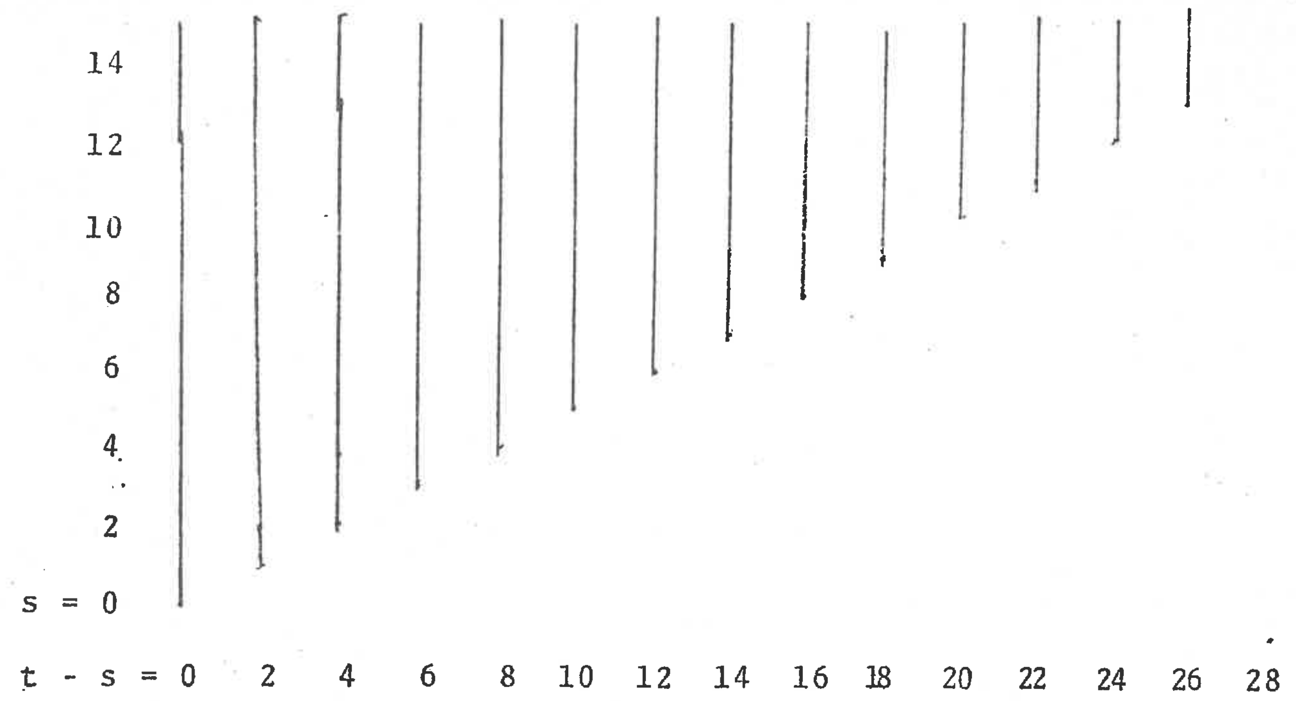
The calculation of $\mathrm{Ext}_{A_1}^{s,t}(C_i,\mathbb{Z}_2)$ is easily accomplished using the following short exact sequences.

$$C_1 \leftarrow C_2 \leftarrow \Sigma^2 C_1$$

$$C_2 \leftarrow C_3 \leftarrow \Sigma^3 C_1.$$

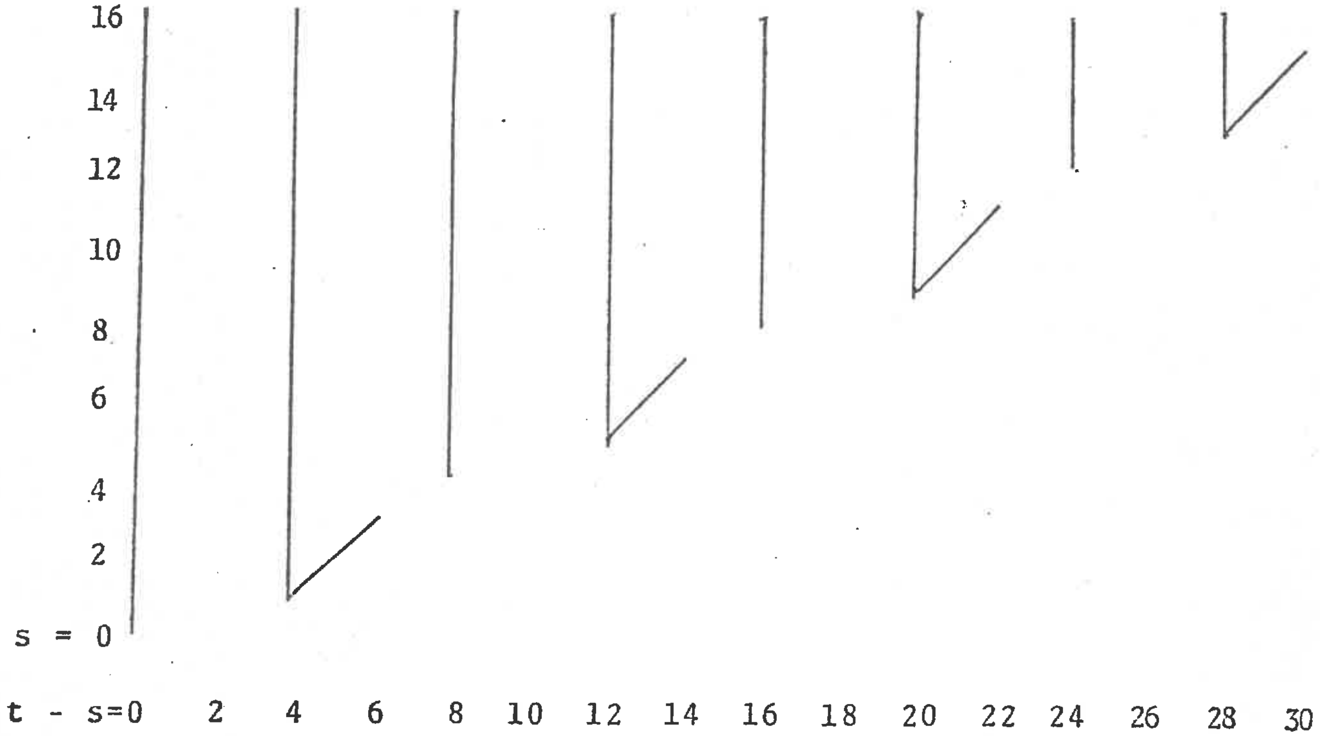
The maps involved are the obvious ones.

These results are summarized in the following charts.



$\mathrm{Ext}_{A_1}^{s,t}(C_2,Z_2)$

Chart 4.3.10

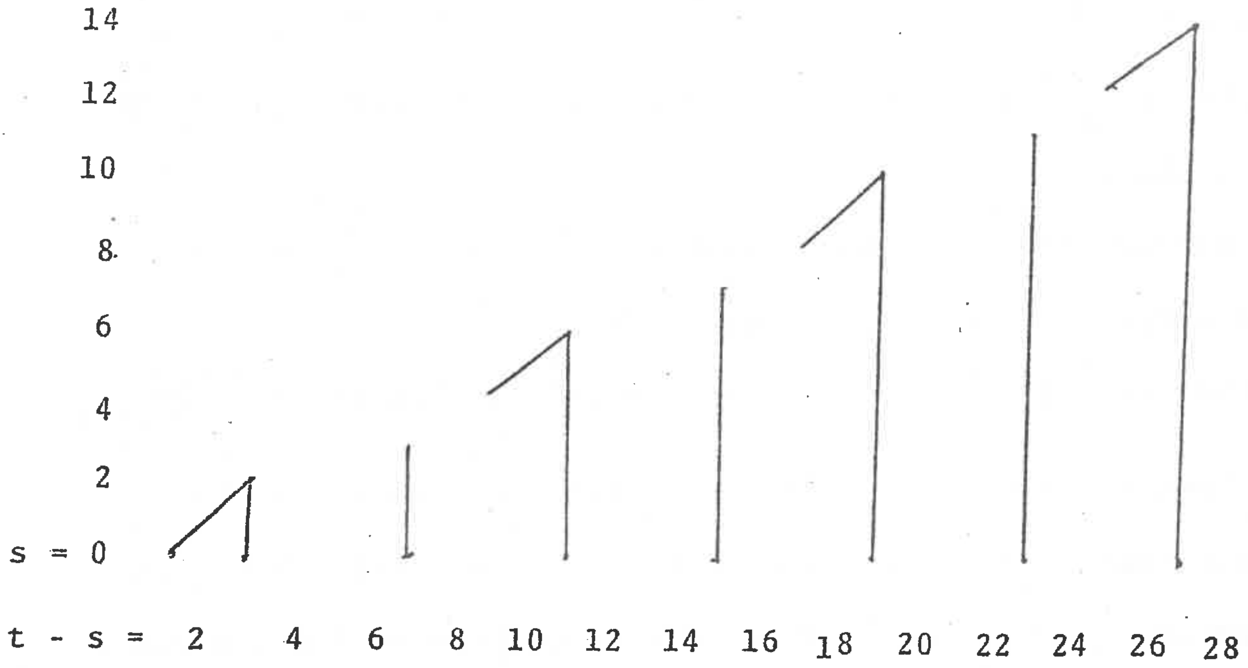


$\mathrm{Ext}_{A_1}^{s,t}(C_3,Z_2)$

Chart 4.3.11

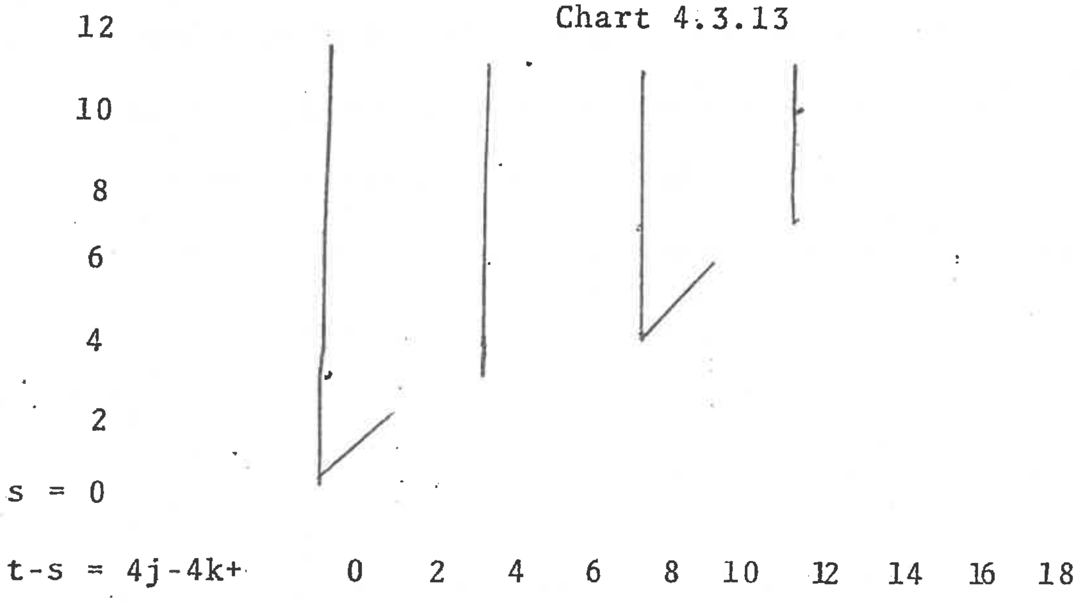
Using these calculations we can easily get

Proposition 4.3.12. $\mathrm{Ext}_{A_1}^{s,t}(\tilde{H}^*(P),\mathbb{Z}_2)$ and $\mathrm{Ext}_{A_1}^{s,t}(\tilde{H}^*(P_n^k),\mathbb{Z}_2)$ are given by the following charts.



$\mathrm{Ext}_{A_1}^{s,t}(\tilde{H}^*(P),Z_2)$

Chart 4.3.13



$\mathrm{Ext}_{A_1}^{s,t}(\tilde{H}^*(P_{4k+1}^{4j}),Z_2)$

Chart 4.3.14

The portion of this chart for s near $(t-s)/2$ is the same as in Chart 4.3.13.

4.4 A_1-free Resolutions

In this section we will present some results analogous to those obtained for A_0-free resolutions. In particular we will prove

Theorem 4.4.1 [20, Corollary 4]. If M is an A_1 module which is A_1-free then $\mathrm{Ext}_A^{s,t}(M,\mathbb{Z}_2) = 0$ if $6s > t + \varepsilon$, $\varepsilon \leq 4$ depends upon the congruence class of s mod 4.

Our methods allow one to easily get the exact edge but we feel that this simpler statement is more useful.

Another result we will include is a calculation of $\mathrm{Ext}_{A_2}^{s,t}(\mathbb{Z}_2,Z_2)$. This has been calculated by many people but first published by Shimada and Iwai [31]. Our calculation is intended to shed light on the methods of Chapter 7. It does seem to have some merit since one of us found it quicker to do this calculation in order to construct the chart than to use [31]. We prefer to give answers in terms of charts like those given because in an algebraic system with generators and relations these objects are extremely complicated. The representation on the charts seems to give a geometric pattern which is possible to comprehend. This may be a matter of taste!

Our proof of 4.4.1 will follow the model of 4.2.1.

Lemma 4.4.2. If the conclusion of 4.4.1 holds for A_1 then it holds for any connected A_1-free module.

The proof is almost identical to 4.2.2.

Lemma 4.4.3. $\mathrm{Ext}_{A_2}^{s,t}(A_1,\mathbb{Z}_2) = 0$ if $6s > t + 4$.

We will delay our proof until after we calculate $\mathrm{Ext}_{A_2}^{s,t}(\mathbb{Z}_2,Z_2)$.

To complete the proof of 4.4.1 we use the spectral sequence $\{E_r(\mathcal{C}(i,2) \otimes A_1)\}$. Then $E_1^{\sigma,s,t} = \mathrm{Ext}_{A_2}^{s-\sigma,t}(I_i^\sigma(A_2) \otimes A_1, \mathbb{Z}_2)$. Since $I_i^\sigma(A_2)$ is 8σ connected $E_1^{\sigma,s,t} = 0$ if $6(s-\sigma) > t + 4$. Taking the worse case $\sigma = 0$, gives the theorem.

Next we wish to calculate $\mathrm{Ext}_{A_2}^{s,t}(\mathbb{Z}_2,\mathbb{Z}_2)$. We will use the spectral sequence 4.1.1 but the complex $\mathcal{C}(2,1)$ is too complicated. We will find a complex similar to the one used in the second proof of 4.3.1. This approach is, in a strong sense, a May spectral sequence approach but it does seem to have some advantage over the straight May approach. There is no doubt that originally our calculations were helped by knowing, in some form, the answer.

Proposition 4.4.4. As A_1 module $I(A_2) = \Sigma^4 C_3 \oplus \Sigma^{10} B_2 \oplus \Sigma^{17} Z_2$ where C_3 is as in 4.3.9 and B_2 is as in 4.3.2.

This is an easy exercise and is left to the reader.

Using C_3 we can construct a sequence of modules analogous to the way B_2 was obtained from A_0. Let C_i, $i = 0,1,2$ be generators of C_3 as a $\mathbb{Z}_2$ vector space. Then $\bigotimes_{i=1}^{s} C_3$ consists of all words involving C_i of length s. Let $N_s \subset \bigotimes_{i=1}^{s} C_3$ be the symmetric subvector space. As before $\bigotimes_{i=1}^{s} C_s$ has a Steenrod algebra action and N_s is a submodule over A. The $\mathbb{Z}_2$ vector space of $A_2 \otimes_{A_1} \mathbb{Z}_2$ is that of an exterior algebra generated by C_i where $|C_0| = 4$, $|C_1| = 6$, $|C_2| = 7$. Thus the standard Koszul resolution result ([10], Chapter VIII, §4) yields

Proposition 4.4.5. The following sequence is exact:

$$0 \to N_s \xrightarrow{f_s} C_3 \otimes N_{s-1} \xrightarrow{g_{s-1}} \Sigma^2 B_2 \otimes N_{s-2} \xrightarrow{h_{s-2}} \Sigma^5 N_{s-3} \to 0.$$

Let $\overline{C}_3$ be the vector space generated by C_0 and C_1. Let $\overline{N}_s$ be the symmetric sub-vector space of $\bigotimes_{i=1}^{s-3} \overline{C}_3 \otimes C_3 \otimes C_3 \otimes C_3$. Then $\overline{N}_s \subset N_s$ as a $\mathbb{Z}_2$-vector space. Since $Sq^1 C_1 = C_2$ and $Sq^3 C_0 = C_2$ no A_1 operation on a class in $\overline{N}_s$ can get out of $\overline{N}_s$. Thus $\overline{N}_s$ is a sub A_1 module of N_s. Notice that $Sq^4(C_1 \otimes C_1 \otimes C_1 \otimes C_1) = C_2 \otimes C_2 \otimes C_2 \otimes C_2$ and so $\overline{N}_s$ is not an A_2 submodule. This fact will be important a little later. It is easy to see that $N_s/\overline{N}_s \cong \Sigma^{12} N_{s-4}$.

Let $\quad {}_s = A_2 \otimes_{A_1} \mathbb{Z}_2 \otimes \Sigma^{4s} N_s \quad s \neq 4$

$$ {}_4 = (A_2 \otimes_{A_1} \mathbb{Z}_2 \otimes \Sigma^{16} \overline{N}_s) \oplus \Sigma^{28} Z_2.$$

Using 4.4.4 and 4.4.5 and the maps f_s, g_s and h_s, we have $d_s: A_2 \otimes_{A_1} \mathbb{Z}_2 \otimes \Sigma^{4s} N_s \to A_2 \otimes_{A_1} \mathbb{Z}_2 \otimes \Sigma^{4s-4} N_{s-1}$ just as 4.3.3. The map d_s restricted to $A_2 \otimes_{A_1} \mathbb{Z}_2 \otimes \Sigma^{4s} \overline{N}_s$ gives a map $\overline{d}_s: C_s \to C_{s-1}$ except for $s = 4$.

Proposition 4.4.6. There is an extension of the definition of $\overline{d}_4$ so that $\mathcal{C} = (C_s, d_s)$ is a convergent acylic chain complex of A_2 modules with augmentation $\mathbb{Z}_2$.

Proof: Following the argument of 4.3.3 we see easily that $(A_2 \otimes_{A_1} \mathbb{Z}_2 \otimes \Sigma^{4s} N_s, d_s)$ is a convergent acylic chain complex of A_2 modules with augmentation $\mathbb{Z}_2$. The quotient map $N_s \to N_s/\overline{N}_s = \Sigma^{12} N_{s-4}$ gives a chain mapping $(A_2 \otimes_{A_1} \mathbb{Z}_2 \otimes \Sigma^{4s} N_s, d_s) \to (A_2 \otimes_{A_1} \mathbb{Z}_2 \otimes \Sigma^{4s+12} N_{s-4})$ augmenting the right hand complex by $\Sigma^{16}\mathbb{Z}_2$ for $s = 3$ making both complexes acylic. The map d_4 is defined to produce this augmentation. This gives 4.4.6.

From 4.1.1 we have

Theorem 4.4.7. There is a spectral sequence such that $E_1^{\sigma,s,t} \cong \mathrm{Ext}_{A_1}^{s-\sigma,t-4\sigma}(N_\sigma, Z_2) \oplus \mathrm{Ext}_{A_2}^{s-4,t-28}(\mathbb{Z}_2, Z_2)$ and whose $E_\infty^{\sigma,s,t} = E_0^\sigma \mathrm{Ext}_{A_2}^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$.

The term $\mathrm{Ext}_{A_2}^{s-4,t-28}(\mathbb{Z}_2, \mathbb{Z}_2)$ gives rise to a virtual polynomial generator. The fact that $\overline{N}_s$ is not an A_2 submodule gives a differential on the class but the square of it in $(8,56)$ is indeed a polynomial generator.

The groups $\mathrm{Ext}_{A_1}^{s,t}(\overline{N}_s, \mathbb{Z}_2)$ are easily calculated.

Proposition 4.4.8. a) $\mathrm{Ext}_{A_1}(N_0, \mathbb{Z}_2) = \mathrm{Ext}_{A_1}(\mathbb{Z}_2, \mathbb{Z}_2)$

b) $\mathrm{Ext}_{A_1}(N_1, \mathbb{Z}_2) = \mathrm{Ext}_{A_1}(C_3, \mathbb{Z}_2)$

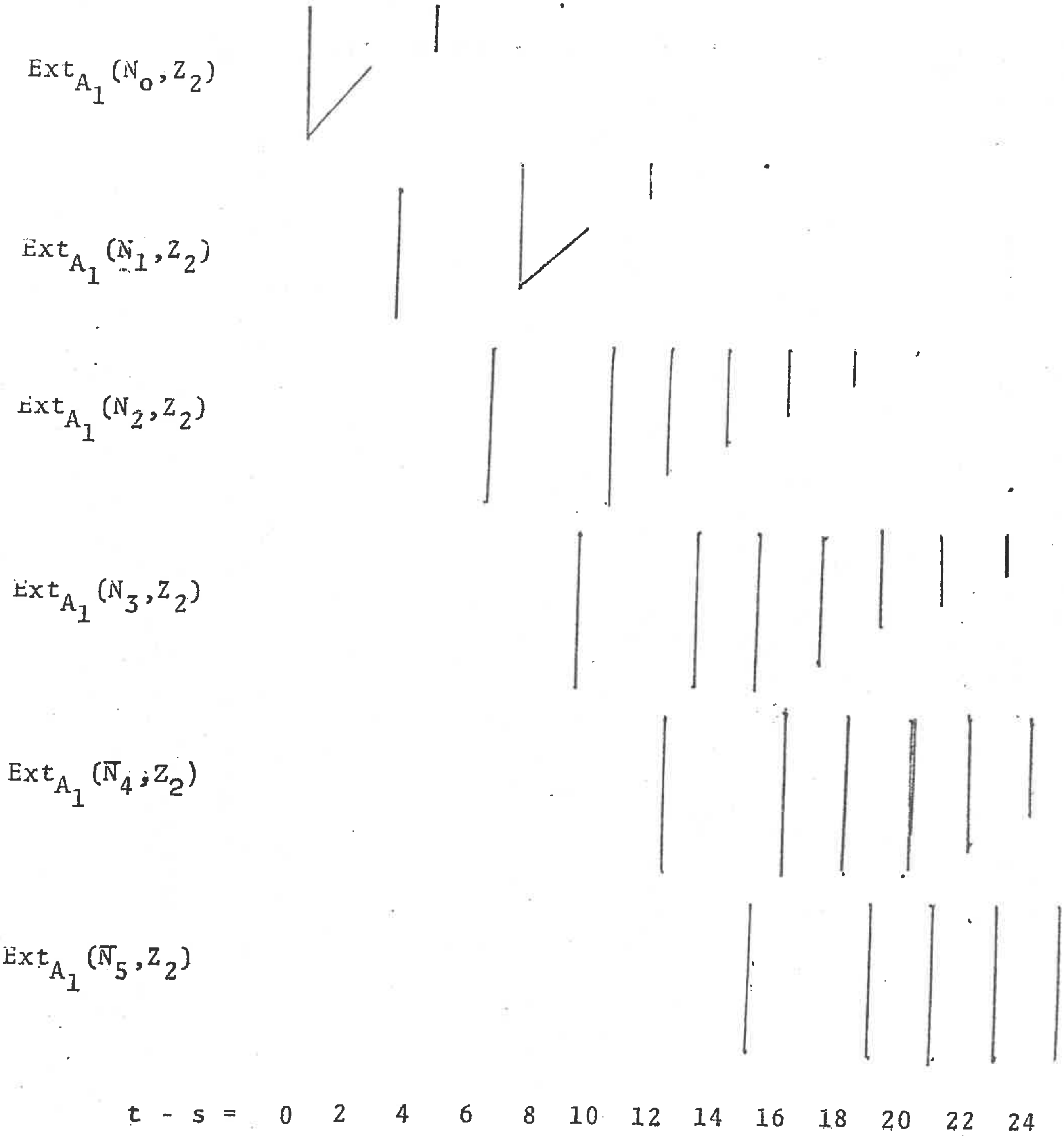
c) $\mathrm{Ext}_{A_1}^{s,t}(\overline{N}_s, \mathbb{Z}_2) \cong$

$$\mathrm{Ext}_{A_0}^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2) \oplus \bigoplus_{j=1}^{s} \mathrm{Ext}_{A_0}^{s,t+2s}(\mathbb{Z}_2, \mathbb{Z}_2) \oplus \mathrm{Ext}_{A_1}^{s,t+2s+2}(C_2, \mathbb{Z}_2) \text{ if } s \geq 2.$$

Proof: Parts a and b are immediate. Part c for $N = 2$ and $N = 3$ are special calculations, which follow easily from the calculations done in 4.3. The rest of part c follows easily by observing that $\overline{N}_s \subset \overline{N}_{s+1} \to \Sigma^{2s+2}B_3$ and a simple induction argument completes the proof.

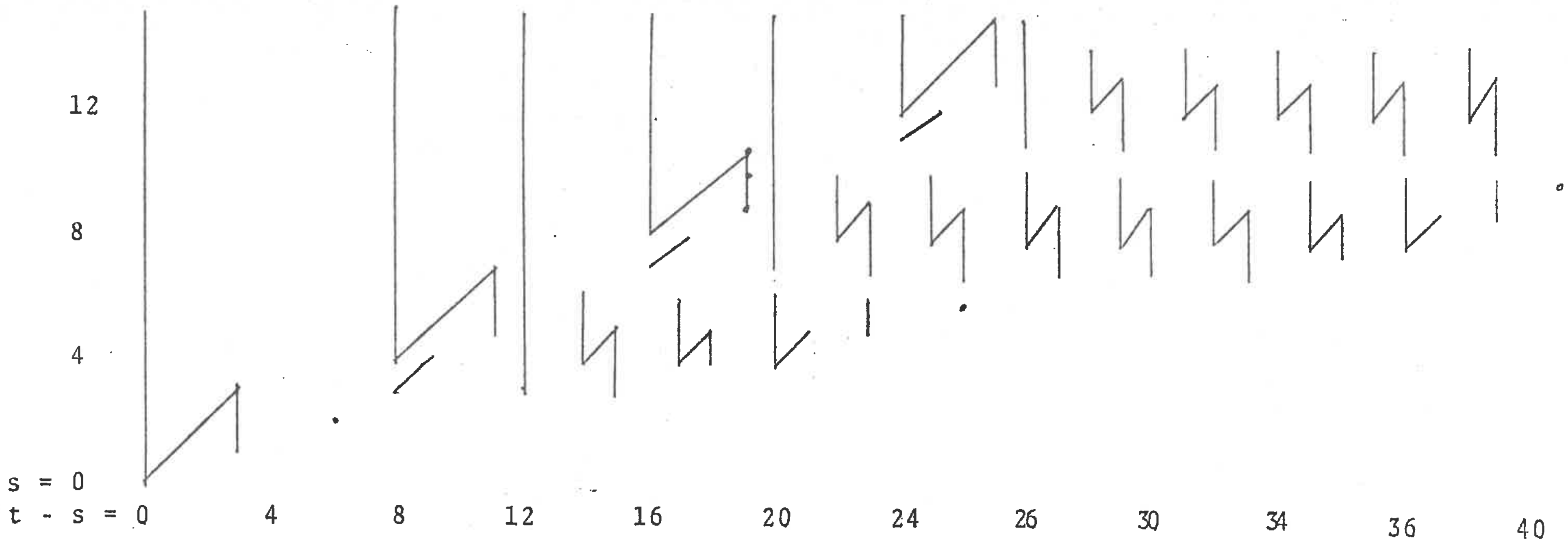
As in 4.3 the differentials in the spectral sequence reflect the A_2 structure of each $\overline{N}_s$. Since $Sq^4 C_0 \otimes C_0 = C_1 \otimes C_1$ and $Sq^6 C_0 \otimes C_0 = C_2 \otimes C_2$ we would expect differentials to occur reflecting this.

We let h_2^s represent the generator of $\mathrm{Ext}_{A_1}^{0,0}(\overline{N}_s, \mathbb{Z}_2)$; let $a_{s,i}$ generate $\mathrm{Ext}_{A_1}^{0,2i}(\overline{N}_s, \mathbb{Z}_2)$; and let b_s generate $\mathrm{Ext}_{A_1}^{0,2s+2}(\overline{N}_s, \mathbb{Z}_2)$. Note that $h_0^j h_2^s$ and $h_0^j a_{s,i}$ are non zero and b_s is free over $\mathrm{Ext}_{A_1}^{s,t}(C_2, \mathbb{Z}_2) \cong P(h_0, v_1)$ where v_1 has bidegree $(1,3)$. Then the presence of Sq^4 gives $d_s^{\#} a_{s,2} = h_2^{s+1}$. The presence of Sq^6 gives $d_s^{\#} a_{s,2} = h_2^{s+1}$. The presence of Sq^6 gives $d_s^{\#} a_{s,5} = a_{s+1,4}$. The class $P \in \mathrm{Ext}_{A_1}^{4,12}(\mathbb{Z}_2, \mathbb{Z}_2)$ acts monomorphically and commutes with $d_s^{\#}$. This allows one to calculate everything in the spectral sequence except for the free generator in $(4,28)$ coming from the free Z_2 in C_4. As noted, this class has a differential since it is in the image of Sq^4. There is one choice and linearity completes the calculations. The following charts illustrate these calculations.



Stages of the calculation for $\mathrm{Ext}_{A_2}(Z_2, Z_2)$

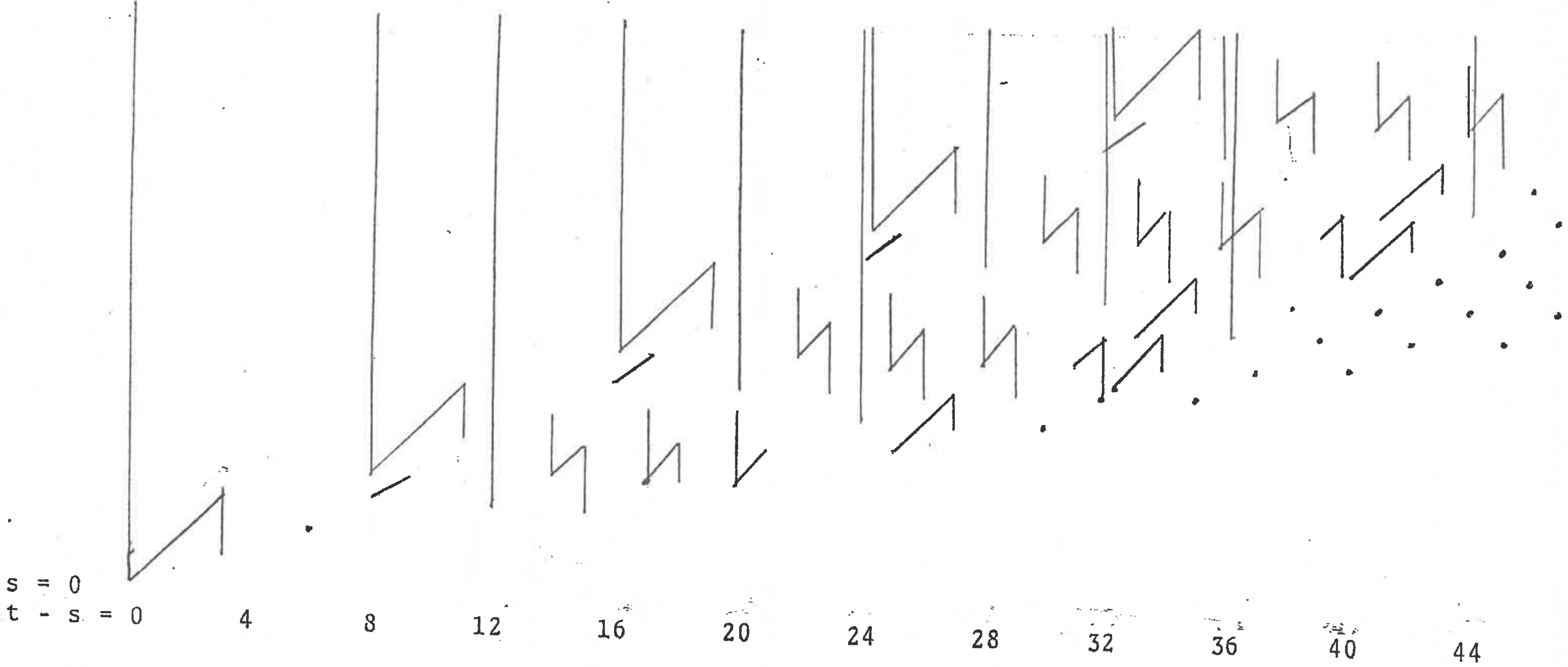
Chart 4.4.8



A portion of the chain complex for 4.4.7

Chart 4.4.9

$\mathrm{Ext}_{A_2}(Z_2, Z_2)$ is periodic on two generators one of (s,t-s) filtration (4,8) and one of filtration (4,48). Some of the first are indicated but none of the second are drawn. There is also a periodicity operator of filtration (1,5) acting on the class in filtration (6,30).



$\mathrm{Ext}_{A_2}(Z_2, Z_2)$

Proof of 4.4.3. Let C be the chain complex of 4.4.6. The chain complex $C \otimes A_1$ is easily seen to be a free A_2 resolution. The peculiar form of C_4 is the problem. The resulting spectral sequence has as its E_1 the following chain complex

$$\begin{tikzcd}
\mathbb{Z}_2 \arrow[r] & \Sigma^4\overline{N}_1 \arrow[r] & \Sigma^8\overline{N}_2 \arrow[r] & \Sigma^{12}\overline{N}_3 \arrow[r] & \Sigma^{16}\overline{N}_4 \arrow[r] & \Sigma^{20}\overline{N}_4 \arrow[r] & \cdots \\
 & & & & \oplus & & \\
 & & & & \mathrm{Ext}_{A_2}^{s-4,t-28}(A,\mathbb{Z}_2) & &
\end{tikzcd}$$

where all groups for $s \neq \sigma$ are zero except in the fourth place. The differentials described above in the proof of 4.4.7 are sufficient to yield the proof of 4.4.3. We include a chart with the result of the complete differential. The thesis of Lin [17] has related results.

The following, which follows immediately from the above calculations, is useful.

Theorem 4.4.12. For each i the spectral sequence $E_r(\mathcal{C}(i,1))$ satisfies $E_2^{\sigma,s,t}(\mathcal{C}(i,1)) = 0$ if $s > \sigma$, $6\sigma > t$ and if $s = \sigma$ then $6\sigma > t+$.

4.5. Stable A_i modules

In this section we present a few odds and ends which we will have occasion to use later.

Theorem 4.5.1. There exists a spectrum, denoted bo, such that $H^*(bo,\mathbb{Z}_2) \cong A \otimes_{A_1} \mathbb{Z}_2$.

A very nice proof is in [6] and we will not present another here. The reader can easily see, with a change of rings theorem, how calculations of $\mathrm{Ext}_{A_1}(H^*(X);\ \mathbb{Z}_2)$ can be changed into calculations

f induces an isomorphism $f : \mathrm{Ext}_A^{s,t}(N,\mathbb{Z}_2) \to \mathrm{Ext}_A^{s,t}(M,\mathbb{Z}_2)$ for $s > 0$ then f is a stable isomorphism.

Proof: Let $M' = M \oplus V$ where V is free and $V \otimes_A \mathbb{Z}_2 =$ $V \otimes_A \mathbb{Z}_2 = \operatorname{coker} f^{\#} \subset \mathrm{Ext}_A^{0,*}(M,\mathbb{Z}_2)$. Then we can modify f to $f': M \to N$ so that $f'^{\#}: \mathrm{Ext}_A^{s,t}(N,\mathbb{Z}_2) \to \mathrm{Ext}_A^{s,t}(M',\mathbb{Z}_2)$ is onto if $s = 0$ and an isomorphism for all other s. By using the cone construction of 3.4 we see that the $E_2^{s,t}(f')$ is 0 if $s > 0$ and hence is free. Thus $M' \to N$ has free coker. Using [24] we see that $N = M' \otimes W$ where W is a free A module.

Another useful fact in this direction was proved by Wall [35] and Anderson, Brown, and Peterson [7]. Let $Q_0 = Sq^1$ and $Q_1 = Sq^3 + Sq^2 Sq^1$. Since Q_0^2 and Q_1^2 are both zero Q_i acts as a boundary operator in any graded module over A_1, M. Let $H_*(M; Q_i)$ be the resulting homology.

Theorem 4.5.5. (Wall; Anderson, Brown and Peterson). If $f: M \to N$ is an A_1 map between two graded A_1 modules then f is a stable A_1 isomorphism if and only if $H_*(f; Q_i) = 0$ for $i = 0$ and 1.

of $\mathrm{Ext}_A(H^*(X \wedge bo); \mathbb{Z}_2)$. This has been considered in great detail by many people.

Theorem 4.5.2. There exists a spectrum bspin whose cohomology is, $H^*(\mathrm{bspin}; \mathbb{Z}_2) = A/A(Sq^1, Sq^5)$. Σ^4bspin is the three connected cover of bo.

Proof: It is easy to verify that $\overline{B}(1)$ exists. It is the stable complex representing the null homotopy of $\eta \circ 2$. In 4.2 we examined a module over A, B_3. In Chapter 6 we define a spectrum $\overline{B}(1)$ such that $H^*(\overline{B}(1)) = B_3$. An easy calculation shows that $H^*(\mathrm{bspin}) \cong H^*(\overline{B}(1) \wedge bo)$. The spectrum $\overline{B}(1)$ is the 3 skeleton of b spin and hence we have $\overline{B}(1) \wedge bo \to b\,spin \wedge bo \to b\,spin$ where the last map uses the ring structure. It is not hard to see that this is an isomorphism in cohomology since $H^*(\overline{B}(1)) = A_1/A_1(Sq^1, Sq^2, Sq^3)$ and $A \otimes_{A_1} \mathbb{Z}_2 \otimes A_1/A_1(Sq^1, Sq^2, Sq^3) \cong A/A(Sq^1, Sq^2, Sq^3) = A/A(Sq^1, Sq^5)$.

The following ideas can be described in a more general setting; however we will only need two special cases. See Adams and Margolis [5] and Margolis [24] for more in this direction.

Definition 4.5.3. Let $R = A$ or A_i. Let M and N be R-modules. M is stably equivalent to N if there exist locally finite projectives P_1 and P_2 such that $M \oplus P_1 \cong N \oplus P_2$.

The following result is useful.

Proposition 4.5.4. If $f: M \to N$ is a map between two A modules and

Chapter 5

The Double Suspension

5.1 Introduction. In this chapter we return to the material of Chapter 3. We need one result from Chapter 4 but otherwise this material is independent of the previous chapter.

Let W_n be the fiber of the map $S^{2n-1} \subset \Omega^2 S^{2n+1}$. Using the unstable Λ algebra as developed in Chapter 3 we can construct a spectral sequence for W_n. See 3.6.8 for a discussion. We will normalize so that $E_2^{1,2}(W_n) = \mathbb{Z}_2$ and $E_2^{s,t}(W_n) = 0$ for all t if $s=0$ and for all s if $t = 0,1$. The main results of this chapter are

Theorem 5.1.1. There are natural maps

$$E_2^{s,t}(W_1) \xrightarrow{f_1} E_2^{s,t}(W_2) \to \cdots \to \mathrm{Ext}_A^{s-1,t-1}$$

so that f_n is an isomorphism for $6s > t + 20 - 4n$.

This result is algebraic in that it asserts only that there is a map between the E_2-terms. This result is proved in [21] and substantial parts of the paper are reprinted here[1]. Since that paper was written Snaith's work [32] and Cohen and Taylor's [12] improvements have appeared. This allows the following strengthening.

Theorem 5.1.2. The maps $E_2^{s,t}(W_n) \to \mathrm{Ext}_A^{s-1,t-1}(A_0, Z_2)$ given by 5.1.1 for each $n \geq 1$ induce maps between the (unstable) spectral sequence $\{E_r(W_n)\}$ and the stable Adams spectral sequence of $\{E_r(\Sigma^{-1}RP^2)\}$.

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[1] "The double suspension homomorphism", Reprinted from the Transactions of the American Mathematical Society, Volume 214, pp. 169-178 by permission of The American Mathematical Society c 1975 by the American

Recent work of Cohen, May and Taylor [11] seems to give

Theorem 5.1.3. There is a geometric map $k_n: W(n) \to \Omega^4 W(n+1)$ which induces the map f_n of 5.1.1.

Theorem 5.1.4 (Theorem 3.1 of [22]). At the E_2 level

$$\mathrm{Ext}_A^{s-1,t-1}(\tilde{H}^*(RP^{2n}),Z_2) \cong E_2^{s,t}(S^{2n+1})$$

for $6s > t + 16$.

The majority of the calculational work of this chapter is done to prove 5.1.1 and occupies sections 5.2-5.5. The proof of 5.1.2 is given in 5.6. The proof of 5.1.4 is contained in 5.5. The balance of this section begins the proof of 5.1.1.

To prove Theorem 5.1.1 we wish to look at the double suspension. Let $\Lambda(W_n) = \varkappa_2\Lambda(4n) \oplus \varkappa_1\Lambda(4n-2)$ and assign $\varkappa_i$ filtration $(1,i+1)$. Then we have

5.1.5. $$0 \to \Lambda^{s,t}(2n-2) \to \Lambda^{s,t}(2n) \xrightarrow{p} \Lambda^{s,t-2n+2}(W_n) \to 0$$

where the first map is the obvious inclusion and the second map satisfies $p(\lambda_{2n}\lambda_I) = \varkappa_2\lambda_I$, $p(\lambda_{2n-1}\lambda_I) = \varkappa_1\lambda_I$ and, if the basis monomial λ_I starts with λ_i for $i < 2n - 1$, then $p(\lambda_I) = 0$. From 5.1.5 we can define a boundary operator, d_1, in $\Lambda(W_n)$, so that the sequence 5.1.5 is a short exact sequence of chain complexes. Theorem 5.1.1 will be proved explicitly by proving the following.

Theorem 5.1.6. There is a natural sequence of chain maps

$\Lambda^{s,t}(W_1) \to \Lambda^{s,t}(W_2) \to \cdots \to K_2\Lambda \oplus K_1\Lambda$, where the last term is associated to P^2 as described in 2.5 and $\Lambda(W_n)/\Lambda(W_{n+1})$ has zero homology if $6s > t + 14 - 4n$ for $n > 1$.

5.2. The Chain Complex $\Lambda(W_n)$.

The first step in proving Theorem 5.1.6 is to determine the differential in $\Lambda(W_n)$.

Proposition 5.2.1. $d(\varkappa_2\lambda_I \oplus \varkappa_1\lambda_J) = \varkappa_2 d\lambda_I \oplus \varkappa_1(\lambda_0\lambda_I + d\lambda_J + a)$ where $a = 0$ if $\lambda_I \in \Lambda(4n-1) \subset \Lambda(4n)$ and $a = (d\lambda_{4n+1})\lambda_{I'}$, if $\lambda_I = \lambda_{4n}\lambda_{I'}$.

The proof is long and so we delay it until the end of this section.

Definition 5.2.2. The map $f_n\colon \Lambda(W_n) \to \Lambda(W_{n+1})$ is given by $f_n(\varkappa_2\lambda_I \oplus \varkappa_1\lambda_J) = \varkappa_2\lambda_I \oplus \varkappa_1(\lambda_J \oplus \varepsilon\lambda_{4n+1}\lambda_{I'})$ where $\varepsilon = 0$ if $\lambda_I \in \Lambda(4n-1)$ and $\varepsilon = 1$ if $\lambda_I = \lambda_{4n}\,\lambda_{I'}$.

Proposition 5.2.3. f_n is a chain mapping.

<u>Proof</u>. The proof is clear from 5.2.1. Indeed,

$$\begin{aligned} f(d(\varkappa_2\lambda_I \oplus \varkappa_1\lambda_J)) &= f(\varkappa_2 d\lambda_I \oplus \varkappa_1(d\lambda_J \oplus \lambda_0\lambda_I \oplus \varepsilon d\lambda_{4n+1}\lambda_{I'})) \\ &= \varkappa_2 d\lambda_I \oplus \varkappa_1(\varepsilon\lambda_{4n+1} d\lambda_{I'} \oplus d\lambda_J \oplus \lambda_0\lambda_I \\ &\qquad \oplus \varepsilon(d\lambda_{4n+1})\lambda_{I'}) \\ &= \varkappa_2 d\lambda_I \oplus \varkappa_1(d(\lambda_J \oplus \varepsilon\lambda_{n+1}\lambda_{I'}) \oplus \lambda_0\lambda_I) \\ &= df_n(\varkappa_2\lambda_I \oplus \varkappa_1\lambda_J). \end{aligned}$$

Sketch of the Proof of Theorem 5.1.6.

We will construct the chain complex $\Lambda(W_{n+1}/W_n)$ and find a complex $\Lambda(C_n)$ which maps into $\Lambda(W_{n+1}/W_n)$. This map will be shown, using an induction hypothesis, to induce an isomorphism in homology. This is done in 5.3. In 5.4 we consider an A_1-free stable complex X and show that $\Lambda(C_n)$ maps into $\Lambda(X)$ and induces an isomorphism in an appropriate range of dimensions. Finally, we recall that $\mathrm{Ext}_A^{s,t}(A_1,\mathbb{Z}_2)$ satisfies the edge given in 4.4.1 and thus completes the proof.

We note, at this point, that $\Lambda(C_n)$ is for our considerations, an algebraic object and not known to be related to any spaces.

Proof of Proposition 5.2.1.

We need to calculate the differential in $\varkappa_2\Lambda(4n) \oplus \varkappa_1(\Lambda(4n-2)$. The differential is evaluated by the following sequence of maps:

5.2.4. $\varkappa_2\Lambda(4n) \oplus \varkappa_1\Lambda(4n-2) \to \Lambda(2n) \xrightarrow{d} \Lambda(2n) \to \varkappa_2\Lambda(4n) \oplus \varkappa_1\Lambda(4n-2)$

where the first map is given by $\varkappa_i \to \lambda_{2n+i-2}$ and the last map is p of 5.1.4. Thus we need to put in admissible form $(d\lambda_{2n})\lambda_I$ and $(d\lambda_{2n+1})\lambda_J$ and determine the coefficient of λ_{2n-1}. We need the following lemma.

Lemma 5.2.5. $\lambda_i\Lambda(k) \subset \Lambda(k-i-1) \cup \Lambda(i)$.

Proof. We wish to look at $\lambda_i\lambda_J$ where $\lambda_J \in \Lambda(k)$. If λ_J is also in $\Lambda(2i)$ then $\lambda_i\lambda_J$ is admissible and in $\Lambda(i)$. We suppose that $\lambda_i\lambda_J$ is not admissible as it stands. If $J = 1$ then

$\lambda_i\lambda_j = \sum_{k\geq 2i+1} a_k\lambda_{i+j-k}\lambda_k$ and this is in $\Lambda(j-i-1)$. Suppose we have established the lemma for all i and J such that $J < n$. Suppose $j_1 = 2\ell$. Then

$$\lambda_{\ell-1}\lambda_J = \lambda_{\ell-1}\lambda_{j_1}\lambda_{J'} = \lambda_\ell\lambda_{j_1-1}\lambda_{J'} \in \lambda_\ell\lambda_{j_1-1}\Lambda^{n-1}(2j_1)$$

$$\subset \lambda_\ell\lambda^n(j_1) \subset \Lambda^{n+1}(\ell)$$

and this is the lemma. Suppose $j_1 = 2\ell + 1$. Then $\lambda_{\ell-1}\lambda_{j_1}\lambda_{J'} = \lambda_{\ell+1}\lambda_{j_1-2}\lambda_{J'} \in \lambda_{\ell+1}\Lambda^n(j_1+1) \in \Lambda(\ell+1)$ which is the lemma. Now suppose we have established the lemma for $\lambda_{\bar{i}}\lambda_{\bar{J}}$ if $\bar{J} < n$ then then $\bar{j}_1 - 2\bar{i} - 1 < q$. Suppose $J = n$ and $j_1 - 2i - 1 = q$. Then $\lambda_i\lambda_J = \sum_{j_1>k>2i+1} a_k\lambda_{i+j_1-k}\lambda_k\lambda_{J'} + \lambda_{j_1-i-1}\lambda_{2i+1}\lambda_{J'}$. If $j_1 \geq k > 2i + 1$ then $\lambda_{i+j_1-k}\lambda_k\Lambda^{n-1}(2j_1) \subset \Lambda_{i+j_1-k}\Lambda^n(2j_1-k-1)$. Since $2j_1 - k - 1 = 2(i\_j_1-k) - 1 < q$ the induction hypothesis implies that the last expression is in $\Lambda^{n+1}(j_1-i-2)$. Finally $\lambda_{j_1-i-1}\lambda_{2i+1}\Lambda^{n-1}(2j_1) \subset \lambda_{j_1-i-1}\Lambda(2j_1-2i-2) \subset \Lambda(j_1-i-1)$. This completes the double induction and the proof of the lemma.

Now we can compute the coefficient of λ_{2n-I} in $(d\lambda_{2n})\lambda_I$. If I begins with $i_1 < 4n$ then

$$d(\lambda_{2n})\lambda_I = \sum_{i\geq 1}\binom{2n-i}{i}\lambda_{2n-i}\lambda_{i-1}\lambda_I\lambda_{2n-1}\lambda_0\lambda_I$$

$$+ \sum_{i>1}\binom{2n-i}{i}\lambda_{2n-i}\lambda_{i-1}\lambda_I$$

and

$$\lambda_{2n-i}\lambda_{i-1}\Lambda(4n-1) \subset \lambda_{2n-i}\Lambda(4n-i-1) \subset \Lambda(2n-2).$$

Thus if $I \in \Lambda(4n-1)$, $d(\varkappa_2\lambda_I) = \varkappa_2 d\lambda_I + \varkappa_1\lambda_0\lambda_I$.

Using 5.2.5 in a similar way, we see that if $i_1 = 4n$ then

$$(d\lambda_{2n})\lambda_I = \sum_{i\geq 1} \binom{2n-i}{i} \lambda_{2n-i}\lambda_{i-1}\lambda_{4n}\lambda_{I'}$$

$$= \sum_{i\geq 1} \binom{2n-i}{i} \lambda_{2n-1}\lambda_{4n-2i+1}\lambda_{2i-1}\lambda_{I'} + c + \lambda_{2n-1}\lambda_0\lambda_I$$

where $c \in \Lambda(2n-2)$.

Lemma 5.2.6. $d\lambda_{4n+1} = \sum_{i\geq 1} \binom{2n-i}{i} \lambda_{4n-2i+1}\lambda_{2i-1}$.

Proof. $d\lambda_{4n+1} = \sum_{j\geq 1} \binom{4n+1-j}{j} \lambda_{4n-j}\lambda_{j-1}$. Thus we need to show that $\binom{4n+1-j}{j} \equiv 0 (\mathrm{mod}\ 2)$ if $j \equiv 1 (\mathrm{mod}\ 2)$ and $\binom{4n+1-2i}{2i} \equiv \binom{2n-i}{i} \mathrm{mod}\ 2$. Note that if α generates $H^1(RP^\infty)$ then $Sq^j \alpha^{4n+1-j} = \binom{4n+1-j}{j} \alpha^{4n+1}$ and if $\varkappa$ generates $H^2(CP^\infty)$ then $Sq^{2i} \varkappa^{2n-i} = \binom{2n-i}{i} \varkappa^{2n}$. Since $\alpha^{4n+1} \notin \mathrm{im}\ Sq^1 \binom{4n+1-j}{j} \equiv 0(2)$ for $j \equiv 1(2)$. Since there is a stable map $f: \Sigma CP^\infty \to RP^\infty$ so that $f^*(\alpha^{2i+1}) = \varkappa^i$ we see that $\binom{4n+1-2i}{2i} \equiv \binom{2n-i}{i} \mathrm{mod}\ 2$.

We return to the proof of 5.2.1. Note that $\lambda_{2n-1}\lambda_{4n-1}\lambda_1\lambda_{I'} = 0$ since $2(2n-1) + 1 = 4n - 1$. Thus $\sum_{i\geq 2} \binom{2n-i}{i} \lambda_{2n-1}\lambda_{4n-2i+1}\lambda_{2i-1}\lambda_{I'} = \lambda_{2n-1}(d\lambda_{4n+1})\lambda_{I'}$ by 4.3. Hence $d(\lambda_{2n})\lambda_I = \lambda_{2n-1}(\lambda_0\lambda_I + (d\lambda_{4n+1})\lambda_{I'}) + c'$ where $c' \in \Lambda(2n-2)$. Putting all of this together, we see that

$d(\lambda_{2n}\lambda_I + \lambda_{2n-1}\lambda_J) = \lambda_{2n-1}(d\lambda_J + \lambda_0\lambda_I + a) + \lambda_{2n} d\lambda_I + c''$ where $c'' \in \Lambda(2n-2)$ and this proves the Proposition.

5.3. The Chain Complex $\Lambda(F_n) = \Lambda(W_{n+1}/W_n)$.

Let $\Lambda(F_n)$ be the quotient chain complex of the map f_n. Then

$\Lambda(F_n) = \kappa_2 \bigoplus_{i=1}^{4} \Lambda_{4n+i}\Lambda(8n+2i) \oplus \kappa_1 \bigoplus_{i=-1}^{2} \lambda_{4n+1}\Lambda(8n+2i)$ and $\Lambda(F_n)$ receives a differential from $\Lambda(W_{n+1})$. The differential is calculated by the following composite

$$\begin{tikzcd} \Lambda(F_n) \arrow[r, "i"] & \Lambda(W_{n+1}) \arrow[r, "d"] & \Lambda(W_{n+1}) \arrow[r, "p"] & \Lambda(F_n) \end{tikzcd}$$

vector space inclusion and p is the projection. The exact form of this differential is very complicated and we will not need it.

Let $\Lambda(C_n) = \kappa_2 \oplus [(\lambda_{4n+1} \oplus \lambda_{4n+2})\Lambda(8n-2) \oplus (\lambda_{4n+2}+\lambda_{4n+4})\Lambda(8n)]$

$+ \kappa_1 \oplus [(\lambda_{4n-1} \oplus \lambda_{4n+1})\Lambda(8n-2) \oplus (\lambda_{4n}+\lambda_{4n+2})\Lambda(8n)]$

Let $g\colon \Lambda(C_n) \to \Lambda(F_n)$ be given by

$$g(\kappa_i(\lambda_{4n+2j}\lambda_I + \lambda_{4n+2j-1}\lambda_J)) = \kappa_i(\lambda_{4n+2j}\lambda_I + \lambda_{4n+2j-1}(\lambda_{8n+1}\lambda_{I'} + \lambda_J))$$

for $j = 1,2$, $i = 2$ and $j = 1$, $i = 1$ and where $\lambda_{I'}$ is zero unless $\lambda_I = \lambda_{8n}\lambda_{I'}$; $\lambda_I \in \Lambda(8n)$ and $\lambda_J \in \Lambda(8n-2)$;
$g(\kappa_1(\lambda_{4n}\lambda_J + \lambda_{4n-1}\lambda_I)) = \kappa_1(\lambda_{4n}\lambda_J + \lambda_{4n-1}\lambda_I)$.

<u>Lemma 5.3.1</u>. $dg \subset \operatorname{im} g$.

<u>Proof</u>. The d in $\Lambda(F_n)$ is calculated by retracting $\Lambda(F_n)$ into $\Lambda(W_{n+1})$, calculating d in $\Lambda(W_{n+1})$ and projecting back to $\Lambda(F_n)$. When this is done for the image of g we get the following formulae:

$$dg\kappa_2\lambda_{4n+4}\lambda_I = \kappa_2(\lambda_{4n+4}d\lambda_I + \lambda_{4n+3}(\lambda_0\lambda_I + (d\lambda_{8n+1})\lambda_{I'} +$$

$$+ \lambda_{8n+1}d\lambda_{I'}) + \lambda_{n+2}\lambda_1\lambda_I) + \kappa_1(\lambda_{4n+2}(\lambda_2\lambda_I + \lambda_1\lambda_{8n+1}\lambda_{I'} +$$

$$+ \epsilon_n\lambda_{4n+1}\lambda_3\lambda_I + \lambda_{4n}(\lambda_{4n}\lambda_I + \lambda_3\lambda_{8n+1}\lambda_{I'}))$$

where $\epsilon_n \equiv (m)\bmod 2$ and where $\lambda_{I'} = 0$ unless $\lambda_I = \lambda_{8n}\lambda_{I'}$.

$$d(g\varkappa_2\lambda_{4n+3}\lambda_J) = \varkappa_1(\lambda_{4n+2}\lambda_1 + \lambda_{4n}\lambda_3)\lambda_J + \varkappa_2\lambda_{4n+3}d\lambda_J$$

$$d(g\varkappa_2\lambda_{4n+2}\lambda_I) = \varkappa_2(\lambda_{4n+2}d\lambda_I + \lambda_{4n+1}(\lambda_0\lambda_I + (d\lambda_{8n+1})\lambda_{I'} + \lambda_{8n+1}d\lambda_{I'}$$

$$+ \varkappa_1(\lambda_{4n+1}\lambda_1\lambda_I + \lambda_{4n}(\lambda_2\lambda_I + \lambda_1\lambda_{8n+1}\lambda_{I'}) + \lambda_{4n-1}\lambda_3\lambda$$

$$d(g\varkappa_2\lambda_{4n+1}\lambda_J) = \varkappa_2\lambda_{4n+1}d\lambda_J + \varkappa_1\lambda_{4n}\lambda_1\lambda_J$$

$$d(g\varkappa_1\lambda_{4n+2}\lambda_I) = \varkappa_1(\lambda_{4n+1}(\lambda_0\lambda_I + d(\lambda_{8n+1})\lambda_{I'}) + \lambda_{4n-1}\lambda_2\lambda_I$$

$$+ \lambda_{4n+2}d\lambda_I) + \lambda_{4n-1}(\lambda_1\lambda_{8n+1} + d\lambda_{8n+3})\lambda_{I'}$$

$$d(g\varkappa_1\lambda_{4n+1}\lambda_J) = \varkappa_1(\lambda_{4n+1}d\lambda_J + \lambda_{4n-1}\lambda_1\lambda_J)$$

$$d(g\varkappa_1\lambda_{4n}\lambda_I) = \varkappa_1(\lambda_{4n}d\lambda_I + \lambda_{4n-1}(\lambda_0\lambda_I + d\lambda_{8n+1}\lambda_{I'}))$$

$$d(g\varkappa_1\lambda_{4n-1}\lambda_J) = \varkappa_1\lambda_{4n-1}d\lambda_J.$$

To see this observe that $d(\varkappa_j\lambda_{4n+i})\lambda_I$ in $\Lambda(W_{n+1})$ involves terms of the form $\varkappa_j\lambda_{4n-p}\lambda_{p+i-1}\lambda_I$ or, if $j = 2$, terms like $\varkappa_{j-1}\lambda_{4n-p}\lambda_{p+i}\lambda_I$. If $j \neq 1$ and i is neither 0 or 2, terms such as these, when made admissible, project to zero in $\Lambda(E_n)$. Indeed, $\lambda_{4n-p}\lambda_{p+i}\lambda_I \subset \Lambda(4n-i-1) \cup \Lambda(4n-p) \subset \Lambda(4n-2)$ except for the above exceptions. In the case of exceptions when $i = 0$ the argument is just that of Section 2. When $i = 2$ we see that

$$p\varkappa_1 d\lambda_{4n+1}\lambda_{8n+1} = \Sigma\binom{4n+1-i}{i}\varkappa_1\lambda_{4n-1}\lambda_{8n-2i+3}\lambda_{2i-1}$$

Also $d\lambda_{8n+3} = \Sigma\binom{8n+3-i}{i}\lambda_{8n+3-i}\lambda_{i-1}$. The argument from Lemma 5.2.5 shows that these are the same and thus

$p\varkappa_1 d(\lambda_{4n+1})\lambda_{8n+1} = \varkappa_1\lambda_{4n-1}(\lambda_1\lambda_{8n+1} + d(\lambda_{8n+3}))$. Thus we see that in all cases the above formulae describe what happens.

It is a simple direct verification now that $dg \subset \operatorname{im} g$. We will do the first one term by term. Suppose that $\lambda_I = \lambda_{8n}\lambda_{I'}$. The other case is easier. Consider

$\varkappa_2(\lambda_{4n+4}((d\lambda_{8n})\lambda_{I'} + \lambda_{8n}d\lambda_{I'}) + \lambda_{4n+3}(\lambda_0\lambda_{8n}\lambda_{I'} + d\lambda_{8n+1}\lambda_{I'} + \lambda_{8n+1}d\lambda_{I'})$. The classes $(d\lambda_{8n})\lambda_{I'} \subset \Lambda(8n-1)$ by 5.2.5. The class $\lambda_0\lambda_{8n}\lambda_{I'} + (d\lambda_{8n+1})\lambda_{I'} \subset \Lambda(8n-2)$ by 5.2.4. Thus the above term is $g(\varkappa_2(\lambda_{4n+4}(a + \lambda_{8n}d\lambda_{I'}) + \lambda_{4n+3}(b))$ where $a \in \Lambda(8n-1)$ and $b \in \Lambda(8n-2)$. The class $\varkappa_2\lambda_{4n+2}\lambda_1\lambda_I$ is handled by noting that $\lambda_1\lambda_I \subset \Lambda(8n-2) \subset \Lambda(8n)$. Continuing with the terms of $dg\varkappa_2\lambda_{4n+4}\lambda_I$ we see $\lambda_2\lambda_I \subset \Lambda(8n-3) \subset \Lambda(8n)$; $\lambda_1\lambda_{8n+1}\lambda_{I'} \subset \lambda_1\Lambda(8n+1) \subset \Lambda(8n-1)$; $\lambda_3\lambda_I \subset \Lambda(8n-4) \subset \Lambda(8n-2)$; $\lambda_4\lambda_I \subset \Lambda(8n-5) \subset \Lambda(8n-1)$; $\lambda_3\Lambda(8n+1) \subset \Lambda(8n-3) \subset \Lambda(8n)$. All the other cases are similarly handled. This proves the lemma.

A key step in the proof of 5.1.1 is the following result.

Lemma 5.3.2. For a fixed t, if Theorem 5.1.6 is true for all $t' < t$, then g induces an isomorphism in homology for $6s > t + 3 - 12n$.

Proof. We will filter the map g in the following fashion.

$$A_1 = \varkappa_1(\lambda_{4n-1}\Lambda(8n-2) \oplus \lambda_{4n}\Lambda(8n)) \xrightarrow{g} \varkappa_1(\lambda_{4n-1}\Lambda(8n-2) \oplus \lambda_{4n}\Lambda(8n)) = B_1$$

$$A_2 = A_1 \oplus \varkappa_1(\lambda_{4n+1}\Lambda(8n-2) \oplus \lambda_{4n+2}\Lambda(8n)) \to \varkappa_1 \bigoplus_{i=-1}^{2} \lambda_{4n+i}\Lambda(8n+2i) = B_2$$

$$A_3 = A_2 \oplus \varkappa_2(\lambda_{4n+1}\Lambda(8n-2) \oplus \lambda_{4n+2}\Lambda(8n) \to B_2 \oplus \varkappa_2 \bigoplus_{i=1} \lambda_{4n+i}\Lambda(8n+2i) = B_3$$

$$A_4 = \Lambda(C_n) \to \Lambda(F_n) = B_4$$

For the resulting spectral sequence we see that

$E_0^{s,t,k}(C) = (A_{i+1}/A_i)^{s,t} = \Lambda^{s-1,t-4n-\varepsilon_i}(W_{2n})$ where $\varepsilon_i = 0,-2,-3,-5$ for $i = 1,2,3,4$ respectively. Also

$E_0^{s,t,i}(F) = (B_{i+1}/B_i)^{s,t} = \Lambda^{s-1,t-4n-\varepsilon_i}(W_{2n+\delta_i})$ where ε_i is as above and $\delta_i = 0,1,1,2$ for $i = 1,2,3,4$ respectively. The map g induces

$$g_i \colon E_0^{s,t,k}(C) \to E_0^{s,t,i}(F)$$

and g_1 is an isomorphism, g_2 and g_3 are f_{2n} and g_4 is $f_{2n+1} \circ f_{2n}$. These are quire easily seen but let us look at g_4.

$$g_4(\varkappa_2\lambda_{4n+3}a + \lambda_{4n+4}\lambda_{8n+b}b + \lambda_{4n+4}c)$$

$$= \varkappa_2(\lambda_{4n+3}a + \lambda_{4n+4}\lambda_{8n+8} + \lambda_{4n+3}\lambda_{8n+9}b + \lambda_{4n+4}c)$$

and this is just what f_{2n} does. The second inclusion is just the identity.

If Theorem 5.1.6 is true for $t' < t$, then g induces an isomorphism at the E_1 level,

$$g_{\#}E_1^{s,t,i}(C) \cong E_1^{s,t,i}(F) \text{ for all } i \text{ if } 6s > t - 12n + 24.$$

Thus g_∞ is an isomorphism for all i, if $6s > t - 12n + 30$.

This proves the lemma.

5.4. The Second Complex.

The complex $\Lambda(C_n)$ is not known to represent any spaces which have been identified. It was introduced because it also is comparable with an identifiable stable complex.

The following is an easy exercise in stable homotopy.

Proposition 5.4.1. Let A_1 be the subalgebra of A, the Steenrod algebra, generated by Sq^1 and Sq^2. There is a space X such that $H^*(X)$ is a free module over A_1 on one generator x.

Proof. Take $K(Z_2,n)$ for $n \geq 6$ and kill Sq^4, Sq^4Sq^2 and everything in dimension above $n + 6$. The resulting space is X.

There is a choice of X so that either $Sq^4Sq^2 = 0$ or $Sq^4Sq^2x = Sq^3Sq^3x$. Let X_1 have $Sq^4Sq^2x \neq 0$ and X_2 have $Sq^4Sq^2x = 0$. In both X_k we require $Sq^6x = 0$.

Proposition 5.4.2. $\Lambda(X_k) = \bigoplus_{i=1}^{2} \bigoplus_{j=2i-3}^{2i} \varkappa_{i,j}\Lambda$ with

$$d(\varkappa_{2,4}) = \varkappa_{2,3}\lambda_0 + \varkappa_{2,2}\lambda_1 + \varkappa_{1,2}\lambda_2 + (k)_{\mathrm{mod}\ 2}\varkappa_{1,1}\lambda_3 + \varkappa_{1,0}\lambda_4$$

$$d(\varkappa_{2,3}) = \varkappa_{1,2}\lambda_1 + \varkappa_{1,0}\lambda_3$$

$$d(\varkappa_{2,2}) = \varkappa_{2,1}\lambda_0 + \varkappa_{1,1}\lambda_1 + \varkappa_{1,0}\lambda_2 + \varkappa_{1,-1}\lambda_3$$

$$d(\varkappa_{2,1}) = \varkappa_{1,0}\lambda_1$$

$$d(\varkappa_{1,2}) = \varkappa_{1,1}\lambda_0 + \varkappa_{1,-1}\lambda_2$$

$$d(\varkappa_{1,1}) = \varkappa_{1,-1}\lambda_1$$

$$d(\varkappa_{1,0}) = \varkappa_{1,-1}\lambda_0$$

Proof: By the results of Chapter 2 the Λ-algebra E_1 term for a stable complex is given by $\tilde{H}_*(X; \mathbb{Z}_2) \otimes \Lambda$ and the differential is given by $d(a \otimes 1) = \sum_i aSq^i \otimes \lambda_{i-1}$ where $Sq^i: H_j(X; \mathbb{Z}_2) \to H_{j-i}(X; \mathbb{Z}_2)$ is the dual Steenrod square [14]. A direct check of the squaring operations in A_1 gives the result. The following picture may help the reader. Each 0 represents a cell and 0-0 represents Sq^1 and 0⌒0 represents Sq^2.

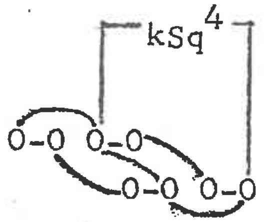

Figure 1. $\tilde{H}^*(X)$

Note that there are several other Sq^4's non-zero in the complex. Since $Sq^5 + Sq^4Sq^1 = Sq^2Sq^3$ and $Sq^2Sq^3 \neq 0$ and $Sq^1Sq^4x = 0$, we see that $Sq^4Sq^1x \neq 0$. Since $Sq^6 = Sq^5Sq^1 + Sq^2Sq^4$ and $Sq^5Sq^1 \neq 0$ we see that $Sq^6 = 0$ implies $Sq^4x \neq 0$. These are reflected in the differentials given above.

Let $\bar{g}\colon \Lambda(C_n) \to \Lambda(X_{(n)})$, where (n) is the congruence class of $n \bmod 2$, be given by:

$$\bar{g}\varkappa_j\lambda_{4n+2i}\lambda_I = \varkappa_{j,2i}\lambda_I + \varkappa_{2,2i-1}\lambda_{8n+1}\lambda_{I'} \qquad j=2,\ i=1,2;\ j=1,\ i=0;$$

$$\bar{g}\varkappa_1\lambda_{4n+2}\lambda_I = \varkappa_{1,2}\lambda_I + \varkappa_{1,1}\lambda_{8n+1}\lambda_{I'} + \varkappa_{1,-1}\lambda_{8n+3}\lambda_{I'}$$

where $\lambda_{I'} = 0$ unless $\lambda_I = \lambda_{8n}\lambda_{I'}$; and $\bar{g}\varkappa_j\lambda_{4n+2i-1}\lambda_I = \varkappa_{j,2i-1}\lambda_I$.

Proposition 5.4.3. $\bar{g}$ is a chain map.

Proof. This is a direct comparison of the two sets of formulae.

Analogously to Lemma 5.3.2 we have

Lemma 5.4.4. For a fixed t, if Theorem 5.1.6 is true for $t' < t$, then induces an isomorphism in homology for $6s > t + 3 - 12n$.

The proof follows closely to that of Lemma 5.3.2.

5.5. Proof of Theorem 5.1.6 and 5.1.4.

The last step in the proof of Theorem 5.1.6 is the following:

Proposition 5.5.1. $H_{s,t}(\Lambda(X_{(n)},d)) = 0$ if $6s > t - 4n + 14$.

This is a special case of Theorem 4.4. Now the proof of Theorem 5.1.5 follows easily. First note that Theorem 5.1.6 is true if $t = 1$. Then note that the case $n = 1$ is not needed in the induction and thus

$$\{(s,t);\ 6s > t+30-12n\} \supset \{(s,t);\ 6s > t-14n+14\} \text{ if } n > 1.$$

The first is when $H_{s,t}(\Lambda(X_{(n)},d)) \cong E_2^{s,t}(F_n)$ (4.2 and 5.4) and the second is when the left hand side is isomorphic to zero (5.5.1).

Proof of 5.1.4. We have seen that $\Lambda(2n+1) = \bigoplus_{i=1}^{2n} \lambda_i\Lambda(2i)$ and $\Lambda(P^{2n}) \cong \oplus\, \varkappa_i\Lambda$. Although the differential on λ_i is given by the same formula as that of $\varkappa_i$ there is no mapping either way of these chain complexes. But we do have the following maps

$$\begin{tikzcd} \oplus\, \varkappa_i\Lambda & \bigoplus_{i=1}^{n} \bar{\varkappa}_{2i-1}\Lambda(2) \oplus \varkappa_{2i}\Lambda(4) \arrow[l, "\tilde{f}"'] \arrow[r, "g"] & \bigoplus_{i=1}^{2n} \lambda_i\Lambda(2i) \end{tikzcd}$$

where for each pair $\varkappa_{2i-1}\Lambda(2) \oplus \varkappa_{2i}\Lambda(4) \xrightarrow{g} \lambda_{2i-1}\Lambda(4i-2) \oplus \lambda_{2i}\Lambda(4i)$. g is the composite $W(1) \to W(i)$ and $\tilde{f}$ is the map $f \cdots f_1$. Using Proposition 5.2.1 it is to verify that both g and $\tilde{f}$ are chain maps and hence induce maps in homology. If we filter the complexes by

$F_j(\bigoplus_{i=1}^{n} \varkappa_{2i-1}\Lambda(2) \oplus \varkappa_{2i}\Lambda(4)) = \bigoplus_{i=1}^{j} \varkappa_{2i-1}\Lambda(2) \oplus \varkappa_{2i}\Lambda(4)$ and analogously for the other two then the resulting spectral sequences have isomorphic E_2 terms by Theorem 5.1.1 in the range $6s > t + 16$ and thus isomorphic E_∞'s for the range $6s > t + 16$ and this is the theorem.

5.6. Proof of 5.1.2.

Recall $W(n)$ is the fiber of $S^{2n-1} \to \Omega^2 S^{2n+1}$. Hence there is a map $K: \Sigma W(n) \to \Omega^2 S^{2n+1}/S^{2n-1}$. Cohen and Taylor [12] show that there is a map $\Sigma^4(\Omega^2 S^{2n+1}/S^{2n-1}) \to S^{4n+2} U_{2i} e^{4n+3} = X$. Thus finally there is a map $W(n) \to \Omega^5 X$. We will use 3.5.2 to show that this map covers a map between the given resolution of $W(n)$ and the unstable resolution for X. The resolution for $W(n)$ is built for a resolution for $\Omega^2 S^{4n-1}$ and one for $\Omega^3 S^{4n+1}$. Let $\{X_i\}$ be the spaces in the resolution for $W(n)$ and $\{Y_i\}$ be the corresponding spaces in $\Omega^5 X$. Using the notation of 3.5.2 we need to verify that $k^* F_i(Y) \subset F_i(X)$. $F_i(X)$ is generated by classes of dimension $\leq 2^i(4n) - 2$. $H^*(X)/F_i(Y)$ is generated by classes of dimension $\geq 2^{i+1}(4n-3)$. If $n > 1$ then $2^i(2n)-2 < 2^{i+1}(4n-3)$ hence $k^* F_i(Y) \subset F_i(X)$ and thus 3.5.2 completes the proof.

Chapter 6

Ring Spectra and Thom Complexes

6.1 Introduction

In this chapter various ring spectra which are Thom complexes of bundles over H-spaces are studied.

Definition 6.1.1. A ring spectrum is a spectrum E with a map of spectra $\mu: E \wedge E \to E$ and a unit $i: S^0 \to E$ such that the following diagrams commute up to homotopy

$$\begin{tikzcd}
E \wedge E \wedge E \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"] & E \wedge E \arrow[d, "\mu"] \\
E \wedge E \arrow[r, "\mu"] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
S^0 \wedge E \arrow[r, "i\wedge 1"] \arrow[dr, "\ell"'] & E \wedge E \arrow[d, "\mu"] & E \wedge S^0 \arrow[l, "1\wedge i"'] \arrow[dl, "r"] \\
 & E &
\end{tikzcd}$$

μ is commutative if

$$\begin{tikzcd}
E \wedge E \arrow[rr, "T"] \arrow[dr, "\mu"'] & & E \wedge E \arrow[dl, "\mu"] \\
 & E &
\end{tikzcd}$$

commutes up to homotopy where T is the map that exchanges factors.

Let L be a space and ξ a bundle over L classified by a map f from L into some H-space (e.g. BO, or BF, the classifying space of stable sphere bundles). We can form the Thom spectrum $T(f)$ of f as a suspension spectrum by letting $(T(f))_n$ be the Thom complex of $L^n \to BF_n$ or $L^n \to BO(n)$. The structure maps for a spectrum are the obvious ones.

Spectra which arise in this fashion have a unit which is the inclusion of the fiber on the Thom class. The following simple theorem is basic.

False

Theorem 6.1.2. Suppose L is an H-space with multiplication μ and $f: L \to BF$ is an H-map. Then the Thom spectrum is a ring spectrum. If L is a double loop space and f is a double loop map then $T(f)$ is a commutative ring spectrum.

We note that the theorem is true for BF replaced by a suitable classifying space, e.g. BO.

Proof: The hypothesis gives a commutative diagram

$$\begin{tikzcd}
L \times L \arrow[r, "f\times f"] \arrow[d, "\mu_L"] & BF \times BF \arrow[d, "\mu_{BF}"] \\
L \arrow[r, "f"'] & BF
\end{tikzcd}$$

Taking Thom complexes we have $T(M_1): T(f) \wedge T(f) \to T(f)$. The Thom class multiplies and so the spectrum has a unit. The commutative conclusion is also immediate from an appropriate diagram at the space level.

The ring of operations of spectra which arise this way is often tractible.

Theorem 6.1.3. If $T(f)$ is a ring spectrum which is the Thom complex of a bundle over an H-space L with an inverse classified by an H-map $f: L \to BF$, then $T(f) \wedge T(f) = L_+ \wedge T(f)$. (+ denotes a disjoint basepoint.)

Proof: Let $\Delta: L \to L \times L$ be the map defined by $\Delta(x) = (x, x^{-1})$. Let

$g: L \times L \to L \times L$ be the composite

$$L \times L \xrightarrow{\Delta \times id} L \times L \times L \xrightarrow{(d,\mu)} L \times L$$

where μ is the multiplication in L. Then, clearly, g is a homotopy equivalence. Consider the bundle over $L \times L$ given by

$$L \times L \xrightarrow{g} L \times L \xrightarrow{(f,f)} BF.$$

The bundle induced by (f,f) is equivalent to the bundle induced by $(f,f) \circ g$. Consider

$$\begin{tikzcd}
L \arrow[r, "i"] & L \times L \arrow[r, "g"] & L \times L \arrow[r, "(f,f)"] \arrow[d, "\mu"] & BF \\
L \arrow[ur, "j"'] & & L \arrow[ur, "f"'] &
\end{tikzcd}$$

where $i: L \to L \times L$ is the left hand inclusion, and j is the right hand inclusion.

The Thom complex of $f\mu g i$ is homotopy equivalent to $T(f)$ while $T(f\mu g j)$ is trivial. Thus, as spectra $L_+ \wedge T(f) \cong T(f) \wedge T(f)$.

6.2 Some examples I

Some very useful spectra are given by taking $L_i = \Omega S^i$ for $i = 2,3,5,9$ and letting f_i be the Ωw where $w: S^i \to B^2O$ is a generator. We will use these spectra frequently and so let $X_i = T(f_i)$, $i = 2,3,5$, and 9. By a different procedure Barratt described similar spectra in 1967. His approach was quite different but he obtained some of the properties we use. Theorems 6.1.1 and 6.1.2 give a much more direct path to these properties. We note several of them.

6.2.1. The ring spectrum X_3 is abelian.

Proof: The map $S^3 \to B^2O$ is equivalent to the loop of $\mathbb{H}P^\infty \xrightarrow{\overline{w}} B^3U$ where $\overline{w}$ is a generator on π_3 and is extended by standard obstruction theory. Then the realification of $\Omega\overline{w}$ is w.

6.2.2. $\mathrm{Ext}_A^{s,t}(H^*(X_2),Z_2)$ contains $\mathbb{Z}_2(v_1,w_5,v_2)$ where v_1,w_5,v_2 have filtration (1,2), (1,6), (1,7) respectively and v_i are related to the BP generators of the same name.

Sketch proof. Using the results of Chapter 4 it is not hard to calculate $\mathrm{Ext}_{A_2}(H^*(X_2),\mathbb{Z}_2)$ and show that it equals $\mathbb{Z}_2(a,v_1,w_5,v_2)$ where a has filtration (0,8). Next one calculates by hand to show that v_1,w_5,v_2 all exist in $\mathrm{Ext}_A(H^*(X_2),Z_2)$. The ring map and the map $\mathrm{Ext}_A(\tilde{H}^*(X_3),\mathbb{Z}_2) \to \mathrm{Ext}_{A_2}(\tilde{H}^*(X_2),\mathbb{Z}_2)$ completes the proof.

6.2.3. From 6.1.2 we have maps $k_j\colon X_i \to \Sigma^{(i-1)j}X_i$ which have degree 1 in dimension $(i-1)j$. The evaluation of these maps in all other dimensions will be important later on. To do so we will describe k_j more explicitly. Let g_i^{-1} be the homotopy inverse of the map g described in 6.1.2 as applied to ΩS^i. Then k_j is the composite

$$X_i \xrightarrow{id\wedge S^0} X_i \wedge X_i \xrightarrow{T(g_i^{-1})} \Omega S^i_+ \wedge X_i = \bigvee_{j=0}^{\infty} \Sigma^{(i-1)j}X_i \to \Sigma^{(i-1)j}X_i$$

The first three maps are the maps induced in Thom complexes by the following space maps

$$\Omega S^i \xrightarrow{id,} \Omega S^i \times \Omega S^i \xrightarrow{\Delta'\times 1} \Omega S^i \times \Omega S^i \times \Omega S^i \xrightarrow{id,} \Omega S^i \times \Omega S^i$$

where $\Delta'(x) = (x,x)$. Let a_j be a class in $H_{(i-1)j}(\Omega S^i)$ then

$$a_\ell \to (a_\ell \otimes 1) \to \sum_{\ell+k=\ell} \binom{\ell}{j} a_j \otimes a_k \otimes 1 \to \sum_{\ell+k=\ell} \binom{\ell}{j} a_j \otimes a_k.$$

Thus

6.2.4. $k_{j*}(a_\ell) = \binom{\ell}{j} a_{\ell-j}$.

If $i = 2$ then everything is with $\mathbb{Z}_2$ for coefficients and this formula is less interesting.

6.2.5. (Brayton Gray and M. G. Barratt). If $\alpha \in \pi_j(S^0)$ let M_α be the complex $S^0 \cup_\alpha e^{j+1}$. Then $X_5 \wedge M_n = X_3$ and $X_3 \wedge M_{2\iota} = X_2$.

Neither of these follow from H maps but up to homotopy equivalence $\Omega S^2 = S^1 \times \Omega S^3$. Note that $X_5 \neq X_9 \wedge M_\nu$. First to see that $X_2 \neq X_3 \wedge M_{2\iota}$ note that $S^3 \xrightarrow[n]{} S^2 \to B^2O$ gives a generator. Thus there is a map $X_3 \to X_2$ of degree 1 on the Thom class. Now it is easy to verify that $M_{2\iota} \wedge X_3 = X_2$. (Note that in X_3 $Sq^{2i}U \neq 0$ for every i).

It is a little harder to verify $M_\nu \wedge X_5 = X_3$. The starting place is the observation that there is a map $M_\nu \to X_3$ with degree 1 on the Thom class. Using the multiplication we have $M_\nu \wedge M_\nu \to X_3$. Using the homotopy commutativity of X_3 we see that $S^4 \to M_\nu \wedge M_\nu \to X_3$ is null homotopic and the cofiber of $S^4 \to M_\nu \wedge M_\nu$ is the 2-skeleton of X_5. Now suppose we have a commutative diagram

$$\begin{tikzcd}
M_\nu \wedge X_5^{4\ell} \arrow[r] \arrow[dr] & X_5^{4\ell+4} \arrow[d] \\
& X_3
\end{tikzcd}$$

Then we have $M_\nu \wedge M_\nu \wedge X_5^{4\ell} \to M_\nu \wedge X_5^{4\ell+4} \to X_3$ and the composite $S^4 \wedge X_5^{4\ell} \to M_\nu \wedge M_\nu \wedge X_5^{4\ell} \to M_\nu \wedge X_5^{4\ell+4}$ has $X_5^{4\ell\ell+8}$ as the cofiber. But as above the composite $S^4 \to M_\nu \wedge M_\nu \to X_3$ is zero and so $M_\nu \wedge X_5^{4\ell+4}$ extends to $X_5^{4\ell+8}$. Hence $X_5 \to X_3$. Now $X_5 \wedge M_\nu = X_3$ by again checking the Steenrod operations. (Everything is still localized at the prime 2.)

6.2.6. Let $L = \Omega^2 S^3$ and let $w\colon S^3 \to B^3 O$ be a generator. Let $f = \Omega^2 w$. Then $T(f) = K(\mathbb{Z}_2, 0)$. This case has received a lot of attention in recent literature [23], [18] and [30].

6.2.7. If F_n is the Milgram filtration of $\Omega^2 S^3$ (see May's paper [38] for a good account and $f_n = f/F_n$ (f is as in 6.2.6). Then Brown and Peterson [36] have shown $T(f_n) = B(n)$, the Brown-Gitler spectra [9]. In particular $\tilde{H}^*(B(n)) \cong M(n)$ where $M(n)$ is defined in 2.4, [23].

Let $W(1)$ be the fiber of the degree 1 map of $\Omega^2 S^3 \to S^1$. ($W(1)$ is related to the $W(1)$ of Chapter 5.) Then f induces a map $\bar{f}\colon W(1) \to BSO$ and $T(\bar{f}) = K(\mathbb{Z}, 0)$ at the prime 2. Snaith [32] has given a stable map of $\Omega^2 S^3 \to \prod_{p\ \text{prime}} Q\Sigma^{2p-2} M_p$ where Q represents $\Omega^\infty \Sigma^\infty$. These maps are just the p-adic part of the Snaith decomposition for each p. For every p there is an essential map of $\Sigma^{2p-2} M_p \to BF$. Thus $\prod Q\Sigma^{2p-2} M_p \xrightarrow{\ell} BF$ is given. Let g be the composite $Y \to \Omega^2 S^3 \to \prod Q\Sigma^{2p-2} M_p \xrightarrow{\ell} BF$.

Proposition 6.2.8. $T(g) = K(\mathbb{Z}, 0)$.

Proof: We will outline the proof since the result is really one dealing with primes other than 2. The proof follows closely that given in [23] for 6.2.6. First note that $\Omega^2 S^3 \to Q\Sigma^{2p-2}M_p$ is part of a commutative diagram

6.2.9

$$\begin{tikzcd}
\Omega^2 S^3 \arrow[r] & Q\Sigma^{2p-2}M_p \arrow[r] & BF \\
\Omega S^2 \arrow[u] \arrow[r] & \Omega S^{2p-1} \arrow[ur, "f_p"'] &
\end{tikzcd}$$

(We will do one prime, they all work the same way.) By using the Cartan formula and $\Pi S^{2p-2} \to \Omega S^{2p-1} \to BF$ we see that in $T(f_p)$ $P^i U \neq 0$ and $\chi P^i U \neq 0$ for all i. (χ is the anti isomorphism.) Next observe that analagously to 2.4 there is a filtration on A_p given by $\mathbb{F}_n A_p$ = vector space generated by χP^I $I = (\epsilon_1, \ldots, \epsilon_{k-1}, \epsilon_k, \epsilon_k, 0, \ldots)$ (see [40] page 77) with $\epsilon_1 \geq n$. Then $\mathbb{F}_1 A_p = H^*(K(Z,0), Z_p)$ and $\mathbb{F}_1 A(P) = \mathbb{F}_2 A(P) \supset \cdots \supset \mathbb{F}_n A(P) \supset \cdots$. Then $\mathbb{F} \cap A(P)/\mathbb{F}_{n+1} A_p = \Sigma^{n(2p-2)}$ (P)/ (P)$\{\chi A^{\epsilon} P^i \mid i > n,\ \epsilon = 0,1\}$. Let $Y_n(P) = i^{-1}(F_{pn})$. Following the product methods of [40] it is easily seen that $\tilde{H}^*(Y_n(P)/Y_{n-1}(P)) \cong \mathbb{F}_n A_p/\mathbb{F}_{n+1} A_p$ as A_p modules. Combining this filtered action of A(P) with the generators given by 6.2.9 gives a proof.

not really

F. Cohen has obtained a more elegant proof of this using more directly the homology operations. This proof appears to be in the spirit of the Madsen-Milgram proof ([18] and [30]) of 6.2.6. The above proof, although admittedly not elegant, does seem to show that theorems of this sort are really theorems about the A structure of $H^*(BF)$ rather than homology statements. (Recently we have received a copy of the thesis of Ralph Cohen [13]. The modules $F_n A_p/F_{n+1}(A_p)$ are discussed there in some detail.)

The Milgram filtration induces a filtration on Y so that $Y_n = i^{-1}(F_2 n)$ where $i\colon Y \to \Omega^2 S^3$. Let $\overline{B}(n) = T(\overline{f}/Y_n)$. Note that $\overline{B}(n) \wedge M_2 = B(2n+1)$. Later we will use

Proposition 6.2.10. $H^*(\overline{B}(n))$ is isomorphic to $M(2n) \otimes_{A_0} \mathbb{Z}_2$.

Proof: Recall that $\overline{B}(n)$ is given as a Thom complex. The right action of Sq^1 is obtained by looking at the classes $Sq^I Sq^1$. Since $Sq^1 U = U \cup \chi_1$ and $Sq^I \chi_1 = 0$ for all I we see that under the map $\overline{B}(n) \xrightarrow{i} B(2n)$, i^* is just the projection $M(2n) \to M(2n) \otimes_{A_0} \mathbb{Z}_2$.

6.2.11. Another collection of interesting spectra result from restricting f of 6.2.6 to $\Omega J_{2^i-1}(S^2) \subset \Omega^2 S^3$ where J_k is the James construction. The homology of $\Omega J_{2^i-1}(S^2)$ is $\mathbb{Z}_2[x_1,\ldots,x_{i-1}]$ and $T(f/\Omega J_{2^i-1}(S^2))$ is a ring spectrum realizing the part of A^* which is $\mathbb{Z}_2(\xi_1,\ldots,\xi_{i-1})$. We leave the details to the interested reader.

6.2.12. As a last example of an interesting spectrum which arises this way we give the following without proof. Consider $S^5 \to B^3F$ which represents a generator. Let $f\colon \Omega^2 S^5 \to BF$ be the double loop map. Then $T(f)$ has the property $\varphi_{j,j} U \neq 0$ for every j where $\varphi_{j,j}$ is the secondary operation described by Adams [1]. The proof is easy but does use homology operations. We do not know of one which does not proceed from the homology point of view.

6.2.13. Finally, having constructed lots of examples of spectra, we would like to note that it seems clear to us that BP, bo and bu cannot be gotten in this fashion.

6.3[2]. Resolutions with respect to ring spectra

The ring spectra which arise from 6.1.1 yield particularly nice resolutions. Before describing these resolutions we fix some notation. Let Ω be an H-space with homotopy inverse and X the Thom spectrum of a bundle over Ω given by an H-map. $\Delta\colon \Omega \times \Omega \to \Omega \times \Omega$ will denote a map which yields the equivalence $\Omega_+ \wedge X \cong X \wedge X$ (6.1.3). By the geometric bar resolution with respect to a spectrum X with unit we mean the tower of fibrations in the stable category

6.3.1
$$\begin{tikzcd}
\vdots \arrow[d, "p_3"'] & \\
X_2 \arrow[r, "1\wedge s^0"] \arrow[d, "p_2"'] & X_1 \wedge X \\
X_1 \arrow[r, "1\wedge s^0"] \arrow[d, "p_1"'] & X_1 \wedge X \\
S^0 \arrow[r, "p_0"] & X
\end{tikzcd}$$

$S^0 \to X$ is the inclusion of the unit. X_1 is the fiber of p_0. In general X_n is the fiber of $X_{n-1} \xrightarrow{1\wedge s^0} X_{n-1} \wedge X$. Associated to this resolution is the cofiber sequence

6.3.2
$$\begin{tikzcd}
 & X \arrow[rr] \arrow[dr, "p_0'"] & & IX \wedge X \arrow[rr] \arrow[dr, "p_1'"] & & I^2X \wedge X \arrow[r] \arrow[dr, "p_2'"] & {} \\
S^0 \arrow[ur] & & IX \arrow[ur, "i_1=1\wedge s^0"] & & I^2X \arrow[ur, "i_2=1\wedge s^0"] & & I^3X
\end{tikzcd}$$

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[2] There is probably much overlap between this section and Adams' article [4].

Here IX is the cofiber of $S^0 \xrightarrow{i_0} X$, the inclusion. i_1: $IX \to IX \wedge X$ is $1 \wedge S^0$. Inductively we define $I^j(X)$ to be the cofiber of i_{j-1}: $I^{j-1}X \to I^{j-1}X \wedge X$. (The notation I^jX is suggestive of the augmentation ideal analogue.) Note that $\Sigma^i X_i = I^i X$.

Applying the functor π_* to 6.3.1 and 6.3.2 the "d_1" of 6.3.1 is the composite $(i_{s+1}P_s)_*$ of 6.3.2.

Associated to 6.3.1 or 6.3.2 is the sequence

6.3.3

$$X \xrightarrow{P_0\wedge S^0} IX \wedge X \xrightarrow{P_1\wedge S^0} I^2X \wedge X \to \cdots \xrightarrow{P_\sigma\wedge S^0} IX^{\sigma+1} \wedge X \to \cdots$$

Let $d_i = P_{i-1} \wedge S^0$. Clearly $d_{i+1}\circ d_i$ is null homotopic. Since this resolution is associated with 6.3.1 we have the stronger condition that brackets of arbitrary length can be formed (and hence contain zero). Indeed $IX \wedge X \cup CS \cong \Sigma S^0 \cup CX_2$; $(I^2X \wedge X) \cup C(IX \wedge X) \cup C\Sigma X \cong \Sigma^2 S^0 \cup CX_3$; etc.

Consider the sequence

6.3.4 $$X \xrightarrow{\overline{d_1}} X \wedge X \xrightarrow{\overline{d_2}} X \wedge X \wedge X \to \cdots \xrightarrow{\overline{d}_{\sigma-1}} X^\sigma$$

where X^σ is $X \wedge \cdots \wedge X$ σ-times and $\overline{d_\sigma} = \sum_{i=1}^{\sigma+1} (-1)^i d^i_\sigma$ for d^i_σ: $X^\sigma \to X^{\sigma+1}$ defined by $1 \wedge \cdots \wedge S^0 \wedge \cdots \wedge 1$ and S^0 occurs in the i^{th} place. (Recall that X is the Thom complex spectrum of a bundle over Ω, an H-space with inverse, induced by an H-map). By standard nonsense we see that $\overline{d}_{\sigma+1}\overline{d}_\sigma$ is null homotopic. The sequence 6.3.4 maps, in an obvious way, to 6.3.3. Indeed, it seems easiest to consider the following diagram displaying these maps

6.3.5

$$\begin{tikzcd}
 & & I^2X \wedge X \arrow[r, "1\wedge \bar{d}_1"] & I^2X \wedge X & \\
 & IX \wedge X \arrow[ur] \arrow[r, "1\wedge \bar{d}_1"] & IX \wedge X \wedge X \arrow[u, "P_1\wedge 1"] \arrow[r, "1\wedge \bar{d}_2"] & IX \wedge X^3 \arrow[u, "P_1\wedge 1"] \arrow[r] & \cdots \\
X \arrow[ur, "P_0\wedge s^0"] \arrow[r, "\bar{d}_1"] & X \wedge X \arrow[u, "P_0\wedge 1"] \arrow[r] & X \wedge X \wedge X \arrow[u, "P_0\wedge 1"] \arrow[r] & X \wedge X \wedge X \wedge X \arrow[u, "P_0\wedge 1"] \arrow[r] & \cdots
\end{tikzcd}$$

Continuing this process yields the desired maps from $X^{\sigma+1} \to I^{\sigma} \wedge X$.

For notational purposes we write it again as

6.3.5'

$$\begin{tikzcd}
X \arrow[r] & IX \wedge X \arrow[r] & I^2X \wedge X & \cdots \arrow[r] & I^{\sigma}X \wedge X \arrow[r] & {} \\
X \arrow[u, "f_1"] \arrow[r, "\bar{d}_1"] & X \wedge X \arrow[u, "f_2"] \arrow[r] & X^3 \arrow[u, "f_3"] \arrow[r] & \cdots \arrow[r, "\bar{d}_{\sigma}"] & X^{\sigma+1} \arrow[u, "f_{\sigma+1}"] \arrow[r] & {}
\end{tikzcd}$$

It seems likely that the bottom row satisfies the stronger condition that brackets of arbitrary length can be formed but this is not known to us.

Next we wish to compare 6.3.4 with what we have using the structure maps of 6.1.2. We have the following diagram

6.3.6

$$\begin{tikzcd}
X \arrow[r, "\bar{d}_1"] & X^2 \arrow[r, "\bar{d}_2"] & X^3 \arrow[r] & \cdots \arrow[r, "\bar{d}_{\sigma}"] & X^{\sigma+1} \arrow[r] & \cdots \\
X \arrow[u, "g_1"] \arrow[r, "\delta_1"] & \Omega_+ \wedge X \arrow[u, "g_2"] \arrow[r, "\delta_2"] & \Omega_+ \wedge \Omega_+ \wedge X \arrow[u, "g_3"] \arrow[r] & \cdots \arrow[r, "\delta_{\sigma}"] & (\Omega_+)^{\sigma} \wedge X \arrow[u, "g_{\sigma+1}"] \arrow[r] & \cdots
\end{tikzcd}$$

where the g_i are homotopy equivalences by $g\colon \Omega \wedge \Omega \to \Omega \wedge \Omega$ (6.1.2). where $\delta_1 = \bar{\Delta} + S^0 \wedge 1$ $\delta_2 = \bar{\Delta} \wedge 1 - 1 \wedge \bar{\Delta} + S^0 \wedge 1$

$\delta_3 = \bar{\Delta} \wedge 1 \wedge 1 - 1 \wedge \bar{\Delta} \wedge 1 + 1 \wedge 1 \wedge \bar{\Delta} - S^0 \wedge 1 \wedge 1$, etc. for $\bar{\Delta}$ the map induced by the usual diagonal.

Proposition 6.3.7. This diagram commutes.

Proof: It is sufficient to look at the space level. The first square becomes

$$\begin{tikzcd}
\Omega \arrow[r, "(1,0)-(0,1)"] & \Omega \times \Omega \\
\Omega \arrow[u, "\mathrm{id}"'] \arrow[r, "\overline{\Delta}-(0,1)"'] & \Omega > \Omega \arrow[u, "g"']
\end{tikzcd}$$

Now $\Delta \circ \overline{\Delta} = (1,0)$ and $\Delta(0,1) = (0,1)$. (Recall g is the composite $\Omega \times \Omega \xrightarrow{\Delta \times 1} \Omega \times \Omega \times \Omega \xrightarrow{1 \times N} \Omega \times \Omega + \Delta'$ is $(1,-1)$.) The general case represents a sequence of similar steps.

Also note that the sequence of maps in 6.3.5 which eliminate the various axes amount to removing the basepoint in 6.3.6. This gives

Proposition 6.3.8. We have the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "P_0 \wedge S^0"] & IX \wedge X \arrow[r] & \cdots \arrow[r, "P_{\sigma-1} \wedge S^0"] & I^\sigma X \wedge X \arrow[r] & \cdots \\
X \arrow[u, "\mathrm{id}"] \arrow[r, "S^0"'] & \Omega \wedge X \arrow[u, "\overline{g}_2"'] \arrow[r, "\delta_2"'] & \cdots \arrow[r, "\delta_\sigma"'] & \Omega^\sigma \wedge X \arrow[u, "\overline{g}_{\sigma+1}"'] \arrow[r] & \cdots
\end{tikzcd}$$

6.4. Some examples II

In this section we apply 6.3 to a few of the spectra described in 6.2.

6.4.1. The theory gives a particularly nice situation when applied to ΩS^i and X_i of 6.2. For each i we have spectral sequence coming from the exact couple of the resolutions whose $E_1^{s,t} = \pi_t((\Omega S^i)^s \wedge X_i) = [\tilde{H}_*(\Omega S^i \wedge \cdots \wedge \Omega^i S; \mathbb{Z}) \otimes \pi_*(X_i)]_t$. The d_1 is induced by δ_s above.

6.4.2. When we apply the theory to $\Omega^2 S^3$ and $K(\mathbb{Z}_2)$ we get the classical bar resolution from 6.3.1. The resolution <u>6.3.1</u> looks slightly different than the bar resolution since it appears to make each of the exterior algebra generators in $H^*(\Omega^2 S^3)$ primitive in the resolution. These generators can be identified with $\xi_i^{2^j} \epsilon A^*$ and $\xi_i^{2^j}$ is not primitive. This apparent discrepancy is cleared up when one recalls that the fact that $\Omega^2 S^3$, as a stable complex, breaks up into parts each of which has a non trivial Steenrod algebra action. The action is given by $x_i \to \sum_{j+k-i} x_j^{2^k} \otimes \xi_k$. When this additional term is added to the primitive term we have the usual bar resolution.

The May spectral sequence seems to be able to be obtained this way also. We look at the resolution

$$\mathbb{Z}_2 \to K(\mathbb{Z}_2,0) \to \Omega^2 S^3 \wedge K(\mathbb{Z}_2,0)$$

$$\to (\Omega^2 S^3)^2 \wedge K(\mathbb{Z}_2,0) \to \cdots \to (\Omega^2 S^3)^\sigma \wedge K(\mathbb{Z}_2,0) \to \cdots$$

Now $\mathrm{Hom}_A(C_s, \mathbb{Z}_2) \cong (\Omega^2 S^3)^s$. The differential in the associated chain complex has two parts, one is the differential in

$$\Omega^2 S^3 \xrightarrow{\bar{\Delta}} (\Omega^2 S^3)^2 \xrightarrow{1\wedge\bar{\Delta}+\bar{\Delta}\wedge 1} (\Omega^2 S^3)^3$$

$$\xrightarrow{1\wedge 1\wedge\bar{\Delta}+1\wedge\bar{\Delta}\wedge 1+\bar{\Delta}\wedge 1\wedge 1} (\Omega^2 S^3)^4 \to \cdots$$

and the second part interprets the action of the Steenrod algebra in $\Omega^2 S^3$. Using the Koszul resolution we see that $H_*(C_1) = \mathbb{Z}_2(R_{i,j})$

$i \geq 0$, $j \geq 1$ where $R_{i,j}$ is represented by $x_j^{2^i}$ and $H_*(\Omega^2S^3) = \mathbb{Z}_2(x_i)$. This is the E_1 term of the May spectral sequence. The d_1 results from identifying $x_j^{2^i}$ with $\alpha \epsilon A$ and asking how $\alpha_{i,j}$ acts on $x_\ell^{2^k}$.

We have $x_j^{2^i} = \alpha_{i,k} X_{j-k}^{2^{i+k}}$ for $k = 1,\ldots,j - 1$. This follows easily from the Brown-Gitler decomposition description of A (see [23]). It probably is easily read from the Nishida relation. Anyway, when dualized this yields $dR_{ij} = \sum_{R=1}^{j-1} R_{i,k}R_{i+k,j-k}$. The higher differentials reflect more complicated squaring operations. The evaluation of differentials seems to be easier in this setting. In particular in Tangora [33], 4.9, the proposition $d_4(b_{03})^2 = h_2b_{12}^2 + h_4b_{02}^2$ is proved. It is apparantly not easy to verify that the term $h_2b_{12}^2$ is present. The statement after 4.4.7 gives a simple proof of its presence. (Note that our development of $\text{Ext}_{A_2}(\mathbb{Z}_2,\mathbb{Z}_2)$ is really a modification of the above and hence a modification of the May spectral sequence. It seems likely that 1.3 of [33] could be proved in this manner.)

6.4.3. An interesting description of the E_2 term for the Novikov spectral sequence results when one applied the theory of 6.3 to BU and MU. The resulting chain complex is

$$MU \xrightarrow{f_1} BU \wedge MU \xrightarrow{\delta_2} BU \wedge BU \wedge MU \rightarrow \cdots$$

where δ_1 is the map of Thom complexes given by $BU \xrightarrow{\Delta} BU \times BU \xrightarrow{0,1} BU$ $\delta_2 = \Delta \wedge 1 - 1 \wedge \delta_1$, $\delta_3 = \Delta \wedge 1 \wedge 1 - 1 \wedge \Delta \wedge 1 + 1 \wedge 1 \wedge \delta$ and so forth. Many standard formulae result.

6.4.4. BO [8,...] and MO [8,...] yield an interesting spectral sequence and recent work of Davis and Mahowald [15] have applied it.

6.4.5. The space $\Omega(J_{2^i-1} S^2)$ where J_k is the James construction yields interesting spectra when one uses the composite $\Omega(J_{2^i-1} S^2) \subset \Omega^2 S^3 \xrightarrow{f} BO$. The homology of $\Omega J_{2^i-1} S^2$ is equal to $P(x_1,\ldots,x_{i-1})$. The resulting resolution seems to give a geometric realization of the various spectral sequence of Adams [1], Chapter 2.

6.5 An interesting spectrum

This section is really a part of the proof of the main results of these lectures. There does not exist an H-space Ω which produces bo as a Thom spectrum. In this section will describe a stable spectrum which looks like the suspension spectrum of a space which, if it did exist, would generate bo. The stable space does exist. From this we will have available the ideas of 6.3 even if we cannot use, directly, the results.

Let $\{Y_i\}$ be the sequence of spaces defined inductively $Y_0 = S^0$, Y_i is the fiber of the map $Y_{i-1} \xrightarrow{1\wedge S^0} Y_{i-1} \wedge K(\mathbb{Z}_2,0)$. Note that $Y_i \wedge Y_j = Y_{i+j}$. Let $\bar{\Omega}_+ = \bigvee_{i=0}^{\infty} \Sigma^{4i} \wedge Y_{2i-\alpha(i)}$. Let $\Delta: \Omega S^5 \to \Omega S^5 \times \Omega S^5$ be the usual diagonal map. We wish to define $\bar{\Delta}$ so that we have a commutative diagram of spectra

6.5.1.

$$\begin{tikzcd}
\bar{\Omega}_+ \arrow[r, "\bar{\Delta}"] & \bar{\Omega}_+ \wedge \bar{\Omega}_+ \\
(\Omega S^5)_+ \arrow[u, "h"] \arrow[r, "\Delta"'] & (\Omega S^5)_+ \wedge (\Omega S^5)_+ \arrow[u, "h\wedge h"']
\end{tikzcd}$$

where $h = \bigvee_{i=0} 1 \wedge S^0$ and $\Omega S^5_+ = \tilde{\bigvee}_{i=0} \Sigma^{4i}$. Consider the composite $\Sigma^{4i} \to \Omega S^5_+ \to \Omega S^5_+ \wedge \Omega S^5_+ \to \Sigma^{4j} \wedge \Sigma^{4k}$ where $j + k = i$. This composite has degree $\binom{i}{j}$. The power of 2 present in $\binom{i}{j}$ is $\alpha^3(j) + \alpha(j-k) - \alpha(i)$. Hence $\bar{\Delta}$ can be made up of composites

$$\begin{tikzcd}
\Sigma^{4i}Y_{2i-\alpha(i)} \arrow[r, "\cong"] & \Sigma^{4j}Y_{2j-\alpha(j)} \wedge \Sigma^{4k}Y_{2k-\alpha(k)} \wedge Y_{\alpha(j)+\alpha(k)-\alpha(i)} \arrow[d] \\
& \Sigma^{4j}Y_{2j-\alpha(j)} \wedge \Sigma^{4k}Y_{2k-\alpha(n)} \wedge S^0 .
\end{tikzcd}$$

The composite $S^0 \to Y_{2(j)+\alpha(k)-\alpha(i)} \to S^0$ is multiplication by $2^{\alpha(j)+\alpha(k)-\alpha(i)}$.

This gives the following commutative diagram

6.5.2

$$\begin{tikzcd}
\Omega S^5 \arrow[r, "\Delta"] \arrow[d] & \Omega S^5 \wedge \Omega S^5 \arrow[r, "d_2"] \arrow[d] & \cdots \arrow[r, "d_\sigma"] & (\Omega S^5)^\sigma \arrow[r] \arrow[d] & \cdots \\
\bar{\Omega} \arrow[r, "\bar{\Delta}"] & \bar{\Omega} \wedge \bar{\Omega} \arrow[r, "\bar{d}_2"] & \cdots \arrow[r, "\bar{d}_\sigma"] & (\bar{\Omega})^\sigma \arrow[r] & \cdots
\end{tikzcd}$$

where $d_\sigma = \Sigma(-1)^i \delta^i_\sigma$ and $\delta^i_\sigma = 1 \wedge \cdots \wedge \Delta \wedge 1 \wedge \cdots \wedge 1$, where the Δ occurs in the i^{th} place. The map $\bar{d}_\sigma$ is analogously defined.

Proposition 6.5.3. Diagram 6.5.2. induces a chain complex

$$\mathcal{C}: \quad \cdots \leftarrow H^*((\Omega S^5)^\sigma) \xleftarrow{d_\sigma} H^*(\Omega S^5)^{\sigma+1}) \leftarrow \cdots$$

of graded groups and $H_*(\mathcal{C}) = Z_2[a_i]$ with bidegree $(1,2^i)$ $i \geq 2$.

This is a very simple calculation.

Proposition 6.5.4. Diagram 6.5.3 induces a chain complex

$$\overline{\mathcal{C}}: \qquad \leftarrow H^*(\overline{\Omega}^{\sigma}) \xleftarrow{\overline{d}_{\sigma}} H^*(\overline{\Omega}^{\sigma+1}) \leftarrow \cdots$$

of graded groups and

$$H_*(\overline{\mathcal{C}}) = \bigoplus_{a^I \in \mathbb{Z}_2[a_i]} a^I \cdot H^*((Y_1)^{i_1} \wedge (Y_3)^{i_2} \wedge \cdots \wedge (Y_{2^{j+1}-1})^{i_j} \wedge \cdots)$$

$$= \bigoplus_{a^I \in \mathbb{Z}_2[a_i]} H^* Y_{\Sigma_{ij}(2^{j+1}-1)}$$

where $I = (i_1, i_2, \ldots, i_j)$.

Proof. The 1-1 correspondence between classes in $H^*(\Omega S^5)$ and modules $H^*(\Sigma^{4k} Y_{2k-2(k)})$ identifies a particular $H^*(Y_{f(a)})$ for each $a \in H^*((\Omega S^5)^{\sigma})$. If $d_{\sigma} a$ is not zero then $\overline{d}_{\sigma} H^*(Y_{f(a)})$ is an isomorphism. If $d_{\sigma} a$ is zero then $\overline{d}_{\sigma} H^*(Y_{f(a)})$ is zero. Hence the homology of $\overline{\mathcal{C}}$ will be a sum of complexes $H^*(Y_{f(a)})$ in 1-1 correspondence with $H_*(\mathcal{C})$.

Let $bo \leftarrow bo^1 \leftarrow bo^2 \leftarrow \cdots$ be any resolution of bo by Eilenberg MacLane space $K(\mathbb{Z}_2)$. That is the fiber of the map $bo^i \xleftarrow{p_i} bo^{i+1}$ is $K(V)$ where V is a graded $\mathbb{Z}_2$ vector space and p_i^* is zero.

Proposition 6.5.5. As A modules, $H^*(bo^{\sigma})$ is stably isomorphic to $H^*(Y_{\sigma} \wedge bo)$.

Proof. By definition $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(bo^{\sigma}), \mathbb{Z}_2) \cong \mathrm{Ext}_A^{s,t}(H^*(Y_{\sigma} \wedge bo), \mathbb{Z}_2)$ for $s > 0$ since both are equal to $\mathrm{Ext}_A^{s+\sigma, t+\sigma}(\tilde{H}^*(bo), \mathbb{Z}_2)$. Two modules with a map between them inducing such an isomorphism are stably equivalent

Let $R(2^i-1)$ and $\overline{R}(2^i-1)$ be as in 4.3.7.

Proposition 6.5.6. If $i \neq 2$ then there is a map $f\colon R(2^i-1) \to Y_{2^i-1}$ such that $f^*\colon H^*(Y_{2^i-1} \wedge bo) \to H^*(R_{2^i-1} \wedge bo)$ is a stable equivalence of A-modules.

Proof: The Adams edge theorem yields the map $f\colon R(2^i-1) \to Y_{2^i-1}$. That f^* is a stable A-isomorphism follows from 4.2.6 and 4.3.5.

We would like to modify $\overline{\Omega}$ to get a second similar spectrum. The diagonal map will be defined in a manner analogous to that for $\overline{\Omega}_+$ but with a crucial difference.

Let $\overline{\Omega}'_+ = \bigvee_{i=0}^{\infty} \Sigma^{4i}\overline{Y}_i$ where

$$\overline{Y}_i = \begin{cases} Y_{2i-\alpha(i)} & \text{for } i \equiv 0 \bmod 2 \\ \overline{B}(1) \wedge Y_{2(i-1)-\alpha(i-1)} & \text{for } i \equiv 1 \bmod 2 \end{cases}$$

An easy calculation gives

Proposition 6.5.7. $H^*(\overline{B}(1) \wedge \overline{B}(1))$ is stably equivalent to $H^*(Y_2)$ as A_1-modules.

We wish to construct a diagonal map for $\overline{\Omega}'_+$ but it will be defined on the cohomology level as A_1-modules.

a) $\Sigma^{8i+4}\overline{Y}_{2i+1} \to \Sigma^{8j+4}\overline{Y}_j \wedge \Sigma^{8k}\overline{Y}_k$

Since $\alpha(2j) + \alpha(2k) - \alpha(2i) = \alpha(2j+1) + \alpha(2k) - \alpha(2i+1)$ this map is defined as above.

b) $\Sigma^{8i}\overline{Y}_{2i} \to \Sigma^{8j}\overline{Y}_{2j-1} \wedge \Sigma^{8k+4}\overline{Y}_{2k+1}$.

This map does not seem to exist with the desired properties. However, we do get the required map in cohomology from the maps.

$$\Sigma^{8i}\overline{Y}_{2i} \xleftarrow{g} \Sigma^{8i}Y_{4i-\alpha(2i)-2} \wedge \overline{B}(1) \wedge \overline{B}(1)$$

6.5.8

$$\to \Sigma^{8j-4}Y_{4(j-1)-\alpha(2j-1)} \wedge \overline{B}(1) \wedge \Sigma^{8k+4}Y_{4k-\alpha(k)} \wedge \overline{B}(1) \wedge$$

$$Y_{\alpha(2j-2)+\alpha(k)-\alpha(4i)+2} \to \Sigma^{8j-4}\overline{Y}_{2j-1} \wedge \Sigma^{8k+4}\overline{Y}_{2k+1} .$$

Since $\alpha(2j-2) + \alpha(2k) + 2 = \alpha(2j-1) + \alpha(2k+1)$ the composite $S^0 \to Y_{\alpha(2j-2)+\alpha(k)+2-\alpha(4i)} \to S^0$ has degree $\binom{2i}{2j-1}$. The map g has an inverse in the following sense.

Since $\overline{Y}_{2i} \wedge bo \xleftarrow{g\wedge 1} Y_{4i-\alpha(2i)-2} \wedge \overline{B}(1) \wedge \overline{B}(1) \wedge bo$ induces a stable isomorphism of A modules, there is a map $g' : \overline{Y}_{2i} \wedge bo \to Y_{4i-\alpha(2i)-2} \wedge \overline{B}(1) \wedge \overline{B}(1) \wedge bo$ which induces a stable isomorphism of A modules. The desired diagonal map is the composite in cohomology of diagram 6.5.8 in which g^*, which is a stable A_1 isomorphism, is replaced by an inverse map which is also a stable A_1-isomorphism.

Hence we have a chain complex $\overline{\mathcal{C}}'_+$:

$$\cdots \leftarrow H^*(\overline{\Omega}'_+)^{\sigma} \xleftarrow{\overline{d}^1_\sigma} H^*(\overline{\Omega}'_+)^{\sigma+1} \leftarrow \cdots$$

Analogously to 6.5.3, again we have

Proposition 6.5.9. $H_*(\overline{\mathcal{C}}'_+) \cong \bigoplus_{a^I \in \mathbb{Z}_2[a_1^2, a_2, \ldots]} a^I H^*(Y_{\Sigma_{ij}(2^{j+1}-1)}) \oplus$

$\bigoplus_{a^I \in a_1 \mathbb{Z}_2[a_1^2, a_2, \ldots]} a^I H^*(Y_{\Sigma_{ij}(2^{j+1}-1)-1} \wedge \overline{B}(1))$ as stable A_1 modules.

Proof. The proof follows closely that of 6.5.4.

Chapter 7

bo Resolution I; Algebraic Version

7.1. Introduction

There does not seem to be an H-space Ω with an H-map $\Omega \to BO$ whose Thom complex is bo. Yet bo resolutions exhibit the same character that resolutions described in Chapter 6 have. Let Ω be the stable spectrum $\Omega_+ = \bigvee_{i=0}^{\infty} \Sigma^{4i}\overline{B}(i)$ where the space $\overline{B}(i)$ are described in 6.2.5. In Chapter 8 we will construct a map $g\colon bo \wedge bo \to \Omega_+ \wedge bo$ which is a homotopy equivalence. The construction of this map will involve a calculation of $\pi_*(bo \wedge bo)$ and this chapter is devoted to, among other things, this calculation. First we will prove the cohomology version.

Theorem 7.1.1. There is a map $g^*\colon H^*(bo \wedge bo) \to H^*(\Omega_+ \wedge bo)$ which is an isomorphism as modules over A.

Using this map we will analyze the chain complex arising from the bo-resolution

$$\to \mathrm{Ext}_A^{s,t}(H^*(I^\sigma bo \wedge bo),\mathbb{Z}_2) \to \mathrm{Ext}_A^{s,t}(H^*(I^{\sigma+1} bo \wedge bo),\mathbb{Z}_2) \to \cdots$$

The results we get are technical and so we will not summarize them.

7.2. The algebraic decomposition theorem

By 6.2.6 $H^*(\overline{B}(n)) \cong M(2n) \otimes_{A_0} \mathbb{Z}_2 = M_1(n)$. Thus 7.1.1 can be restated as the following

Proposition 7.2.1. Let $g^*\colon \oplus\, \Sigma^{4k} M_1(k) \to A \otimes_{A_1} Z_2$ be defined by $\Sigma^{4k}M_1(k) \to M_1(k)\chi Sq^{4k} \subset A \otimes_{A_1} Z_2$. Then g^* is an isomorphism of

A_1-modules.

We will give two proofs of this result. The one which follows is self contained. We give another in 7.5 as a corollary of another development containing some other results we also need.

Proof of 7.2.1. In §2.4 we discussed a filtration of A which we will use here. Let $\mathfrak{F}_n(A) = \{\chi Sq^I | I \text{ admissible and } i_1 \geq n\}$. Then $\mathfrak{F}_n(A) \supset \mathfrak{F}_{n+1}(A)$ and $\mathfrak{F}_n(A)/\mathfrak{F}_{n+1}(A) = \Sigma^n M([\frac{n}{2}])\chi Sq^n$. Under the natural map of bo → $K(Z_2)$ we have $i: A \to A \otimes_{A_1} Z_2$ and this map is an epimorphism. The filtration $\mathfrak{F}$ filters $A \otimes_{A_1} Z_2$. We will prove 7.7.1 by showing $E^0 A \otimes_{A_1} Z_2 = \bigoplus_{i \geq 0} \overline{M}_1(i)\chi Sq^{4i}$. We will show this by showing

a) if $k \not\equiv 0(4)$ then $\chi Sq^k = 0$ in $A \otimes_{A_1} Z_2$;

b) if $k \equiv 0(4)$ then $\Sigma^k M([\frac{k}{2}])$ is mapped isomorphically to $\Sigma^k M_1(k)$; and

c) as left A_1 modules $E^0 H^*(bo) = H^*(bo)$.

Proof of a: If we apply χ to $Sq^1 Sq^{2n}$ and $Sq^2 Sq^{4n} = Sq^{4n+2} + Sq^1 Sq^{4n} Sq^1$ we see that in $A \otimes_{A_1} Z_2$ $\chi Sq^k = 0$ if $k \not\equiv 0(4)$.

Proof of b: We need to show that under i $M(2i)\chi Sq^{4i}$ maps into $(M(2i) \otimes_{A_0} Z_2)\chi Sq^{4i}$. To see this it is sufficient to verify that $Sq^1 \chi Sq^{4i} = 0$. But $Sq^2 Sq^{4i-1} = Sq^1 Sq^{4i} + Sq^{4i} Sq^1$, applying χ completes the argument.

Proof of c: Let $\overline{B}(1)$ be as in §6.2. There is a map $g': \Sigma^6 \overline{B}(1) \to B^3 0$ so that $S^6 \to \Sigma^6 \overline{B}(1) \to B^3 0$ is a generator. Let g

be the double loop map and let $T(g)$ be the Thom complex of g and let $h\colon T(g) \to bo$ be the K-theory orientation. Let $\{\overline{F}_k\}$ be the Milgram filtration [38] of $\Omega^2\Sigma^6\overline{B}(1)$, $\chi Sq^{4k} \in H^*(T(g/F_k))$. It is an easy calculation of the kind done in [23] to see that $H^*(T(g/F_k)/T(g/F_{k-1})) \leftarrow M_1(k)\chi Sq^{4k}$ is a monomorphism. Thus the representation $A \otimes_{A_1} \mathbb{Z}_2$ as $\oplus\, \Sigma^{4k} M_1(k)$ is compatible with $\oplus\, H^*(T(g/\overline{F}_k)/T(g(\overline{F}_{k-1}))$. Since g is a spin bundle A_1 acts on $H^*(T(g))$ exactly as it does in $H^*(\Omega^2\Sigma^6\overline{B}(1))$. Thus as A_1 modules $H^*(T(g)) \cong \oplus\, H^*(T(g/\overline{F}_k)/T(g/\overline{F}_{k-1}))$.

Note that the following proposition can be proved in essentially the same way. Part c in the proof, of course, requires more work. The proof in 7.5 is probably easier to generalize.

Proposition 7.2.2. Let $M_{i+1}(k)$ be the image of $M(k\cdot 2^i)$ in $A \otimes_{A_i} \mathbb{Z}_2$ and let $f\colon \bigoplus_{k=2}^{\infty} \Sigma^{(2^{i+1})k} M_{i+1}(k) \to A \otimes_{A_{i+1}} \mathbb{Z}_2$ be given by $\Sigma^{2^{i+1}k} M_{i+1}(k) \to M_{i+1}(k)\chi Sq^{2^i k} \in A \otimes_{A_{i+1}} \mathbb{Z}_2$. Then f is a left A_{i+1} isomorphism.

We leave the proof to the interested reader. This decomposition should have some applications, but that is another story.

7.3. The functor $\mathrm{Ext}_A(\ ,\mathbb{Z}_2)$ applied to the bo resolution.

Armed with 7.2.1 we now can calculate $\mathrm{Ext}_A(H^*(I^\sigma bo \wedge bo),\mathbb{Z}_2)$. Using the standard change of rings theorem (compare the proof of 4.1.2) we see

7.3.1. $$\mathrm{Ext}_A(H^*(I^\sigma bo \wedge bo),\mathbb{Z}_2) \cong \mathrm{Ext}_{A_1}(H^*(I^\sigma bo),\mathbb{Z}_2)$$

$$\cong \mathrm{Ext}_{A_1}(\tilde{H}^*(\overbrace{\Omega_+ \wedge \cdots \wedge \Omega_+}^{\sigma \text{ factors}}),\mathbb{Z}_2).$$

We will be content to determine these groups for $s > 0$. What we miss this way "essentially" will be the A_1 free parts of these cohomology modules. Thus we can replace $M_1(k)$ by something which is stably isomorphic as A_1 modules.

Proposition 7.3.2. There is a map $g\colon \Omega_+ \wedge bo \to \overline{\Omega}'_+ \wedge bo$ whose induced map in cohomology is a stable A isomorphism. ($\overline{\Omega}'_+$ is defined in 6.5.)

Proof. The Adams edge theorem gives immediately the maps $\overline{B}(i) \to Y_{4i-\alpha(2i)}$ for i a power of 2. Here $\alpha(n)$ is the number of ones in the dyadic expansion of n. If $2i-2^j < 2^j < 2i$ then there is a map $B(2i) \to B(2^j) \wedge B(2i-2^j)$ of degree one on the Thom class. Thus if $B(2i-2^j) \to Y_{4i-\alpha(2i)+1}$ then

$$B(2i) \to B(2^j) \wedge B(2i-2^j) \to Y_{2^{j+1}-1} \wedge Y_{4i-2^{j+1}-\alpha(2i)+1} = Y_{4i-\alpha(2i)}.$$

There does not seem to be a map of $\overline{B}(2i+1) \to Y_{4i-\alpha(2i)} \wedge \overline{B}(1)$ but the following maps $\overline{B}(2i+1) \leftarrow \overline{B}(2i) \wedge \overline{B}(1) \to Y_{4i-\alpha(2i)} \wedge \overline{B}(1)$ each induce stable A_1 isomorphisms in cohomology. Hence there are maps $\overline{B}(2i+1) \wedge bo \to \overline{B}(2i) \wedge \overline{B}(1) \wedge bo \to Y_{4i-\alpha(2i)} \wedge \overline{B}(1) \wedge bo$ and this composite induces the stable A isomorphism in cohomology.

Corollary 7.3.3. $H^*(bo \wedge bo)$ is stably isomorphic to $H^*(\overline{\Omega}'_+ \wedge bo)$ as A-modules.

This yields immediately

Theorem 7.3.4. If $s > 0$ then $\mathrm{Ext}_A^{s,t}(H^*(bo \wedge bo), Z_2) \cong$

$$\bigoplus_{i=0}^{\infty} [\mathrm{Ext}_{A_1}^{s+4i-\alpha(i),\,t-4i-\alpha(i)}(Z_2, Z_2) \oplus \mathrm{Ext}_{A_1}^{s+4i-\alpha(i),\,t-4i-4-\alpha(i)}(C_3, Z_2)].$$

If $t-s \equiv 3(4)$, $s > 0$ then $\mathrm{Ext}_A^{s,t}(H^*(bo \wedge bo), Z_2) = 0$. On the other hand classes in this group for $t-s \equiv 1,2 \bmod 4$, if non zero, are h_1 composition from a class with $t-s \equiv 0 \pmod 8$. Hence, there can be no differentials in the Adams spectral sequence. Thus, we have

Theorem 7.3.5. In the Adams spectral sequence for $bo \wedge bo$ $E_\infty^{s,t} \cong \mathrm{Ext}_A^{s,t}(H^*(bo \wedge bo), \mathbb{Z}_2)$.

This effectively calculates $\pi_*(bo \wedge bo)$. As an A_1 module $H^*(I^\sigma bo)$ is stably isomorphic to $H^*((\overline{\Omega}')^\sigma)$. Since $\overline{\Omega}'$ is a wedge of Y_j's and $Y_j \wedge B(1)$ (6.5) and $\mathrm{Ext}_{A_1}^{s,t}(H^*(Y_j), \mathbb{Z}_2)$ and $\mathrm{Ext}_{A_1}(H^*(Y_j \wedge \overline{B}(1), \mathbb{Z}_2)$ are calculated in Chapter 4, we have calculated all the groups which arise in a bo-resolution. We will be content to describe explicitly a much smaller calculation in 7.4.

7.4. The algebraic E_2 term for bo resolutions

We have all the pieces to begin the investigation of the chain complex which results from a bo resolution. Consider the following chain complex

7.4.1. $\mathrm{Ext}_{A_1}^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2) \to \mathrm{Ext}_{A_1}^{s,t}(H^*(\Omega), \mathbb{Z}_2) \to \cdots \to \mathrm{Ext}_{A_1}^{s,t}(H^*(\Omega \wedge \cdots \wedge \Omega), \mathbb{Z}_2) \to \cdots$

which results from the bo resolution after repeated use of 7.3.3 by applying $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(\ ), \mathbb{Z}_2)$ as the functor and using the change of rings theorem.

We will analyze this complex by studying the corresponding cohomology complex

7.4.2. $\mathbb{Z}_2 \leftarrow H^*(\Omega) \leftarrow \cdots \leftarrow \tilde{H}^*(\Omega \wedge \cdots \wedge \Omega) \leftarrow \cdots$

as left A_1 modules. There is a subtle point here which the reader should note. The chain complex 7.4.1 is not the one induced by applying $\mathrm{Ext}_{A_1}^{s,t}(\ ,\mathbb{Z}_2)$ to 7.4.2. There is an additional component in the differential of 7.4.1 which arises from the action of the coefficients, $\mathrm{Ext}_{A_1}^{s,t}(\mathbb{Z}_2,\mathbb{Z}_2)$. This action induces a term in the differential of 7.4.1 which does not arise from an A_1 map in 7.4.2. How this term behaves is illustrated nicely in the calculation of 4.4 and also in the discussion of 6.4.2. In addition this term will be crucial in Chapter 8. In this section, then, we will only analyze 7.4.2 and we will show that when the complex 7.4.2 is tensored with A_0 and the functor $\mathrm{Ext}_{A_1}^{s,t}(\ ,\mathbb{Z}_2)$ is applied the resulting homology is the homology of 7.4.1. The key idea will be to show that 7.4.2 is just $\overline{C}'$ of 6.5.9.

Let X_5 be the spectrum of 6.4.1.

Proposition 7.4.3. There is a map $h\colon X_5 \to bo$ which is a rational equivalence and induces an isomorphism

$h_*\colon H_*(X_5,\hat{Z}) \to H_*(bo,\hat{Z})/T$. ($\hat{Z}$ denotes the 2-adic integers).

Proof. The map h is the K-theory orientation. It is then sufficient to note $\chi Sq^{4k}U \neq 0$ in $H^*(X_5)$, U the Thom class, to complete the proof.

Let $f\colon \Omega_+S^5 \to \Omega_+$ be given by the composite

$\Omega_+S^5 \to V\Sigma^{4i} \to V\Sigma^{4i} \wedge \overline{B}(i)$.

Proposition 7.4.4. The following diagram commutes

$$\begin{tikzcd}
H^*(bo) \otimes H^*(bo) \arrow[d, "f^* \otimes h^*"] & H^*(\Omega_+) \otimes H^*(bo) \arrow[l, "g^*"'] \arrow[d, "f^* \otimes h^*"] \\
H^*(X_5) \otimes H^*(X_5) & H^*(\Omega S^5_+) \otimes H^*(X_5) \arrow[l, "g_{X_5}"]
\end{tikzcd}$$

<u>Proof</u>. This is immediate from the definition of the maps g_X and g^*.

The map h induces a map between the X_5-resolution and the bo resolution. This gives (we suppress the subscript on X_5)

7.4.5.
$$\begin{tikzcd}
H^*(X) & H^*(\Omega S^5) \otimes H^*(X) \arrow[l] & \cdots \arrow[l] & H^*(\Omega S^5 \wedge \cdots \wedge S^5) \otimes H^*(X) \arrow[l] \\
H^*(bo) \arrow[u] & H^*(\Omega) \otimes H^*(bo) \arrow[l, "d_1"'] \arrow[u] & \cdots \arrow[l, "d_2"'] & H^*(\Omega \wedge \cdots \wedge \Omega) \otimes H^*(bo). \arrow[l] \arrow[u]
\end{tikzcd}$$

We also have a stable A isomorphism from 7.3.2

$$H^*(\Omega \wedge \cdots \wedge \Omega) \otimes H^*(bo) \cong H^*(\bar{\Omega}' \wedge \bar{\Omega}' \cdots \wedge \bar{\Omega}') \otimes H^*(bo)$$

Hence from 7.4.5 we get a chain complex of A_1 modules

7.4.6. $$Z_2 \xleftarrow{d_1} H^*(\bar{\Omega}') \leftarrow \cdots \leftarrow H^*((\bar{\Omega}')^\sigma) \leftarrow \cdots$$

Proposition 7.4.7. The chain complex 7.4.6 is $\mathcal{C}'$ of 6.5.9.

<u>Proof</u>. The complex $\mathcal{C}'$ is constructed by using the complex which results from ΩS^5 and this is just what 7.4.5 asserts.

The homology is given by 6.5.9. If we apply $\mathrm{Ext}_{A_1}^{s,t}(\ ,Z_2)$ to the chain complex 7.4.6 we get a chain complex

7.4.8. $$\mathrm{Ext}_{A_1}^{s,t}(Z_2,Z_2) \to \cdots \to \mathrm{Ext}_{A_1}^{s,t}(H^*((\bar{\Omega}')^\sigma),Z_2) \to \cdots$$

This complex is not 7.4.1 but is related to it.

Let $\mathcal{J} = \{I; I = \{i_j\}, i_j$ are non-negative integers with $\Sigma i_j < \infty\}$. Let $\mathcal{J}^+ = \{I \in d; i_1 \equiv o(2)\}$ and $\mathcal{J}^- = \{I \in d; i_1 \equiv 1(2)\}$. Let $\mathcal{J}_\sigma = \{I; \Sigma i_j = \sigma\}$. Let $\rho(I) = \delta(I) + \Sigma i_j(2^j - 1)$ where $\delta(I) = -1$. if $I \in \mathcal{J}^-$ and 0 if $I \in \mathcal{J}^+$. Let $\gamma(I) = \Sigma i_j \cdot 2^{j+1}$.

Theorem 7.4.9. For $s > \sigma$ the homology of the complex 7.4.8 in dimension σ is

$$\bigoplus_{I \in \mathcal{J}^+_\sigma} \mathrm{Ext}_{A_1}^{s-\sigma+\rho(I),t+\rho(I)-\gamma(I)}(\mathbb{Z}_2,\mathbb{Z}_2) \oplus$$

$$\bigoplus_{I \in \mathcal{J}^-_\sigma} \mathrm{Ext}_{A_1}^{s-\sigma+\rho(I),t+\rho(I)-\gamma(I)}(C_3,\mathbb{Z}_2).$$

Proof. This is now immediate from 6.5.9.

If we tensor the complex 7.4.6 with A_0 we obtain

Theorem 7.4.10. The E_2 term of the bo resolution for A_0 for $s > 0$ is

$$E_2^{\sigma,s,t} = \bigoplus_{I \in \mathcal{J}^+_\sigma} \mathrm{Ext}_{A_1}^{s-\sigma+\rho(I),t+\rho(I)-\gamma(I)}(A_0,\mathbb{Z}_2) \oplus$$

$$\bigoplus_{I \in \mathcal{J}^-_\sigma} \mathrm{Ext}_{A_1}^{s-\sigma+\rho(I),t+\rho(I)-\gamma(I)}(C_3 \otimes A_0,\mathbb{Z}_2)$$

Proof. All that remains is to show that the portion of the complete differential not covered by 7.4.9 does not contribute anything. This follows easily since no differential is possible for reasons of filtration.

7.5. Alternate discussion of 7.1.1.

In this section we will produce another proof of 7.1.1 together with some other results which we were not able to get in a fashion more completely in the spirit of earlier sections. As before it is sufficient to look at $A \otimes_{A_1} \mathbb{Z}_2$ as a right A_1 module. This section is heavily influenced by Peterson's lectures [39]. Recall Milnor's result $A^* \cong \mathbb{Z}_2(\xi_1, \xi_2, \ldots)$.

Proposition 7.5.1. As left A modules $\chi(A \otimes_{A_1} \mathbb{Z}_2)^* = \mathbb{Z}_2(\xi_1^4, \xi_2^2, \xi_3, \ldots)$.

<u>Proof</u>. Since $A \otimes_{A_1} \mathbb{Z}_2 \cong A/A(Sq^1, Sq^2)$ we have

$$A \oplus A \xrightarrow{R(sq^2) \oplus R(Sq^1)} A \rightarrow A/A(Sq^1, Sq^2) \rightarrow 0$$ which gives

$$A^* \oplus A^* \xleftarrow{LSq^1 \oplus LSq^2} A^* \leftarrow [A/A(Sq^1, Sq^2)]^* \leftarrow 0$$ and finally

$$A^* \oplus A^* \xleftarrow{RSq^1 \oplus RSq^2} A^* \leftarrow \chi(A/A(Sq^1, Sq^2))^* \leftarrow 0.$$

But $\xi_k Sq = \xi_k + \xi_{k-1}$ where $Sq = \sum_{i=0}^{\infty} Sq^i$. Hence $\chi(A/(ASq^1, Sq^2)) = \mathbb{Z}_2(\xi_1^4, \xi_2^2, \xi_3, \ldots) = \ker RSq^1 \oplus RSq^2$. This completes the proof.

Assign to each ξ_i degree 2^{i-1} and each monomial $\xi^I = \xi_1^{i_1} \xi_2^{i_2} \ldots$ degree $\Sigma\, i_j\, 2^{j-1}$. Let N_{4n} be the $\mathbb{Z}_2$ vector space generated by monomials of degree 4n. Then $\mathbb{Z}_2(\xi_1^4, \xi_2^2, \xi_3, \ldots) \cong \bigoplus_n N_{4n}$.

Proposition 7.5.2. As left A_1 modules $\mathbb{Z}_2(\xi_1^4 \xi_2^2 \xi_3 \cdots) \cong \bigoplus_n N_{4n}$.

<u>Proof.</u> The left A action is given by $Sq\xi_k = \xi_k + \xi_{k-1}^2$. In the absence of $\xi_1, \xi_1^2, \xi_1^3, \xi_2$ and products, degree $(Sq^1 \xi^I)$ = degree ξ^I and degree $(Sq^2 \xi^I)$ = degree ξ^I (of course 0 has every degree).

Proposition 7.5.3. $\chi N_{4n}^* = \overline{M}(n)$.

Proof. Using the multiplication in A^* and the multiplicative nature of the degree we have maps

$$\chi N_{4n}^* \to \chi N_{4j}^* \otimes \chi N_{4k}^* \qquad j + k = n$$

which are monomorphism if $n \neq 2^i$ and $4j = 2^i$ and 2^i is such that $4n < 2^i < 4n + 1$. If $n = 2^i$ then the class corresponding to ξ_{i+2} generates the kernel. Now using the obvious isomorphism $\chi N_4^* = \overline{M}(1)$ and the kind of argument of [23] §4 we get the result.

Combining 7.5.2 and 7.5.3 we get the proof of 7.1.1. Using this explicit calculation we also can get the following. Let $Q_0 = Sq^1$ and $Q_1 = Sq^3 + Sq^2Sq^1$. Then Q_j acts as a differential in M for any A (or A_1) module M.

Proposition 7.5.4. $H_*(\chi(A \underset{A_1}{\otimes} \mathbb{Z}_2)^*, Q_0) = \mathbb{Z}_2(\xi_1^4)$ and $H_*(\chi(A \underset{A_1}{\otimes} \mathbb{Z}_2)^*, Q_1) \cong E(\xi_2^2, \xi_3^2, \ldots)$.

Proof. Both of these are easy calculations from 7.5.2.

Proposition 7.5.5. As stable A_1 modules $H^*(R(2^i-1))$ and $\overline{M}(2^i)$ are isomorphic.

Proof. We have the A module map $f: \overline{M}(2^i) \to H^*(R(2^i-1))$ given in 6.5.8. The map is degree 1 on the bottom class and hence induces a Q_0 homology isomorphism. The Q_1 homology is easily seen to be generated by the cohomology class in dimension 2^i-2 in both cases and f is an isomorphism in this dimension.

Corollary 7.5.7. Let V be a graded $\mathbb{Z}_2$ vector space. Then $\overline{B}(2^i) \wedge bo = R(2^i-1) \wedge bo \vee K(V)$.

Proof. Proposition 7.5.5 and 4.5.6 imply this immediately.

Using the ideas of this section a neat proof of 7.7.2 is possible.

Chapter 8

bo Resolutions; Geometric Version

8.1. The decomposition of bo $\wedge$ bo

In this chapter we will show that much of the algebraic material of Chapter 7 can be done geometrically. We will use the explicit calculation of $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(bo \wedge bo), Z_2)$ to do this. Among the corollaries of this approach is a new proof of the Adams-Priddy theorem about the uniqueness of bo. The result which is central is

Theorem 8.1.1. $bo \wedge bo \cong \Omega_+ \wedge bo$.

This result was first proved by the first author and dates from the original lectures. Later Milgram [27] found a very nice proof which does not use the results of Chapter 7. His proof does not seem to yield the Adams-Priddy theorem [6]. The proof given here is essentially the same as the proof given in the 1969 lectures.

8.2. Proof of 8.1.1.

The first step will be to construct a map $\Sigma^{2^i} R(2^i-1) \xrightarrow{f_i} bo \wedge bo$ so that f_i^* is an epimorphism and so that

$$\begin{tikzcd}
H^*(\Sigma^{2^i} R(2^i-1)) \arrow[d] & H^*(bo \wedge bo) \arrow[l, "f_i^*"'] \arrow[d] \\
H^*(\Sigma^{2^i} \wedge S^0) & H^*(\Omega S^5_+ \wedge X) \arrow[l, "\bar{f}_i^*"]
\end{tikzcd}$$

commutes. R(k) is defined in 4.4.7. We will do this by showing

Lemma 8.2.1. There is a modification of the composite

$\Sigma^{2^i} \wedge S^0 \to \Omega S^5_+ \wedge X \to bo \wedge bo$, f_i', by a homotopy class of filtration ≥ 1 so that the composite $\Sigma^{2^i} P^{2^i-2} \xrightarrow{\lambda} \Sigma^{2^i} \to bo \wedge bo$ is null homotopic.

Proof: Let $A^{s,t}(i) = \mathrm{Ext}_{A_1}^{s+4i-\alpha(i),t-4i-\alpha(i)}(\mathbb{Z}_2,\mathbb{Z}_2) \oplus \mathrm{Ext}_{A_1}^{s+4i-\alpha(i),t-4i-4-\alpha(i)}(C_3,\mathbb{Z}_2)$ then $\mathrm{Ext}_A^{s,t}(H^*(bo \wedge bo),\mathbb{Z}_2) = \bigoplus_{i\geq 0} A^{s,t}(i)$. We write this in two parts as $\bigoplus_{j\leq 2^{i-2}} A^{s,t}(j) \oplus \bigoplus_{j>2^{i-2}} A^{s,t}(j)$.

There is a graded finitely generator free abelian group V, such that $\mathrm{Ext}_{A_0}^{s,t}(V,\mathbb{Z}_2) \cong \bigoplus_{j>2^{i-2}} A^{s,t}(j)$ if $s > 0$ and $t - s < 2^{i+1}$. There is a map $bo \wedge bo \to K(V)$ which induces this isomorphism. Let b be the fiber. The composite $\Sigma^{2^i} \xrightarrow{f_i} bo \wedge bo \to K(V)$ is zero and so f_i lifts to $\tilde{f}_i : \Sigma^{2^i} \to b$. We now wish to consider the composite $\Sigma^{2^i} P^{2^i-2} \xrightarrow[\lambda]{} \Sigma^{2^i} \xrightarrow[\tilde{f}_i]{} b$. Because of the splitting of $bo \wedge bo$ as A modules there is no obstruction to extending $\tilde{f}_i$ to a map of $\Sigma^{2^i} R(2^i-1)$ at filtration 1. Suppose the composite $\tilde{f}_i\lambda$ lifts to Adams filtration s but does not lift further. Then there is a k such that $\Sigma^{2^i} P^{k-1} \to \Sigma^{2^i} \to b^s$[4] is trivial but $\Sigma^{2^i} P^k \to \Sigma^{2^i} \to b^s$ is not. This identifies a particular element in $\mathrm{Ext}_A^{s,t}(H^*(bo \wedge bo),\mathbb{Z}_2)$. This element belongs to one of the summands in $\bigoplus_{j\leq 2^i-2} A(j)$, say $A^{s,t}(j)$. For $2^i \leq t-s < 2^{i+1}$ $A^{s,t}(j) = \mathrm{Ext}_{A_1}^{s+\bar{s},t+\bar{t}}(\mathbb{Z}_2,\mathbb{Z}_2) \oplus \mathrm{Ext}_{A_1}^{s+\bar{\bar{s}},t+\bar{\bar{t}}}(\mathbb{Z}_2,\mathbb{Z}_2)$ for $\bar{s},\bar{\bar{s}},\bar{t}$ and $\bar{\bar{t}}$ chosen appropriately. Thus the essential map

---

[4] b^s is the sth stage of an Adams resolution.

$\Sigma^{2^i}P^k \to \Sigma^{2^i} \to b^s$ represents a map $\Sigma^{2^i}P^k \to \Sigma^{t'-s'}bo^{s+s'}$ where $s' = \overline{s}$ or $\overline{\overline{s}}$ and $t' = \overline{t}$ or $\overline{\overline{t}}$. (bo^σ is defined in 6.5.5.) A simple check shows $t'-s' \equiv 0 \bmod 8$. The following lemma completes the proof.

Lemma 8.2.2. Every map of $\Sigma^{8j}P^k \to bo^\sigma$ of filtration $s > 0$ can be factored through $\Sigma^{8j}P^k \xrightarrow{\lambda} S^{8j} \xrightarrow{g} bo^\sigma$ where g has filtration $s - 1$.

Proof: Note that $[\Sigma^{8j}P^k, bo] = [\Sigma^{8j}P^k, bo[8j,\ldots]] = [P^k, bo]$ where $bo[8j,\ldots]$ denotes bo-8j connected. Note also that $[\Sigma^{8j}P^k, bo^\sigma] = [P^k, bo^{\sigma-4j}]$. Thus, it suffices to prove that a map $f \epsilon [P^k, bo^{\sigma-4j}]$ of filtration $s > 0$ factors through S^0. This we will do by induction on σ.

A generator of $[P^k, bo]$ is given by

$$\begin{tikzcd}
P^k \arrow[r] & P \arrow[r, "\lambda"] \arrow[d] & S^0 \arrow[d] \\
& P \wedge bo \arrow[r, "\lambda\wedge 1"] & S^0 \wedge bo = bo
\end{tikzcd}$$

Thus the case $\sigma = 0$ is true. Suppose that any essential map $P^k \to bo^{\tau-4j}$, $\tau < \sigma$, of filtration $s > 0$, factors through S^0. Let $f\colon P^k \to bo^{\sigma-4j}$ be an essential map of filtration $s > 0$. Consider the composite

$$\begin{tikzcd}
P^k \arrow[r, "f"] \arrow[dr, "\lambda"'] & bo^{\sigma-4j} \arrow[r, "i"] & bo^{\sigma-4j-1} \\
& S^0 \arrow[ur, "g"'] &
\end{tikzcd}$$

8.3. Calculation of $E_2^{\sigma,t}(S^0,\pi)$.

In this section we will do most of the work to prove Theorem 1.3.1. Using 8.1.1 we have

8.3.1
$$\begin{tikzcd}
bo \arrow[r] & Ibo \wedge bo \arrow[r] & I^2bo \wedge bo \arrow[r] & \cdots \arrow[r] & I^\sigma bo \wedge bo \arrow[r] & \cdots \\
bo \arrow[r] \arrow[u] & \Omega \wedge bo \arrow[r] \arrow[u] & \Omega \wedge \Omega \wedge bo \arrow[r] \arrow[u] & \cdots \arrow[r] & \Omega^\sigma \wedge bo \arrow[r] \arrow[u] & \cdots \\
X \arrow[r] \arrow[u] & \Omega S^5 \wedge X_5 \arrow[r] \arrow[u] & \Omega S^5 \wedge \Omega S^5 \wedge X_5 \arrow[r] \arrow[u] & \cdots \arrow[r] & (\Omega S^5)^\sigma \wedge X_5 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

where the first two rows are homotopically equivalent.

In Chapter 7 we analyzed what happened to the chain complex induced from the top row in cohomology. In §6.5 we studied the middle row and in 6.4 we studied the bottom row. Recall that if $X \to Y$ induces a stable isomorphism over A in cohomology there exists V and V', $\mathbb{Z}_2$ vector spaces, so that $0 \to K(V) \to X \to K(V') \to 0$ is an exact sequence of spectra. Let $\Omega(I) = \Sigma^{|\mathcal{I}|}\overline{B}(1)^{i_1} \wedge \cdots \wedge \overline{B}(2^{j-1})^{i_j}$ where $\mathcal{I} = \{i_1,\ldots,i_j,0\}$, $|\mathcal{I}| = \Sigma i_j \cdot 2^{j+1}$. Let $\mathcal{I}_\sigma = \{\mathcal{I} : \Sigma\, i_j = \sigma\}$.

We will use 8.1.1 to construct maps

$$\bigvee_{I\in\mathcal{I}_\sigma} \Omega(I) \wedge bo \xrightarrow{\rho} \Omega^\sigma \wedge bo \xrightarrow{\tau} \bigvee_{I\in\mathcal{I}_\sigma} \Omega(I) \wedge bo.$$

whose composite is the identity. Here $\bigvee_{I\in\mathcal{I}_\sigma} \Omega(I) \wedge bo = \Omega(I_\sigma) \wedge bo$.

The map ρ is defined as follows.

Let $i_k\colon \Sigma^{2^{k_r-1}}\overline{B}(2^k) \to \Omega$ be the obvious inclusion. Then $\rho = i_1 \wedge \cdots \wedge i_1 \wedge i_2 \wedge \cdots \wedge i_2 \wedge \cdots \wedge i_j \wedge \cdots \wedge i_j$. τ is made

By the induction hypothesis there exists a map $g: S^0 \to bo^{\sigma-4j-1}$ factoring $i \circ f$ through S^0. Thus we will be finished once we factor g through $bo^{\sigma-4j}$. Since $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(bo^{\sigma-4j}),\mathbb{Z}_2) \cong \mathrm{Ext}_{A_1}^{s+\sigma-4j,t}(\mathbb{Z}_2,\mathbb{Z}_2)$ and so g factors through $bo^{\sigma-4j}$.

In Section 7.5.6 it is shown that $\overline{B}(2^i) \wedge bo = R(2^i-1) \wedge bo \vee K(V)$ where V is some graded $\mathbb{Z}_2$-vector space. (In what follows V is some graded $\mathbb{Z}_2$ vector space and may be different in each case). Proposition 6.5.6 asserts that $R(2^i-1) \wedge bo \vee K(V) = Y_{2^i-1} \wedge bo$. Combining these we have for $k = \Sigma 2^{i_j}$, $i_j < i_{j+1}$ $X_{2k-\alpha(k)} \wedge bo = \bigwedge_j R(2^{i_j}-1) \vee K(V)$. Since $X_{2k-\alpha(k)} \wedge bo = \overline{B}(k) \wedge bo\ K(V)$ the map $\Sigma^{4k} \wedge \bigwedge_j R(2^{i_j}-1) \xrightarrow{\wedge f'_{i_j}} \bigwedge_j bo \to bo$ gives a map of $\Sigma^{4k}\overline{B}(k) \wedge bo \to bo \wedge bo$. Finally we get $\Omega_+ \wedge bo = V\Sigma^{4k}\overline{B}(k) \wedge bo \to bo \wedge bo$ and by this map is a stable A equivalence. As $\mathbb{Z}_2$ vector spaces $\tilde{H}^*(\Omega_+ \wedge bo) \cong \tilde{H}^*(bo \wedge bo)$ hence the wedge of Eilenberg-MacLane spectra, $(K(V))$, V a $\mathbb{Z}_2$ vector space, on each side can be matched up to give 8.1.1.

Remark 8.2.3. Note that the proof of this theorem only used the calculation of $\mathrm{Ext}_A^{s,t}(H^*(bo \wedge bo),\mathbb{Z}_2)$ of Chapter 7 and thus uses only the cohomology of bo as an A_1 module. Thus suppose bo and bo' were two spaces whose cohomology is $A \otimes_{A_1} \mathbb{Z}_2$ and suppose bo is a ring spectrum. Then 8.1.1 is valid and asserts $\Omega_+ \wedge bo \overset{h.e.}{=} bo' \wedge bo$. Thus the composite $bo' \xrightarrow{1 \wedge S^0} bo' \wedge bo \to \Omega_+ \wedge bo \to bo$ is a homotopy equivalence and this is essentially the main result of [6].

up in a similar manner by the projections $\Omega \xrightarrow{\tau_k} \Sigma^{2^{k+}} \overline{B}(2^k)$. Let $\overline{d}_\sigma : \bigvee_{I \in \mathcal{I}_\sigma} \Omega(I) \wedge bo \to \bigvee_{I \in \mathcal{I}_{\sigma+1}} \Omega(I) \wedge bo$ be defined by $\tau\, d_\sigma\, \rho$.

Proposition 8.3.2. The two chain complexes $(H^*(I^\sigma bo \wedge bo), d_\sigma^*)$ and $(H^*(\bigvee_{I \in \mathcal{I}_\sigma} \Omega(I) \wedge bo), \overline{d}_\sigma^*)$ have isomorphic homologies as stable A modules.

This is just a restatement of 7.4.7.

Thus in order to understand the homology of the chain complex $(\pi_*(I^\sigma bo \wedge bo), d_{\sigma_*})$ as a $\pi_*(bo)$ module it is only necessary to understand $(\pi_*(\Omega(I_\sigma) \wedge bo), \overline{d}_{\sigma_*})$ as a $\pi_*(bo)$ module.

Theorem 8.3.3.

$$
\begin{aligned}
H_0^t(\pi_*(\Omega(I_\sigma) \wedge bo), \overline{d}_{\sigma_*})) &= Z & t &= 0 \\
&= Z_2 & t &\equiv 1,2 \pmod 8
\end{aligned}
$$

$$
\begin{aligned}
H_1^t(\pi_*(\Omega(I_\sigma) \wedge bo), \overline{d}_{\sigma_*}) &= Z_2\rho(t) & t &\equiv 0(4) \text{ and} \\
& & t &\equiv 2^{\rho(t)-1} \bmod 2^{\rho(t)} \\
&= Z_2 & t &\equiv 1,2(8) \\
&= 0 \text{ otherwise}
\end{aligned}
$$

for $\sigma > 1$ $H_\sigma^t(\pi_*(\Omega(I_\sigma) \wedge bo), \overline{d}_{\sigma_*}) = V_\sigma$, a Z_2 vector space. $\pi_*(bo)$ acts non-trivially on H_0 and H_1 and acts trivially on H_σ for $\sigma > 1$.

Proof. Let $I = \{i_j\}$ with $i_k \equiv 1(2)$ for a particular $k \neq 1$. Let $K' = \{i_j'\}$ where $i_j' = i_j$ $j \neq k$ or $k-1$ and $i_{k-1}' = i_{k+1} + 2$ and $i_k' = i_k - 1$. If $I \in {}_\sigma$ then $I' \in I_{\sigma+1}$. Let $i: \Omega(I) \to \Omega(I_\sigma)$ and $j: \Omega(I_{\sigma+1}) \to \Omega(I')$ be the obvious inclusion and projection.

Lemma 8.3.4. The kernel and coker of the composite

$$\pi_*(\Omega(I) \wedge bo) \xrightarrow{i_*} \pi_*(\Omega(I_\sigma) \wedge bo) \xrightarrow{\overline{d}_{\sigma *}}$$

$$\pi_*(\Omega(I_{\sigma+1}) \wedge bo) \xrightarrow{j_*} \pi_*(\Omega(I') \wedge bo)$$

are Z_2 vector spaces as π_*bo modules.

Proof. From the bottom the composite map $\Omega(I) \wedge bo \to \Omega(I_\sigma) \wedge bo \xrightarrow{\overline{d}_\sigma} \Omega(I_{\sigma+1}) \wedge bo \to \Omega(I') \wedge bo$ is multiplication by $\binom{2^{k+1}}{2^k}$ on the cell in dimension I by commutativity of 8.3.1. Thus is a map of filtration 1 and by 6.5 induces an isomorphism of E_s terms $E_\infty^{s,t}(\Omega(I) \wedge bo) \to E_\infty^{s+1,t+1}(\Omega(I') \wedge bo)$ for $s > 0$ and an epimorphism if $s = 0$. Hence the kernel and cokernel represent classes of a filtration 0 in $E_\infty^{s,t}(\Omega(I) \wedge bo)$ and $E_\infty^{s,t}(\Omega(I') \wedge bo)$ respectively.

Coming back to the proof of 8.3.3 we note that the sequences $I \in \mathcal{I}_\sigma$ are in 1-1 correspondence with monomials in $Z_2(a_1,\ldots)$. The correspondence of 8.3.4 corresponds to a differential in $Z_2(a_1,\ldots, a)$ generated by $d\, a_i = a_{i-1}^2 \quad i \neq 1$. Since

$$H_i(Z_2(a_1,\ldots, a_i),d) = Z_2 \quad i = 0,1$$
$$= 0 \quad i > 1$$

we have that as $\pi_*(bo)$ modules the chain complex $\pi_*(\Omega(\mathcal{I}_\sigma) \wedge bo,\overline{d}_{\sigma*})$ has non-trivial homology only in gradation 0 and 1. In these dimensions the homology is the homology of $\tau'_*: \pi_*bo \to \pi_*\Sigma^4\overline{B}(1) \wedge bo$ where τ' is the composite given by 8.3.5 $bo \to \Omega \wedge bo \to \Sigma^4\overline{B}(1) \wedge bo$. Thus 8.3.6 completes the proof.

Lemma 8.3.6. The ker of τ'_* is H_1^t of 8.3.3 and coker of τ'_* is H_1 of 8.3.3.

Proof. We have the following diagram

$$\begin{tikzcd}
bo \arrow[r] & \Omega \wedge bo \arrow[r] & \Sigma^4\overline{B}(1) \wedge bo \\
X \arrow[r] \arrow[u] & \Omega S^5 \wedge X \arrow[r] \arrow[u] & \Sigma^4 X. \arrow[u]
\end{tikzcd}$$

The top row is τ'. The infinite cyclic classes in $H_*(X)$ are mapped isomorphically to those of H_*bo and likewise those of $H_*(\Sigma^4 X) \to H_*(\Sigma^4\overline{B}(1) \wedge bo)$. In 6.2 we see that $H_{4k}(X) \to H_{4k}(\Sigma^4 X)$ is of degree k. Hence $\tau'_* H_{4k}(bo) \to H'_{4k}(\Sigma^4 B(1) \wedge bo)$ has degree k. Hence, in the spectral sequence described in 3.4 for $(\Sigma^4 B(1) \wedge bo) \cup C(bo)$ we have a differential $\delta_{i(k)}$ in $t-s = 4k$ and $i(k)$ satisfies $k \equiv 2^{i(k)-1} \bmod 2^{i(k)}$. A simple comparison of the charts 8.3.7 gives the desired result.

8.4. v_1-periodicity.

Note that Theorem 8.3.3 somewhat describes $E_2^{\sigma,t}(S^0, bo, \pi)$. We need to show that all higher differentials on $E_2^{0,t}$ and $E_2^{1,t}$ are zero. Since they can not be boundaries this will imply that they survive to E_∞.

Proposition 8.4.1. The classes in H_0^t (of 8.3.3) are cycles.

Proof. Consider the map $S^0 \to bo$. This induces $f\colon \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2,\mathbb{Z}_2) \to \mathrm{Ext}_{A_1}^{s,t}(\mathbb{Z}_2,\mathbb{Z}_2)$ and the results in Chapter 4 imply that if $s = 4i + \varepsilon$, $\varepsilon = 1,2$; $t = 12i + 2\varepsilon$ then f is an isomorphism.

If there were an $\alpha \in E_r^{s',t'}$ such that $\delta_r \alpha \neq 0 \in E_r^{4i+\varepsilon,12i+2\varepsilon}$ then $f\alpha \neq 0$. But $f\alpha = 0$ and so these classes are never boundaries. By the edge theorem they are cycles so they project to non-trivial homotopy classes.

Proposition 8.4.2. The classes in H_1^t (of 8.3.3) are cycles.

Proof. We have Spin $\xrightarrow{J} \Omega^\infty S^\infty \xrightarrow{p}$ BO where p is the "looped" version of $S^0 \to bo$. $\Omega^3 \mathrm{Spin} \to \Omega^3 BO$ is easily seen to be zero since $\Omega^3 \mathrm{Spin} = B$ Simplectic and $H^j(B\mathrm{Spin}; \pi_j(\Omega^3 BO)) = 0$. Thus J lifts to the fiber of p and this gives $\Omega^3 \mathrm{Spin} \to \Omega^3(\Omega^\infty I^\sigma bo) \xrightarrow{g} \Omega^3 \mathrm{Spin}$ where g is the "looped" version of $\Omega I^\sigma bo \to \Sigma^3 b\mathrm{spin}$ given by $I^\sigma bo \cong \Omega \wedge bo \to \Sigma^4 \overline{B}(1) \wedge bo$. On the class in dimension 4 of $\Omega^3 \mathrm{Spin}$ this composite is an isomorphism. By Bott periodicity this gives an isomorphsim in every dimension. Thus all the classes in H_1^t are cycles.

This completes the proof of Theorem 1.

Now we can define v_1 periodicity.

Proposition 8.4.3. [3]. For each $j \geq 3$ there is a map $\Sigma^{2^j} Y_{2^j} \xrightarrow{v_1^{2^{j-1}}} Y_{2^j}$ and if $j < 3$, $\Sigma^8 Y_{2^j} \xrightarrow{v_1^4} Y_{2^j}$ such that all iterates are non zero.

Proof. Theorem 8.3.3 asserts that in $\pi_{k(2^j)-1}(S^0)$ there is a class a_j of order 2^j and $\pi_{8i}(bo)$ acts non trivially on it. The map $v_1^{2^{j-1}}$ is the coextension such that the composite $\Sigma^{2^j} Y_{2^j} \to Y_{2^j} \to S'$

is a_j. If $j < 3$ we have 4σ or σ as our maps.

Definition 8.4.4. A family of classes $\beta_k \in \pi_{k-2^j+n+q}(S^n)$ are v_1-periodic if there is a K such that for $k > K, 2^j\ \beta_k = 0$ and β_{k+1} is the composite

$$S^{(k+1)\cdot 2^j+n+q} \xrightarrow{i} \Sigma^{(k+1)\cdot 2^j+n+q} Y_{2^j} \xrightarrow{v_1^{2^{j-1}}} \Sigma^{k\cdot 2^j+n+q} Y_{2^j} \xrightarrow{\beta_k^{\#}} S^n .$$

i is the inclusion of the bottom cell and $\beta_k^{\#}$ is the extension of the map $S^{k\cdot 2^j+n+q} \xrightarrow[\beta_k]{} S^n$ to all of $\Sigma^{k\cdot 2^j+n+q} Y_{2^j}$.

Chapter 9

Applications

9.1. The Moore space and Theorem 1.1.1

In this chapter in addition to proving Theorem 1.1 we will give details on some of the results given in [22]. The starting point is the following.

Theorem 9.1.1. (Theorem 1 of [20] and Theorem 5 of [22]). In the Adams' spectral sequence for the stable Z_2 Moore space M if $6s > t + 18$ then $E_\infty^{s,t} = E_2^{s,t}(M \wedge bo) \oplus E_2^{s-2,t-9}(M \wedge bo)$.

<u>Proof</u>. There is a mapping of resolutions which is induced by $bo \to K(\mathbb{Z}_2,0)$ given by 9.1.2

$$\begin{tikzcd}
M \wedge I^\sigma bo \arrow[r] \arrow[d] & M \wedge I^\sigma K(Z_2) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
M \wedge Ibo \arrow[r] \arrow[d] & M \wedge IK(Z_2) \arrow[d] \\
M \arrow[r] & M
\end{tikzcd}$$

If we use π_* as the functor there is a filtration respecting map

$$E_2^{s,t}(M,bo) \to E_2^{s,t}(M,K) \cong \mathrm{Ext}_A^{s,t}(\tilde{H}^*(M),Z_2).$$

Recall that $E_2^{s,t}(M,bo)=0$ for $3s-2 > t$. If there is a class in $E_2^{s,t}(M,K)$ with $3s-2 > t$ which projects to a non-zero class in E_∞ then it must have filtration $s' < s$ in the bo resolution. The classes described by the theorem belong to $s' = 0$ and 1. The edge theorem of 4.4.12 asserts that the class corresponding to $\langle 1,2i,\nu^2 k\rangle$ is the highest filtration possible this way. This completes the proof.

Theorem 9.1.2. (Theorem 2.3 of [22]). There is an isomorphism $q\colon E_\infty^{s,t}(W_n) \cong E_\infty^{s,t}(M)$ if $6s > t + 18$.

Proof. For $n > 1$ this follows from 5.1.2 and 9.1.1. The important point is that in either W_n or M, v_1^4 is a homotopy class and composition with it, if defined, commute with Adams differentials. For $n = 1$ we need to work a little harder. We first note that $W(1)$ is the loop four times of a space $B^4W(1)$ which is the fiber of $\mathbb{H}P \to K(Z,4)$. By direct calculation we see that $f\colon V_{7,2} \to B^4W(1)$ exists with f^* being an epimorphism. Let $\overline{V}$ be the fiber of $g_2\colon S^5 \to S^5$. There is a map $\overline{g}\colon \overline{V} \to \Omega V_{7,2}$. Finally there are maps Adams spectral sequences

9.1.3
$$\begin{tikzcd}
E_r(f) \arrow[r, "\overline{g}"] \arrow[d, "p"'] & E_r(V_{7,2}) \arrow[r, "f'"] & E_r(W_5(1)) \\
E_r(M) & &
\end{tikzcd}$$

where p is the stabilization map.

If we use the Λ algebra approach it is easy to see that the fiber of $\overline{V} \to B^3W(1)$ is $W(2)$. We have maps $E_\infty(W(2)) \to E_\infty(\overline{V}) \to E_\infty(B^3W(1))$. Above the 1/5 line the fiber F of $\overline{V} \to Q\Sigma^4 M$ also looks like $W(2)$. Since v_1^4 commutes with Adams differentials we have that above the 1/5 line $E_\infty^{s,t}(\overline{V}) = E_\infty^{s,t}(M) \oplus E_\infty^{s,t}(W(2))$. Hence the fibration $W(2) \to \overline{V} \to B^3W(1)$ the classes above the 1/5 line behave just as they do in $F \to \overline{V} \to Q\Sigma^4 M$. This proves the theorem.

Corollary 9.1.4. There is an isomorphism of the v_1-periodic elements of $\pi_*(W_n)$ and $\pi_*(M)$.

This corollary allows us to prove part of Theorem 1.1.1.

Theorem 9.1.5. There is an isomorphism of the v_1-periodic elements of $\pi_*(S^{2n+1})$ and $\pi_*(P^{2n})$.

Proof. For $n = 1$ this is just 9.1.4 for $n = 1$. Suppose we have the result for $n - 1$. Consider the fibration

$$\begin{tikzcd}
QP^{2n-2} \arrow[r] & QP^{2n} \arrow[r] & QP^{2n}_{2n-1} \\
\Omega^{2n-1}S^{2n-1} \arrow[r] \arrow[u, "f_{n-1}"] & \Omega^{2n+1}S^{2n+1} \arrow[r] \arrow[u, "f_n"] & \Omega^{2n}W(n) \arrow[u, "f_n'"'] \\
A \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & C \arrow[u]
\end{tikzcd}$$

where A, B, and C are fibers of the maps f_{n-1}, f_n and f_n' respectively. (Recall $Q(X) = \Omega^\infty \Sigma^\infty X$.) The bottom row is again a fibration. The hypothesis implies that in $\pi_*(A)$ and $\pi_*(C)$ there are no v_1 periodic elements. By exactness there are none in $\pi_*(B)$ which is what we wanted to prove.

9.2. v_1-periodic homotopy of $M = M_{2\iota}$

First it is necessary to name the family of elements given by Theorem 9.1.1. We first label them in the E_2 term and finally will identify them as homotopy classes. The labels which we use are as consistent as we can be with those of May and Tangora [33]. We will use the following exact sequence

$$\cdots \to \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2,\mathbb{Z}_2) \xrightarrow{i_\#} \mathrm{Ext}_A^{s,t}(A_0,\mathbb{Z}_2) \xrightarrow{p_\#} \mathrm{Ext}_A^{s,t-1}(\mathbb{Z}_2,\mathbb{Z}_s) \xrightarrow{A_0} \mathrm{Ext}_A^{s+1,t}(\mathbb{Z}_2,\mathbb{Z}_2) \to$$

to both calculate $E_2^{s,t}(M)$ and name the elements. The convention will be to identify $i_\# a$ with a and label a class (a coset really) as $\overline{a}$ if $p_\# \overline{a} = a$. Theorem 9.1.1 describes two families and in the chart below we separate them also. The first family is given by:

$$\begin{array}{lcccccc}
s \;=\; 4k+0 & a_k & & & & & \\
 +1 & & p^k h_1 & \overline{p^k h_1} & & & \\
 +2 & & & p^k h_1^2 & \overline{p^k h_1^2} & & \\
 +3 & & & & & \overline{p_k h_1^3} & \\
t-s = 8k + & 0 & 1 & 2 & 3 & 4 & 5\text{-}7
\end{array}$$

where $a_0 = 1$; and a_k, $k > 0$ has the property $p_\# a_k \neq 0$ and represents the unique solution to this equation. The second family given by:

$$\begin{array}{lccccc}
1 & & & & & \\
s = 4k+1 & & & & p^k h_2 \ (k > 0) & \\
2 & & & & & \overline{p^{k-1} h_2^2} \\
3 & p^{k-1} c_0 & \overline{p^{k-1} c_0} & & & \\
4 & & p^{k-1} h_1 c_0 & \overline{p^{k-1} h_1 c_0} & & \\
t-s = 8k+0 & & 1 & 2 & 3 & 7
\end{array}$$

In homotopy the elements of the first group represent the μ-family $(\{p^k h_1\}, \{p^k h_1^2\})$ and the elements of order 2 in the $4k-1$ stem image of the J-homomorphism. The second family represents the generators of the image of the J-homomorphism. In each case the elements of order 2 which are summands are counted twice, one from each sphere, and otherwise the generator is counted from the 0-sphere and the element of order 2 is counted from the 1-sphere. There is a further confusion with $\overline{p^{k-1} h_2^2}$. The above claims this should represent the generator of the image of J in stem 8k-1. If $k > 0$, this is indeed true but if $k = 0$ we should use h_3. Also c_0 and $h_1 c_0$ are not proper names for $\eta\sigma$ and $\eta^2\sigma$. Still if you keep these exceptions in mind the theory works as if this were a correct description. For further convenience we label elements as follows. α_i is the homotopy class in filtration i for the first family which is either the element of order 2 in the image of the J-homomorphism in stem 2i-1 ($i \equiv 0(4)$) or stem 2i-3 ($i \equiv 3(4)$) or μ ($i \equiv 1(4)$) or $\eta\mu$ ($i \equiv 2(4)$). The element β_i is the generator of the ith nonzero image of the J homomorphism with $\nu = \beta_1$, $\sigma = \beta_2$, etc. Note that in each case the filtration assigned to α_i or β_i is i (with the difficulties about σ already noted). It will be convenient to have functions giving the stem of α_i and β_i. If $\alpha \in \pi_{n+j}(S^n)$ then $|\alpha| = j$. If $i = 4a + b$ with $0 \leq b \leq 3$ then

$$\begin{aligned} |\alpha_i| &= 8a + b, \ b = 1, 2, \text{ or } 3 \\ &= 8a - 1, \ b = 0, \end{aligned}$$

and

$$\begin{aligned} |\beta_i| &= 8a + 2^{b+1} - 1, \ b = 0,1,2 \\ &= 8(a+1), \qquad b = 3. \end{aligned}$$

9.3. The v_1-periodic homotopy of P^{2n}.

In this section we will describe the v_1-periodic structure of

P^{2n}. We will introduce a spectrum J which includes the v_1 periodic homotopy of S^0 and a filtration on $\pi_* J$ which will allow us to describe quite completely the v_1-periodic structure of P^{2n} and hence S^{2n+1}.

9.3.1. Let J be the fiber of the map $\tau'\colon bo \to \Sigma^4 bspin$ given by 8.3.5. The homotopy groups of J are given by

Proposition 9.3.2. (Lemma 3.3 of [22]). $\pi_j(J) = E_2^{0,j}(S,bo) \oplus E_2^{1,j+1}(S,bo)$.

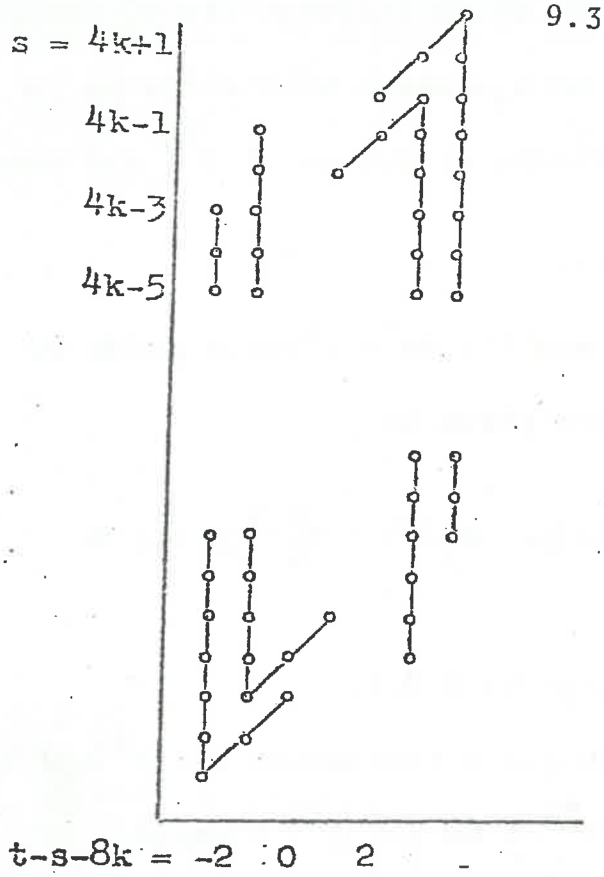
This is just the calculation given by 8.3.6.

Using the theory of 3.6 we can get a resolution for $P^{2n} \wedge J$ by using ordinary Adams resolution for $P^{2n} \wedge bo$ and $P^{2n} \wedge bspin$. If we use minimal resolution of the spaces then the charts 9.3.6 describe $E_1^{s,t}(P^{2n} \wedge J)$.

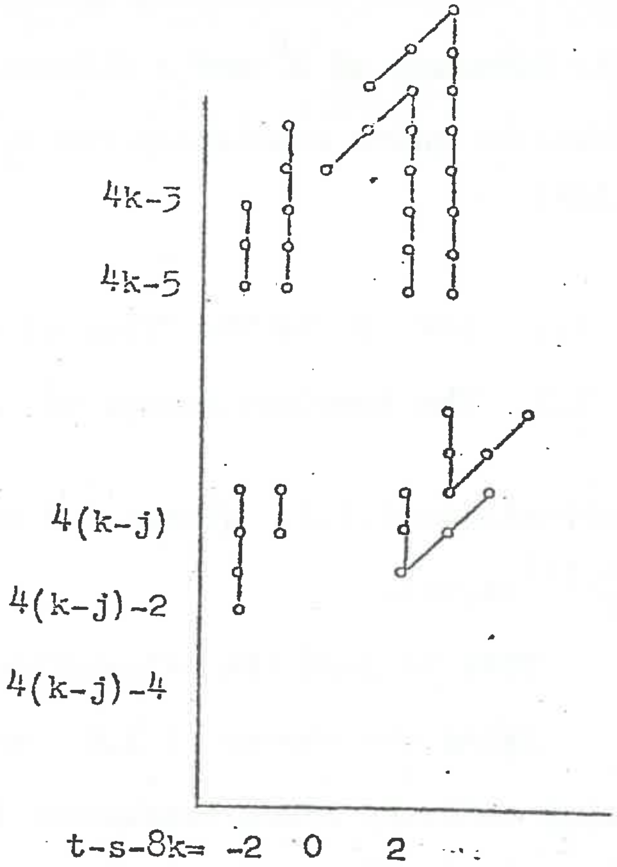
Let $\rho(k)$ be defined by $4k \equiv 2^{\rho(k)-1} \bmod 2^{\rho(k)}$. The homotopy calculations follow from the following

Theorem 9.3.3. (Theorem 3.6 of [22]). The homotopy of $P^{2n} \wedge J$ results from the above charts by a differential $\delta_{\rho(k)-2} a_{4k-1} \neq 0$ if possible for any element a_{4k-1} in $t - s = 4k - 1$.

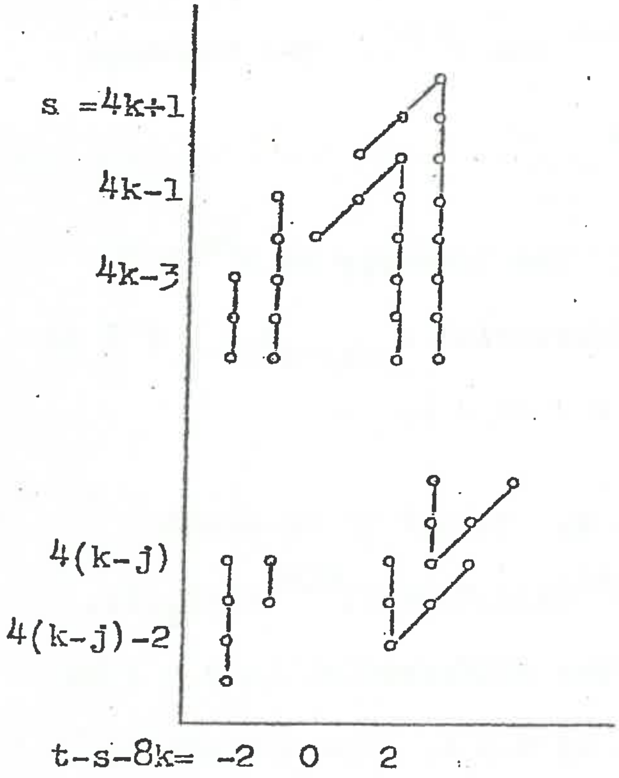
Proof. Consider the sequence $P \to S^0 \to R$. In 4.2 it is shown $R \wedge bo = V\Sigma^{4i} K(Z,0)$. $R \wedge \Sigma^4 bspin = V\Sigma^{4i}(K(Z,0) \oplus V\Sigma^{4i+2}(K(Z_2,0)$. Thus we have $P \wedge J \to S^0 \wedge J \to R \wedge J$. The differential in $R \wedge J$ is given by that of $S^0 \wedge J$ which is given by 8.3.6. The connecting homomorphism from $E_1(R \wedge J) \to E_1(P \wedge J)$ is onto and this gives the result.



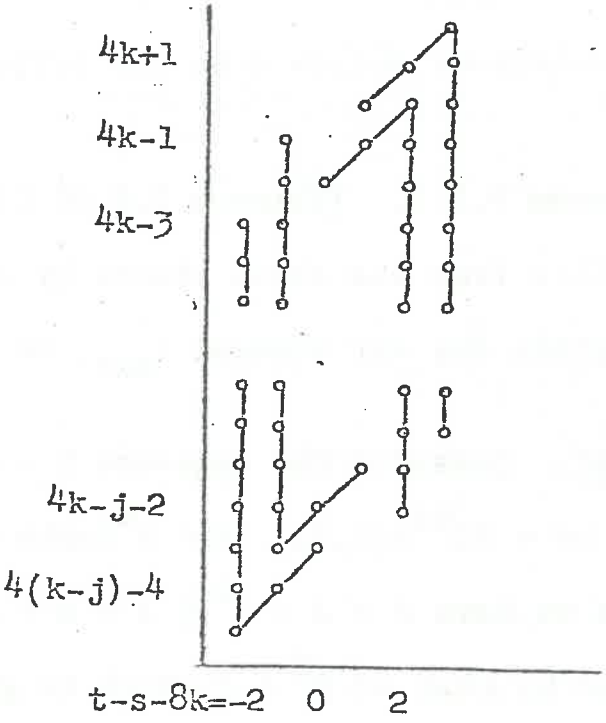
$E_1^{s,t}(P^{8j})$, $3+8k \geq 8j$



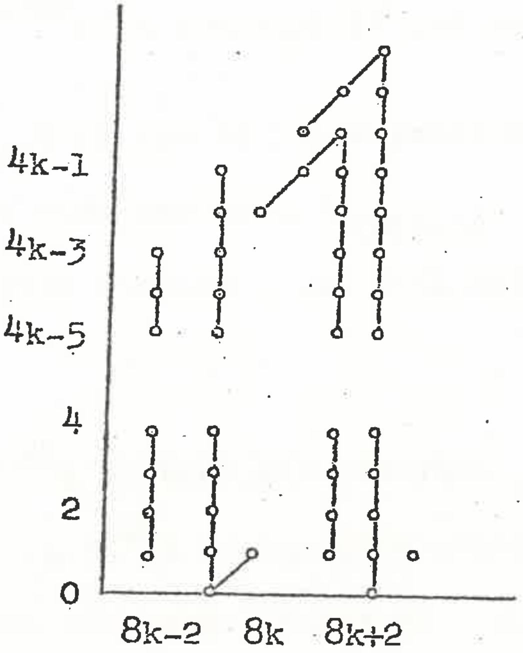
$E_1^{s,t}(P^{8j+2})$, $8j \leq 8k+3$



$E_1(P^{8j+4})$, $8j \leq 8k+3$



$E_1(P^{8j+6})$, $8j \leq 8k+3$



$E_1^{8k+2}(P^{8j})$, $8j > 8k+3$

The theorem was first proved using the work of Toda and the Adams' resolution of the vector field problem. This proof is clearly independent of that work and thus gives an independent proof also of the vector field problem. Let $\varphi(k) = 8a + 2^b$ where $k \equiv 2^i \bmod 2^{i+1}$ and $i + 2 = 4a + b$, $0 \leq b \leq 3$.

Theorem 9.3.4. There is no map of degree 1 of $S^{4k-1} \to P_{4k-\varphi(k)}$.

Proof. If there were then there would be one in $P_{4k-\varphi(k)} \wedge J$. But if we look at $P \wedge J \to P_{4k-\varphi(b)} \wedge J$ we see that the class of filtration zero in dimension $4k-1$ has a nonzero differential.

Finally we note

Proposition 9.3.5. The v_1 periodic homotopy of P^{2n} is mapped isomorphically to the v_1 periodic homotopy of $P^{2n} \wedge J$.

The proof is immediate. We should note that not all of the homotopy groups of $P^{2n} \wedge J$ are parts of v_1-periodic families.

Proposition 9.3.5 completes the proof of 1.1.1.

9.4. Whitehead product structure and composition properties.

The EHP sequence from Chapter 1 gives diagram

9.4.1.

$$\begin{tikzcd}
QP^{2n-2} \arrow[r, "i"] & QP^{2n} \arrow[r, "j"] & Q(P^{2n}_{2n-1}) \\
\Omega^{2n}S^{2n-1} \arrow[r, "E"'] \arrow[u] & \Omega^{2n+1}S^{2n+1} \arrow[r, "H"'] \arrow[u] & \Omega^{2n}W(n) \arrow[u]
\end{tikzcd}$$

and where P is the boundary homomorphism in homotopy of the bottom sequence. (E and H are used both to present the map and the induced map in homotopy.)

We wish to restrict our attention to odd spheres for simplicity. We call S^{2n+1} the sphere of origin of a non zero class $\alpha \in \pi_{j+2k+1}(S^{2k+1})$ if $\alpha \in \operatorname{im} \pi_j(\Omega^{2n+1}S^{2n+1}) \to \pi_j(\Omega^{2k+1}S^{2k+1})$ and n is the smallest integer with this property. The Hopf invariant of α is the coset $H(\alpha')$ for all $\alpha' \in \pi_j(\Omega^{2n+1}S^{2n+1})$ which map to α under $\Omega^{2n+1}S^{2n+1} \subset \Omega^{2k+1}S^{2k+1}$. The central result of these notes asserts that with respect to v_1 periodic elements the two sequences of 9.4.1 are the same. So the Hopf invariants of v_1 periodic classes among odd spheres is the same as for the corresponding stable class in $\{P^{2n}\}$. Theorems 4.1 through 4.8 of [22] list the results which follow immediately from this observation and the charts of the previous section.

Adams in [3] gives some stable compositions involving v_1-periodic elements. His calculations can be summarized by

Proposition 9.4.2 (Adams). If $i \equiv 1,2 \bmod 4$ and $|\alpha_i| + |\beta_j| = |\beta_{i+j}|$ then $\alpha_i \circ \beta_j = \beta_{i+j}$.

In the stable Moore space the composition properties of the elements described in 9.2 are as follows.

Theorem 9.4.3. Suppose $f\colon \Sigma^k M \xrightarrow{f} M$ is a mapping of stable Z_2 Moore space so that $S^k \xrightarrow{i} \Sigma^k M \xrightarrow{f} M$ is one of the elements described in 9.2 and suppose the Adams' filtration of f is the same as f_i and is s'. Then whenever $f_\#\colon E_\infty^{s,t}(M) \to E_\infty^{s+s',t+s'+k}(M)$ can be non-zero if $6s > t + 18$ it is non-zero.

Proof. The theorem follows easily by checking cases after observing that if f: $\Sigma^{8k}M \to M$ represents v_1^4(bo-periodicity) then the composite $S^{8k+8j-1}M \xrightarrow{\beta_{4j-2}} \Sigma^{8k}M \to M$ represents $\beta_{4k+4j-2}$. Now various compositions with η, ν, and the secondary composition $\langle\ ,2i,n\rangle$ give all remaining possibilities.

The crucial steps for proving the results of [22] are now complete. The results given there for $\pi_*(S^{2k+1})$ are direct consequences of the calculations of $\pi_*^S(P^{2k} \wedge J)$ via the exact couple spectral sequence which results from $P^2 \subset P^4 \subset P^6 \subset P^8 \subset \cdots \subset P^{2k}$.

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