Integrable systems

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Problem Set

Here are some problems to try out. I will keep adding more problems until the end of the semester; please submit at least $\lceil n/3 \rceil$ of them (where n is the number of problems below)! Some of these problems are easier than others.

(1) something about legendre transform. I'll add this later

(2) In class, we constructed a symplectic form on complex projective space $\mathbf{C}P^{n-1}$ by Hamiltonian reduction of the "diagonal" S^1-action on $T^*\mathbf{R}^n \cong \mathbf{C}^n$. (That is, $\lambda \in S^1$ sends $(z_1, \cdots, z_n) \mapsto (\lambda z_1, \cdots, \lambda z_n)$.) Namely, $\mathbf{C}P^{n-1} = \mu^{-1}(1/2)/S^1$. Let us try to describe this more explicitly. What our formula tells us is that if $\omega_{\mathbf{C}P^{n-1}}$ and $\omega_{\mathbf{C}^n}$ are the symplectic forms on $\mathbf{C}P^{n-1}$ and $\mathbf{C}^n$, respectively, and $q : S^{2n-1} \to \mathbf{C}P^{n-1}$ and $i : S^{2n-1} \subseteq \mathbf{C}^n$ are the natural maps, then

$$i^*\omega_{\mathbf{C}^n} = q^*\omega_{\mathbf{C}P^{n-1}}.$$

Now:

- Let $\omega' = \frac{\omega_{\mathbf{C}^n}}{\|z\|^2}$. Show that $i^*\omega' = i^*\omega_{\mathbf{C}^n}$, and that ω' is $\mathbf{C}^\times$-invariant (the action being by rescaling).
- Recall that $\mathbf{C}P^{n-1} = (\mathbf{C}^n - \{0\})/\mathbf{C}^\times$. Using the previous part, show that if I use homogeneous coordinates $[z_1 : \cdots : z_n]$ for $\mathbf{C}P^{n-1}$, then

$$\omega_{\mathbf{C}P^{n-1}} = \frac{i}{2}\sum_{j,k} \frac{z_j\overline{z_k}}{\|z\|^4} dz_j \wedge d\overline{z_k}.$$

  This might not be exactly right... (You'll get full credit if you do it for $\mathbf{C}P^2$!)
- When $n = 2$, we can identify $\mathbf{C}P^1 = S^2$. Show that $\omega_{\mathbf{C}P^1}$ is just the area form for S^2.

(3) If you haven't done the previous part, take as given the calculation of $\omega_{\mathbf{C}P^{n-1}}$ stated above. Let U_0 denote the locus of points $[z_1 : \cdots : z_n] \in \mathbf{C}P^{n-1}$ with $z_1 \neq 0$. Then U_0 can be identified with $\mathbf{C}^{n-1}$ via $[z_1 : \cdots : z_n] \mapsto (z_2/z_1, \cdots, z_n/z_1) =: y$, so we get an inclusion $i : \mathbf{C}^{n-1} \subseteq \mathbf{C}P^{n-1}$. What is $i^*\omega_{\mathbf{C}P^{n-1}}$ in terms of y?

Here is another way to construct this 2-form on $U_0 \cong \mathbf{C}^{n-1}$ that you can write down explicitly. Let $\|y\|$ denote the norm of y, and let $\partial = \sum \partial_{y_j}$

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and $\overline{\partial} = \sum \partial_{\overline{y_j}}$ denote the holomorphic and antiholomorphic derivatives. Use your calculation above to show that

$$i^* \omega_{\mathbf{C}P^{n-1}} = \frac{i}{2} \partial \overline{\partial} \log(1 + \|y\|^2).$$

The symplectic form $\omega_{\mathbf{C}P^{n-1}}$ is called the *Fubini-Study form.*

(4) Equip $\mathbf{C}^3$ with the symplectic form $\frac{i}{2} \sum_{j=1}^{3} dz_j \wedge d\overline{z_j}$. What is this symplectic form if I view $\mathbf{C}^3$ as $\mathbf{R}^6$? Consider the action of T^2 where $(\lambda_1, \lambda_2) \in T^2$ sends

$$(z_1, z_2, z_3) \mapsto (\lambda_1 z_1, \lambda_2 z_2, \lambda_1^{-1} \lambda_2^{-1} z_3).$$

Is this action Hamiltonian? What is the moment map? More generally, say that the compact torus T^n acts on $\mathbf{C}^n$ by

$$(z_j) \mapsto (\lambda_j^{a_j} z_j)$$

for some integers a_j. What is the moment map for this action?

Can you write down an example of a group acting on a symplectic manifold (M, ω) by symplectomorphisms, but which is *not* Hamiltonian? As a complement to this, show that if M is simply-connected, then every group action by symplectomorphisms is Hamiltonian.

(5) There are a number of "exceptional isomorphisms" in small dimensions. Let $\mathfrak{sl}_n$ denote the Lie algebra of $n \times n$-matrices with zero trace, and let $\mathfrak{so}_n$ denote the Lie algebra of $n \times n$-matrices A such that $A + A^T = 0$. Show that there is an isomorphism $\mathfrak{sl}_2 \cong \mathfrak{so}_3$. (Hint: you want to exhibit $\mathfrak{sl}_2$ as a subalgebra of the Lie algebra $\mathfrak{gl}_3$ of 3×3-matrices, so you want a map $\mathfrak{sl}_2 \to \mathfrak{gl}_3$. To do this, consider the action of $\mathfrak{sl}_2$ on itself by the Lie bracket, i.e., $y \mapsto [x, y]$.) Is this isomorphism true at the level of Lie groups?

Let $\mathfrak{sp}_2$ denote the Lie algebra of the group Sp_2 of automorphisms of $\mathbf{R}^2$ preserving the symplectic form. Can you identify $\mathfrak{sp}_2 \cong \mathfrak{sl}_2$? Is this isomorphism going to hold at the level of Lie groups?

(6) Can you identify $\mathfrak{so}_4$ with $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$? (Hint: again, you want to exhibit $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ as a subalgebra of the Lie algebra $\mathfrak{gl}_4$ of 4×4-matrices, so you want a map $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{gl}_4$. To do this, consider the action of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ on $\mathfrak{gl}_2$ by left and right multiplication.)

You showed in the first part that $\mathfrak{sl}_2 \cong \mathfrak{so}_3$, and also that $\mathfrak{so}_4 \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$. There is an embedding $\mathfrak{so}_3 \subseteq \mathfrak{so}_4$ (just add zeros at the end of your 3×3-matrix). If you combine the isomorphisms above, you get a map

$$\mathfrak{so}_3 \subseteq \mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3.$$

Can you identify this map?

(7) Let A be a symplectic matrix acting on $\mathbf{R}^{2n}$, and let $f(t)$ denote its characteristic polynomial. Show that $f(t) = t^{2n} f(1/t)$.

(8) Suppose X is a smooth manifold, and let G be a compact Lie group acting freely on X. Then G acts on T^*X in a Hamiltonian way, and so we get a moment map $\mu : T^*X \to \mathfrak{g}^*$. Suppose $0 \in \mathfrak{g}^*$ is a regular value of μ and that G acts freely on $\mu^{-1}(0)$. What is $\mu^{-1}(0)/G$ in terms of X/G?

References

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