Splitting cobordism spectra

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Outline

Cobordism invariants

Let M be a closed n-manifold. Then:

- If $\alpha \in \mathrm{H}^n(M; \mathbf{F}_2)$, then we can define $\int_M \alpha \in \mathbf{F}_2$; if α is the top Steifel-Whitney class w_n, then this is the mod 2 Euler characteristic.
- Suppose M is oriented (and $4n$-dimensional). If $\alpha \in \mathrm{H}^n(M; \mathbf{Z})$, then we can define $\int_M \alpha \in \mathbf{Z}$; if α is the Hirzebruch L-genus polynomial, this recovers the signature of M.
- Suppose M is spin (and $4n$-dimensional). If $\alpha \in \mathrm{KO}^n(M)$, then we can define $\int_M \alpha \in \mathrm{KO}^n$; if α is (dual to) the class of the Dirac operator (viewing K-homology as represented by elliptic differential operators), then this recovers the index of the Dirac operator on M.

Maps of spectra

Each of these cobordism invariants can be refined to a spectrum map from a certain cobordism spectrum to the corresponding cohomology theory. More precisely:

- the mod 2 Euler characteristic can be viewed as a map $\mathrm{MO} \to \mathrm{H}\mathbf{F}_2$, where MO is the spectrum representing the cohomology theory of unoriented cobordism.
- the signature can be viewed as a map $\mathrm{MSO} \to \mathrm{H}\mathbf{Z}$, where MSO is the spectrum representing the cohomology theory of oriented cobordism.
- the A-hat genus can be viewed as a map $\mathrm{MSpin} \to \mathrm{KO}$, where MSpin is the spectrum representing the cohomology theory of spin cobordism.

These can be thought of as refinements of the cobordism invariants above to families of manifold living over some base space. These spectrum maps are also "canonical" in a very precise sense.

Some splittings

In his now-famous thesis, Thom showed:

Theorem (Thom)

The map $\mathrm{MO} \to \mathrm{H}\mathbf{F}_2$ admits a splitting. Similarly, the map $\mathrm{MSO} \to \mathrm{H}\mathbf{Z}$ admits a 2-local splitting.

(The "2-local splitting" means that the map admits a splitting after inverting all primes but 2.) It turns out that the theory is easier — and most interesting — once we 2-localize, so we will implicitly do so from now on.

There is a similar result for spin cobordism, too:

Theorem (Anderson-Brown-Peterson)

The map $\mathrm{MSpin} \to \mathrm{bo}$ admits a (2-local) splitting.

We will now describe a modern proof of Thom's result, and see that it cannot be naively adapted to the Anderson-Brown-Peterson splitting. Our goal in this talk will be to give a program for a modern proof of the Anderson-Brown-Peterson splitting.

The string orientation

Reproving old results is always fun, but our modern approach has other payoffs, too.

Recall that orientations and spin structures can be viewed as liftings of the map $M \to \mathrm{BO}$ classifying the tangent bundle to higher connected covers of BO. The next step up after $\mathrm{BSpin} = \tau_{\geqslant 4}\mathrm{BO}$ classifies "string orientations", and is represented by a spectrum $\mathrm{MString}$.

Theorem (Ando-Hopkins-Rezk)

There is a map $\mathrm{MString} \to \mathrm{tmf}$ to the spectrum of topological modular forms, which refines the Witten genus (the "index of the Dirac operator on the free loop space").

Splitting the string orientation

It has long been expected that the map $\mathrm{MString} \to \mathrm{tmf}$ admits a splitting. Our program for a modern proof of the Anderson-Brown-Peterson splitting also works for tmf, and we get:

Theorem (D.)

A certain conjecture in unstable homotopy theory and a "centrality" conjecture imply that the map $\mathrm{MString} \to \mathrm{tmf}$ admits a splitting.

Moreover, the map $\mathrm{MString}_*(*) \to \mathrm{tmf}_*(*)$ splits (*unconditionally*) as a map of graded abelian groups.

The latter result was stated by Hopkins and Mahowald, but a full proof remained unpublished.

Since it requires less background, we will only focus on the A-hat genus in this talk.

Thom spectra

Let us first focus on splitting $e : \mathrm{MO} \to \mathrm{H}\mathbf{F}_2$; the hard part is constructing a map in the other direction.

For this, we need to recall that MO is the Thom spectrum of the universal real (virtual) bundle over BO, i.e., is obtained by the colimit of $\mathrm{MO}(n)$, where $\mathrm{MO}(n) = \Sigma^{-n}\Sigma^{\infty}\mathrm{Thom}(\mathrm{BO}(n); \xi_n)$. We will write this as $\mathrm{MO} = \mathrm{BO}^{\xi}$.

In general, if $f : X \to Y$ is a map of spaces, and γ is a (virtual) bundle over Y, then there is an induced map $X^{f^*\gamma} \to Y^{\gamma}$ of Thom spectra. So:

Upshot

If we could find a space H and a map $\mu : H \to \mathrm{BO}$ such that the Thom spectrum $H^{\mu^*\xi}$ is $\mathrm{H}\mathbf{F}_2$, then we would get a map $\mathrm{H}\mathbf{F}_2 \to \mathrm{MO}$.

Hopkins-Mahowald

Let us now construct H. Let $\mu : S^1 \to \mathrm{BO}$ be the Mobius bundle. By Bott periodicity, BO is an infinite loop space, so this can be viewed as a map $S^3 \to \mathrm{B}^3\mathrm{O}$. Taking the double loop space gives a map $\mu : \Omega^2 S^3 \to \mathrm{BO}$.

Theorem (Hopkins-Mahowald)

The Thom spectrum $(\Omega^2 S^3)^\mu$ is equivalent to $\mathrm{H}\mathbf{F}_2$.

The induced map $\mathrm{H}\mathbf{F}_2 = (\Omega^2 S^3)^\mu \to \mathrm{BO}^\xi = \mathrm{MO}$ is the desired splitting.

The proof of this result boils down via the Thom isomorphism to a (structured) isomorphism between

$$\mathrm{H}_*(\Omega^2 S^3; \mathbf{F}_2) \simeq \mathbf{F}_2[x_1, x_3, x_7, ...]$$

and Milnor's computation of the dual Steenrod algebra

$$\mathrm{H}_*(\mathrm{H}\mathbf{F}_2; \mathbf{F}_2) \simeq \mathbf{F}_2[\xi_1, \xi_2, \xi_3, ...].$$

Hopkins-Mahowald, continued

In fact, this suggests an approach to splitting the signature $s : \mathrm{MSO} \to \mathrm{H}\mathbf{Z}$, too. Again, because MSO is the Thom spectrum of the universal oriented bundle over BSO, if we could find a space Z and a map $\mu : Z \to \mathrm{BSO}$ such that the Thom spectrum $Z^{\mu^* \xi}$ is $\mathrm{H}\mathbf{Z}$, then we would get a map $\mathrm{H}\mathbf{Z} \to \mathrm{MSO}$.

One can calculate that $\mathrm{H}_*(\mathrm{H}\mathbf{Z}; \mathbf{F}_2) \simeq \mathbf{F}_2[\xi_1^2, \xi_2, \xi_3, ...]$; this is missing the ξ_1 in the dual Steenrod algebra, corresponding to the Bockstein.

This space Z would satisfy the Thom isomorphism $\mathrm{H}_*(Z; \mathbf{F}_2) \simeq \mathrm{H}_*(\mathrm{H}\mathbf{Z}; \mathbf{F}_2)$, so based off Hopkins-Mahowald, we might expect that Z is obtained by killing the degree 1 class in $\mathrm{H}_*(\Omega^2 S^3; \mathbf{F}_2)$.

Theorem (Hopkins-Mahowald)

Let $\Omega^2 S^3\langle 3\rangle$ denote the fiber of the canonical map $\Omega^2 S^3 \to S^1$, so there is a map $\Omega^2 S^3\langle 3\rangle \to \mathrm{BSO}$. Then the Thom spectrum of the induced bundle over $\Omega^2 S^3\langle 3\rangle$ is $\mathrm{H}\mathbf{Z}$.

As before, the induced map $\mathrm{H}\mathbf{Z} = (\Omega^2 S^3\langle 3\rangle)^\mu \to \mathrm{BSO}^\mu = \mathrm{MSO}$ splits the signature.

Fail.

Can we run the same sort of argument to split the A-hat genus? Namely, is there a space K and a bundle μ over K such that $K^{\mu} \simeq \mathrm{bo}$?

Theorem (Mahowald, Rudyak)

NO.

In the case of the string orientation, Chatham has proved that tmf also can't be a Thom spectrum.

Our approach to splitting the A-hat genus gets around this problem by using a more general notion of bundle, developed in modern language by Ando-Blumberg-Gepner-Hopkins-Rezk (ABGHR).

Intuition

To motivate this more general construction from the point of view of splitting the A-hat genus, we state the following "intuition", which is basically the Thom isomorphism:

"Intuition"

The ring $\mathrm{H}_*(\mathrm{H}\mathbf{Z}; \mathbf{F}_2) \cong \mathbf{F}_2[\xi_1^2, \xi_2, \xi_3, \cdots]$ is obtained from $\mathrm{H}_*(\mathrm{point}; \mathbf{F}_2) = \mathbf{F}_2$ by "gluing on" the homology of $\Omega^2 S^3\langle 3\rangle$ in a "twisted" way.

There is an isomorphism

$$\mathrm{H}_*(\mathrm{bo}; \mathbf{F}_2) \cong \mathbf{F}_2[\xi_1^4, \xi_2^2, \xi_3, \cdots],$$

and so what the Mahowald-Rudyak result tells us is that there is no space K such that $\mathrm{H}_*(\mathrm{bo}; \mathbf{F}_2)$ is obtained from $\mathrm{H}_*(\mathrm{point}; \mathbf{F}_2) = \mathbf{F}_2$ by "gluing on" the homology of K in a "twisted" way.

Algebraic observation

The previous page stated two isomorphisms:

$$\mathrm{H}_*(\mathrm{H}\mathbf{Z}; \mathbf{F}_2) \cong \mathbf{F}_2[\xi_1^2, \xi_2, \xi_3, \cdots],$$
$$\mathrm{H}_*(\mathrm{bo}; \mathbf{F}_2) \cong \mathbf{F}_2[\xi_1^4, \xi_2^2, \xi_3, \cdots].$$

Note that the subring $\mathbf{F}_2[\xi_2^2, \xi_3, \cdots] \subseteq \mathrm{H}_*(\mathrm{bo}; \mathbf{F}_2)$ looks exactly like $\mathrm{H}_*(\mathrm{H}\mathbf{Z}; \mathbf{F}_2)$. We know that there is a space Z such that $\mathrm{H}_*(Z; \mathbf{F}_2) = \mathrm{H}_*(\mathrm{H}\mathbf{Z}; \mathbf{F}_2)$; so is there is a space K such that

$$\mathrm{H}_*(K; \mathbf{F}_2) \cong \mathbf{F}_2[\xi_2^2, \xi_3, \cdots]$$

as graded abelian groups?

Cohen-Moore-Neisendorfer, Gray, Selick, Theriault, ...

Yes. (But there are subleties since we are working with mod 2 homology.)

Splitting the A-hat genus

So, if there is also a spectrum A such that $\mathrm{H}_*(A) = \mathbf{F}_2[\xi_1^4]$, then perhaps bo can be obtained by "gluing on" K to A in a "twisted" way.

If we then had a map $A \to \mathrm{MSpin}$ which extended across this "gluing" $A \to \mathrm{bo}$, then we would have a splitting of the A-hat genus.

Fact

This spectrum A exists (and is rather easy to construct), and there is a canonical map $A \to \mathrm{MSpin}$.

We therefore need to describe how to "glue" K to A — this is where ABGHR's "generalized Thom spectra" comes in.

Splitting the string orientation

Let me briefly indicate the necessary modifications in the case of the string orientation. There is an isomorphism

$$\mathrm{H}_*(\mathrm{tmf}; \mathbf{F}_2) \cong \mathbf{F}_2[\xi_1^8, \xi_2^4, \xi_3^2, \xi_4, \cdots].$$

Moreover, there is a space K_3 such that

$$\mathrm{H}_*(K_3; \mathbf{F}_2) \cong \mathbf{F}_2[\xi_3^2, \xi_4, \cdots]$$

as graded abelian groups. There is also a spectrum B (which is not as easy to construct as A) such that $\mathrm{H}_*(B) \cong \mathbf{F}_2[\xi_1^8, \xi_1^4]$. There is a canonical map $B \to \mathrm{MString}$.

Then, tmf can be obtained by "gluing on" K_3 to B in a "twisted" way.

So, if the canonical map $B \to \mathrm{MString}$ extended across this "gluing" $B \to \mathrm{tmf}$, then we would have a splitting of the string orientation.

A table

We have the following table.

"Universal base"	Cobordism	Cohomology	Cobordism invariant
S^0, sphere spectrum	MSO	$\mathrm{H}\mathbf{Z}$	Signature
A	MSpin	bo	A-hat genus
B	$\mathrm{MString}$	tmf	String orientation

Moreover, there are always maps

$$\text{Universal base} \to \text{Cobordism} \to \text{Cohomology},$$

and the "cohomology" is obtained from the "universal base" by a gluing procedure.

Gluing, finally

We will now describe ABGHR's approach to "gluing on" a space X to a spectrum R in a twisted way. The "untwisted" gluing is just the chains $C_*(X; R)$ with coefficients in R.

If one thinks of the chains as similar to the group ring $k[G]$ of a group G with coefficients in a ring k, then the "twisted" gluing of X to R is supposed to be like the twisted group ring.

Recall that if $\tau \in Z^2(G; k^\times)$ is a 2-cocycle, then we can define the twisted group ring $k^\tau[G]$. A 2-cocycle gives a class in $\mathrm{H}^2(G; k^\times)$, i.e., a map $G \to Bk^\times$.

Similarly, given an associative ring spectrum R, we can form its units $\mathrm{GL}_1(R)$, which can be delooped to a space $\mathrm{BGL}_1(R)$.

ABGHR

If $\tau : X \to \mathrm{BGL}_1(R)$ is a map, then we can form the "twisted gluing"/generalized Thom spectrum X^τ. If τ is nullhomotopic, then this is $C_*(X; R)$.

Extending across gluings

Again, let R be an associative ring spectrum and X a space.

ABGHR

Suppose R' is an associative ring spectrum with a ring map $f : R \to R'$. If $\tau : X \to \mathrm{BGL}_1(R)$ is a map, then f extends along $R \to X^\tau$ if and only if the composite

$$X \xrightarrow{\tau} \mathrm{BGL}_1(R) \xrightarrow{f} \mathrm{BGL}_1(R')$$

is nullhomotopic.

Splitting the A-hat genus, redux

Recall that for the A-hat genus, we had a map $A \to \mathrm{MSpin}$, and we wanted to:

1. "glue" the space K to A, via some map $\mu : K \to \mathrm{BGL}_1(A)$;
2. extend the map $A \to \mathrm{MSpin}$ along $A \to K^\mu$.
3. show that K^μ is bo;

ABGHR tells us that (1) and (2) follow if we construct μ and show that

$$K \xrightarrow{\mu} \mathrm{BGL}_1(A) \to \mathrm{BGL}_1(\mathrm{MSpin})$$

is null. Neither step is obvious, but a certain conjecture in unstable homotopy theory allows us to construct μ, and it is almost immediate from its construction that the composite is null.

Step (3) is a calculation which utilizes a result of Adams and Priddy, which states that bo is uniquely determined by its homology.

Again, essentially the same steps gets the splitting of the string orientation (conditional on the aforementioned conjectures).

Some questions

Thom's result about MO and MSO is actually stronger than we stated: not only does the map $\mathrm{MO} \to \mathrm{H}\mathbf{F}_2$ split, but MO is actually a wedge sum of copies of $\mathrm{H}\mathbf{F}_2$. Similarly, MSO is actually a wedge sum of copies of $\mathrm{H}\mathbf{Z}$ and $\mathrm{H}\mathbf{F}_2$. Anderson-Brown-Peterson also showed that MSpin is a wedge sum of copies of connected covers of bo, $\mathrm{H}\mathbf{Z}$, and $\mathrm{H}\mathbf{F}_2$.

Question

Suppose the conjectures assumed in our program are true, so the map $\mathrm{MString} \to \mathrm{tmf}$ admits a splitting. Then, is $\mathrm{MString}$ a wedge sum of covers of tmf, bo, $\mathrm{H}\mathbf{Z}$, and $\mathrm{H}\mathbf{F}_2$?

Some questions

If X is any space, then there are also isomorphisms

$$\mathrm{MO}_*(X) \otimes_{\mathrm{MO}_*(*)} \mathbf{F}_2 = \mathrm{H}_*(X; \mathbf{F}_2),$$
$$\mathrm{MSO}_*(X) \otimes_{\mathrm{MSO}_*(*)} \mathbf{Z} = \mathrm{H}_*(X; \mathbf{Z}),$$
$$\mathrm{MSpin}_*(X) \otimes_{\mathrm{MSpin}_*(*)} \mathrm{KO}_*(*) = \mathrm{KO}_*(X),$$

where the latter was proved by Hopkins and Hovey.

Question

Suppose the conjectures assumed in our program are true, so the map $\mathrm{MString} \to \mathrm{tmf}$ admits a splitting. Is there an isomorphism

$$\mathrm{MString}_*(X) \otimes_{\mathrm{MString}_*(*)} \mathrm{TMF}_*(*) = \mathrm{TMF}_*(X)?$$

This is work in progress.

Thank you!

Thank you for listening to this talk, and for the invitation to speak.

I'd like to thank Jeremy Hahn, Mark Behrens, Peter May, and Haynes Miller (to name a few) for many discussions and advice over the years. I'd also like to apologize to anyone whose names I've accidentally omitted either in the above list or in the talk.