Integrable systems

S. K. Devalapurkar

Lecture 5: Rigid body motion

In this lecture, I want to focus on the example of *rigid body* dynamics. A rigid body is a system of point masses (in $\mathbf{R}^3$) such that the distance between two points is constant. The position of such a rigid body is determined by three points on it which are not collinear. Indeed, the position of any other point is determined by three such points because of the condition that the distance between two points is constant. Now three such points in $\mathbf{R}^3$ which are not collinear determine an affine 2-space in $\mathbf{R}^3$, and the space of such affine 2-spaces is precisely $\mathbf{R}^3 \times \mathbf{R}P^3 = \mathbf{R}^3 \times \mathrm{SO}_3$. The $\mathbf{R}^3$ accounts for the "origin" of this affine space, and the SO_3 tells us how to get from one chosen such 2-space to any other. We will consider rigid bodies which are "centered" at the origin, so that the configuration space of positions of such an object is given by SO_3. That is, we will consider the Euler-Lagrange equation on $T\mathrm{SO}_3$, or equivalently Hamilton's equations on $T^*\mathrm{SO}_3$. Said differently, the dynamics of a rigid body in $\mathbf{R}^3$ is given by the dynamics of a point particle in SO_3.

Let us actually consider more generally dynamics on a compact Lie group G. It will be most convenient to assume that there is no "potential energy", so we will just consider the motion of a *free* particle in G. (I will also work in the Lagrangian formulation, but it is not very difficult to do things in the Hamiltonian picture.) To make sense of this, we need to fix a metric on G. The most natural such ones are (left) G-invariant, hence are determined by a nondegenerate symmetric bilinear form $\langle -, - \rangle$ on $\mathfrak{g}$. Let us fix such a pairing. Then the Lagrangian in question is

$$L(\gamma, \dot{\gamma}) = \frac{1}{2} \|\dot{\gamma}(t)\|^2_{\gamma(t)} \tag{1}$$

for a path $\gamma : \mathbf{R} \to G$. We can now apply the Euler-Lagrange equations to get the equations of motion for such a particle. Recall that the Euler-Lagrange equations for a free particle on a manifold are precisely the geodesic equations! In other words, we're studying geodesic motion on G.

Let us just run through the derivation of the equations of motion. Note that $\|\dot{\gamma}(t)\|^2_{\gamma(t)}$ can be expressed as $\|\dot{\gamma}(t)\gamma^{-1}(t)\|^2$, where $\dot{\gamma}(t)\gamma^{-1}(t)$ is now viewed as a path in $\mathfrak{g} = T_1(G)$, and the norm is taken using the fixed bilinear form $\langle -, - \rangle$ on $\mathfrak{g}$. Consider a variation $\gamma_s(t)$, so that $\gamma_0(t) = \gamma(t)$ is our original path. Let me write

Part of this work was done when the author was supported by NSF DGE-2140743.

$\xi = \dot{\gamma}\gamma^{-1}$. Then

$$\frac{1}{2}\delta\|\dot{\gamma}\gamma^{-1}\|^2 = \langle \delta(\xi), \xi\rangle.$$

Using the chain rule, we have

$$\begin{aligned}\delta(\xi) &= \delta(\dot{\gamma}\gamma^{-1})\\ &= \partial_s\partial_t(\gamma)\gamma^{-1} - \partial_t(\gamma)\gamma^{-1}\partial_s(\gamma)\gamma^{-1}.\end{aligned}$$

But also,

$$\partial_t(\partial_s(\gamma)\gamma^{-1}) = \partial_t\partial_s(\gamma)\gamma^{-1} - \partial_s(\gamma)\gamma^{-1}\partial_t(\gamma)\gamma^{-1}.$$

Adding these two together, we find that

$$\begin{aligned}\delta(\xi) &= \partial_t(\partial_s(\gamma)\gamma^{-1}) + [\partial_s(\gamma)\gamma^{-1}, \partial_t(\gamma)\gamma^{-1}]\\ &= \partial_t(\partial_s(\gamma)\gamma^{-1}) + [\partial_s(\gamma)\gamma^{-1}, \xi].\end{aligned}$$

Therefore,

$$\int \langle \delta(\xi), \xi\rangle dt = \int \langle \partial_t(\partial_s(\gamma)\gamma^{-1}), \xi\rangle dt + \int \langle [\partial_s(\gamma)\gamma^{-1}, \xi], \xi\rangle dt.$$

By integrating by parts, the first term becomes

$$-\int \langle \partial_s(\gamma)\gamma^{-1}, \partial_t(\xi)\rangle dt = -\int \langle \partial_s(\gamma), \partial_t(\xi)\gamma\rangle_\gamma dt.$$

What about the second term?

Definition 1. Let $b : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ denote the transpose of the Lie bracket, so that it is defined by the formula

$$\langle x, b(y,z)\rangle = \langle [x,y], z\rangle.$$

Then, the second term is given by

$$\int \langle [\partial_s(\gamma)\gamma^{-1}, \xi], \xi\rangle dt = \int \langle \partial_s(\gamma)\gamma^{-1}, b(\xi,\xi)\rangle dt = \int \langle \partial_s(\gamma), b(\xi,\xi)\gamma\rangle_\gamma dt.$$

That is,

$$\int \langle \delta(\xi), \xi\rangle dt = \int \langle \partial_s(\gamma)\gamma^{-1}, b(\xi,\xi) - \partial_t(\xi)\rangle dt,$$

which means that:

Proposition 2. *If $\gamma : \mathbf{R} \to G$ is a curve, and $\xi = \dot{\gamma}\gamma^{-1}$, then γ is a geodesic if and only if*

$$\dot{\xi} = b(\xi,\xi).$$

This is sometimes called the *Euler-Arnold equation*; "Euler" for reasons we will discuss below, and "Arnold" because he was the first to realize that Euler's equations were the special case of the above equations when $G = \mathrm{SO}_3$. We will specialize to that case in a moment and explicate Proposition 2, but first let us massage the calculation further. The discussion below is essentially an implementation of the Legendre transform in the context of compact Lie groups.

The Lie algebra $\mathfrak{g}$ has a canonical symmetric bilinear form defined on it, namely the Killing form $\kappa(-,-)$, given by the formula

$$\kappa(x,y) = \mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_y).$$

This is an invariant bilinear form, which for $\mathfrak{sl}_n(\mathbf{R})$, for instance, is given by $2n\,\mathrm{Tr}(xy)$, and for $\mathfrak{so}_n(\mathbf{R})$, it is $(n-2)\,\mathrm{Tr}(xy)$. Note that in general,

$$\kappa([x,y],z) = -\kappa(y,[x,z]).$$

Example 3. Recall that $\mathfrak{so}_3$ identifies with $\mathbf{R}^3$, where the Lie bracket becomes the cross product. Explicitly, $(x,y,z) \in \mathbf{R}^3$ defines the matrix

$$\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \in \mathfrak{so}_3.$$

The formula

$$x \times (y \times z) = (x \cdot z)y - (x \cdot y)z,$$

and so

$$\mathrm{ad}_x \mathrm{ad}_y = x \otimes y - (x \cdot y)\mathrm{id}_{\mathbf{R}^3}$$

as an endomorphism of $\mathfrak{so}_3 = \mathbf{R}^3$. Therefore,

$$\kappa(x,y) = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y) = x \cdot y - 3x \cdot y = -2x \cdot y.$$

Notice that this is a nondegenerate bilinear form on $\mathfrak{so}_3$.

Theorem 4 (Cartan)**.** *The Lie algebra $\mathfrak{g}$ is semisimple if and only if κ is nondegenerate.*

Proof sketch. The important input is Cartan's criterion for solvability, which states that if V is a vector space and $\mathfrak{g}$, then $\mathfrak{g}$ is solvable if and only if $\kappa(x,y) = 0$ for all $x \in \mathfrak{g}$ and all $y \in [\mathfrak{g},\mathfrak{g}]$. (I will not prove this, which is why this is a proof sketch and not a proof.)

Let J denote the kernel of κ. Suppose first that $\mathfrak{g}$ is semisimple. Recall that this means that if $\mathrm{rad}(\mathfrak{g})$ is the maximal solvable ideal of $\mathfrak{g}$, then $\mathrm{rad}(\mathfrak{g}) = 0$. By definition, $\mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y) = 0$ for all $x \in J$ and $y \in \mathfrak{g}$, hence in particular for all $y \in [J,J]$. By Cartan's criterion, J is solvable. But also J is an ideal in $\mathfrak{g}$, so $J \subseteq \mathrm{rad}(\mathfrak{g}) = 0$.

Now suppose $J = 0$. To show that $\mathfrak{g}$ is semisimple, it suffices to show that every abelian ideal $I \subseteq \mathfrak{g}$ is contained in J. Let $x \in I$. For any $y \in \mathfrak{g}$, the operator $\mathrm{ad}_x \mathrm{ad}_y$ sends $\mathfrak{g}$ to I because I is an ideal; and it sends I to zero because I is abelian. So $\mathrm{ad}_x \mathrm{ad}_y$ squares to zero, hence has vanishing trace, and so $\kappa(x,y) = 0$. Therefore $x \in J$, which is zero. $\square$

Suppose $\mathfrak{g}$ is semisimple. Then our bilinear form $\langle -,- \rangle$ on $\mathfrak{g}$ can be written as

$$\langle x, Iy \rangle = \kappa(x,y)$$

for some invertible linear transformation $I : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$. This transformation is self-adjoint with respect to $\langle -,- \rangle$ and $\kappa(-,-)$. Note that because κ is nondegenerate, specifying $\langle -,- \rangle$ is equivalent to specifying I.

Using I, we can rewrite b:

$$\begin{aligned} \langle x, b(Iy,Iy) \rangle &= \langle [x,Iy], y \rangle = \kappa([x,Iy],y) \\ &= -\kappa([Iy,x],y) = \kappa(x,[Iy,y]). \end{aligned}$$

On the other hand,

$$\langle x, b(Iy,Iy) \rangle = \kappa(x, I^{-1}b(Iy,Iy)),$$

and since κ is nondegenerate, we have

$$I^{-1}b(Iy,Iy) = [Iy,y].$$

So, if $\gamma : \mathbf{R} \to G$ is a curve, and as before, we set $\xi = \dot{\gamma}\gamma^{-1} : \mathbf{R} \to \mathfrak{g}$, then we can apply I to ξ to restate Proposition 2 as follows.

Corollary 5. *Let $M = I^{-1}\xi : \mathbf{R} \to \mathfrak{g}$; then the geodesic equation of Proposition 2 is equivalent to*

$$\dot{M} = [IM, M].$$

PROOF. Indeed,

$$\dot{M} = I^{-1}\dot{\xi} = I^{-1}b(\xi, \xi) = I^{-1}b(IM, IM) = [IM, M]. \quad \square$$

Example 6. Suppose that $G = \mathrm{SO}_3$, so that $\mathfrak{so}_3^*$ is canonically $\mathbf{R}^3$, but we will use the Killing form to identify $\mathfrak{so}_3 \cong \mathfrak{so}_3^* \cong \mathbf{R}^3$. The Lie bracket goes to the cross product. Let I denote the isomorphism $\mathbf{R}^3 \xrightarrow{\sim} \mathbf{R}^3$ given by the diagonal matrix $\mathrm{diag}(I_1, I_2, I_3)$ for some nonzero I_1, I_2, I_3. If $M \in \mathfrak{so}_3$ corresponds to $(M_1, M_2, M_3) \in \mathbf{R}^3$, then $IM = (I_1M_1, I_2M_2, I_3M_3)$, and Corollary 5 gives the equations

$$\begin{aligned} I_1\dot{M}_1 &= (I_2 - I_3)M_2M_3 \\ I_2\dot{M}_2 &= (I_3 - I_1)M_2M_1 \\ I_3\dot{M}_3 &= (I_1 - I_2)M_1M_2. \end{aligned}$$

These are Euler's equations for rigid body dynamics; it describes what's known as the "Euler top".

Returning to the general story: the quantity ξ is the *intrisic velocity* of our particle on G. Note that the map $I : \mathfrak{g} \to \mathfrak{g}$ defines a composite

$$\mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g} \xrightarrow{I} \mathfrak{g},$$

where the isomorphism is given by κ. This composite can be identified with the adjoint to the bilinear form $\langle -, - \rangle$ on $\mathfrak{g}$. If you recall how the Legendre transform worked for quadratic Lagrangians, you'll see that the above map $\mathfrak{g}^* \to \mathfrak{g}$ is precisely the Legendre transform on $TG \cong G \times \mathfrak{g}$ for the Lagrangian (1). Therefore, $M = I^{-1}\xi$ can be regarded as an element of $\mathfrak{g}^*$, and it should be viewed as the *intrinsic momentum* of our particle on G. But it will be convenient to use the Killing form to identify $\mathfrak{g}^*$ and $\mathfrak{g}$ (otherwise I'd have to say a bit more about what the symbol $[IM, M]$ means).

Remark 7. The equation of Corollary 5 is in *Lax form*, meaning that it is a matrix differential equation consisting of two matrices $L(t)$, $A(t)$ (or more generally, paths in a Lie algebra) satisfying an equation

$$\dot{L} = [A, L].$$

We will have much more to say about Lax pairs in future lectures.

Example 8. Let us continue Example 6. If $M \in \mathfrak{so}_3$ corresponds to $(M_1, M_2, M_3) \in \mathbf{R}^3$, then

$$-\frac{1}{2}\,\mathrm{Tr}(M^2) = -\frac{1}{2}\,\mathrm{Tr}\begin{pmatrix} -M_2^2 - M_3^2 & M_1M_2 & M_1M_3 \\ M_1M_2 & -M_1^2 - M_3^2 & M_2M_3 \\ M_1M_3 & M_2M_3 & -M_1^2 - M_2^2 \end{pmatrix} = M_1^2 + M_2^2 + M_3^2 = \|M\|^2,$$

and in fact $\mathrm{Tr}(M^n)$ for $n \geq 3$ can be expressed as a polynomial in the above expression. This has a name in physics: it is the *magnitude of angular momentum.*

There is also another conserved quantity, namely the total energy

$$H(M) = \langle M, M\rangle = \kappa(M, I^{-1}M) = 2M \cdot I^{-1}M$$
$$= 2\left(\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}\right).$$

We therefore see that the trajectories can be described as intersections of level sets of the "Casimir" $-\frac{1}{2}\operatorname{Tr}(M^2)$ (which are spheres) with level sets of the Hamiltonian $\frac{H}{2}$ (which are "ellipsoids", i.e., quadric surfaces).

In fact, the discussion here is a special case of our work on symplectic reduction. Namely, recall that $IM = \xi = \dot\gamma\gamma^{-1}$. We claim:

Lemma 9 (Conservation of angular momentum)**.** *The quantity $\gamma^{-1}M\gamma$ is conserved (i.e., has vanishing derivative). This is called the* extrinsic angular momentum*.*

Proof. Note that since $IM = \dot\gamma\gamma^{-1}$, we have

$$\gamma^{-1}\dot{M}\gamma = \gamma^{-1}[IM, M]\gamma = \gamma^{-1}\dot\gamma\gamma^{-1}M\gamma - \gamma^{-1}M\dot\gamma.$$

But then

$$\partial_t(\gamma^{-1}M\gamma) = -\gamma^{-1}\dot\gamma\gamma^{-1}M\gamma + \gamma^{-1}\dot{M}\gamma + \gamma^{-1}M\dot\gamma,$$

and the outer two terms cancel the inner term out, so we are left with zero. □

Now, the fact that $\operatorname{Tr}(M^2)$ is conserved is just a consequence of the fact that M and $\gamma^{-1}M\gamma$ are conjugate. In any case, this *is* the moment map

$$\mu : T^*\mathrm{SO}_3 \to \mathbf{R}^3 \cong \mathfrak{so}_3^*,\ \mu(\gamma, M) = \gamma^{-1}M\gamma.$$

More generally, the map

$$T^*G \to \mathfrak{g}^*,\ (g, v) \mapsto \mathrm{Ad}_g(v)$$

is the moment map for the Hamiltonian action of G on T^*G coming from right-translation of G on itself.

Remark 10. Fix some (nonzero) $v \in \mathfrak{so}_3^*$ as a chosen value for our extrinsic angular momentum. Since v is nonzero, its stabilizer is SO_2, and so we get to consider the quotient $\mu^{-1}(v)/\mathrm{SO}_2$, which, by our discussion on symplectic reduction from last time, identifies precisely with the coadjoint orbit of v, i.e., the sphere of radius given by $\|v\| = \|M\|$. The Hamiltonian H descends to a Hamiltonian function on $\mu^{-1}(v)/\mathrm{SO}_2 = S^2_{\|v\|}$, and its level sets are precisely the aforementioned intersection of an ellipsoid with a sphere.

Note that this means that the 2-dimensional submanifold $V := \{H(q,p) = c,\ \mu(q,p) = v\} \subseteq T^*\mathrm{SO}_3$ has an SO_2-action, and the quotient by this action is the intersection of an ellipsoid with a sphere. In particular, V is diffeomorphic to an SO_2-bundle over (two) circles, at least if v is nonzero and $c > 0$. This identifies V with a 2-torus (or two copies of it). This observation is a special case of the Arnold-Liouville theorem on integrable systems (of which the Euler top is an example).

The above observations give us a lot of insights into the nature of the solutions to Euler's equations. We will explore these concepts in more detail once we have built up more tools, so our analysis for now will look a bit crude.

Let us begin by looking at the trajectory of our particle. The intersection of an ellipsoid with a sphere gives a curve which looks a bit like the boundary of a taco shell. What the intersection looks like will depend on the axes of the ellipsoid, i.e., the ordering of I_1, I_2, and I_3 given by their magnitudes. Namely, suppose $I_1 > I_2 > I_3$, corresponding (say) to the z, y, and x planes. Then the x-axis is called the "major axis", the y-axis is called the "intermediate axis", and the z-axis is called the "minor axis". (This is a quirk of the fact that $I_1 > I_2 > I_3$ means that $I_3^{-1} > I_2^{-1} > I_1^{-1}$, and these inverses are what show up in our Hamiltonian.)

The curve traced out by the momentum sphere intersecting the energy ellipsoid will stick close to the z and x axes. See, for instance, Figure 1. However, near the y-axis, the curve on the ellipsoid will go from one side of the sphere to the other (as shown in Figure 2). (I made these figures using GeoGebra.) This means that the axis of rotation will flip from one side of the sphere to the other if you perturb either the energy or the angular momentum. This is a very interesting observation, due to Poinsot from the 1830's, and it was made publicly famous in a demonstration by Soviet cosmonaut Vladimir Dzhanibekov. This is also sometimes known as the tennis racket theorem, and I will demonstrate it in class!

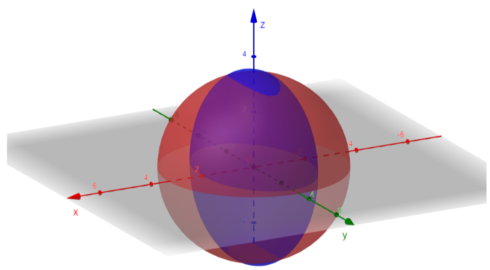

FIGURE 1. Intersection of momentum sphere and energy ellipsoid near the minor axis (in this case, the z-axis). Note that the curve traced out sticks close to the z-axis. The picture would look similar for the major axis (the ellipsoid would just be bigger).

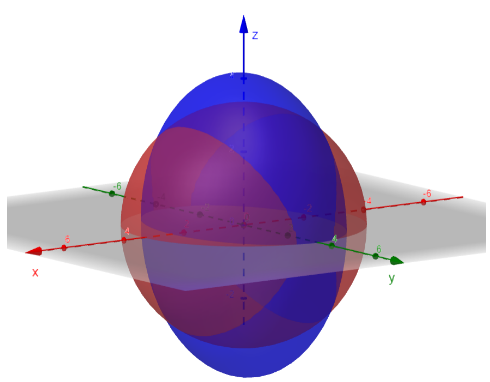

FIGURE 2. Intersection of momentum sphere and energy ellipsoid near the intermediate axis (in this case, the y-axis). Note that the curve traced out is a long arc, and does not stick close to the y-axis.

Let us now turn to the observation that since $\|M\|^2$ and H are conserved, we can express M_2 and M_3 in terms of M_1. This is sort of a gigantic mess, and you can try to write it out explicitly if you're so inclined. You'll find that since

$$M_3^2 = \|M\|^2 - M_1^2 - M_2^2,$$

one has

$$M_2^2 = \frac{I_2 I_3}{I_2 - I_3}\left(\frac{(I_3 - I_1)M_1^2}{I_1 I_3} - \frac{H}{2} + \frac{\|M\|^2}{I_3}\right).$$

Now using the equation

$$\dot{M}_1 = \frac{I_2 - I_3}{I_1} M_2 M_3,$$

you will finally see that M_1 satisfies an equation of the form

$$\dot{M}_1^2 = a + bM_1^2 + cM_1^4,$$

which means that it cuts out an *elliptic curve* in phase space. Could we have predicted, without gory calculation, why one gets an elliptic curve? The answer is yes, and we will discuss this in more detail when we talk about "spectral curves". This is also related to the question of whether we can express both H and $\|M\|^2$ as traces of powers of certain matrices (we were only able to get $\|M\|^2$, remember); this, too, will be discussed later.

References

1 Oxford St, Cambridge, MA 02139
Email address: sdevalapurkar@math.harvard.edu, February 23, 2024