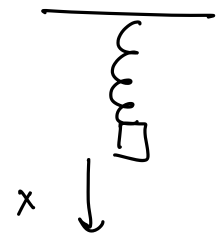
Please fill out the when2meet !!!

- CMSA Special Semester on Arithmetic QFT
  Lecture series Feb 5 - 9
- Textbook: Arnold, "Mathematical methods of classical mechanics"

Integrable system: "System w/ ~~lots of~~ n conserved quantities" of diff eqs in 2n variables

Simple Harmonic oscillator: (SHO)



Hooke's law:
$\exists$ constant ω s.t.
$F = -\omega^2 x$
Force

Newton $F = m\ddot{x}$
acceleration $x = x(t)$, and $\ddot{x} = \frac{\partial^2 x(t)}{dt^2}$

$\rightsquigarrow \cancel{m}\ddot{x} = -\omega^2 x$
(Assume that $m=1$)

Rmk Rewrite as

$$\left(\frac{\partial x}{\partial t} = \right) \dot{x} = p \quad \text{momentum}$$
$$\dot{p} = -\omega^2 x$$
integrable system

Solve: $x(t) = a\cos(\boxed{\omega t - t_0})$
$p(t) = -\omega a \sin(\omega t - t_0)$

t_0, a are constants, determined by $x(0)$, $p(0)$.

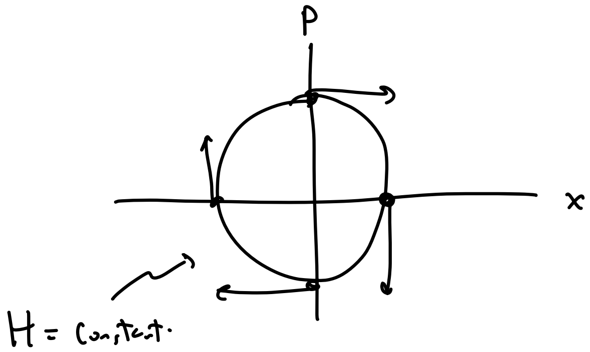
Observe that $p(t)^2 + \omega^2 x(t)^2 = 2\omega^2 a^2 =: H$ (Hamiltonian)
is independent of t!

Goof! accidentally omit this
$\left(a \rightsquigarrow \frac{a}{\sqrt{2}}\right)$

This is the $\boxed{\text{conserved quantity}}$.

$$\mathbb{R}^2 \xrightarrow{H} \mathbb{R}$$
$$(x,p) \longmapsto \omega^2 x^2 + p^2$$

- Constant on sol'n to the SHO
- fibers are ellipses.
  ($H=0$ degenerate ellipse)



[Exercise: Try to do this for
$$\ddot{x} = -\sin(x)$$
(ideal planar pendulum)]

Area of the ellipse $H = \omega^2 a^2$

$$\leftrightsquigarrow \quad \frac{p^2}{\omega^2 a^2} + \frac{x^2}{a^2} = 1 \quad \text{is} \quad \pi\, a \cdot \omega a = \pi a^2 \omega$$
$$= \frac{\pi H}{\omega} \overset{\text{def}}{=} I$$

action

For any given sol'n, $I = \text{const}$
(ie on any fiber of H)

Let's also set $\theta = \omega t - t_0$, so (angle)

$$x(t) = a\cos(\omega t - t_0) = \sqrt{\frac{I}{\pi\omega}}\,\cos(\theta)$$
$$p(t) = -a\omega\,\sin(\omega t - t_0) = -\sqrt{\frac{I}{\pi a}}\,\sin(\theta)$$

$(x,p) \leftrightsquigarrow (I,\theta)$ new coordinates on $\mathbb{R}^2$
(polar coords, basically)

eqns of motion are equiv. to $\dot{I} = 0$, $\dot{\theta} = \omega$
$\left(\frac{\partial I}{\partial t} =\right)$

So H determines the ellipse of area I on which the particle moves, and it moves along at constant speed along θ.

Also note $H = \omega^2 x^2 + p^2$, so

$$\frac{\partial H}{\partial p} = 2p = \dot{x}$$
$$\frac{\partial H}{\partial x} = 2\omega^2 x = -\dot{p}$$

ignore

Def A Hamiltonian System in $2n$-variables is:

- $U \subseteq \mathbb{R}^{2n}$ open
- $H : U \longrightarrow \mathbb{R}$

and $\dot{x}_i = \dfrac{\partial H}{\partial p_i}$, $\dot{p}_i = -\dfrac{\partial H}{\partial x_i}$

Such a system is called integrable if it has "n conserved quantities" $f_1, \dots, f_n$. "$\dot{f}_i = 0$"

Thm (Arnold-Liouville thm) $M_{\vec{c}}$ = level set, cut out by $f_1 = c_1, \dots, f_n = c_n$

Suppose that $M_{\vec{c}}$ = cpct + connected smooth manifold

Then:

- $M_{\vec{c}} \underset{\text{diffeo}}{\cong} (S^1)^n$
- in a nbhd of $M_{\vec{c}}$ inside U, there are canonical variables $\underbrace{I_1, \dots, I_n}_{\text{action}}$, $\underbrace{\theta_1, \dots, \theta_n}_{\text{angle}}$ s.t. the Hamilt. system is equiv to
$$\dot{\theta}_j = \frac{\partial H}{\partial I_j}, \quad \dot{I}_j = 0$$

[Eg $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ $(z_0, \dots, z_n)$
$\cup$
$S^{2n+1} \xrightarrow{\text{Hopf}} \mathbb{C}P^n$ $\quad (z_0, \dots, z_n) \mapsto [z_0 : \dots : z_n]$
Can get this from $(n+1)$ pendulums]

Recall from above $\underset{\text{(action)}}{I} = \pi \omega a^2$ = area of the ellipse

Observe:
$$\oint p\, dx = \oint (-\omega a \sin(\omega t - t_0))\, d(a \cos(\omega t - t_0))$$
$$= \omega a^2 \int_0^{2\pi} \sin^2(\omega t - t_0)\, d(\omega t)$$
$$= \pi \omega a^2 = I$$

why? $\oint p\, dx = \int_{\text{interior}} \underbrace{dp \wedge dx}_{\text{area form}}$ $\quad$ see that $dp \wedge dq = dI \wedge d\theta$ (equality of 2-forms)

Change of coords $(p, x) \rightsquigarrow (\theta, I)$ preserves this closed 2-form

These sort of considerations will lead us to the thy of symplectic geometry $\quad (\mathbb{R}^2_{x,p}, \underbrace{dp \wedge dq}_{\text{Symplectic form}})$

By contemplating symmetries of symplectic manifolds, led to things like Noether's thm

(Eg Cons. of momentum (p) $\longleftrightarrow$ translation invariance)

- Naive: via Euler-Lagrange + Lagrangian mechanics
- momentum maps

every symmetry of M (symp) $\rightsquigarrow$ $M \xrightarrow[\text{conserved qty}]{} \mathbb{R}$, and so $G \xrightarrow{\text{Lie}} \mathfrak{g} \curvearrowright M$

$$M \xrightarrow{\mu} \mathrm{Lie}(G)^*$$

$\xi \in \mathrm{Lie}(G)$ "infinitesimal symmetry"

$$\begin{tikzcd} \rightsquigarrow M \arrow[r, "\mu"] \arrow[rr, bend right] & \mathrm{Lie}(G^*) \arrow[r, "\langle -, \xi \rangle"'] & \mathbb{R} \end{tikzcd}$$

Use the thy of moment maps to construct "new" integrable systems

Eg Kazhdan-Kostant-Sternberg Use this to reconstruct Calogero-Moser system

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$$\text{action} \quad I = \int_{t_i}^{t_f} p\,dx = \int p\dot{x}\,dt$$

So let's set $L = \underbrace{p\dot{x} - H(x,p)}_{\text{Legendre transform}}$ Lagrangian

$$\rightsquigarrow \frac{\partial L}{\partial \dot{x}} = p$$

$$\underset{\text{(functional)}}{\overset{S}{\text{action}}} = \int L\,dt = \int (p\dot{x} - H)\,dt = \int p\dot{x}\,dt - \underbrace{H(t_{final} - t_{initial})}_{\text{constant}}$$

Quick observation:

$$\frac{\partial L}{\partial \dot{x}} = p \rightsquigarrow -\frac{\partial H}{\partial x} = \dot{p} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\| \qquad \qquad =$$

$$\frac{\partial L}{\partial x} \qquad \boxed{\text{Euler-Lagrange eq'n}}$$

Originally: came from analyzing

$S = \int L\,dt$ describe the minimizer of S (next week) (Chapter 3 of Arnold)

(principle of least action)

Really fun lecture

in Vol II of Feynman Lectures