REPRESENTATIONS OF FROBENIUS KERNELS

These are notes for a talk that I was coerced into giving for the Langlands support group on modular representation theory. I am certain that this talk will go horribly — there are too many indices flying around! I should've paid more attention to how the physicists teach general relativity.

Let $S = \mathrm{Spec}(k)$ with k a perfect field of characteristic p. Recall that if X is an S-scheme, then the pullback diagram

$$\begin{tikzcd}
X \arrow[dr, "F_{X/S}"] \arrow[drr, "F_X"] \arrow[ddr] & & \\
& X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\
& S \arrow[r, "\mathrm{Frob}"'] & S
\end{tikzcd}$$

defines the absolute Frobenius F_X and the relative Frobenius $F_{X/S} : X \to X^{(p)}$. In this talk, we will be concerned with the relative Frobenius, so we shall simply denote it by $F : X \to X^{(p)}$.

Definition 1. Let G be an S-group, and let $F^n : G \to G^{(p^n)}$ denote the nth iterate of the relative Frobenius. The nth Frobenius kernel of G is the S-subgroup of G defined by $G_n = \ker(F^n)$.

Remark 2. There are inclusions $G_n \subseteq G_{n+1}$, and that $(G_n)_{n'}$ is either $G_{n'}$ (if $n' \leq n$) or G_n (if $n \leq n'$).

Example 3. Suppose $S = \mathrm{Spec}(k)$, and let $G = \mathbf{G}_m = \mathrm{Spec}(k[t^{\pm 1}])$. Then $F : \mathbf{G}_m \to \mathbf{G}_m^{(p)}$ sends t to t^p, and so $(\mathbf{G}_m)_n = \mathrm{Spec}\, k[t]/(t^{p^n} - 1) = \mu_{p^n}$. Similarly, if $G = \mathbf{G}_a = \mathrm{Spec}(k[t])$, then $(\mathbf{G}_a)_n = \mathrm{Spec}\, k[t]/t^{p^n} = \alpha_{p^n}$.

Remark 4. If G is a reduced affine S-group scheme, then F^n induces an isomorphism $G/G_n \to G^{(p^n)}$. Indeed, this follows from the fact that G/G_n is the closed subgroup of $G^{(p^n)}$ given by the kernel of $k[G]^{(p^n)} \to k[G]$ sending f to f^{p^n}.

Last time, Kevin talked about the equivalence of categories between finite S-group schemes and finite-dimensional cocommutative Hopf k-algebras.

Example 5. Let $\mathfrak{g}$ be a restricted Lie algebra (so there is a map $X \mapsto X^{[p]}$, referred to as a p-derivation). If $\mathfrak{g}$ is the Lie algebra of an algebraic group over k, then the pth power of a derivation is a derivation (because we are in characteristic p), and the operation of taking the pth power of a derivation defines a restricted Lie algebra structure on $\mathfrak{g}$. For instance, if $\mathfrak{g} = \mathrm{Lie}(\mathbf{G}_a)$, then $U(\mathfrak{g}) = k[x]$, and the p-derivation is trivial.

Recall that $U^{[p]}(\mathfrak{g})$ denotes the quotient $U(\mathfrak{g})/(x^p - x^{[p]})$ of the universal enveloping algebra of $\mathfrak{g}$ by the relation which forces the pth power of any element to be equal to its pth derivation. This is a finite-dimensional k-algebra, of dimension $p^{\dim(\mathfrak{g})}$. The k-algebra $U^{[p]}(\mathfrak{g})$ admits the structure of a cocommutative Hopf algebra, given by sending $x \in \mathfrak{g}$ to $x \otimes 1 + 1 \otimes x$ under the diagonal, to $-x$ under the antipode, and to 0 under the counit. By the equivalence of categories discussed last time, there exists a finite algebraic k-group with associated Hopf algebra $U^{[p]}(\mathfrak{g})$. For instance, if $\mathfrak{g} = \mathrm{Lie}(\mathbf{G}_a)$, then $U^{[p]}(\mathfrak{g}) = k[x]/x^p$ — notice that this is exactly the Hopf algebra associated to α_p. This is a special case of the following more general observation:

Proposition 6. *Let $\mathfrak{g} = \mathrm{Lie}(G)$. The finite k-group corresponding to $U^{[p]}(\mathfrak{g})$ is G_1.*

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Proof. Recall that if H is any algebraic k-group, then $U^{[p]}(\mathfrak{h})$ injects into the algebra $\mathrm{Dist}(H)$ of distributions. In particular, $U^{[p]}(\mathrm{Lie}(G_1))$ injects into $\mathrm{Dist}(G_1)$. Now, $\dim U^{[p]}(\mathfrak{g}) = p^{\dim(\mathfrak{g})}$, so since $\mathrm{Lie}(G_n) = \mathfrak{g}$ for any $n \geq 1$, we must have $\dim U^{[p]}(\mathrm{Lie}(G_1)) = p^{\dim(\mathfrak{g})}$. But $\dim \mathrm{Dist}(G_1) = \dim k[G_1] \leq p^m$, and so the inclusion of $U^{[p]}(\mathrm{Lie}(G_1))$ into $\mathrm{Dist}(G_1)$ must be an isomorphism. Because G_1 is infinitesimal (the augmentation ideal is nilpotent), $\mathrm{Dist}(G_1)$ is the Hopf algebra associated to G_1. We conclude that the finite k-group corresponding to $U^{[p]}(\mathfrak{g})$ is G_1. $\square$

In particular, the representation theory of G_1 is the same as the representation theory of $U^{[p]}(\mathfrak{g})$, which is in turn the same as the representation theory of $\mathfrak{g}$ as a restricted Lie algebra.

In characteristic p, the cohomology of groups (with coefficients in some representation) is quite interesting, so it would be useful to understand how to relate the group cohomology of G with the inverse limit of the group cohomology of its Frobenius kernels.

Proposition 7. *Let G be irreducible and reduced, and suppose that $\mathrm{H}^*(G;k)$ is concentrated in degree 0. Suppose further that for every finite-dimensional G-representation V, the cohomology groups $\mathrm{H}^i(G;V)$ are finite-dimensional k-vector spaces. Then for every finite-dimensional G-representation W, the map $\mathrm{H}^i(G;W) \to \lim \mathrm{H}^i(G_n;W)$ is an isomorphism.*

Proof. For each $n \geq 1$, we have the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{i,j}(r) = \mathrm{H}^i(G/G_n; \mathrm{H}^j(G_n;W)) \Rightarrow \mathrm{H}^{i+j}(G;W).$$

Because G is reduced, there is an isomorphism $G/G_r \cong G^{(p^n)}$. Moreover, each $E_r^{i,j}(n)$ is finite-dimensional. Moreover, the maps $G_n \to G_{n+1}$ and $G/G_n \to G/G_{n+1}$ induce a map $\{E_r^{i,j}(n+1)\} \to \{E_r^{i,j}(n)\}$ of spectral sequences. Inverse limits are exact on filtered colimits of finite-dimensional vector spaces, so there is a spectral sequence $\{E_r^{i,j}\} := \{\lim_n E_r^{i,j}(n)\}$ converging to $\mathrm{H}^*(G;W)$.

We claim that $E_2^{i,j} = 0$ for $i > j$. To see this, it suffices to show that for any n, there exists some $m \geq n$ (depending only on n and j) such that the map $E_2^{i,j}(m) \to E_2^{i,j}(n)$ is zero for all $i > 0$. To see this, note that the map $E_2^{i,j}(m) \to E_2^{i,j}(n)$ factors as

$$\mathrm{H}^i(G/G_m; \mathrm{H}^j(G_m;W)) \to \mathrm{H}^i(G/G_n; \mathrm{H}^j(G_m;W)) \to \mathrm{H}^i(G/G_n; \mathrm{H}^j(G_n;W)^{G_m}) \to \mathrm{H}^i(G/G_n; \mathrm{H}^j(G_n;W)).$$

It suffices to show that for each n and j, there is some m such that $\mathrm{H}^j(G_n;W)^{G_m}) = \mathrm{H}^j(G_n;W)^G$; indeed, then

$$\mathrm{H}^i(G/G_n; \mathrm{H}^j(G_n;W)^{G_m}) \cong \mathrm{H}^i(G/G_n; \mathrm{H}^j(G_n;W)^G) \cong \mathrm{H}^i(G;k) \otimes \mathrm{H}^j(G_n;W)^G,$$

which vanishes because we assumed that $\mathrm{H}^*(G;k)$ is concentrated in degree 0.

If M is a G-module and G is irreducible, then $M^G = \bigcap_{n>0} M^{G_n}$. There is a descending chain $\subseteq M^{G_2} \subseteq M^{G_1}$, so if M is moreover finite-dimensional, then there exists some $N \gg 0$ such that $M^G = M^{G_N}$. In particular, it follows that for each n and j, there is some m such that $\mathrm{H}^j(G_n;W)^{G_m}) = \mathrm{H}^j(G_n;W)^G$, as desired.

Since $E_2^{i,j} = 0$ for $i > j$, we have $\mathrm{H}^i(G;W) \cong \lim \mathrm{H}^i(G_n;W)^G$. Since there exists some $m \geq n$ such that the map $\mathrm{H}^i(G_m;W) \to \mathrm{H}^i(G_n;W)$ takes values in $\mathrm{H}^i(G_n;W)^G$, we find that $\lim \mathrm{H}^i(G_n;W)^G \cong \lim \mathrm{H}^i(G_n;W)$. This proves the desired result. $\square$

We now turn to the May spectral sequence to calculate the cohomology of these Frobenius kernels. Recall how this goes:

Definition 8. Let A be a commutative Hopf algebra over k, and let I denote the kernel of the augmentation $A \to k$. If M is an A-comodule, then the Hochschild complex $A^{\otimes \bullet} \otimes M$ computes the cohomology of M as an A-module. The powers of the augmentation ideal give a filtration of the Hochschild complex, where the nth filtered piece in degree j is $\sum I^{a_1} \otimes \cdots \otimes I^{a_j} \otimes M$ with $\sum a_i \geq n$. The associated graded of this filtration in degree j is therefore $\bigoplus I^{a_1}/I^{a_1+1} \otimes \cdots \otimes I^{a_j}/I^{a_j+1} \otimes M$, where the sum is taken over all a_i such that $\sum a_i = n$.

It's an easy calculation that, in fact, the differentials respect the filtration, and hence descend to the associated graded. Moreover, if $\mathrm{gr}(A)$ is defined to be $\bigoplus_{n \geq 0} I^n/I^{n+1}$, then $\mathrm{gr}(A)$ is a Hopf algebra over k, and M can be given the structure of a trivial $\mathrm{gr}(A)$-module. The associated graded of the

Hochschild complex of M as a G-module is then isomorphic (as a complex) to the Hochschild complex of M as a $\mathrm{gr}(A)$-module. In particular, if $\bigcap I^n = 0$, then we obtain we obtain a convergent spectral sequence

$$E_1^{i,j} = \mathrm{H}^{i+j}(\mathrm{gr}(A); k)_j \otimes M \Rightarrow \mathrm{H}^{i+j}(A; M).$$

This is known as the May spectral sequence.

Example 9. If G is an algebraic k-group, setting $A = k[G]$ produces a commutative Hopf algebra $\mathrm{gr}(A)$, and hence a k-group scheme $\mathrm{gr}(G)$ such that $k[\mathrm{gr}(G)] = \mathrm{gr}(A)$. It follows that if G is irreducible, then there is a May spectral sequence

$$E_1^{i,j} = \mathrm{H}^{i+j}(\mathrm{gr}(G); k)_j \otimes M \Rightarrow \mathrm{H}^{i+j}(G; M).$$

This spectral sequence is multiplicative.

Proposition 10. *Let G be reduced and irreducible. Then there is an isomorphism*

$$E_1^{i,j} \cong \bigoplus M \otimes (\mathrm{Sym}^{a_1} \mathfrak{g}^*)^{(p)} \otimes (\mathrm{Sym}^{a_2} \mathfrak{g}^*)^{(p^2)} \otimes \cdots \otimes \Lambda^{b_1} \mathfrak{g}^* \otimes (\Lambda^{b_2} \mathfrak{g}^*)^{(p)} \otimes \cdots$$

for p odd, where the sum is over all finite sequences $\{a_n\}$ and $\{b_n\}$ with $i + j = \sum 2a_n + b_n$ and $i = \sum a_n p^n + b_n p^{n-1}$.

If $p = 2$, then

$$E_1^{i,j} \cong \bigoplus M \otimes \mathrm{Sym}^{a_1} \mathfrak{g}^* \otimes (\mathrm{Sym}^{a_2} \mathfrak{g}^*)^{(p)} \otimes \cdots ,$$

where the sum is over all finite sequences $\{a_n\}$ such that $i + j = \sum a_n$ and $i = \sum a_n 2^{n-1}$.

Proof sketch. Obviously, we only need to understand $\mathrm{H}^*(\mathrm{gr}(G); k)$. We will work at an odd prime; the argument is the same for the even prime. There is a surjection $\mathrm{Sym}(I/I^2) \to k[\mathrm{gr}(G)]$, and this is an isomorphism if G is reduced. In particular, the cohomology of $\mathrm{gr}(G)$ is precisely the cohomology of the Hopf algebra $\mathrm{Sym}(I/I^2) = \mathrm{Sym}(\mathfrak{g}^*)$ (with trivial diagonal). The claimed result now follows from the general claim that if V is a finite-dimensional k-vector space, then

$$\mathrm{H}^*(\mathrm{Sym}(V^*); k) \cong \mathrm{Sym}\left(\bigoplus_{n \geq 1} (V^*)^{(p^n)}\right) \otimes \Lambda \left(\bigoplus_{n \geq 0} (V^*)^{(p^n)}\right),$$

if I haven't messed up the indexing. One can check this by writing down a Koszul resolution for k as a $\mathrm{Sym}(V^*)$-comodule. ☐

Remark 11. Recall that in characteristic 0, we have $\mathrm{H}^*(\mathrm{Sym}(V^*); k) \cong \Lambda V$.

The spectral sequence of Proposition 10 also works for the cohomology of the Frobenius kernels; the idea here is that one truncates the contributions coming from a large enough I-adic filtration.

Proposition 12. *Let G be reduced and irreducible. There is a May spectral sequence*

$$E_1^{i,j} = \mathrm{H}^{i+j}(\mathrm{gr}(G); k)_j \otimes M \Rightarrow \mathrm{H}^{i+j}(G_N; M).$$

There is an isomorphism

$$E_1^{i,j} \cong \bigoplus M \otimes (\mathrm{Sym}^{a_1} \mathfrak{g}^*)^{(p)} \otimes (\mathrm{Sym}^{a_2} \mathfrak{g}^*)^{(p^2)} \otimes \cdots \otimes \Lambda^{b_1} \mathfrak{g}^* \otimes (\Lambda^{b_2} \mathfrak{g}^*)^{(p)} \otimes \cdots$$

for p odd, where the sum is over all finite sequences $\{a_n\}_{1 \leq n \leq N}$ and $\{b_n\}_{1 \leq n \leq N}$ with $i+j = \sum 2a_n + b_n$ and $i = \sum a_n p^n + b_n p^{n-1}$.

If $p = 2$, then

$$E_1^{i,j} \cong \bigoplus M \otimes \mathrm{Sym}^{a_1} \mathfrak{g}^* \otimes (\mathrm{Sym}^{a_2} \mathfrak{g}^*)^{(p)} \otimes \cdots ,$$

where the sum is over all finite sequences $\{a_n\}_{1 \leq n \leq N}$ such that $i + j = \sum a_n$ and $i = \sum a_n 2^{n-1}$.

Example 13. Let $G = \mathbf{G}_a$. Then

$$\mathrm{H}^*(\mathbf{G}_a; k) \cong \begin{cases} k[x_1, x_2, \cdots] & \mathrm{char}(k) = 2 \\ k[y_1, y_2, \cdots] \otimes \Lambda(x_1, x_2, \cdots) & \mathrm{char}(k) > 2, \end{cases}$$

where $|x_i| = 1$ and $|y_i| = 2$. Similarly,

$$\mathrm{H}^*(\alpha_{p^n}; k) \cong \begin{cases} k[x_1, \cdots, x_n] & \mathrm{char}(k) = 2 \\ k[y_1, \cdots, y_n] \otimes \Lambda(x_1, \cdots, x_n) & \mathrm{char}(k) > 2. \end{cases}$$

(This is the cohomology of $B(\mathbf{Z}/p)^n$.)

Example 14. Suppose $N = 1$, and work at an odd prime. Then one can check that

$$E_1^{i,j} = \begin{cases} M \otimes (\mathrm{Sym}^s\, \mathfrak{g}^*)^{(p)} \otimes \Lambda^{t-s}\mathfrak{g}^* & \text{if } i = s(p-1) + t,\ j = -(p-2)s \\ 0 & \text{else.} \end{cases}$$

This spectral sequence is rather sparse: because $E_1^{i,j}$ vanishes for $(p-2) \not| j$, we find that the d_r-differential vanishes for $r \neq 1 \pmod{p-2}$. Let us define a reindexed spectral sequence $\{\widetilde{E}_r^{i,j}\}$ by setting $\widetilde{E}_r^{i,j} = E_{(p-2)r+1}^{i+j,-(p-2)i}$. One then ends up with a spectral sequence

$$\widetilde{E}_0^{i,j} = M \otimes (\mathrm{Sym}^i\, \mathfrak{g}^*)^{(p)} \otimes \Lambda^{j-i}\mathfrak{g}^* \Rightarrow \mathrm{H}^{i+j}(G_1; M).$$

This is jarring to write down; I've never written down the E_0-page of a spectral sequence.

Example 15. One can explicitly write down the 0-line of the E_1-page of the spectral sequence in Example 14. Since $d_0^{0,j} : E_0^{0,j} \to E_0^{0,j+1}$ and $E_0^{0,j} = M \otimes \Lambda^j \mathfrak{g}^*$, we find that

$$d_0^{0,j} : M \otimes \Lambda^j \mathfrak{g}^* \to M \otimes \Lambda^{j+1}\mathfrak{g}^*.$$

A rather tedious calculation shows that the complex $(E_0^{0,*}, d_0^{0,*})$ is isomorphic to the Chevalley-Eilenberg complex $M \otimes \Lambda^\bullet \mathfrak{g}$ computing the cohomology of $\mathfrak{g}$. I haven't had time to go through this in detail myself. In any case, one ends up with an isomorphism $\widetilde{E}_1^{0,j} \cong \mathrm{H}^j(\mathfrak{g}; M)$.

In fact, one can calculate the entire E_1-page of this spectral sequence.

Theorem 16 (Friedlander-Parshall, Jantzen)**.** *The E_1-page of the spectral sequence in Example 14 can be described as follows:*

$$\widetilde{E}_1^{i,j} = \mathrm{H}^{j-i}(\mathfrak{g}; M) \otimes (\mathrm{Sym}^*\, \mathfrak{g}^*)^{(p)}.$$

Proof sketch. The d_0-differential (still jarring) goes $d_0^{i,j} : \widetilde{E}_0^{i,j} \to E_0^{i,j+1}$, i.e.,

$$d_0^{i,j} : M \otimes \Lambda^{j-i}\mathfrak{g}^* \otimes (\mathrm{Sym}^s\, \mathfrak{g}^*)^{(p)} \to M \otimes \Lambda^{j-i+1}\mathfrak{g}^* \otimes (\mathrm{Sym}^s\, \mathfrak{g}^*)^{(p)}.$$

This differential is a derivation, so

$$d_0^{i,j}(m \otimes x \otimes y) = d^{0,j-i}(m \otimes x) \otimes y + m \otimes x \otimes d_0^{i,i}(y).$$

Here, $d_0^{i,i}$ denotes the differential appearing in the E_0-page of the spectral sequence for $M = k$. Example 15 implies that in order to get the statement of the theorem, it suffices to show that $d_0^{i,i}$ vanishes for all i. In turn, it suffices to show that $d_0^{1,1} = 0$.

It is rather easy to observe that if H is a closed subgroup of G, then the vanishing of $d_0^{i,j}$ for H implies the vanishing of $d_0^{i,j}$ for G. Since we can always choose some large enough n such that G is a closed subgroup of SL_n, it suffices to show that $d_0^{1,1}$ vanishes for SL_n (at least, for certain n).

Recall that

$$d_0^{1,1} : (\mathfrak{g}^*)^{(p)} \to \mathfrak{g}^* \otimes (\mathfrak{g}^*)^{(p)}.$$

There is a natural G-action on the spectral sequence of Proposition 10, and so this is a G-equivariant, and hence $\mathfrak{g}$-equivariant, homomorphism. One now argues that if G is defined over $\mathbf{F}_p$, then the $\mathfrak{g}$-action on $(\mathfrak{g}^*)^{(p)}$ is trivial. Therefore, $d_0^{1,1}$ factors through $(\mathfrak{g}^*)^{\mathfrak{g}} \otimes (\mathfrak{g}^*)^{(p)} = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* \otimes (\mathfrak{g}^*)^{(p)}$. One now concludes using the observation that if $p \not| n$, then $\mathfrak{sl}_n = [\mathfrak{sl}_n, \mathfrak{sl}_n]$. $\square$

Remark 17. The Friedlander-Parshall spectral sequence is usually written with another indexing: suppose we reindex again, and write

$$'E_{2r}^{i,j} = {}'E_{2r-1}^{i,j} = \begin{cases} \widetilde{E}_r^{m,m+j} & i = 2m \\ 0 & i = m' + 1 \end{cases},$$

so that $d_{2r}^{2m,j} = d_r^{m,m+j}$. Then Theorem 16 implies that

$$'E_2^{2i,j} \cong \mathrm{H}^j(\mathfrak{g}; M) \otimes (\mathrm{Sym}^i \, \mathfrak{g}^*)^{(p)} \Rightarrow \mathrm{H}^{i+j}(G_1; M).$$

Remark 18. These spectral sequences imply that $\mathrm{H}^*(G_1; k)$ is a finitely generated k-algebra. This was generalized by Friedlander and Suslin, who showed that if G is a finite group scheme, then $\mathrm{H}^*(G; k)$ is a finitely generated k-algebra. Moreover, they showed that if M is a finite-dimensional G-module, then $\mathrm{H}^*(G; M)$ is a finitely generated $\mathrm{H}^*(G; k)$-module.

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