Integrable systems

S. K. Devalapurkar

Lecture 4: Hamiltonian reduction

One of the motivations for moving to the Hamiltonian picture is that it placed the coordinates q and p on a more equal level. It is natural to expect that this leads to further symmetries which are harder to see at the Lagrangian level. For example, if M is a smooth manifold, and G was a Lie group acting on M, one can see that the induced action of G on T^*M was Hamiltonian (and so one obtains a moment map $T^*M \to \mathfrak{g}^*$). Here is a quick review of how this goes: the action of G on M defines a symplectic action of G on T^*M; namely, if $g \in G$ induces an automorphism $f_g : M \to M$, then the induced automorphism of T^*M sends $(q,p) \mapsto (f_g(q), (f_g^{-1})^*(p))$. (Why the inverse? It's because the action of G on T^*M should preserve the symplectic form, or equivalently the Liouville 1-form pdq.) We therefore get a map $\mathfrak{g} \to \mathrm{Vect}(T^*M)$, and pairing a vector field on T^*M with the Liouville 1-form defines a map $\mathfrak{g} \to C^\infty(T^*M)$. It is not difficult to convince yourself that this does in fact satisfy the conditions necessary to call it a moment map.

One might hope that there are *more* Hamiltonian symmetries of T^*M than ones coming from symmetries of M, and that these might play a role in mechanics.

Example 1. There is an action of the entire symplectic group Sp_{2n} on $\mathbf{R}^{2n} \cong \mathbf{C}^n$, which obviously preserves the symplectic form, and it has a maximal torus given by $(S^1)^n$, which rotates each complex factor. The moment map $\mu : \mathbf{R}^{2n} \to \mathfrak{sp}_{2n}^*$ is given by

$$\mu : v \mapsto [\xi \mapsto \tfrac{1}{2}\omega(v, \xi v)].$$

This is pretty much definitional (but note that one could actually shift this moment map by any constant, and it would still be a moment map). But note that this Sp_{2n}-action on $\mathbf{R}^{2n}$ *definitely* does not come from an Sp_{2n}-action on $\mathbf{R}^n$.

The restriction of the Sp_{2n}-action to the diagonal circle $S^1 \subseteq (S^1)^n \subseteq \mathrm{Sp}_{2n}$ is clearly still Hamiltonian, and its moment map is given by

$$\mu : \mathbf{R}^{2n} \to \mathfrak{sp}_{2n}^* \to \mathrm{Lie}(S^1) = \mathbf{R},\ v \mapsto \tfrac{|v|^2}{2}.$$

This can be viewed as the Hamiltonian of a system of n harmonic oscillators in $\mathbf{R}^2$. Now, suppose we are studying the dynamics of some Hamiltonian $H : \mathbf{R}^{2n} \to \mathbf{R}$, and we knew that it was invariant under the S^1-action on $\mathbf{R}^{2n}$ described above. Then, Noether's theorem, as rephrased in the previous lecture, tells us that the

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particle (in the phase space $\mathbf{R}^{2n}$) is constrained to move around only on level sets of the moment map μ. If we take some $x \in \mathbf{R}$ (this is the value of the momentum associated to the S^1-action), the preimage $\mu^{-1}(x)$ has an S^1-action, but since H is S^1-invariant, the motion of the particle is really living on something like the quotient space $\mu^{-1}(x)/S^1$.

There are a couple of points to make here about taking quotients, but let us for the moment observe that $\mu^{-1}(x)/S^1$ need not be a cotangent bundle. It is not even clear whether it is a symplectic manifold (but it will turn out to be, at least if x is nonzero). But it can sometimes be identified with more familiar spaces. For example, say $x = 1$. Then $\mu^{-1}(1/2)$ is the subset of $\mathbf{R}^{2n}$ of those vectors such that $|v|^2 = 1$, but this is just a $(2n-1)$-sphere of radius 1. The quotient $\mu^{-1}(1/2)/S^1$ can be identified with $\mathbf{C}P^{n-1}$. In fact, we mentioned above that these quotients $\mu^{-1}(x)/S^1$ admit symplectic structures, and one can verify that the symplectic structure thus obtained on $\mathbf{C}P^{n-1}$ agrees with the Fubini-Study one (when $n = 2$, this was the area form on $\mathbf{C}P^1 = S^2$).

This procedure of using the moment map to construct new symplectic manifolds is called *Hamiltonian reduction*, and it will be our focus this week. Before going to the general theory, let us see a related example in the setting of solving the Hamiltonian flow equation.

Namely, let $H : \mathbf{R}^{2n} \to \mathbf{R}$ be a Hamiltonian function, and let ξ be a nonzero vector field on $\mathbf{R}^{2n}$ which preserves the standard symplectic form and also preserves H. Let $\mu : \mathbf{R}^{2n} \to \mathbf{R}$ denote the conserved quantity corresponding to ξ. (One does not need to work on $\mathbf{R}^{2n}$, but then the discussion below will only hold locally.) Then:

Proposition 2 (Elimination of variables)**.** *In the above setup, one can reduce the Hamiltonian system $\dot{f} = X_H(f)$ to a Hamiltonian system in **two** fewer variables such that the solutions to our original system can be obtained from the new system by an integral.*

Proof. This is very similar to the Darboux theorem. If $(x_1, \cdots, x_{2n})$ were our original coordinates on $\mathbf{R}^{2n}$, let us choose coordinates $(q(x), p(x), y_1(x), \cdots, y_{2n-2}(x))$ such that $p = \mu(x)$ and the vector field ξ is ∂_q. Then the symplectic form can be written as $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & v \\ 0 & v^T & \omega' \end{pmatrix}$, where $v = v(p, \vec{y})$, and $\omega' = \omega'(p, \vec{y})$ is a symplectic form on $\mathbf{R}^{2n-2}$. For any fixed value of $p = \mu$, we can write down a Hamiltonian system on $\mathbf{R}^{2n-2}$, namely the flow of the vector field X_H which is defined using ω' and the restricted map $H|_{\mathbf{R}^{2n-2}} : \mathbf{R}^{2n-2} \to \mathbf{R}$. This is our "reduced" system.

Now, since ξ preserves H and we chose coordinates so that $\xi = \partial_q$, we know that $\partial_q H = 0$. But in the new coordinates $(q, p, \vec{y})$, our original Hamiltonian system becomes

$$\begin{aligned} \dot{p} &= -\partial_q H = 0, \\ \dot{q} &= \partial_p H + v \cdot \nabla H \\ \dot{y} &= X_H(y), \end{aligned}$$

where we remember that X_H is defined using ω' on $\mathbf{R}^{2n-2}$. (The last equation is our reduced system.) The first equation just says that p is a constant. If we can solve the third equation, the second equation allows us to determine q by a single integral. $\square$

Example 3 (Calogero-Moser)**.** Suppose we have a Hamiltonian function $H : \mathbf{R}^4 \to \mathbf{R}$ given by

$$H(q_1, p_1, q_2, p_2) = \frac{p_1^2 + p_2^2}{2} + V(q_1 - q_2),$$

where $V : \mathbf{R} \to \mathbf{R}$ is some function. The Hamiltonian above describes the Calogero-Moser system for two particles interacting on the line with potential V. The Hamiltonian system is

$$\dot{q}_1 = p_1,\ \dot{q}_2 = p_2,\ \dot{p}_1 = -V'(q_1 - q_2),\ \dot{p}_2 = V'(q_1 - q_2).$$

Clearly, the action of $\mathbf{R}$ sending $(q_1, p_1, q_2, p_2) \mapsto (q_1+s, p_1, q_2+s, p_2)$ is a symmetry preserving this Hamiltonian. The associated vector field is $\partial_{q_1} + \partial_{q_2}$, and the corresponding conserved quantity is the total momentum $\mu = p_1+p_2$ of the system. So, let us take new coordinates where $p = p_1+p_2$, $q = q_1+q_2$, and set $y_1 = p_1-p_2$, $y_2 = q_1 - q_2$. (Other choices of $\vec{y}$ also work, of course.) One then finds

$$\dot{p} = 0,\ \dot{q} = p,\ \dot{y}_1 = -2V'(y_2),\ \dot{y}_2 = y_1.$$

Of course, p is constant; q is a constant shift of pt; and the final two equations for y_1, y_2 define another Hamiltonian system.

Here is another way to say this. The moment map $\mu : \mathbf{R}^4 \to \mathbf{R}$ sends $(q_1, p_1, q_2, p_2) \mapsto p = p_1 + p_2$, and the fiber $\mu^{-1}(c) = \{(q_1, p, q_2, c-p)\}$ has a free $\mathbf{R}$-action. The quotient $\mu^{-1}(c)/\mathbf{R}$ is isomorphic to $\mathbf{R}^2$ with coordinates y_1, y_2, and the Hamiltonian system

$$\dot{y}_1 = -2V'(y_2),\ \dot{y}_2 = y_1$$

is defined on this quotient. In other words, the *reason* that we were able to eliminate two variables is that we first took a level set of μ, and then quotiented out by the 1-dimensional group $\mathbf{R}$.

This is very helpful, but one runs into issues when trying to do elimination of variables for more than one vector field. For example, if ξ_1 and ξ_2 are vector fields which are symmetries of a Hamiltonian system, one might think that solving the original system would reduce to solving a Hamiltonian system in *four* fewer variables. But this need not be true: for example, if one does the reduction procedure of Proposition 2 for the first vector field ξ_1, the second vector field ξ_2 may no longer be a symmetry of this reduced system. Here is another way of saying this.

Let G be a Lie group acting on M by Hamiltonian vector fields, and suppose that the G-action preserves a Hamiltonian $H : M \to \mathbf{R}$. Let $\mu : M \to \mathfrak{g}^*$ denote the moment map. Take some $v \in \mathfrak{g}^*$ (so it is a vector describing choices of values for the conserved quantities). Then one can consider $\mu^{-1}(v)$, but this need not be preserved by all of G. Namely, the coadjoint action of G on $\mathfrak{g}^*$ can move v around. We can therefore only sensibly take the quotient of $\mu^{-1}(v)$ by the *stabilizer* of v. So there will be no problem in this reduction procedure if the symmetry group G is abelian: in this case, the coadjoint action is trivial, and life is simpler.

Let us now try to make this precise. Most of the following theorem is linear algebra.

Theorem 4 (Marsden-Weinstein)**.** *Let G be a compact Lie group with a Hamiltonian action on M, and let $\mu : M \to \mathfrak{g}^*$ denote the moment map. Suppose $\alpha \in \mathfrak{g}^*$, and let $G_\alpha \subseteq G$ denote its stabilizer group under the coadjoint action of G on $\mathfrak{g}^*$. Assume that G_α acts freely on $\mu^{-1}(\alpha)$, and that α is a regular value of μ. Then*

- *$\mu^{-1}(\alpha)/G_\alpha$ is a smooth symplectic manifold, and its symplectic form is a descent of the pullback of the 2-form ω on M to $\mu^{-1}(\alpha)$. It is $\dim(M) - \dim(G) - \dim(G_\alpha)$-dimensional.*
- *If $H : M \to \mathbf{R}$ is a G-invariant Hamiltonian, then it descends to a function $\mu^{-1}(\alpha)/G_\alpha \to \mathbf{R}$ whose associated vector field is the descent of the vector field X_H on M.*

PROOF SKETCH. Suppose Y is a smooth manifold, and let $f : Y \to \mathbf{R}^n$ be a smooth map. Let $\alpha \in \mathbf{R}^n$. If $df : T_yY \to \mathbf{R}^n$ is surjective at every fiber in $f^{-1}(\alpha)$ (i.e., α is a regular value of f), then $f^{-1}(\alpha)$ is a smooth manifold. (This is a version of the implicit function theorem.) So, $\mu^{-1}(\alpha)$ is a smooth manifold since α is assumed to be a regular value of μ. Now the assumption that G_α acts freely on $\mu^{-1}(\alpha)$ implies that $\mu^{-1}(\alpha)/G_\alpha$ is a smooth manifold, and the map $\mu^{-1}(\alpha) \to \mu^{-1}(\alpha)/G_\alpha$ is smooth.[1]

Let us now try to construct the symplectic structure on $\mu^{-1}(\alpha)/G_\alpha$. Let $x \in \mu^{-1}(\alpha) \subseteq M$, and let $[x] \in \mu^{-1}(\alpha)/G_\alpha$ denote its orbit, then

$$T_{[x]}(\mu^{-1}(\alpha)/G_\alpha) \cong T_x(\mu^{-1}(\alpha))/T_x(G_\alpha \cdot x).$$

We would like to define a symplectic form $\overline{\omega}$ on this vector space by the formula

$$\overline{\omega}([v],[w]) = \omega(v,w) \text{ for } [v],[w] \in T_x(\mu^{-1}(\alpha))/T_x(G_\alpha \cdot x),$$

where ω is our symplectic form on $T_x(M)$. For this to be well-defined, we need to know that ω vanishes on $T_x(G_\alpha \cdot x)$. We claim:

(a) Let $G \cdot x$ denote the orbit of x. Then $T_x(\mu^{-1}(\alpha))$ is the symplectic complement of $T_x(G \cdot x)$.
(b) One can identify $T_x(G_\alpha \cdot x)$ with $T_x(\mu^{-1}(\alpha)) \cap T_x(G \cdot x)$.

Together, this implies that $\overline{\omega}$ is well-defined. Let us now see that it is closed and nondegenerate.

- To see that $\overline{\omega}$ is closed, note that the pullback of $\overline{\omega}$ to $\mu^{-1}(\alpha)$ is the restriction of the closed 2-form ω; so the pullback of $\overline{\omega}$ is closed, which implies that $\overline{\omega}$ is closed since $\mu^{-1}(\alpha) \to \mu^{-1}(\alpha)/G_v$ is surjective.
- To see that $\overline{\omega}$ is nondegenerate, suppose $\overline{\omega}([v],[w]) = 0$ for all $w \in T_x\mu^{-1}(\alpha)$. By definition, this means that $\omega(v,w) = 0$, and so v is in the symplectic complement of $T_x\mu^{-1}(\alpha)$. By (a) above, this means that $v \in T_x(G \cdot x)$, and hence
$$v \in T_x(\mu^{-1}(\alpha)) \cap T_x(G \cdot x) = T_x(G_\alpha \cdot x).$$
But then $[v] = 0$.

Let us now show (a) and (b). First, let's do (a). Suppose $x \in \mu^{-1}(\alpha)$, and let $v \in T_xM$. Note that μ induces a map $d_x\mu : T_xM \to T_\alpha\mathfrak{g}^* \cong \mathfrak{g}^*$. One can identify $T_x\mu^{-1}(\alpha)$ with the kernel of this map. Now, $v \in T_x\mu^{-1}(\alpha) = \ker(d_x\mu)$ if and only if $\langle (d_x\mu)(v), \xi\rangle = 0$ for all $\xi \in \mathfrak{g}$. The defining equation of moment maps says that

$$\omega(\xi, v) = \langle (d_x\mu)(v), \xi\rangle,$$

[1]Note that it is important that G_α is compact. For example, consider $\mathbf{R}^\times$ acting on $\mathbf{R}$ by multiplication. Then $\mathbf{R}/\mathbf{R}^\times$ has two points, namely the orbit $[0]$ of zero and the orbit $[\mathbf{R}^\times]$ of everything else. This is not Hausdorff, because any neighborhood of $[\mathbf{R}^\times]$ has to contain $[0]$ since there are real numbers arbitrarily close to zero.

so $v \in T_x\mu^{-1}(v) = \ker(d_x\mu)$ if and only if $\omega(\xi, v) = 0$ for all $\xi \in \mathfrak{g}$. Since there is a surjection $T_1G \cong \mathfrak{g} \twoheadrightarrow T_x(G \cdot x)$, this happens if and only if $\omega(u, v) = 0$ for all $v \in T_x(G \cdot x)$; but this is just the definition of the symplectic complement.

Let us now do (b). Since $T_x\mu^{-1}(\alpha) = \ker(d_x\mu)$, the intersection $T_x(\mu^{-1}(\alpha)) \cap T_x(G \cdot x)$ can be identified with the kernel of the composite

$$T_x(G \cdot x) \to T_xM \xrightarrow{d_x\mu} T_\alpha\mathfrak{g}^* \cong \mathfrak{g}^*.$$

There is a surjection $T_1G \cong \mathfrak{g} \twoheadrightarrow T_x(G \cdot x)$, and so we get a map $\mathfrak{g} \to \mathfrak{g}^*$. An element $\xi \in \mathfrak{g}$ is in the kernel of this map if and only if $\langle \xi, \alpha \rangle = 0$. But this is equivalent to ξ being in the Lie algebra $\mathfrak{g}_\alpha = \mathrm{Lie}(G_\alpha)$. Since there is a surjection $\mathfrak{g}_\alpha \twoheadrightarrow T_x(G_\alpha \cdot x)$, we see that $T_x(\mu^{-1}(\alpha)) \cap T_x(G \cdot x)$ can be identified with $T_x(G_\alpha \cdot x)$, as desired.

I will leave the final part of the theorem (concerning H) to the reader. □

Theorem 4 is a beautiful result, and immediately gives us a ton of ways to produce new examples of Hamiltonian spaces. I will explain the example of the rigid body system in three dimensions in detail next time. Let us content ourselves for now with some examples.

Example 5. Let $j > n$, and consider the action of the unitary group $\mathrm{U}(n)$ on $\mathrm{Hom}(\mathbf{C}^j, \mathbf{C}^n)$ (the vector space of linear maps $\mathbf{C}^j \to \mathbf{C}^n$) via $g \cdot A = Ag^{-1}$. There is a symplectic form on $\mathrm{Hom}(\mathbf{C}^j, \mathbf{C}^n) \cong \mathbf{C}^{nj}$ given by the imaginary part of the Hermitian inner product on $\mathbf{C}^{nj}$, and the above action is Hamiltonian with moment map

$$\mu : \mathbf{C}^{nj} \to \mathfrak{u}(n)^*, \; A \mapsto [\xi \mapsto \tfrac{i}{2}\,\mathrm{Tr}(A\xi A^*)],$$

where A^* is the conjugate transpose of A. Said differently, if we use the Killing form $\kappa(\xi, \zeta) = \mathrm{Tr}(\xi^*\zeta)$ to identify $\mathfrak{u}(n)^* \cong \mathfrak{u}(n)$, this is the map

$$\mu : \mathbf{C}^{nj} \to \mathfrak{u}(n), \; A \mapsto \frac{i}{2}A^*A.$$

The preimage of $\frac{i}{2}\mathrm{id}$ is the collection of those $j \times n$-matrices which define a unitary n-frame in $\mathbf{C}^j$. One can check that $\frac{i}{2}\mathrm{id}$ is a regular value of μ, and $\mathrm{U}(n)$ acts freely on $\mu^{-1}(\frac{i}{2}\mathrm{id})$, so the quotient $\mu^{-1}(\frac{i}{2}\mathrm{id})/\mathrm{U}(n)$ admits a symplectic structure by Theorem 4. Note that it can be identified with $\mathrm{Gr}_n(\mathbf{C}^j)$. This procedure generalizes the construction of the symplectic structure on $\mathbf{C}P^{j-1}$.

Example 6. Let G act on G by left translations, so that it induces a Hamiltonian action on $T^*G \cong G \times \mathfrak{g}^*$. Then the moment map $\mu : T^*G \to \mathfrak{g}^*$ amounts to projection onto the second factor. (Exercise!) This means that any $\alpha \in \mathfrak{g}^*$ is a regular value. Also, $\mu^{-1}(\alpha) \cong G \times \{\alpha\}$, and so G_α acts freely. Therefore, $\mu^{-1}(\alpha)/G_\alpha \cong G/G_\alpha$, which is precisely the *orbit* of α under the coadjoint action of G on $\mathfrak{g}^*$. In particular, coadjoint orbits admit the structure of a Hamiltonian G-space by Theorem 4.

What is the symplectic form? This is an easy thing to compute, but it is not entirely straightforward. Instead of giving the details, I'll refer you to [**AM78**, Page 303]. The final answer is quite elegant: if $v \in G/G_\alpha$, so that there is a surjection $\mathfrak{g} \twoheadrightarrow T_v(G/G_\alpha) \cong \mathfrak{g}/\mathfrak{g}_\alpha$, then

$$\omega_v(\xi, \zeta) = \langle v, [\xi, \zeta] \rangle$$

for $\xi, \zeta \in T_v(G/G_\alpha)$. (I am representing these elements by choices of lifts to $\mathfrak{g}$, and $[\xi, \zeta]$ is the Lie bracket in $\mathfrak{g}$.) The moment map for the G-action on G/G_α is very simple to write down; it is just the inclusion $G/G_\alpha \hookrightarrow \mathfrak{g}^*$. The above Hamiltonian G-structure on coadjoint orbits is due to Kirillov, Kostant, and Souriau, and plays an extremely important role in representation theory. We will study such coadjoint orbits in greater detail later.

For example, supppose $G = \mathrm{SO}_3$. Then one can identify $\mathfrak{so}_3^* \cong \mathbf{R}^3$, and the coadjoint action of SO_3 is just by rotations. Therefore, the orbit of any vector in $\mathfrak{so}_3^*$ is either zero (if the vector was zero), or is a 2-sphere. In other words, coadjoint orbits of SO_3 are just spheres of varying radii. The symplectic structure on such a 2-sphere coming from Theorem 4 is just the area form. The Hamiltonian SO_3-action on such a coadjoint orbit then gives the inclusion $S^2 \subseteq \mathfrak{so}_3^* \cong \mathbf{R}^3$, as we have mentioned in a previous lecture.

Exercise 7. In Theorem 4, suppose $\mathcal{O}_\alpha$ is the orbit of $\alpha \in \mathfrak{g}^*$. Then, check that G_α acts freely on $\mu^{-1}(\alpha)$ if and only if G acts freely on $\mu^{-1}(\mathcal{O}_\alpha)$, and moreover, the canonical map $\mu^{-1}(\alpha)/G_\alpha \to \mu^{-1}(\mathcal{O}_\alpha)/G$ is a diffeomorphism.

References

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1 Oxford St, Cambridge, MA 02139
Email address: `sdevalapurkar@math.harvard.edu`, February 17, 2024