A SMALL SAMPLING OF NONABELIAN HODGE THEORY

1. Introduction

In these notes, we will give an overview of the nonabelian Hodge correspondence. Let us begin by contemplating classical Hodge theory. Let X be a compact complex manifold. Then every C^∞-n-form on X can be written uniquely as a sum of (p,q)-forms with $p+q=n$. If X is furthermore Kähler, then the (p,q)-component of a harmonic form is harmonic, and so the space of harmonic n-forms splits as a sum of harmonic (p,q)-forms with $p+q=n$. This gives a decomposition of $\mathrm{H}^n(X;\mathbf{C})$ as $\bigoplus_{p+q=n}\mathrm{H}^q(X;\Omega^p_X)$; this is the Hodge decomposition.

Let's assume X is a smooth $\mathbf{C}$-variety. One might think of $\mathrm{H}^n(X;\mathbf{C})$ as $\mathrm{R}^n\pi_*(\mathbf{C})$, where $\pi : X \to *$ is the projection, and $\mathbf{C}$ is viewed as the trivial rank one local system $\mathbf{C}$ on X. Similarly, $\bigoplus_{p+q=n}\mathrm{H}^q(X;\Omega^p_X)$ is the cohomology of the complex $\bigoplus_i \Omega^i_X[-i]$; the Hodge decomposition is an isomorphism $\mathrm{R}\pi_*(\mathbf{C}) \cong \mathrm{R}\pi_*\left(\bigoplus_i \Omega^i_X[-i]\right)$ in the derived category of $\mathbf{C}$-vector spaces. One might therefore hope for a categorification of the Hodge decomposition; namely, one might hope that the Hodge decomposition is a special case of a more general theorem relating local systems on X with complexes of vector bundles on X.

One such connection comes from the Riemann-Hilbert correspondence: this relates local systems on X with vector bundles on X: a classical result of Deligne's says that if X is a smooth complex variety, then:

$$\{\text{Local systems on } X\} \xrightarrow{\sim} \left\{\begin{matrix}\text{Vector bundles on } X\ + \\ \text{a flat connection}\end{matrix}\right\}.$$

This, however, is not a categorification of the Hodge decomposition: rather, one should think of it as a categorification of the de Rham isomorphism $\mathrm{H}^*_{\mathrm{dR}}(X) \cong \mathrm{H}^*(X;\mathbf{C})$.

We therefore need to search harder. As we mentioned before, the purpose of the nonabelian Hodge correspondence is to categorify the Hodge decomposition. The nonabelian Hodge correspondence states, roughly, that there is an equivalence of categories

$$\{\text{Local systems on } X\} \xrightarrow{\sim} \left\{\begin{matrix}\text{Certain vector bundles on } X\ + \\ \text{a ``Higgs field''}\end{matrix}\right\}.$$

We shall see the definition (and the restrictions necessary on the vector bundle) of a Higgs field in more detail below. For the moment, let us mention that a Higgs field on a vector bundle $\mathcal{F}$ on X is a 1-form $\phi \in \Gamma(X;\mathrm{End}(\mathcal{F})\otimes\Omega^1_X)$ which commutes with itself (i.e., $\phi\wedge\phi = 0$). A bundle equipped with a Higgs field is often just referred to as a Higgs bundle.

Under the nonabelian Hodge correspondence, the trivial local system $\mathbf{C}$ on X gets sent to the vector bundle $\mathcal{O}_X$ along with the trivial Higgs field $\phi = 0$. If $(\mathcal{F},\phi)$ is a Higgs bundle, one obtains a $\mathcal{O}_X$-linear morphism $\phi : \mathcal{F} \to \mathcal{F}\otimes\Omega^1_X$, and one can then form the associated Dolbeaut complex

$$\mathrm{Dol}(\mathcal{F},\phi) = [\mathcal{F} \xrightarrow{\phi} \mathcal{F}\otimes\Omega^1_X \xrightarrow{\phi} \mathcal{F}\otimes\Omega^1_X \to \cdots];$$

this is a complex because of the Higgs condition. Therefore, the Dolbeaut complex of the Higgs bundle $(\mathcal{O}_X, 0)$ associated to the trivial local system $\mathbf{C}$ on X is simply $\bigoplus_i \Omega^i_X[-i]$. If the nonabelian Hodge correspondence is sufficiently natural in X, and (derived) pushforward of the Higgs bundle to a point is given by taking the (derived global sections of the) Dolbeaut complex, then the Hodge decomposition for cohomology follows by pushing forward along the projection morphism $\pi : X \to *$. It is in this sense that the nonabelian Hodge correspondence is, indeed, a categorification of the Hodge decomposition.

The proof of the nonabelian Hodge correspondence passes through Deligne's equivalence. Namely, we will actually show that there is an equivalence

$$\begin{Bmatrix}\text{Vector bundles on } X + \\ \text{a flat connection}\end{Bmatrix} \simeq \begin{Bmatrix}\text{Certain vector bundles on } X + \\ \text{a “Higgs field”}\end{Bmatrix}. \tag{1}$$

We can try to guess how such an equivalence might go. Recall that a connection on a vector bundle $\mathcal{F}$ is just a map $\mathrm{D} : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$ satisfying the Leibniz rule $\mathrm{D}(fs) = s \otimes df + f\mathrm{D}(s)$, where f is a section of $\mathcal{O}_X$ and s is a section of $\mathcal{F}$. This is quite similar to the definition of a Higgs field, except that a Higgs field on $\mathcal{F}$ is a $\mathcal{O}_X$*-linear* morphism $\phi : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$. In other words, $\mathrm{D}(fs) = f\mathrm{D}(s)$, where f is a section of $\mathcal{O}_X$ and s is a section of $\mathcal{F}$.

In light of this, one might try to prove (1) by a “straightline homotopy”: namely, construct a one-parameter deformation of the notion of a connection along some parameter λ, so that when $\lambda = 0$ one recovers Higgs fields, and when $\lambda = 1$, one recovers connections. Then, one can try to determine which Higgs fields arise as degenerations at $\lambda = 0$ of such a one-parameter family. This turns out to be quite subtle: for the appropriate notion of “λ-connection” on a complex Kähler manifold, the degeneration at $\lambda = 0$ of a one-parameter family just turns out to be trivial — unless one allows the holomorphic structure on the bundle to vary with λ, too.

2. λ-connections

In this brief section, we shall introduce the notion of a λ-connection. We shall return to a more detailed study after sketching the proof of the nonabelian Hodge correspondence.

Definition 2.1. Let $\lambda \in \mathbf{C}$. A λ-connection on a vector bundle $\mathcal{F}$ over X is a map $\mathrm{D}_\lambda : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$ satisfying

$$\mathrm{D}_\lambda(fs) = \lambda s \otimes df + f\nabla(s),$$

where s is a section of $\mathcal{F}$, and f is a section of $\mathcal{O}_X$. Say that a λ-connection is flat if $\mathrm{D}_\lambda \circ \mathrm{D}_\lambda : \mathcal{F} \to \mathcal{F} \otimes \Omega^2_X$ is zero.

Example 2.2. If $\lambda = 0$, then a λ-connection is precisely a $\mathcal{O}_X$-linear map $\mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$, i.e., a global section ϕ of $\mathrm{End}(\mathcal{F}) \otimes \Omega^1_X$. This λ-connection is flat if and only if $\phi \wedge \phi = 0$. The data of a 0-connection ϕ on a vector bundle $\mathcal{F}$ is called a Higgs field, and the pair $(\mathcal{F}, \phi)$ is called a Higgs bundle.

Example 2.3. If $\lambda = 1$, then a λ-connection is precisely a connection on $\mathcal{F}$. It is flat if and only if the connection is flat in the usual sense.

Remark 2.4. Let D_λ be a λ-connection on a vector bundle $\mathcal{F}$. If $\lambda' \in \mathbf{C}$, then $\lambda'\mathrm{D}_\lambda$ is a $\lambda\lambda'$-connection on $\mathcal{F}$. In particular, there is a canonical action of $\mathbf{C}^\times$ on the category of vector bundles equipped with a 0-connection (i.e., the category of Higgs bundles). Furthermore, there is a canonical equivalence of categoris between the category of vector bundles equipped with a 1-connection and the category of vector bundles equipped with a λ-connection for nonzero λ.

3. A precise statement

The following is a precise statement of the nonabelian Hodge correspondence, as stated in [Sim92].

Theorem 3.1. *Let X be a smooth complex projective variety. Then there is an equivalence of categories between:*

- *semisimple flat bundles on X (i.e., every subbundle preserved by the connection is a direct summand);*

- *Higgs bundles $(\mathcal{F}, \phi)$ on X which are direct sums of stable Higgs bundles and have vanishing first and second Chern classes.*

Remark 3.2. Restricting to simple Higgs bundles is the same as restricting to simple flat bundles on X; under the Riemann-Hilbert correspondence, these are irreducible representations of $\pi_1(X)$.

Before we proceed, we make an important observation.

Remark 3.3. In Remark 2.4, we observed that there is a natural action of the torus $\mathbf{C}^\times$ on the category of Higgs bundles. This action preserves semistability, and respects the vanishing of the first and second Chern classes, and therefore defines an action of $\mathbf{C}^\times$ on the category of semisimple flat bundles on X. The Riemann-Hilbert correspondence further gives an action of $\mathbf{C}^\times$ on the category of semisimple representations of $\pi_1(X)$. For instance, the element $-1 \in \mathbf{C}^\times$ sends a representation of $\pi_1(X)$ to its complex conjugate.

The $\mathbf{C}^\times$-action is extremely interesting. It is hard to describe the effect of the $\mathbf{C}^\times$-action on local systems, but the fixed points turn out to admit a simple characterization. The following result is just linear algebra.

Proposition 3.4. *A Higgs bundle $(\mathcal{F}, \phi)$ is fixed by the $\mathbf{C}^\times$-action if and only if it can be written as $\bigoplus_{i=1}^k \mathcal{F}_k$ satisfying Griffiths transversality:*

$$\phi : \mathcal{F}_i \to \mathcal{F}_{i-1} \otimes \Omega^1_X.$$

Proof. Let f is an isomorphism $(\mathcal{F}, \phi) \to (\mathcal{F}, t\phi)$ for t not a root of unity. Then the coefficients of the characteristic polynomial of f are holomorphic functions on X (and therefore are constant). The decomposition of $\mathcal{F}$ into eigenbundles for f is $\bigoplus_\lambda \mathcal{F}_\lambda$, where $\mathcal{F}_\lambda = \ker((f-\lambda)^n)$ if λ is an eigenvalue of multiplicity n. Notice that because $t^n\phi(f-\lambda)^n = (f-t\lambda)^n$, the map θ sends $\mathcal{F}_\lambda$ to $\mathcal{F}_{t\lambda}$. Because t is not a root of unity, the set S of eigenvalues of f can be decomposed into strings of the form $\lambda, t\lambda, \cdots, t^k\lambda$. In particular, $S = \coprod_{i=1}^k S_i$, and one then defines $\mathcal{F}_i = \bigoplus_{\lambda \in S_i} \mathcal{F}_\lambda$. $\square$

As one might expect from Proposition 3.4, the $\mathbf{C}^\times$-fixed points in Higgs bundles are related to variations of Hodge structures. Recall:

Definition 3.5. Let X be a smooth projective variety. A complex variation of Hodge structures is the datum of:

- a vector bundle $\mathcal{V} = \bigoplus_{p+q=n} \mathcal{V}^{p,q}$;
- a flat connection D on $\mathcal{V}$ such that

$$\mathrm{D} : \mathcal{V}^{p,q} \to \mathcal{A}^1(\mathcal{V}^{p,q}) \oplus \mathcal{A}^{1,0}(\mathcal{V}^{p-1,q+1}) \oplus \mathcal{A}^{0,1}(\mathcal{V}^{p+1,q});$$

- a Hermitian form h on $\mathcal{V}$ which makes the decomposition orthogonal, and which is positive (resp. negative) definite on $\mathcal{V}^{p,q}$ if p is even (resp. odd).

Remark 3.6. Recall that a Hodge structure (of weight n) is an abelian group H along with a decomposition $\mathrm{H}_\mathbf{C} = \bigoplus_{p+q=n} \mathrm{H}^{p,q}$ such that $\mathrm{H}^{p,q} = \overline{\mathrm{H}^{q,p}}$. One gets a decreasing filtration by looking at $F_i\mathrm{H}_\mathbf{C} = \bigoplus_{p\geq i} \mathrm{H}^{p,q}$; the conjugate of this filtration is $\overline{F}_i\mathrm{H}_\mathbf{C} = \bigoplus_{q\geq i} \mathrm{H}^{p,q}$. This filtration specifies the Hodge decomposition, because $\mathrm{H}_\mathbf{C} = F_i\mathrm{H}_\mathbf{C} \oplus \overline{F}_{n-i+1}\mathrm{H}_\mathbf{C}$. A variation of Hodge structure on X is a $\mathbf{Z}$-local local system H on X along with a decreasing filtration $F_p\mathrm{H}_X$ of $\mathrm{H}_X := \mathrm{H} \otimes_\mathbf{Z} \mathcal{O}_X$ such that this filtration defines a Hodge structure on each fiber of H_X, and such that Griffiths transversality is satisfied:

$$\nabla : F_p\mathrm{H}_X \to F_{p-1}\mathrm{H}_X \otimes \Omega^1_X.$$

Example 3.7. Suppose $f : Y \to X$ is a smooth projective morphism. Then $\mathcal{V} = \mathrm{R}^n f_*(\mathbf{C}) \otimes_{\mathbf{C}} \mathcal{O}_X$ admits a Hodge decomposition

$$\mathcal{V} \cong \bigoplus_{p+q=n} \mathrm{R}^q f_*(\Omega^p_{Y/X}).$$

The Hermitian form on $\mathcal{V}$ is given by pairing with the Kähler form ω: on each fiber $\mathrm{H}^n(Y_x; \mathbf{C})$, the pairing is defined by

$$\langle \alpha, \beta \rangle = \int_{Y_x} \alpha \wedge \overline{\beta} \wedge \omega^{\dim(Y_x)-n},$$

up to some constant factor. The Gauss-Manin connection gives the connection D, and the condition required of D comes from Griffiths transversality.

We shall now describe how to construct a $\mathbf{C}^\times$-fixed point in Higgs bundles from a complex variation of Hodge structures.

Construction 3.8. Suppose we are given a complex variation of Hodge structures $(\mathcal{V} = \mathcal{V}^{p,q}, \mathrm{D}, h)$. We can decompose D as a map

$$\mathrm{D} = \partial \oplus \overline{\partial} \oplus \theta \oplus \overline{\theta} : \mathcal{V}^{p,q} \to \mathcal{A}^{1,0}(\mathcal{V}^{p,q}) \oplus \mathcal{A}^{0,1}(\mathcal{V}^{p,q}) \oplus \mathcal{A}^{1,0}(\mathcal{V}^{p-1,q+1}) \oplus \mathcal{A}^{0,1}(\mathcal{V}^{p+1,q}).$$

The operator $\overline{\partial}$ equips $\mathcal{V}^{p,q}$ with a holomorphic structure, and the operator θ equips $\mathcal{V}^{p,q}$ with a map $\mathcal{V}^{p,q} \to \mathcal{V}^{p-1,q+1} \otimes \Omega^1_X$. Therefore, the bundle $\mathcal{V}$ can be written as a direct sum $\bigoplus_{i=1}^n \mathcal{F}_i$ (where n is the weight of $\mathcal{V}$), with $\mathcal{F}_i = \bigoplus_{p \geq i} \mathcal{V}^{p,q}$. Since D is assumed to be flat, we find that $\theta \wedge \theta = 0$, so $(\mathcal{V}, \theta)$ is a Higgs bundle. By Proposition 3.4, it is a fixed point of the $\mathbf{C}^\times$-action on Higgs bundles.

Example 3.9. Consider the complex variation of Hodge structures defined in Example 3.7. The vector bundle is $\mathcal{V} = \mathrm{R}^n f_*(\mathbf{C})$, and the associated Higgs field sends $\mathrm{R}^q f_*(\Omega^p_{Y/X}) \to \mathrm{R}^{q+1} f_*(\Omega^{p-1}_{Y/X}) \otimes \Omega^1_X$. On each fiber $x \in X$, this morphism is given by pairing with the Kodaira-Spencer class

$$\eta_x \in \mathrm{Hom}(\mathrm{T}_{X,x}, \mathrm{R}^1 f_*(\mathrm{T}_{Y_x})) \cong \mathrm{R}^1 f_*(\mathrm{T}_{Y_x}) \otimes (\Omega^1_X)_x.$$

The mechanism of Construction 3.8 in fact characterizes the fixed points of the $\mathbf{C}^\times$-action on semisimple flat bundles on X:

Theorem 3.10. *The fixed points of the $\mathbf{C}^\times$-action on semisimple flat bundles on X are precisely those bundles admitting a complex variation of Hodge structures.*

4. An unfairly brief proof sketch

In this section, we give an unreasonably brief proof sketch of the nonabelian Hodge correspondence. This is a talk for algebraic geometers, and so I didn't want to talk about all the analysis that goes into the proof (which is really interesting).

Recall that the proof of the Hodge decomposition passed through the intermediate notion of harmonic forms. The corresponding intermediate object in the proof of the nonabelian Hodge correspondence is the notion of a harmonic bundle. In order to prove Theorem 3.1, one shows that both categories in question are in turn equivalent to the category of harmonic bundles.

We shall content ourselves in these notes with just the definition of a harmonic bundle; showing that the categories in Theorem 3.1 are equivalent to the category of harmonic bundles is the real heart of the proof, but we will not go into that in these notes. To motivate the definition of harmonic bundles, let us begin by taking a look at what a Higgs bundle really is.

Construction 4.1. Suppose $(\mathcal{F}, \theta)$ is a Higgs bundle on a compact Kähler manifold X, so θ takes sections of $\mathcal{F}$ to $(1,0)$-forms with coefficients in $\mathcal{F}$. By the Newlander-Nirenberg theorem, the holomorphic structure on X is determined by an operator $\overline{\partial}$ which kills the holomorphic sections; in other words, $\overline{\partial}$ takes sections of $\mathcal{F}$ to $(0,1)$-forms. The condition that θ be a holomorphic $\mathcal{O}_X$-linear map means that $\overline{\partial}\theta + \theta\overline{\partial} = 0$. In other words, if we define $\mathrm{D}'' = \overline{\partial} + \theta$, then $(\mathrm{D}'')^2 = 0$.

Conversely, if one has an operator D'' satisfying the Leibniz rule such that $(\mathrm{D}'')^2 = 0$, then we get (by decomposing D'' into its $(0,1)$ and $(1,0)$ components) operators $\overline{\partial}$ and θ such that

$$\overline{\partial}^2 = 0,\ \theta \wedge \theta = 0,\ \overline{\partial}\theta + \theta\overline{\partial} = 0.$$

We will write the associated Higgs bundle via $(\mathcal{F}, \overline{\partial}, \theta)$ or $(\mathcal{F}, \mathrm{D}'')$

To get the nonabelian Hodge correspondence, we'd therefore like to be able to extract such an operator D'' from a vector bundle with flat connection.

Construction 4.2. Suppose that $(\mathcal{F}, \mathrm{D})$ is a vector bundle with a connection on X, so $\mathrm{D} : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$. By splitting D into its $(0,1)$ and $(1,0)$ components, we can write $\mathrm{D} = d_1 + d_2$, with d_1 an operator of type $(0,1)$, and d_2 an operator of type $(1,0)$. If D is flat, then $\mathrm{D}^2 = 0$, and so

$$d_1^2 = 0,\ d_1 d_2 + d_2 d_1 = 0,\ d_2^2 = 0.$$

Let's fix a Hermitian metric K on $\mathcal{F}$ (i.e., a unitary isomorphism between $\mathcal{F}$ and $\overline{\mathcal{F}}^{\vee}$). One can then show that there is a unique operator δ_1 (resp. δ_2) of type $(1,0)$ (resp. type $(0,1)$) such that $\delta_1 + d_2$ (resp. $d_1 + \delta_2$) preserves the metric K. We may then define the following four operators, which send sections of $\mathcal{F}$ to 1-forms with values in $\mathcal{F}$:

$$\partial_K = \frac{d_1 + \delta_1}{2}, \quad \overline{\partial}_K = \frac{d_2 + \delta_2}{2}$$
$$\theta_K = \frac{d_1 - \delta_1}{2}, \quad \overline{\theta}_K = \frac{d_2 - \delta_2}{2}.$$

In particular, ∂_K and θ_K take sections of $\mathcal{F}$ to $(1,0)$-forms valued in $\mathcal{F}$, while $\overline{\partial}_K$ and $\overline{\theta}_K$ take sections of $\mathcal{F}$ to $(0,1)$-forms valued in $\mathcal{F}$. Further define

$$\mathrm{D}'_K = \partial_K + \overline{\theta}_K = \frac{d_1 + d_2 + \delta_1 - \delta_2}{2},\ \mathrm{D}''_K = \overline{\partial}_K + \theta_K = \frac{d_1 + d_2 - \delta_1 + \delta_2}{2}.$$

Note that

$$\mathrm{D}'_K + \mathrm{D}''_K = d_1 + d_2 = \mathrm{D}.$$

The operator D''_K looks a lot like an operator which defines a Higgs structure on $\mathcal{F}$. It satisfies the Leibniz rule. However, it might not satisfy the flatness condition, i.e., $(\mathrm{D}''_K)^2$ might be nonzero. We may summarize this observation in the following lemma.

Lemma 4.3. *If $(\mathrm{D}''_K)^2 = 0$, then $(\mathcal{F}, \mathrm{D}''_K)$ is a Higgs bundle.*

We also need to be able to go from Higgs bundles to vector bundles with flat connection.

Construction 4.4. Suppose that $(\mathcal{F}, \mathrm{D}'') = (\mathcal{F}, \overline{\partial}, \theta)$ is a Higgs bundle. Again, there is a unique operator ∂_K taking sections of $\mathcal{F}$ to $(1,0)$-forms valued in $\mathcal{F}$ such that $\partial_K + \overline{\partial}$ preserves the metric K. Define $\overline{\theta}_K$ as the adjoint of θ with respect to K, and define

$$\mathrm{D}'_K = \partial_K + \overline{\theta}_K,\ \mathrm{D}_K = \mathrm{D}'_K + \mathrm{D}'' = \overline{\partial} + \partial_K + \theta + \overline{\theta}_K.$$

Note that

$$\mathrm{D}_K - \mathrm{D}'_K = \mathrm{D}''.$$

The operator D_K sends sections of $\mathcal{F}$ to 1-forms valued in $\mathcal{F}$, and satisfies the Leibniz rule. However, as before, D_K is not necessarily flat, so $(\mathcal{F}, \mathrm{D}_K)$ does not necessarily define a vector bundle with a flat connection. We may summarize this observation in the following lemma.

Lemma 4.5. *If $\mathrm{D}_K^2 = 0$, then $(\mathcal{F}, \mathrm{D}_K)$ is a vector bundle with flat connection.*

This motivates the definition of a harmonic bundle. From both Higgs bundles and vector bundles with flat connection, we extracted a pair of operators D and D'' (essentially determined by the operator we called D'), one of which depended on the choice of metric.

Definition 4.6. A harmonic bundle on X is a tuple $(\mathcal{F}, \mathrm{D}, \mathrm{D}'')$, where $\mathcal{F}$ is a vector bundle, D is a flat connection on $\mathcal{F}$, D'' defines a Higgs structure on $\mathcal{F}$, such that there is a Hermitian metric K on $\mathcal{F}$ for which

$$\mathrm{D}'' = \mathrm{D}''_K, \ \mathrm{D} = \mathrm{D}_K$$

via the above construction.

Remark 4.7. Note that the datum of the Hermitian metric K is not included in the definition of a harmonic bundle.

Remark 4.8. Notice the analogy between harmonic bundles and complex variations of Hodge structures: in both cases, one has a vector bundle equipped with a flat connection D, along with a Hermitian metric and a decomposition of D as $\partial + \overline{\partial} + \theta + \overline{\theta}$.

Remark 4.9. By Lemma 4.3, a vector bundle $(\mathcal{F}, \mathrm{D})$ determines a harmonic bundle if and only if there is a Hermitian metric K on $\mathcal{F}$ such that $(\mathrm{D}''_K)^2 = 0$. Similarly, Lemma 4.5 implies that a Higgs bundle $(\mathcal{F}, \mathrm{D}'')$ determines a harmonic bundle if and only if there is a Hermitian metric K on $\mathcal{F}$ such that $\mathrm{D}_K^2 = 0$.

Here is the result which allows us to interpolate between flat connections and Higgs bundles:

Proposition 4.10. *Let $(\mathcal{F}, \mathrm{D}, \mathrm{D}'')$ be a harmonic bundle on X. Then there is a family $(\mathcal{F}_\lambda, \mathrm{D}_\lambda)$ of flat λ-connections on X such that*

$$(\mathcal{F}_1, \mathrm{D}_1) = (\mathcal{F}, \mathrm{D}), \ (\mathcal{F}_0, \mathrm{D}_0) = (\mathcal{F}, \mathrm{D}'').$$

Proof. Define

$$\mathrm{D}' = \mathrm{D} - \mathrm{D}''.$$

Let us write $\mathrm{D}'' = \overline{\partial} + \theta$, and $\mathrm{D} = d_1 + d_2$. Because $(\mathcal{F}, \mathrm{D}, \mathrm{D}'')$ is a harmonic bundle, there is a Hermitian metric K on $\mathcal{F}$ such that $\overline{\partial}_K = \overline{\partial}$ and $\theta_K = \theta$. As before,

$$\mathrm{D} = \overline{\partial} + \partial_K + \theta + \overline{\theta}_K = \mathrm{D}'' + \partial_K + \overline{\theta}_K,$$

so $\mathrm{D}' = \partial_K + \overline{\theta}_K$. We'll now omit the subscript K. Define

$$\mathrm{D}_\lambda = \mathrm{D}'' + \lambda \mathrm{D}' = \overline{\partial} + \theta + \lambda\partial + \lambda\overline{\theta}.$$

Then the $(0,1)$-component of D'_λ is $\overline{\partial} + \lambda\overline{\theta}$, while the $(1,0)$-component of D'_λ is $\partial + \lambda\theta$. Because $(\mathcal{F}, \mathrm{D}, \mathrm{D}'')$ is a harmonic bundle, we know that $\mathrm{D}_\lambda^2 = 0$, so these two components commute. It follows that $\partial + \lambda\theta$ defines a flat λ-connection on $\mathcal{F}$, where the holomorphic structure on $\mathcal{F}$ is determined by $\overline{\partial} + \lambda\overline{\theta}$. $\square$

We conclude from Remark 4.9 and Proposition 4.10 that in order to prove Theorem 3.1, we must determine when $(\mathrm{D}''_K)^2 = 0$ and $\mathrm{D}_K^2 = 0$. This is where the analysis comes in; we shall only state the relevant results.

Theorem 4.11 (Siu, Sampson, Corlette, Deligne)**.** *Let $(\mathcal{F}, \mathrm{D})$ be a vector bundle equipped with a flat connection. Then there exists a Hermitian metric K on $\mathcal{F}$ such that $(\mathrm{D}''_K)^2 = 0$ if and only if $\mathcal{F}$ is semisimple.*

Theorem 4.12 (Narasimhan-Seshadri, Donaldson, Uhlenbeck-Yau, Beilinson-Deligne, Hitchin, Simpson)**.** *Let $(\mathcal{F}, \mathrm{D}'')$ be a Higgs bundle. Then there exists a Hermitian metric K on $\mathcal{F}$ such that $\mathrm{D}_K^2 = 0$ if and only if:*

- *$\mathcal{F}$ is polystable, meaning that it is a direct sum of stable Higgs bundles of the same slope; and*

- *the first two Chern classes vanish:*
$$c_1(\mathcal{F}) \cdot [\omega]^{\dim(X)-1} = c_2(\mathcal{F}) \cdot [\omega]^{\dim(X)-2} = 0.$$

Piecing together Theorem 4.11 and Theorem 4.12 along with the preceding discussion yields Theorem 3.1. (We are omitting many details: this argument only shows essential surjectivity; one needs to prove full faithfulness.)

5. Back to λ-connections

Having sketched the proof of Theorem 3.1, we shall now return to studying λ-connections. From now, we shall work entirely in the algebraic setting, so X will be an algebraic variety. To motivate our discussion, observe that if $(\mathcal{F}, \phi)$ is a Higgs bundle, then the $\mathcal{O}_X$-linear coaction of Ω^1_X on $\mathcal{F}$ (defined by ϕ) is equivalent an action of $\mathrm{Sym}(\mathrm{T}_X) = \mathrm{Sym}((\Omega^1_X)^\vee)$ on $\mathcal{F}$. In other words, a Higgs bundle is essentially the datum of a coherent sheaf on the cotangent bundle T^*X. There is a similar characterization of vector bundles with flat connection. To define this, we recall the definition of the de Rham space.

Definition 5.1. The de Rham space X_{dR} is the functor $\mathrm{CAlg}_{\mathbf{C}} \to \mathrm{Set}$ defined by $X_{\mathrm{dR}}(R) = X(R/I)$, where I is the nilradical of R.

Then (see [GR14], for instance):

Theorem 5.2 (Grothendieck)**.** *There is an equivalence of categories* $\mathrm{QCoh}(X_{\mathrm{dR}}) \simeq \mathrm{Mod}(\mathcal{D}_X)$.

Recall how this equivalence goes[1]. A quasicoherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X_{\mathrm{dR}})$ is the data of a quasicoherent sheaf $\mathcal{F}$ on X along with compatible isomorphisms $\mathcal{F}(x) \to \mathcal{F}(y)$ for every pair of "infinitesimally close" R-points $x, y \in X(R)$ (i.e., points whose image under $X(R) \to X(R/I)$ are the same, where I is the nilradical of R). More precisely, if the pair (x, y) is thought of as an R-point of $X \times X$, then x and y are infinitesimally close if and only if they are the same in some thickening of the diagonal $\Delta : X \to X \times X$. Therefore, if $\mathcal{I}$ denotes the ideal sheaf of Δ, then x and y are infinitesimally close if and only if for every $\mathbf{C}$-algebra R, the ideal $(x,y)^*\mathcal{I}^n$ is zero in R for $n \gg 0$, where $(x, y) : \mathrm{Spec}(R) \to X \times X$.

Let $X^{(n)}$ denote the closed subscheme of $X \times X$ defined by $\mathcal{I}^{n+1}$. Let p_i denote the projections $(X \times X)^\wedge_X = \mathrm{colim}\, X^{(n)} \to X$, and let $p_i^{(n)}$ denote the induced maps $X^{(n)} \to X$. A quasicoherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X_{\mathrm{dR}})$ is therefore a quasicoherent sheaf $\mathcal{F}$ on X along with the data of compatible isomorphisms $(p_1^{(n)})^*\mathcal{F} \to (p_2^{(n)})^*\mathcal{F}$. This, in turn, is the same as a map $\mathcal{F} \to (p_1^{(n)})_*(p_2^{(n)})^*\mathcal{F} \cong \mathcal{O}_{X^{(n)}} \otimes_{\mathcal{O}_X} \mathcal{F}$.

The key point, now, is that there is a canonical pairing $F_n\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{(n)}} \to \mathcal{O}_X$. Given a differential operator D and a function $f(x, y)$ defined up to order $n+1$ (i.e., a section of $\mathcal{O}_{X^{(n)}}$), we obtain a function on X by applying D to f (keeping the variable y constant) and evaluating on (x, x) (i.e., $(Df)(x, x)$). When $X = \mathbf{A}^1 = \mathrm{Spec}\,\mathbf{C}[t]$, we know that $F_n\mathcal{D}_X$ is the free $\mathbf{C}[t]$-module generated by $\frac{\partial_t^k}{k!}$ for $1 \leq k \leq n$, and that $\mathcal{O}_{X^{(n)}}$ is $\mathbf{C}[t, z]/(t-z)^{n+1}$. Applying $\frac{\partial_t^k}{k!}$ to the function $t^k z^j$ in the manner described above produces the function t^j on $\mathbf{A}^1$. In particular, the pairing can be checked to be perfect (and this is true over any smooth variety X). Therefore the maps $\mathcal{F} \to \mathcal{O}_{X^{(n)}} \otimes_{\mathcal{O}_X} \mathcal{F}$ are the same as maps $F_n\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$, and these assemble into an action of $\mathcal{D}_X$ on $\mathcal{F}$.

Returning to Theorem 5.2, recall that $\mathcal{D}_X$-modules which are coherent as $\mathcal{O}_X$-modules are precisely vector bundles with flat connection. Therefore, the category of vector bundles with flat connection sits as a full subcategory of $\mathrm{Coh}(X_{\mathrm{dR}})$. The discussion at the beginning of

[1] See these notes by Jacob Lurie: `http://people.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/Nov17-19(Crystals).pdf`.

this section concluded that a Higgs bundle is essentially the datum of a coherent sheaf on the cotangent bundle T^*X. Based on this discussion, it is natural to wonder if λ-connections sit as a full subcategory of coherent sheaves on a certain stack X_λ. This is in fact the case, and we shall devote the rest of this section to describing X_λ.

We first begin by espousing a perspective on λ-connections (and in fact, one-parameter families of deformations in general).

Proposition 5.3. *Let $\mathbf{A}^1/\mathbf{G}_m$ denote the stacky quotient of the affine line by its $\mathbf{G}_m$-action. Then $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m) \simeq \mathrm{Fun}(\mathbf{Z}, \mathrm{Vect})$, where $\mathbf{Z}$ is the poset viewed as a category.*

In more concrete terms, a quasicoherent sheaf over $\mathbf{A}^1/\mathbf{G}_m$ is precisely a collection $\{V_n\}_{n\in\mathbf{Z}}$ of vector spaces along with maps $V_n \to V_{n+1}$ (or V_{n-1}, depending on which ordering on $\mathbf{Z}$ you choose). We shall refer to this data as a filtered vector space. If we ask that the quasicoherent sheaf be torsion-free, then the maps $V_n \to V_{n+1}$ will in fact be inclusions.

Proof. A quasicoherent sheaf on $\mathbf{A}^1/\mathbf{G}_m$ is just a $\mathbf{G}_m$-equivariant sheaf on $\mathbf{A}^1$. If we write $\mathbf{A}^1 = \operatorname{Spec}\mathbf{C}[t]$, then this is equivalently the datum of a $\mathbf{G}_m$-equivariant $\mathbf{C}[t]$-module. In turn, this is just a $\mathbf{C}$-vector space V which admits a $\mathbf{G}_m$-action along with an endomorphism $t : V \to V$ which shifts the $\mathbf{G}_m$-weight by 1. Let $\bigoplus_{n\in\mathbf{Z}} V_n$ denote the weight decomposition; then the functor $\mathbf{Z} \to \mathrm{Vect}$ sends n to V_n, and the morphism $V_n \to V_{n+1}$ is induced by t; clearly, the datum of such a functor is equivalent to a $\mathbf{C}$-vector space V which admits a $\mathbf{G}_m$-action along with an endomorphism $t : V \to V$ which shifts the $\mathbf{G}_m$-weight by 1. $\square$

Remark 5.4. Let $\mathcal{F}$ be a quasicoherent sheaf over $\mathbf{A}^1/\mathbf{G}_m$. There are exactly two points in the underlying space of $\mathbf{A}^1/\mathbf{G}_m$: these correspond to the orbits of 1 and 0. The orbit of 1 defines a map $\mathbf{G}_m/\mathbf{G}_m = \mathrm{Spec}(\mathbf{C}) \to \mathbf{A}^1/\mathbf{G}_m$, and pulling back $\mathcal{F}$ along this morphism simply amounts to viewing a filtered vector space as a vector space (i.e., it sends $\{V_n\}$ to $\bigoplus_n V_n$). The orbit of 0 defines a map $B\mathbf{G}_m \to \mathbf{A}^1/\mathbf{G}_m$, and pulling back $\mathcal{F}$ to $B\mathbf{G}_m$ produces a $\mathbf{G}_m$-representation, i.e., a graded vector space. This procedure sends a filtered vector space $\{V_n\}$ to its associated graded $\{V_n/V_{n-1}\}$.

From this perspective, it is most natural to think of vector bundles with flat connection (or, more generally, quasicoherent sheaves on X_{dR}) as living over the point corresponding to 1 (i.e., $\mathrm{Spec}(\mathbf{C})$), and Higgs bundles (or, more generally, quasicoherent sheaves on T^*X) as living over the stacky point corresponding to 0, i.e., over $B\mathbf{G}_m$. Indeed, recall that the order filtration on $\mathcal{D}_X$ had associated graded $\mathcal{O}_{\mathrm{T}^*X}$; this jibes well with the perspective on filtrations provided by Remark 5.4. Harmonic bundles/λ-connections (or, more generally, quasicoherent sheaves on the stack X_λ) should therefore be thought of as living over $\mathbf{A}^1/\mathbf{G}_m$.

We begin by defining the analogue $\mathcal{D}_X^\lambda$ of the sheaf $\mathcal{D}_X$. As the above paragraph suggests, it literally captures the order filtration on $\mathcal{D}_X$.

Definition 5.5. Let $\widetilde{\mathcal{D}}_X^\lambda$ denote the sheaf of algebras over $X \times \mathbf{A}^1$ defined by

$$\widetilde{\mathcal{D}}_X^\lambda = \sum_{k\geq 0} \mathcal{O}_X[t] t^k \mathcal{D}_X^{\leq k},$$

where t is the coordinate on $\mathbf{A}^1$. The coordinate t acts as λ. There is a $\mathbf{G}_m$-action on $\widetilde{\mathcal{D}}_X^\lambda$ compatible with the $\mathbf{G}_m$-action on $\mathbf{A}^1$, given by scaling t. Let $\mathcal{D}_X^\lambda$ denote the induced sheaf of algebras on $X \times \mathbf{A}^1/\mathbf{G}_m$.

Remark 5.6. Since the coordinate t acts as λ, the datum of a $\widetilde{\mathcal{D}}_X^\lambda$-module is precisely a sheaf along with an action of vector fields on X, such that a differential form of order k acts with weight k (witnessed by t^k). In particular, a vector bundle with a flat λ-connection on a vector bundle equips it with the structure of a $\widetilde{\mathcal{D}}_X^\lambda$-module.

Remark 5.7. The discussion in Remark 5.4 implies that the fiber of $\mathcal{D}_X^\lambda$ over $1 : X \to X \times \mathbf{A}^1/\mathbf{G}_m$ is precisely $\mathcal{D}_X$ itself. The fiber of $\mathcal{D}_X^\lambda$ over $0 : X \times B\mathbf{G}_m \to X \times \mathbf{A}^1/\mathbf{G}_m$ is the associated graded of $\widetilde{\mathcal{D}}_X^\lambda$ with respect to the t-adic filtration. But

$$\mathrm{gr}(\widetilde{\mathcal{D}}_X^\lambda) = \sum_{k\geq 0} \mathcal{O}_X[t]t^k F_k \mathcal{D}_X / \sum_{k\geq 0} \mathcal{O}_X[t]t^{k+1}\mathcal{D}_X^{\leq k} = \sum_{k\geq 0} \mathcal{O}_X t^{k+1} \mathcal{D}_X^{\leq k}/\mathcal{D}_X^{\leq k-1}.$$

Since the associated graded of the order filtration on $\mathcal{D}_X$ is just $\mathcal{O}_{\mathrm{T}^*X}$, we find that the fiber of $\mathcal{D}_X^\lambda$ over $0 : X \times B\mathbf{G}_m \to X \times \mathbf{A}^1/\mathbf{G}_m$ is just $\mathcal{O}_{\mathrm{T}^*X}$.

In other words, $\mathcal{D}_X^\lambda \in \mathrm{CAlg}(\mathrm{QCoh}(X \times \mathbf{A}^1/\mathbf{G}_m))$ is the desired deformation of $\mathcal{D}_X$. We now turn to the deformation of X_{dR}. This is given by what is known as the deformation to the normal cone.

Construction 5.8. Let $\widetilde{\mathcal{B}}_\bullet$ be the cosimplicial scheme defined by

$$\widetilde{\mathcal{B}}_\bullet : \Delta \to \mathrm{Aff}_{/\mathbf{A}^1},\ [n] \mapsto \mathrm{Spec}(\mathbf{C}[x,y]/(x^n - y^n)) = \widetilde{\mathcal{B}}_n.$$

There is a canonical map $\widetilde{\mathcal{B}}_n \to \mathbf{A}^1$ detecting the function x, and this morphism is $\mathbf{G}_m$-equivariant for the canonical scaling action on x and y. Taking quotients produces a cosimplicial stack

$$\mathcal{B}_\bullet : \Delta \to \mathrm{Aff}_{/(\mathbf{A}^1/\mathbf{G}_m)},\ [n] \mapsto \mathrm{Spec}(\mathbf{Z}[x,y]/(x^n - y^n))/\mathbf{G}_m = \mathcal{B}_n.$$

Define a cosimplicial stack $\mathrm{D}_\bullet$ over $\mathbf{A}^1/\mathbf{G}_m$ via

$$\mathrm{D}_\bullet = \mathrm{Hom}_{\mathbf{A}^1/\mathbf{G}_m}(\mathcal{B}_\bullet, X \times \mathbf{A}^1/\mathbf{G}_m).$$

Remark 5.9. The fiber of $\mathrm{Hom}_{\mathbf{A}^1}(\mathcal{B}_\bullet, X \times \mathbf{A}^1)$ over $\mathbf{A}^1 - \{0\}$ is simply $X^{\times n} \times (\mathbf{A}^1 - \{0\})$, while the fiber over 0 is $\mathrm{T}X \times_X \cdots \times_X \mathrm{T}X$. In particular, the fiber of D_n at $1 : \mathrm{Spec}(\mathbf{C}) \to \mathbf{A}^1/\mathbf{G}_m$ is just $X^{\times n}$, while the fiber over $0 : \mathrm{Spec}(\mathbf{C}) \to B\mathbf{G}_m \to \mathbf{A}^1/\mathbf{G}_m$ is $(\mathrm{T}X)^{\times_X n}$. In particular, there is a diagonal morphism $X \times \mathbf{A}^1/\mathbf{G}_m \to \mathrm{D}_\bullet$, which is given away from zero by the diagonal on X, and at zero by the inclusion of X into $\mathrm{T}X$ by the zero section.

Definition 5.10. Define X_λ to be the geometric realization of the stack (we are abusing terminology here; this is just a functor) over $\mathbf{A}^1/\mathbf{G}_m$ given by

$$X_{\lambda,\bullet} = \mathrm{D}_\bullet \times_{(\mathrm{D}_\bullet)_{\mathrm{dR}}} (X \times \mathbf{A}^1/\mathbf{G}_m)_{\mathrm{dR}}.$$

In other words, $X_{\lambda,\bullet}$ is the formal completion of $\mathrm{D}_\bullet$ along the diagonal $X \times \mathbf{A}^1/\mathbf{G}_m \to \mathrm{D}_\bullet$. Let $\widetilde{X}_\lambda = X_\lambda \times_{\mathbf{A}^1/\mathbf{G}_m} \mathbf{A}^1$.

Just as with Theorem 5.2 (in fact, as a consequence of it), one can show:

Theorem 5.11. *There is an equivalence* $\mathrm{QCoh}(X_\lambda) \simeq \mathrm{Mod}(\mathcal{D}_X^\lambda)$.

Remark 5.12. In particular, we find that λ-connections (or rather, a $\mathbf{G}_m$-equivariant family $(\mathcal{D}_\lambda, \mathrm{D}_\lambda)$ of λ-connections) give rise to quasicoherent sheaves on X_λ. In other words, $\mathbf{G}_m$-equivariant families $(\mathcal{D}_\lambda, \mathrm{D}_\lambda)$ of λ-connections are just points in the stack $\underline{\mathrm{Coh}}(X_\lambda)$ of coherent sheaves on X_λ; note that this stack lives over $\mathbf{A}^1/\mathbf{G}_m$. By Proposition 4.10, a harmonic bundle $(\mathcal{F}, \mathrm{D}, \mathrm{D}'')$ on X gives rise to a map $\mathbf{A}^1/\mathbf{G}_m \to \underline{\mathrm{Coh}}(X_\lambda)$: this is induced by the $\mathbf{G}_m$-equivariant map $\mathbf{A}^1 \to \underline{\mathrm{Coh}}(\widetilde{X}_\lambda)$ sending $\lambda \in \mathbf{A}^1$ to $(\mathcal{F}_\lambda, \mathrm{D}_\lambda)$.

Remark 5.13. A particular substack of the stack $\underline{\mathrm{Coh}}(\widetilde{X}_\lambda)$ is often denoted $\mathcal{M}_{\mathrm{Hod}}(X)$ in the literature. Similarly, a particular substack of the stack $\underline{\mathrm{Coh}}(X_{\mathrm{dR}})$ is often denoted $\mathcal{M}_{\mathrm{DR}}(X)$, while a particular substack of the stack $\underline{\mathrm{Coh}}(\mathrm{T}^*X)$ is often denoted $\mathcal{M}_{\mathrm{Dol}}(X)$. These substacks are characterized by the properties appearing in Theorem 3.1. They satisfy the property that the fiber of the stacky quotient $\mathcal{M}_{\mathrm{Hod}}(X)/\mathbf{G}_m$ (which lives over $\mathbf{A}^1/\mathbf{G}_m$) over 1 is $\mathcal{M}_{\mathrm{DR}}(X)$, while the fiber over 0 is $\mathcal{M}_{\mathrm{Dol}}(X)$.

Simpson calls the stack $\mathcal{M}_{\mathrm{Hod}}(X)/\mathbf{G}_m$ (which is a substack of $\underline{\mathrm{Coh}}(X_\lambda)$) the *Hodge filtration on nonabelian cohomology*.

Construction 5.14. In the analytic world, there is an isomorphism $\mathcal{M}_{\mathrm{DR}}(X)^{\mathrm{an}} \simeq \mathcal{M}_{\mathrm{DR}}(\overline{X})^{\mathrm{an}}$, where $\overline{X}$ is the conjugate variety. This isomorphism passes through the Riemann-Hilbert correspondence: given a flat vector bundle $(\mathcal{F}, \mathrm{D}_1)$, identify it with a representation of $\pi_1(X)$ via the Riemann-Hilbert correspondence; since $\pi_1(X) \cong \pi_1(\overline{X})$, another application of the Riemann-Hilbert correspondence gives the bundle $(\overline{\mathcal{F}}, \mathrm{D}_2)$ on $\overline{X}$.

In particular, since the fiber of $\mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{an}} \to \mathbf{A}^1$ over $\mathbf{A}^1 - \{0\}$ is $\mathcal{M}_{\mathrm{DR}}(X)^{\mathrm{an}}$, it agrees with the fiber of $\mathcal{M}_{\mathrm{Hod}}(\overline{X})^{\mathrm{an}} \to \mathbf{A}^1$ over $\mathbf{A}^1 - \{0\}$. Therefore, we may glue the stack $\mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{an}} \to \mathbf{A}^1$ with $\mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{an}} \to \mathbf{A}^1$ over $\mathbf{A}^1 - \{0\}$; this produces the *Deligne-Hitchin twistor space* $\mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{an}} \to \mathbf{P}^1$.

It follows from Remark 5.12 that every harmonic bundle $(\mathcal{F}, \mathrm{D}, \mathrm{D}'')$ on X gives a morphism $\mathbf{A}^1 \to \mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{an}}$, by producing the family $(\mathcal{F}_\lambda, \mathrm{D}_\lambda)$ of λ-connections on X. Similarly, one may produce the family $(\mathcal{F}_{-\overline{\lambda}^{-1}}, \mathrm{D}_{-\overline{\lambda}^{-1}})$ of λ-connections on $\overline{X}$, which in turn produces a morphism $\mathbf{A}^1 \to \mathcal{M}_{\mathrm{Hod}}(\overline{X})^{\mathrm{an}}$. These two morphisms patch together over $\mathbf{A}^1 - \{0\} \subseteq \mathbf{A}^1$, and so we obtain a section $\mathbf{P}^1 \to \mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{an}}$. These are known as "preferred sections". I'll stop here, now, though, because the notes already go through more information than can reasonably be covered in a single talk.

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