Integrable systems

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Lecture 7: Integrability

Let us leave Lax pairs aside for a little bit and return to our study of Hamiltonian systems. We saw briefly in the previous lecture that writing a system in Lax form automatically led to a bunch of conserved quantities. It turns out that there are many interesting examples of Hamiltonian systems with lots of conserved quantities, and these are what are called *integrable systems*. I want to introduce this concept today, and explore some examples in the next few lectures.

Recall from a previous lecture that when we had a Hamiltonian system (on $\mathbf{R}^{2n}$) with a Hamiltonian symmetry (by a vector field), we could reduce the number of degrees of freedom by 2, to get a Hamiltonian system in $2n-2$ variables. Therefore, one could hope to keep reducing symmetries in this way by continually reducing along symmetries. As we saw in the lectures on Hamiltonian reduction, it is necessary that these vector field symmetries commute with one another in order to do this reduction. In other words, if we have a Hamiltonian action of an n-torus T^n on our symplectic manifold, then we get a moment map $\mu : M \to (\mathfrak{t}^n)^* \cong \mathbf{R}^n$; then the Hamiltonian reduction $\mu^{-1}(v)/T^n$ will be a zero dimensional (symplectic, but that's meaningless here) manifold, and we can solve the Hamiltonian system entirely by just doing a bunch of integrals. In fact, this is exactly what we will do; to formalize this, we need a definition.

Definition 1. A (Liouville) *integrable system* on a symplectic manifold M of dimension $2n$ is a collection $F_1, \cdots, F_n : M \to \mathbf{R}$ of n *functionally independent* smooth functions on M which Poisson commute, i.e., $\{F_i, F_j\} = 0$. Here, "functionally independent" means that you cannot express F_j in terms of each other, i.e., the differentials $d_x F_j \in T_x^* M$ are linearly independent at each point $x \in M$.

You might as well treat F_1 as a Hamiltonian function $M \to \mathbf{R}$, and then you get a Hamiltonian differential equation

$$\dot{f} = \{H, f\}$$

on M.

Example 2. If M is a 2-dimensional symplectic manifold and $H : M \to \mathbf{R}$ is a function, then it is automatically integrable in the above sense. Let us recall the

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example when $M = T^*\mathbf{R}$ and $H(q,p) = \frac{1}{2}(p^2 + \omega^2 q^2)$. We saw in the very first lecture that the Hamiltonian system can be solved by

$$q = \sqrt{H}\sin(\omega t - t_0),\ p = \sqrt{H}\omega\cos(\omega t - t_0),$$

and the function H is conserved. In the phase space $T^*\mathbf{R}$, H is fixed (it is the area of the ellipse in question), and the particle moves along this ellipse. The angle coordinate on this ellipse is $\theta = \omega t - t_0$. Since H is the area of this ellipse, it can be obtained as

$$H = \oint pdq.$$

Actually, this is something quite general for integrable systems. Let us just look at the 2-dimensional case first, and then generalize to arbitrary dimension. Suppose M is a 2-dimensional symplectic manifold and $H : M \to \mathbf{R}$ is a function. Let $v \in \mathbf{R}$ be such that $H^{-1}(v) =: M_v$ is smooth, compact, and connected. Suppose that M_v contains no critical points of H. Then:

- M_v is diffeomorphic to a circle. This is just because the only smooth, compact, and connected 1-manifold is a circle.
- The Hamiltonian flow on M_v is simply given by $\dot{\theta} = 0$, i.e., $\theta(t) = \theta_0 + ct$ for some constants θ_0 and c. To see this, consider the Hamiltonian vector field X_H. It is tangent to M_v, because the derivative of H in the direction of X_H vanishes:

$$\langle dH, X_H\rangle = \{X_H, X_H\} = 0.$$

Also, X_H vanishes on M_v if and only if $\omega(X_H, -) = dH$ vanishes on M_v. But we assumed that M_v contains no critical points of H, and so dH (and hence X_H) cannot vanish on M_v. This means, by the first part, that X_H is a nonvanishing vector field on a circle.

But it is now a simple fact about circles that if you have a nonvanishing vector field on M_v, then there is some choice of coordinate $\theta : \mathbf{R}/\mathbf{Z} \xrightarrow{\sim} M_v$ such that $X_H = \partial_\theta$. Indeed, suppose we write $X_H = f(x)\partial_x$. If we express $x = x(\theta)$, then we're asking that

$$\partial_\theta = f(x)\partial_x = f(x)\frac{dx}{d\theta}\partial_\theta,$$

or in other words that $d\theta = \frac{dx}{f(x)}$. So we may take $\theta = \int \frac{dx}{f(x)}$. Since $X_H|_{M_v} = \partial_\theta$ is the constant vector field, and the Hamiltonian flow preserves X_H, this means that $\dot{\theta} = c$ for some constant c, as desired.

Essentially the same argument will prove the following:

Theorem 3 (Arnold-Liouville). *Suppose $\mu = (F_1, \cdots, F_n) : M \to \mathbf{R}^n$ is a Liouvlle integrable system on a $2n$-dimensional symplectic manifold. Let $H = F_1$, so that we obtain a Hamiltonian system on M. Then for any given $v \in \mathbf{R}^n$:*

- *The fiber $\mu^{-1}(v) =: M_v$ is a smooth manifold.*
- *If M_v is compact and connected, it is diffeomorphic to an n-torus with coordinates $\theta_1, \cdots, \theta_n$.*
- *There is an open neighborhood U of v, and a coordinate transformation $(q_j, p_j) \mapsto (\theta_j, I_j)$ called* action-angle coordinates *such that Hamilton's equation is given by*

$$\dot{\theta}_j = \omega_j(v),\ \dot{I}_j = 0$$

where $\omega_j(v)$ is some function of v.

Proof. Suppose X is a compact, connected, smooth (closed) n-manifold which admits n pairwise commuting and linearly independent vector fields. Then X is diffeomorphic to an n-torus. Here is a sketch of why this is true: if $\xi_1, \cdots, \xi_n$ are the aforementioned vector fields, the flows $f_{1,t}, \cdots, f_{n,t}$ of these vector fields define an action of $\mathbf{R}^n$ on X. In other words, each $x \in X$ defines a map $g_x : \mathbf{R}^n \to X$. The image of this map is the orbit of x under the $\mathbf{R}^n$-action.

Since the vector fields $\xi_1, \cdots, \xi_n$ are linearly independent at each point, the map $\mathbf{R}^n \to X$ is a local diffeomorphism. Therefore, its image is an open subset of X. The $\mathbf{R}^n$-orbits of points of X therefore define disjoint open subsets of X, and since X is connected, it must consist of a single orbit. If Γ denotes the stabilizer of $x \in X$, then we get a diffeomorphism $\mathbf{R}^n/\Gamma \xrightarrow{\sim} X$. But Γ is necessarily a discrete subgroup of $\mathbf{R}^n$, because the map $\mathbf{R}^n/\Gamma \to X$ induces an isomorphism on tangent spaces. It is now an algebraic lemma that every discrete subgroup of $\mathbf{R}^n$ is isomorphic to a lattice $\mathbf{Z}^j$ for $j \leq n$.

Here is how you'd see this. Suppose first that $n = 1$, so we're looking at discrete subgroups of $\mathbf{R}$. Take some nonzero vector $e_1 \in \Gamma$ with smallest length. If there was $w \in \Gamma$ such that $w = re_1$ with $j < r < j+1$ (for $j \in \mathbf{Z}$), then $w - je_1$ would have length strictly smaller than that of e_1, which is a contradiction. So $\Gamma = \mathbf{Z}e_1$. Now suppose $n = 2$, and let $e_1 \in \Gamma \subseteq \mathbf{R}$ be nonzero of minimal length. If $\Gamma \subseteq \mathbf{R}e_1$, the case $n = 1$ tells us that $\Gamma = \mathbf{Z}e_1$. So suppose Γ is not contained in $\mathbf{R}e_1$. Then take some e_2 of minimal (nonzero) distance from $\mathbf{R}e_1$; we claim $\Gamma = \mathbf{Z}\{e_1, e_2\}$. Indeed, if w is not contained in $\mathbf{Z}\{e_1, e_2\}$, then we can translate w by some $\mathbf{Z}$-linear combination of e_1 and e_2 to get a vector w' which is of smaller distance from $\mathbf{R}e_1$, which is a contradiction. (The case of general n follows similarly by induction.)

Anyway, since $\Gamma = \mathbf{Z}^j$, this means that

$$T^j \times \mathbf{Z}^{n-j} \cong \mathbf{R}^n/\mathbf{Z}^j \xrightarrow{\sim} X.$$

But X is compact, and so $j = n$, i.e., X is an n-torus.

In order to use this, we need to show that the fiber M_v admits n pairwise commuting and linearly independent vector fields. We already have a candidate for these: namely, we would like to restrict $X_{F_1}, \cdots, X_{F_n}$ to M_v. To do this, we need to know that these vector fields are tangent to M_v. But the derivative of F_i in the direction of X_{F_j} is zero because

$$\langle dF_i, X_{F_j} \rangle = \{F_i, F_j\} = 0,$$

which means that X_{F_j} is tangent to M_v.

Remark 4. Actually, we have shown a bit more: the entire symplectic form on M vanishes when restricted to the tangent space of M_v. This is because the functions F_j are linearly independent, i.e., X_{F_j} span the tangent spaces of M_v, and $\omega(X_{F_i}, X_{F_j}) = \{F_i, F_j\} = 0$. So $M_v \subseteq M$ is a half-dimensional torus on which the symplectic form restricts to zero; it is sometimes called a *Lagrangian torus*.

How about the "action-angle coordinates" we were after? I will just consider the case when $M = T^*\mathbf{R}^n$, for simplicity. These action-angle coordinates are not some random change of coordinates: they are going to be related to the standard symplectic form $\omega = \sum_j dq_j \wedge dp_j$ by

$$\omega = \sum_j dI_j \wedge d\theta_j.$$

In other words, the transformation $(q,p) \mapsto (I,\theta)$ is *canonical*. Motivated by our discussion for the harmonic oscillator, let us fix some integral basis $\gamma_1, \cdots, \gamma_n$ of 1-cycles on M_v which generate $\mathrm{H}_1(M_v; \mathbf{Z})$. (We can do this because M_v is a torus). Recall that $F_j(q,p) = v_j$; we can solve for p to express each p_j as a function $p_j(v,q)$. Then, define

$$I_j(v) = \frac{1}{2\pi} \oint_{\gamma_j} \lambda,$$

where $\lambda = \sum_i p_i dq_i$ is the canonical 1-form. Note that we have integrated out the q-dependence of p_i, and so $I_j(v)$ really is just a function of v. But $v \in \mathbf{R}^n$ is t-independent, and so the same is true of the I_j; i.e.,

$$\dot{I}_j = 0.$$

Now, whatever the θ_j are, they have to satisfy the property that

$$\sum_j dI_j \wedge d\theta_j = \sum_j dq_j \wedge dp_j = \omega.$$

Equivalently, we need some function S such that

$$dS = \sum_j p_j dq_j + \theta_j dI_j.$$

This suggests finding $S = S(I,q)$ such that $p_j = \frac{\partial S}{\partial q_j}$, and then we could define

$$\theta_j = \frac{\partial S}{\partial I_j}.$$

Let us just take

$$S = \int_{q_0}^{q} \sum_i p_i(q', I) dq_i',$$

where $q_0 \in M_v$ is some point (in the neighborhood of which we're trying to construct a coordinate), and the integral is taken over some path q_0 to q; and define θ_j as above. The function S is sometimes called a *generating function* for the canonical transformation $(q,p) \mapsto (I,\theta)$.

How well-defined is θ_j? We can answer this by thinking about how the integral defining S depends on the choice of path q_0 to q. Let's close up the path, so we consider a path γ from q_0 to q back to q_0. If we tack this path onto a chosen path from q_0 to q, then S varies by

$$\Delta_\gamma S = \int_\gamma \lambda.$$

Since the γ_i form a basis for $\mathrm{H}_1(M_v; \mathbf{Z})$, this integral depends only on $I_j = \int_{\gamma_j} \lambda$. Given the above variation in S, we see that θ_j varies by

$$\Delta_\gamma \theta_j = \frac{\partial}{\partial I_j} \Delta_\gamma S = \frac{\partial}{\partial I_j} \int_\gamma \lambda.$$

In particular, we see that

$$\Delta_{\gamma_i}\theta_j = \frac{\partial}{\partial I_j} I_i = 2\pi\delta_{ij}.$$

In particular, the θ_j define independent "angle" coordinates on the generating cycles γ_j, and hence form coordinates on M_v. Moreover, under this canonical change of coordinates, we have

$$\begin{aligned} \dot{I}_j &= -\frac{\partial H}{\partial \theta_j} = 0, \\ \dot{\theta}_j &= \frac{\partial H}{\partial I_j} =: \omega_j(I) = \omega_j(v), \end{aligned}$$

as desired. □

Exercise 5. For the harmonic oscillator with $H = \frac{1}{2}(p^2 + \omega^2 q^2)$, so that $p(H, q) = \sqrt{2H - \omega^2 q^2}$, one has

$$I = \frac{1}{2\pi}\oint_E \sqrt{2H - \omega^2 q^2}dq = \frac{1}{2\pi}\int_{-\sqrt{2H}/\omega}^{\sqrt{2H}/\omega} \sqrt{2H - \omega^2 q^2}dq = \frac{H}{\omega}.$$

Show that

$$\theta(I, q) = \arctan\left(\frac{\omega q}{\sqrt{2I\omega - \omega^2 q^2}}\right).$$

Verify also that $dq \wedge dp = dI \wedge d\theta$.

Remark 6. It follows from Theorem 3 that the dynamics of the integrable system is *periodic*.

Remark 7. The only manipulations used in the proof involved inverting diffeomorphisms, and integrating. The original system of equations is often said to be *integrated by quadratures*. Obviously, once one finds I and θ, the entire integrable system is solved.

I will discuss the example of the *Kepler problem* next time.

References

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