Integrable systems

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Lecture 6: Lax pairs

In the previous lecture, we saw that geodesic motion on a semisimple compact Lie group G equipped a left-invariant metric (identified, by comparison to the Killing form, with a linear isomorphism $I : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$) can be described by the differential equation

$$\dot{M} = [IM, M]$$

for $M : \mathbf{R} \to \mathfrak{g}$. (This is the "Euler-Arnold equation".) This is an equation in *Lax form*, meaning that it is a matrix differential equation consisting of two matrices $L(t)$, $A(t)$ (or more generally, paths in a Lie algebra) satisfying an equation

$$\dot{L} = [A, L].$$

Many other Hamiltonian systems we have studied can be written in this form.

Remark 1. Before proceeding, I want to quickly remind you of something about our analysis of the Euler top/rigid body from last time. (I don't believe I emphasized this point in the lecture.) Recall that the moment map for the G-action on T^*G induced from the right G-action on G was given by

$$\mu : T^*G \cong G \times \mathfrak{g}^* \to \mathfrak{g}^*, \ (g, x) \mapsto \mathrm{Ad}_g(x).$$

When $G = \mathrm{SO}_3$, this map extracted the (extrinsic) angular momentum of our rigid body in $\mathbf{R}^3$. In general, if we fix a value $v \in \mathfrak{g}^*$ for the "angular momentum", our analysis from before can be viewed as describing Hamiltonian mechanics on the symplectic reduction $\mu^{-1}(v)/G_v$; but this is the coadjoint orbit of $v \in \mathfrak{g}^*$. In particular, when $G = \mathrm{SO}_3$, this coadjoint orbit is S^2 when $v \neq 0$. So our analysis on Tuesday could be understood as Hamiltonian mechanics on a symplectic manifold of *non-cotangent* type. In other words, such non-cotangent symplectic manifolds do in fact play an important role in classical mechanics.

Let's now return to the Lax pair story.

Example 2. Recall that the equations for the harmonic oscillator with Hamiltonian $H(q, p) = \frac{p^2+\omega^2 q^2}{2}$ were given by

$$\dot{q} = p, \ \dot{p} = -\omega^2 x.$$

Part of this work was done when the author was supported by NSF DGE-2140743.

Let $L = \left(\begin{smallmatrix} p & \omega q \\ \omega q & -p \end{smallmatrix}\right)$ and $A = \begin{pmatrix} 0 & -\omega/2 \\ \omega/2 & 0 \end{pmatrix}$. Then

$$\dot{L} = \begin{pmatrix} \dot{p} & \omega\dot{q} \\ \omega\dot{q} & -\dot{p} \end{pmatrix} = \begin{pmatrix} -\omega^2 q & \omega p \\ \omega p & \omega^2 q \end{pmatrix} = [A, L].$$

Note that $L^2 = (p^2 + \omega^2 q^2)\mathrm{id}_{\mathbf{R}^2}$, and so $\frac{1}{2}\operatorname{Tr}(L^2)$ is precisely the Hamiltonian, which is a conserved quantity!

The final observation above can be generalized: if you have a Lax pair, you automatically get a lot of conserved quantities. For example, in the case of the Euler-Arnold equation, we saw that $\operatorname{Tr}(M^2) = \|M\|^2$ was conserved (this is the total angular momentum). For a general Lax pair, these conserved quantities are given by the invariant polynomials in the eigenvalues of L. Namely:

Lemma 3. *The quantity* $\operatorname{Tr}(L^n)$ *is conserved for all* $n \geq 0$.

Proof. Indeed:

$$\begin{aligned} \partial_t \operatorname{Tr}(L^n) &= n \operatorname{Tr}(L^{n-1}\dot{L}) = n \operatorname{Tr}(L^{n-1}[A, L]) \\ &= \operatorname{Tr}([A, L^n]) = 0, \end{aligned}$$

because the trace of any commutator vanishes. □

Remark 4. Actually, more is true: the eigenvalues of L are conserved. Indeed, suppose we define an invertible matrix $U(t)$ by the equation $\dot{U} = AU$, where $U(0) = \mathrm{id}$. Then the solution of the Lax equation with initial value $L(0)$ is $L(t) = U(t)L(0)U(t)^{-1}$; since $L(t)$ is conjugate to $L(0)$, the eigenvalues of $L(t)$ must be the same as those of $L(0)$, i.e., they are conserved. Indeed, since $\partial_t(U^{-1}) = -U^{-1}\dot{U}U^{-1}$, we have:

$$\begin{aligned} \dot{L}(t) &= \partial_t(UL(0)U^{-1}) \\ &= \dot{U}L(0)U^{-1} - UL(0)U^{-1}\dot{U}U^{-1} \\ &= AUL(0)U^{-1} - UL(0)U^{-1}AUU^{-1} \\ &= AUL(0)U^{-1} - UL(0)U^{-1}A = AL - LA = [A, L]. \end{aligned}$$

This is not really relevant to our discussion below, but it's a nice piece of mathematics: how does one construct this matrix $U(t)$? (It implements "time evolution".) This can be done using "Dyson's formula". The idea is simple: $U(0) = \mathrm{id}$, so

$$U(t) = \mathrm{id} + \int_0^t A(t_1)U(t_1)dt_1.$$

Iterating,

$$U(t) = \mathrm{id} + \int_0^t A(t_1)\left(\mathrm{id} + \int_0^{t_1} A(t_2)U(t_2)dt_2\right)dt_1,$$

and so on. Now, define the *time-ordering* operator by

$$\mathcal{T}(A(t_1)A(t_2)) = \begin{cases} A(t_1)A(t_2) & t_1 \geq t_2 \\ A(t_2)A(t_1) & t_1 \leq t_2. \end{cases}$$

Then

$$\int_0^t \int_0^{t_1} A(t_1)A(t_2)dt_2dt_1 = \frac{1}{2}\int_0^t \int_0^t \mathcal{T}(A(t_1)A(t_2))dt_2dt_1,$$

and more generally, one finds that

$$U(t) = \sum_{n\geq 0} \frac{1}{n!} \int_0^t \cdots \int_0^t \mathcal{T}(A(t_1)\cdots A(t_n))dt_1 \cdots dt_n.$$

This is sometimes called a *time-ordered exponential*, and is denoted

$$U(t) = \mathcal{T}\exp\left(\int_0^t A(\tau)d\tau\right).$$

This formal manipulation plays an important role in quantum mechanics.

More generally, if L and A are elements of $\mathfrak{g}$, then any invariant polynomial in $\mathfrak{g}$ (applied to L) will be conserved under the equation $\dot{L} = [L, A]$, where $[L, A]$ is the Lie bracket in $\mathfrak{g}$. This can be shown by exactly the same argument as above.

Remark 5. If we can write a Hamiltonian system in Lax form, such a representation is not unique! It can fail to be unique in many ways. For one, you could have a representation by $n \times n$-matrices for different values of n. Another example is that one can shift A by any polynomial in L, and the resulting pair $(L, A + f(L))$ would still be a Lax pair representation. One could also *conjugate* L and A: namely, if $L, A : \mathbf{R} \to \mathfrak{g}$ and $\mathfrak{g}$ is the Lie algebra of a Lie group G, then any $g : \mathbf{R} \to G$ defines

$$L' = \mathrm{Ad}_g(L),\ A' = \mathrm{Ad}_g(A) + \dot{g}g^{-1},$$

and one has

$$\begin{aligned}\dot{L}' &= \dot{g}Lg^{-1} + g\dot{L}g^{-1} - gLg^{-1}\dot{g}g^{-1}\\ &= \dot{g}Lg^{-1} + g[A, L]g^{-1} - gLg^{-1}\dot{g}g^{-1}\\ &= [gAg^{-1} + \dot{g}g^{-1}, gLg^{-1}] = [A', L'].\end{aligned}$$

If $\mathfrak{g}$ is semisimple of rank n, then there will be n linearly independent invariant polynomials on $\mathfrak{g}$. This is a consequence of the following results of Chevalley and Chevalley-Shephard-Todd. (I might prove these results later, if there is interest.)

Proposition 6 (Chevalley restriction theorem)**.** *Let $T \subseteq G$ be a maximal torus with Lie algebra $\mathfrak{t}$, and let $W = N_G(T)/T$ denote its Weyl group. Then there is an isomorphism* $\mathrm{Sym}(\mathfrak{g})^G \cong \mathrm{Sym}(\mathfrak{t})^W$.

For example, if $\mathfrak{g} = \mathfrak{gl}_n$, one can identify W acting on $\mathfrak{t}$ with the symmetric group Σ_n acting on $\mathbf{C}^n$, and the statement is that the only GL_n-invariant polynomials in the entries of an $n \times n$-matrix A are given by symmetric polynomials in the eigenvalues of A. These polynomials can be identified with $\frac{1}{j!}\mathrm{Tr}(A^j)$ for $1 \leq j \leq n$ (up to a change of basis for the space of symmetric polynomials). One could, of course, consider $\mathrm{Tr}(A^j)$ with $j > n$; but this would be expressible as a linear combination of the polynomials already constructed. In general, the G-invariant polynomials in $\mathfrak{g}$ are sometimes referred to as *Casimirs*.

Proposition 7 (Chevalley-Shephard-Todd)**.** *The algebra* $\mathrm{Sym}(\mathfrak{t})^W$ *is a* polynomial *ring (with the number of generators given by* $\dim(\mathfrak{t})$*, and the inclusion* $\mathrm{Sym}(\mathfrak{t})^W \subseteq \mathrm{Sym}(\mathfrak{t})$ *exhibits* $\mathrm{Sym}(\mathfrak{t})$ *as a free* $\mathrm{Sym}(\mathfrak{t})^W$*-module of rank* $|W|$.

For example, for Σ_n acting on $\mathbf{C}^n$, this is the statement that the algebra $\mathbf{C}[x_1, \cdots, x_n]^{\Sigma_n}$ is polynomial in n generators; the polynomial generators of this invariant ring can be taken to be the elementary symmetric polynomials in x_i.

The above discussion is somewhat deficient in the example of the rigid body system/Euler-Arnold equation. Indeed, if we view $M = (M_1, M_2, M_3) \in \mathbf{R}^3$ as the corresponding matrix

$$M = \begin{pmatrix} 0 & -M_3 & M_2 \\ M_3 & 0 & -M_1 \\ -M_2 & M_1 & 0 \end{pmatrix} \in \mathfrak{so}_3,$$

then $\frac{1}{2}\operatorname{Tr}(M^2)$ recovers $\|M\|^2$, but $\operatorname{Tr}(M^n)$ is either zero (if n is odd) or is a function of $\|M\|^2$. In particular, the Casimirs do not produce all the conserved quantities of the Euler-Arnold system: we are missing the Hamiltonian itself! It's impossible to get the Hamiltonian just from M, because we'd be missing the moment of inertia $I = (I_1, I_2, I_3)$. However, remarkably enough, it turns out that there is a way to modify our Lax pair so as to get both the Hamiltonian and $\|M\|^2$ as traces of powers of our matrices.

Let me try to motivate the Lax pair we will write down as follows. It will be convenient to work with a slight variant of I: namely, let

$$\begin{aligned} \mathcal{I}_1 &= \tfrac{I_2^{-1}+I_3^{-1}-I_1^{-1}}{2}, \\ \mathcal{I}_2 &= \tfrac{I_3^{-1}+I_1^{-1}-I_2^{-1}}{2}, \\ \mathcal{I}_3 &= \tfrac{I_1^{-1}+I_2^{-1}-I_3^{-1}}{2}. \end{aligned}$$

Let $\mathcal{I}$ denote the diagonal matrix $\operatorname{diag}(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$. Then, you can check that

$$M = \mathcal{I}(IM) + (IM)\mathcal{I}.$$

Lemma 8. *One has*

$$[IM, \mathcal{I}^2] + [\mathcal{I}, M] = 0.$$

Proof. Indeed,

$$\begin{aligned} [IM, \mathcal{I}^2] + [\mathcal{I}, M] &= (IM)\mathcal{I}^2 - \mathcal{I}^2(IM) + \mathcal{I}M - M\mathcal{I} \\ &= (IM)\mathcal{I}^2 - \mathcal{I}^2(IM) + \mathcal{I}(\mathcal{I}(IM) + (IM)\mathcal{I}) - (\mathcal{I}(IM) + (IM)\mathcal{I})\mathcal{I} \\ &= 0. \end{aligned}$$ □

We would like to modify our M in such a way that it knows about the moment of inertia; that is, by adding $\mathcal{I}$ to M. This will mess up the commutator $[IM, M]$, and so we have to modify IM, too. Here is how to do this; we will introduce a new parameter λ, called the *spectral parameter*. Assume that all the $\mathcal{I}_j$ are distinct, and let

$$L(\lambda) = \mathcal{I}^2 + \lambda^{-1}M, \;\; A(\lambda) = \lambda\mathcal{I} + IM.$$

See **[Man76]**. Then:

$$\begin{aligned} \dot{L}(\lambda) = \lambda^{-1}\dot{M} &= \lambda^{-1}[IM, M] \\ &= \lambda^{-1}[IM, M] + [IM, \mathcal{I}^2] + [\mathcal{I}, M] \\ &= [\lambda\mathcal{I} + IM, \mathcal{I}^2 + \lambda^{-1}M] \\ &= [A(\lambda), L(\lambda)]. \end{aligned}$$

So the equation described by the Lax pair $L(\lambda)$ and $A(\lambda)$ is exactly the same as the Euler-Arnold equation. The above pair $(L(\lambda), A(\lambda))$ is much better for describing the conserved quantity in the Euler-Arnold system. Indeed:

Proposition 9. *One has*

$$\operatorname{Tr}(L(\lambda)^2) = \operatorname{Tr}(\mathcal{J}^4) - \frac{2}{\lambda^2}\|M\|^2$$
$$\operatorname{Tr}(L(\lambda)^3) = \operatorname{Tr}(\mathcal{J}^6) - \frac{3}{\lambda^2}\left(\frac{(\operatorname{Tr}\mathcal{J})^2\|M\|^2}{4} - I_1I_2I_3H\right).$$

Proof. Exercise. □

Since we know $\mathcal{J}$ beforehand, the symmetric polynomials $\operatorname{Tr}(L(\lambda)^2)$ and $\operatorname{Tr}(L(\lambda)^3)$ can be used to recover $\|M\|^2$ and H. Note that all the higher traces $\operatorname{Tr}(L(\lambda)^j)$ for $j > 3$ can be recovered from these two. Another way of saying this is that:

Proposition 10. *The coefficients of η in the characteristic polynomial $p(\eta) = \det(L(\lambda) - \eta\mathrm{id})$ are all conserved quantities.*

So, what is this polynomial?

finish; elliptic curve, and describe relation to euler's solution via elliptic integrals

Remark 11. Just as with the case of SO_3, one can rewrite the Euler-Arnold equation for $G = \mathrm{SO}_n$ using a Lax pair $(L(\lambda), A(\lambda))$; and this will actually produce all the conserved quantities. If you replace $\mathcal{J}$ by a diagonal matrix, let $IM \in \mathfrak{so}_n$, and set $M = \mathcal{J}(IM) + (IM)\mathcal{J}$. Note that $M^T = -M$, so M is still in $\mathfrak{so}_n$. The analogously defined pair $(L(\lambda), A(\lambda))$ will have Lax equation given by the Euler-Arnold equation for SO_n. So, what are the conserved quantities here? They will come from the coefficients of λ^i in $\operatorname{Tr}(L(\lambda)^j)$ for $j = 2, \cdots, n$.

show that there are $\frac{1}{2}\left(\binom{n}{2} - \lfloor \frac{n}{2} \rfloor\right)$ many conserved quantities. This is half the dimension of the regular nilpotent orbit in SO_n

Remark 12. You could ask: what is the physical meaning of this spectral parameter λ? It does not have any; you could think of it as something like a Lagrange multiplier in a variational problem, which does not have any physical meaning (because it does not appear as a variable in the equations of motion).

References

[Man76] S. V. Manakov. A remark on the integration of the Eulerian equations of the dynamics of an n-dimensional rigid body. *Funkcional. Anal. i Priložen.*, 10(4):93–94, 1976.

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