Orientations of derived formal groups

Sanath Devalapurkar

1. Introduction

In previous lectures, we discussed the spectral deformation theory of p-divisible groups. The main result we proved was (see [**Lur16**, Theorem 3.0.11]):

Theorem 1.1. *Let $\mathbf{G}_0$ be a nonstationary p-divisible group over a Noetherian F-finite $\mathbf{F}_p$-algebra R_0* [1]*. Then there is a universal deformation of $\mathbf{G}_0$: in other words, there is a Noetherian connective $\mathbf{E}_\infty$-ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ equipped with a universal deformation $\mathbf{G}$ of $\mathbf{G}_0$.*

In analogy with the classical story, one might hope that the universal deformation of a p-divisible formal group $\mathbf{G}_0$ over a field k of characteristic p would give Morava E-theory $E(k, \mathbf{G}_0)$ — but this is not true! Morava E-theory is 2-periodic, but $R^{\mathrm{un}}_{\mathbf{G}_0}$ is a connective $\mathbf{E}_\infty$-ring.

The reason for this apparent failure can be boiled down to a very simple problem: we did not ask that these deformations of $\mathbf{G}_0$ have anything to do with topology. At the moment, this a rather vague statement, but later in this lecture we will make it more precise. For now, let us illustrate with the concrete example of $\mathbf{G}_0 = \mu_{p^\infty}$ (over an algebraically closed field k of characteristic p). The Cartier dual of $\mathbf{G}_0$ is just the constant group scheme $\mathbf{Q}_p/\mathbf{Z}_p$ (if k was not algebraically closed, this would just be an étale group scheme), and the deformation theory of the constant group scheme is trivial. It follows that $\mathrm{Def}_{\mu_{p^\infty}}$ is representable by $\mathrm{Spf}\, S_p$, so that $R^{\mathrm{un}}_{\mu_{p^\infty}} = S_p$, the p-complete sphere.

We already know that $E(k, \mu_{p^\infty})$ is supposed to be p-adic K-theory, so we would like a way of constructing (via an algebro-geometric procedure) K_p from S_p. To do this, we take a hint from a classical result of Snaith's (see [**Sna81**]):

Theorem 1.2 (Snaith)**.** *There is an equivalence $\Sigma^\infty_+ \mathbf{C}P^\infty[\beta^{\pm 1}] \simeq K$.*

There is therefore a canonical map of $\mathbf{E}_\infty$-rings $\Sigma^\infty_+ \mathbf{C}P^\infty \to K$, given by localization at the Bott element.

Remark 1.3. This map of $\mathbf{E}_\infty$-rings can be constructed without ever having to refer to Snaith's theorem: the inclusion $\mathbf{C}P^\infty \hookrightarrow \mathrm{GL}_1 K$ is adjoint to the $\mathbf{E}_\infty$-ring map $\Sigma^\infty_+ \mathbf{C}P^\infty \to K$.

We are left with accomplishing the following two tasks:

[1] This just means that $\mathbf{G}_0$ is classified by an unramified map $\mathrm{Spec}\, R_0 \to \mathscr{M}_{\mathrm{BT}}$ over a ring R_0 with a finite Frobenius map $\phi : R_0 \to R_0$.

(1) Construct (again, via an algebro-geometric procedure) $\Sigma^\infty_+ \mathbf{C}P^\infty$ from S_p.
(2) Define the Bott element in $\pi_2 \Sigma^\infty_+ \mathbf{C}P^\infty$.

We will accomplish both of these tasks (and more) in this lecture, where S_p is replaced by a general $\mathbf{E}_\infty$-ring, and μ_{p^∞} is replaced by a general formal group. For the purpose of concreteness, we will illustrate (almost) everything with the example of the formal multiplicative group throughout these notes.

Remark 1.4. We used Snaith's theorem as a motivating construction, but one can actually easily recover his result from the content of this and the following lectures.

2. Dualizing sheaves on formal groups

In the previous lecture, Robert defined the dualizing line of a formal group $\mathbf{G}_0 : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ (with underlying formal hyperplane $X = \Omega^\infty \mathbf{G}_0$) over an $\mathbf{E}_\infty$-ring R, with a fixed basepoint $\eta \in X(R)$. This required us to be fairly careful: the naïve definition as the pullback $\eta^* \mathbf{L}_{X/R}$ of the cotangent complex is not sufficient. The primary issue with this construction is that if R is an ordinary ring, then $\eta^* \mathbf{L}_{X/R}$ is *not* concentrated in degree 0, so it does not agree with the cotangent space $R \otimes_{\mathscr{O}_X} \Omega_{\mathscr{O}_X/R}$. These problems are remedied by the dualizing line, whose definition and key properties we will now recall.

We will fix an $\mathbf{E}_\infty$-ring R and a formal hyperplane (which will always be one-dimensional) X over R, with a basepoint $\eta \in X(\tau_{\geq 0} R)$. In all cases of interest, X will arise as $\Omega^\infty \mathbf{G}_0$.

Definition 2.1. Define $\mathscr{O}_X(-\eta)$ by the cofiber sequence

$$\mathscr{O}_X(-\eta) \to \mathscr{O}_X \xrightarrow{\eta} R;$$

then the dualizing line $\omega_{X,\eta}$ is defined to be $\mathscr{O}_X(-\eta) \otimes_{\mathscr{O}_X} R$.

Proposition 2.2. *The dualizing line satisfies the following properties:*

(1) $\omega_{X \otimes_R R', \eta \otimes_R R'} \simeq \omega_{X,\eta} \otimes_R R'$ *for any* $\mathbf{E}_\infty$*-ring map* $R \to R'$.
(2) *A map* $f : X \to X'$ *of hyperplanes is an equivalence if and only if the map* $\omega_{X',\eta'} \to \omega_{X,\eta}$ *is an equivalence.*
(3) $\omega_{X,\eta}$ *sits in a fiber sequence of* R*-modules*

$$\Sigma \omega_{X,\eta} \to R \otimes_{\mathscr{O}_X} R \xrightarrow{m} R.$$

Remark 2.3. When R is a classical ring, and X is a formal hyperplane over R, we may identify $\omega_{X,\eta}$ with $\ker(\epsilon)/\ker(\epsilon)^2$, where $\epsilon : \mathscr{O}_X \to R$ is the augmentation. This is exactly the cotangent space.

Construction 2.4 (Linearization)**.** Using Proposition 2.2, we obtain a map, natural in the connective $\mathbf{E}_\infty$-R-algebra A:

$$\begin{tikzcd}
\Omega X(A) \arrow[r, equal] \arrow[dr, "\text{linearization}"'] & \mathrm{Map}_{\mathrm{CAlg}_R}(R \otimes_{\mathscr{O}_X} R, A) \arrow[r] & \mathrm{Map}_{\mathrm{Mod}_R}(R \otimes_{\mathscr{O}_X} R, A) \arrow[d] \\
& \mathrm{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, \Sigma^{-1} A) \arrow[r, equal] & \mathrm{Map}_{\mathrm{Mod}_R}(\Sigma \omega_{X,\eta}, A)
\end{tikzcd}$$

The linearization map is particularly important when $A = \tau_{\geq 0} R$.

Example 2.5. The strict multiplicative group $\mathbf{G}_m : \mathrm{CAlg} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ is defined via

$$\mathbf{G}_m(R) = \mathrm{Map}_{\mathrm{Sp}}(\mathrm{H}\mathbf{Z}, \mathrm{GL}_1(R)) \simeq \mathrm{Map}_{\mathrm{CAlg}}(\Sigma^\infty_+ \mathbf{Z}, R).$$

The last identification above shows that $R \mapsto \Omega^\infty \mathbf{G}_m(R)$ is represented by $\mathrm{Spec}\, \Sigma^\infty_+ \mathbf{Z} \simeq \mathrm{Spec}\, S[t^{\pm 1}]$. Of course, one can now define $\mathbf{G}_m$ over any $\mathbf{E}_\infty$-ring by base change. Let $\mathbf{G}_0$ be the formal multiplicative group $\widehat{\mathbf{G}}_m$. This is defined to be the formal completion of the strict multiplicative group $\mathbf{G}_m$; in other words, $\widehat{\mathbf{G}}_m$ is defined by the fiber sequence

$$\widehat{\mathbf{G}}_m \to \mathbf{G}_m(R) \to \mathbf{G}_m(R^{\mathrm{red}}).$$

By construction, this is representable by $S[t^{\pm 1}]^\wedge_{(t-1)}$. Therefore,

$$S \otimes_{\mathscr{O}_{\widehat{\mathbf{G}}_m}} S \simeq S \otimes_{\Sigma^\infty_+ \mathbf{Z}} S \simeq \Sigma^\infty_+ B\mathbf{Z} \simeq \Sigma^\infty_+ S^1 \simeq \Sigma^\infty S^1 \vee S.$$

By Proposition 2.2, we learn that $\omega_{\widehat{\mathbf{G}}_m} \simeq S$. It follows that the diagram defining the linearization map becomes (our base scheme here is S, so A is any connective $\mathbf{E}_\infty$-ring)

$$\begin{tikzcd}
\Omega^{\infty+1}\widehat{\mathbf{G}}_m(A) \arrow[r, equal] \arrow[d, "\text{linearization}"'] & \mathrm{Map}_{\mathrm{CAlg}}(\Sigma^\infty_+ S^1, A) \arrow[r] & \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+ S^1, A) \simeq \Omega^{\infty+1}\mathfrak{gl}_1(A) \arrow[d, "\Omega(x \mapsto x-1)"', "\sim"] \\
\Omega^{\infty+1} A & \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty S^1, \Sigma^{-1} A) \arrow[l] \arrow[r, equal] & \mathrm{Map}_{\mathrm{Sp}}(\Sigma\Sigma^\infty S^1, A)
\end{tikzcd}$$

The linearization map is therefore aptly named.

3. Classifying orientations

In order to proceed, we will need to recall a classical bit of algebraic topology; namely, the following statements are equivalent for a spectrum E:

(1) the Atiyah-Hirzebruch spectral sequence computing $E^*(\mathbf{C}P^\infty)$ degenerates.
(2) the canonical unit element of $\widetilde{E}^2(S^2) \simeq E^0(*) \simeq \pi_0 E$ lies in the image of $\widetilde{E}^2(\mathbf{C}P^\infty) \to \widetilde{E}^2(S^2)$.

The unit element can be thought of as a pointed map $S^2 \to \Omega^\infty E$ (however, this is dependent on the choice of a basepoint of $S^2 \subseteq \mathbf{C}P^\infty$). This motivates:

Definition 3.1. A preorientation of a formal hyperplane $X \to \mathrm{Spec}\, R$ is a pointed map $S^2 \to X(\tau_{\geq 0} R)$.

In particular, the space $\mathrm{Pre}(X)$ of preorientations is exactly $\Omega^2 X(\tau_{\geq 0}R)$. Note that space this is functorial in R. The linearization map above gives a map:

$$\mathrm{Pre}(X) \simeq \Omega(\Omega X(\tau_{\geq 0}R)) \to \Omega \mathrm{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, \Sigma^{-1}R) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(\omega_{X,\eta}, \Sigma^{-2}R).$$

The choice of a preorientation of X therefore determines a map $\omega_{X,\eta} \to \Sigma^{-2}R$ of R-modules; this is called the Bott map.

If X arises as $\Omega^\infty \circ \mathbf{G}_0$ for some formal group $\mathbf{G}_0$, then

$$\mathrm{Pre}(\mathbf{G}_0) = \Omega^{\infty+2}\mathbf{G}_0(\tau_{\geq 0}R) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma^2 \mathbf{Z}, \mathbf{G}_0(\tau_{\geq 0}R)).$$

Example 3.2. By the above discussion, we know that $\mathrm{Pre}(\widehat{\mathbf{G}}_m) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma^2\mathbf{Z}, \widehat{\mathbf{G}}_m(\tau_{\geq 0}R))$. In the fiber sequence

$$\widehat{\mathbf{G}}_m(\tau_{\geq 0}R) \to \mathbf{G}_m(\tau_{\geq 0}R) \to \mathbf{G}_m(\pi_0(R)^{\mathrm{red}}),$$

the third term is discrete. It follows that

$$\begin{aligned}\mathrm{Pre}(\widehat{\mathbf{G}}_m) &\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\Sigma^2\mathbf{Z}, \mathbf{G}_m(\tau_{\geq 0}R)) \\ &\simeq \mathrm{Map}_{\mathrm{CAlg}}(\Sigma^\infty_+\Omega^\infty\Sigma^2\mathbf{Z}, R) \\ &= \mathrm{Map}_{\mathrm{CAlg}}(\Sigma^\infty_+\mathbf{C}P^\infty, R).\end{aligned}$$

Therefore the functor CAlg $\to$ Top given by $R \mapsto \mathrm{Pre}(\widehat{\mathbf{G}}_m)$ is representable the affine scheme $\mathrm{Spec}\,\Sigma^\infty_+\mathbf{C}P^\infty$. We've now accomplished task (1).

Remark 3.3. Note that a preorientation of $X = \Omega^\infty \circ \widehat{\mathbf{G}}_m$ gives a map $\omega_{\widehat{\mathbf{G}}_m,\eta} \simeq R \to \Sigma^{-2}R$ of R-modules, i.e., an element of $\pi_2 R$.

This representability result holds in general:

Proposition 3.4. *Let R be an $\mathbf{E}_\infty$-ring. Suppose X is a formal hyperplane over R. The functor* $\mathrm{CAlg}_R \to \mathrm{Top}$ *given by* $R' \mapsto \mathrm{Pre}(X_{R'})$ *is representable by an affine scheme* $\mathrm{Spec}\,A$.

Proof. The functor $\Omega X : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathrm{Top}$ is corepresentable by the connective $\mathbf{E}_\infty$-ring $B = R \otimes_{\mathscr{O}_X} R$. We noted above that $\mathrm{Pre}(X) \simeq \Omega^2 X(\tau_{\geq 0}R)$, so the functor in the proposition is corepresentable by the connective $\mathbf{E}_\infty$-ring $A = R \otimes_B R$, as desired. $\square$

Remark 3.5. In particular, there is an $\mathbf{E}_\infty$-ring A with a ring map $R \to A$ such that there is a universal preorientation of X_A. This gives a universal Bott map $\omega_{X_A,\eta} \to \Sigma^{-2}A$ of A-modules.

Let E be an even periodic complex oriented $\mathbf{E}_\infty$-ring; then $\widehat{\mathbf{G}}_0 = \mathrm{Spf}\, E^0(\mathbf{C}P^\infty)$ is a formal group over $\pi_0 E$. Picking a coordinate t for $\widehat{\mathbf{G}}_0$, we learn that the cotangent space to $\widehat{\mathbf{G}}_0$ is exactly $(t)/(t)^2$, which is isomorphic to $\pi_2 E$. One should therefore think of an identification of the cotangent space with $\pi_0\Sigma^{-2}E$ as providing a complex orientation (and not just a "preorientation") of E. In fact, this comes from a spectral identification, as we will now discuss.

Example 3.6. Let R be a complex oriented weakly even periodic $\mathbf{E}_\infty$-ring, i.e., what Jacob calls a complex periodic $\mathbf{E}_\infty$-ring. We will denote by $\widehat{\mathbf{G}}^Q_R$ the Quillen formal group; this is the functor $\mathrm{Lat}^{\mathrm{op}}_{\mathbf{Z}} \to \mathrm{coCAlg}^{\mathrm{sm}}_R$ defined by sending M to $R \otimes \Sigma^\infty_+\mathbf{C}P^\infty$. Last time, we proved that this is a smooth formal group over R of dimension 1. Then

$$\mathscr{O}_{\widehat{\mathbf{G}}^Q_R} \simeq \underline{\mathrm{Map}}_{\mathrm{Sp}}(\Sigma^\infty_+\mathbf{C}P^\infty, R) =: C^*(\mathbf{C}P^\infty; R).$$

There is a canonical base point $\eta \in \widehat{\mathbf{G}}^Q_R(\tau_{\geq 0}R)$, given by the map $C^*(\mathbf{C}P^\infty; R) \to R$ defined by evaluation on the basepoint of $\mathbf{C}P^\infty$. It follows from Proposition 2.2 that there is a fiber sequence

$$\begin{tikzcd} \Sigma\omega_{\widehat{\mathbf{G}}^Q_R,\eta} \arrow[r] \arrow[d] & R \otimes_{C^*(\mathbf{C}P^\infty;R)} R \arrow[r] \arrow[d, "\sim"] & R \arrow[d, equal] \\ \Sigma^{-1}R \simeq C^*_{\mathrm{red}}(S^1; R) \arrow[r] & C^*(S^1; R) \arrow[r, "\mathrm{evaluate}"'] & R \end{tikzcd}$$

It follows that there is a *canonical* equivalence $\omega_{\widehat{\mathbf{G}}^Q_R,\eta} \xrightarrow{\sim} \Sigma^{-2}R$.

Remark 3.7. If $\widehat{\mathbf{G}}_0$ is a formal group over a complex periodic $\mathbf{E}_\infty$-ring R, then an identification of $\omega_{\widehat{\mathbf{G}}_0,\eta}$ with $\Sigma^{-2}R$ (via a preorientation) is canonically the same as an identification of $\omega_{\widehat{\mathbf{G}}_0,\eta}$ with $\omega_{\widehat{\mathbf{G}}^Q_R,\eta}$. By Proposition 2.2, this is the same as an identification of $\widehat{\mathbf{G}}_0$ with $\widehat{\mathbf{G}}^Q_R$.

Remark 3.8. The astute reader might argue that we were initially talking about an identification of the cotangent space with $\pi_0\Sigma^{-2}R = \pi_2 R$, which is *a priori* not the same as an identification of the *spectral* R-modules $\omega_{\widehat{\mathbf{G}}_0,\eta}$ with $\Sigma^{-2}R$. This will be made clear in Theorem 3.12.

Our discussion above motivates the following definition.

Definition 3.9. An orientation of a formal hyperplane $X \to \operatorname{Spec} R$ is a preorientation for which the associated Bott map $\omega_{X,\eta} \to \Sigma^{-2}R$ is an equivalence.

As we proved above, this is the same as an identification of X with $\Omega^\infty \circ \widehat{\mathbf{G}}^Q_R$.

Remark 3.10. As $\omega_{X,\eta}$ is locally free of rank 1 as an R-module, R must be weakly even periodic in order for X to admit an orientation. In particular, although preorientations of $X \to \operatorname{Spec} R$ are equivalent to preorientations of $X_{\tau_{\geq 0}R} \to \operatorname{Spec} \tau_{\geq 0}R$, it is not true that orientations of $X \to \operatorname{Spec} R$ are the same as orientations of $X_{\tau_{\geq 0}R} \to \operatorname{Spec} \tau_{\geq 0}R$.

Lemma 3.11. *Let $\mathbf{G}_0$ be a formal group over an $\mathbf{E}_\infty$-ring R. Then there is an equivalence*

$$\operatorname{Pre}(\mathbf{G}_0) \simeq \operatorname{Map}(\widehat{\mathbf{G}}^Q_R, \mathbf{G}_0).$$

Proof. We argued above that

$$\operatorname{Pre}(\mathbf{G}_0) \simeq \operatorname{Map}_{\operatorname{Mod}_{\mathbf{Z}}}(\Sigma^2\mathbf{Z}, \mathbf{G}_0(\tau_{\geq 0}R)) \simeq \operatorname{Map}_{\operatorname{Ab}(\operatorname{Top})}(\mathbf{C}P^\infty, \operatorname{Map}_{\operatorname{coCAlg}_R}(R, \mathscr{O}^\vee_{\mathbf{G}_0})).$$

This reflects the slogan "$\mathbf{C}P^\infty$ is generated by $\mathbf{C}P^1$ as a topological abelian group". Therefore

$$\operatorname{Pre}(\mathbf{G}_0) \simeq \operatorname{Map}_{\operatorname{Ab}(\operatorname{coCAlg}_R)}(R \otimes \Sigma^\infty_+ \mathbf{C}P^\infty, \mathscr{O}^\vee_{\mathbf{G}_0}) \simeq \operatorname{Map}(\widehat{\mathbf{G}}^Q_R, \mathbf{G}_0).$$

$\square$

The following result makes everything run.

Theorem 3.12. *Fix an $\mathbf{E}_\infty$-ring R.*

(1) *Let X be a formal hyperplane over R. Then there is an $\mathbf{E}_\infty$-ring*[2] *R^{or} with a ring map $R \to R^{\mathrm{or}}$ such that there is a universal orientation of $X_{R^{\mathrm{or}}}$.*
(2) *Suppose $\widehat{\mathbf{G}}$ is a formal group over R with a preorientation $e \in \operatorname{Pre}(\widehat{\mathbf{G}})$. Then e is an orientation if and only if*
   (a) *R is complex periodic.*
   (b) *The associated map $\widehat{\mathbf{G}}^Q_R \to \widehat{\mathbf{G}}$ is an equivalence.*

Proof. We begin by proving (1); this is equivalent to proving that the functor $\operatorname{CAlg}_R \to \operatorname{Top}$ given by $R' \mapsto \{\text{orientations of } X_{R'}\}$ is corepresentable. In Proposition 3.4, we showed that the functor $R' \mapsto \operatorname{Pre}(X_{R'})$ is corepresented by an $\mathbf{E}_\infty$-R-algebra A. By construction, this is equipped with a universal Bott map $\omega_{X_A,\eta} \to \Sigma^{-2}A$. In order to prove (1), it therefore suffices to prove the following result: let R be an $\mathbf{E}_\infty$-ring, and suppose $u : L \to L'$ is a map of invertible

[2] Jacob denotes this by $\mathfrak{O}_X$, but I do not know how to draw a fraktur O on the chalkboard.

R-modules. Then there is an object $R[u^{-1}]$ such that for every $A \in \mathrm{CAlg}_R$, we have:

$$\mathrm{Map}_{\mathrm{CAlg}_R}(R[u^{-1}], A) \simeq \begin{cases} * & \text{if } u : A \otimes_R L \xrightarrow{\sim} A \otimes_R L' \\ \emptyset & \text{else.} \end{cases}$$

The proof of this result is just algebra, so we will omit it. There is an equivalence of R-modules

$$\mathrm{colim}(R \xrightarrow{u} L^{-1} \otimes_R L' \xrightarrow{u} (L^{-1})^{\otimes 2} \otimes_R L'^{\otimes 2} \xrightarrow{u} \cdots) \simeq R[u^{-1}].$$

Applying this to the Bott map $\beta : L = \omega_{X_A,\eta} \to \Sigma^{-2}A = L'$, we get the $\mathbf{E}_\infty$-R-algebra $R^{\mathrm{or}} = A[\beta^{-1}]$.

Let us now turn to the proof of (2). Our discussion above establishes that if R is complex periodic and the associated map $\widehat{\mathbf{G}}_R^Q \to \mathbf{G}_0$ (from Lemma 3.11) is an equivalence, then e is an orientation. It suffices to prove the other direction.

Suppose e is an orientation. As (b) is equivalent to the map $\widehat{\mathbf{G}}_R^Q \to \mathbf{G}_0$ being an equivalence (by Proposition 2.2), it suffices to show that R is complex periodic. As R is weakly even periodic by Remark 3.10, it suffices to show that R is complex oriented. In other words, we need to show that the map $\pi_{-2}C^*_{\mathrm{red}}(\mathbf{C}P^\infty; R) \to \pi_{-2}C^*_{\mathrm{red}}(\mathbf{C}P^1; R)$ is surjective. To prove this, we will use the following diagram:

$$\begin{tikzcd}
\mathscr{O}_{\mathbf{G}_0}(-1) \arrow[r] \arrow[d] & \mathscr{O}_{\mathbf{G}_0} \arrow[r] \arrow[d] & R \arrow[d, equal] \\
C^*_{\mathrm{red}}(\mathbf{C}P^\infty; R) \arrow[r] \arrow[d] & C^*(\mathbf{C}P^\infty; R) \arrow[r] \arrow[d] & R \arrow[d, equal] \\
C^*_{\mathrm{red}}(\mathbf{C}P^1; R) \arrow[r] & C^*(\mathbf{C}P^1; R) \arrow[r] & R
\end{tikzcd}$$

where $\mathscr{O}_{\mathbf{G}_0}(-1)$ is defined as the fiber of the augmentation $\mathscr{O}_{\mathbf{G}_0} \to R$. The map $C^*_{\mathrm{red}}(\mathbf{C}P^\infty; R) \to C^*_{\mathrm{red}}(\mathbf{C}P^1; R)$ therefore factors the map $\mathscr{O}_{\mathbf{G}_0}(-1) \to C^*_{\mathrm{red}}(\mathbf{C}P^1; R)$, so it suffices to prove that the latter map is surjective on π_{-2}. This map can be identified with the composite

$$\mathscr{O}_{\mathbf{G}_0}(-1) \to R \otimes_{\mathscr{O}_{\mathbf{G}_0}} \mathscr{O}_{\mathbf{G}_0}(-1) = \omega_{\mathbf{G}_0} \xrightarrow{\beta} \Sigma^{-2}R \simeq C^*_{\mathrm{red}}(\mathbf{C}P^1; R).$$

The Bott map β is an equivalence since e is an orientation. The proof is now completed by observing that the map $\mathscr{O}_{\mathbf{G}_0}(-1) \to \omega_{\mathbf{G}_0}$ is surjective on homotopy. □

Remark 3.13. Let R be an $\mathbf{E}_\infty$-ring and $\widehat{\mathbf{G}}$ a preoriented formal group over R. Denote by $\widehat{\mathbf{G}}_0$ the underlying classical formal group of $\widehat{\mathbf{G}}$, living over $\pi_0 R$. It follows from Theorem 3.12 that a preorientation $e \in \Omega^2\widehat{\mathbf{G}}(R)$ is an orientation if and only if:

(1) $\widehat{\mathbf{G}}_0 \to \mathrm{Spec}\, \pi_0 R$ is smooth of relative dimension 1.
(2) The map $\omega_{\widehat{\mathbf{G}}_0} \to \pi_2 R$ induces isomorphisms

$$\omega_{\widehat{\mathbf{G}}_0} \otimes_{\pi_0 R} \pi_n R \xrightarrow{\beta} \pi_2 R \otimes_{\pi_0 R} \pi_n R \to \pi_{n+2} R$$

for every integer n.

See [**Lur09**, Definition 3.3] for this definition of an orientation.

Example 3.14. It follows from Example 3.2 and Remark 3.3 that the universal Bott element β for $\widehat{\mathbf{G}}_m \to \operatorname{Spec} S$ lies in $\pi_2\Sigma^\infty_+\mathbf{C}P^\infty$. By Example 2.5, we learn that β is exactly given by the inclusion of $S^2 = \mathbf{C}P^1$ into $\mathbf{C}P^\infty$; in other words, β is the usual Bott map. We've now accomplished task (2) as well. It follows from Theorem 3.12 that $S^{\mathrm{or}} = \Sigma^\infty_+\mathbf{C}P^\infty[\beta^{-1}]$.

Example 3.15. Let us return to the discussion in the introduction. Fix a non-stationary p-divisible group $\mathbf{G}_0$ over a Noetherian F-finite $\mathbf{F}_p$-algebra R_0. Denote by $\mathbf{G}$ the universal deformation of $\mathbf{G}_0$ over the $\mathbf{E}_\infty$-ring $R^{\mathrm{un}}_{\mathbf{G}_0}$, and let $\mathbf{G}^\circ$ be the connected component of the identity. Then $\mathbf{G}^\circ$ is a formal group over $R^{\mathrm{un}}_{\mathbf{G}_0}$. By Theorem 3.12, there is an $\mathbf{E}_\infty$-$R^{\mathrm{un}}_{\mathbf{G}_0}$-algebra $R^{\mathrm{or}}_{\mathbf{G}_0}$ such that there is a universal orientation of $\mathbf{G}^\circ \otimes_{R^{\mathrm{un}}_{\mathbf{G}_0}} R^{\mathrm{or}}_{\mathbf{G}_0}$. This $\mathbf{E}_\infty$-ring is the desired analogue of Morava E-theory for p-divisible groups (compare with Example 3.14 and Snaith's theorem).

It is not clear that $R^{\mathrm{or}}_{\mathbf{G}_0}$ agrees with Morava E-theory when R_0 is an algebraically closed field of characteristic p and $\mathbf{G}_0$ is a p-divisible formal group over R_0; this will be the content of the following two lectures. The method of proof of this result is a generalization of the moduli-theoretic proof of Snaith's theorem (see [**Mat12**]). In order to prove this result, it will be simpler to work in the $K(n)$-local category: it turns out that this does not lose any information since one can prove that $R^{\mathrm{or}}_{\mathbf{G}_0}$ is itself $K(n)$-local. We will now develop some methods allowing us to prove that an $\mathbf{E}_\infty$-ring is $K(n)$-local, which will be useful in the sequel.

4. $K(n)$-locality of complex periodic $\mathbf{E}_\infty$-rings

Let us begin with a classical observation[3].

Proposition 4.1. *Let R be a complex oriented ring spectrum (not necessarily an $\mathbf{E}_\infty$-ring). Then there is an equivalence*

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & L_{K(n)}R \arrow[d, dashed, "\sim"] \\ & \operatorname{holim}_{J\in\mathbf{N}^n} v_n^{-1}R/I_n^J =: R_{v_n}, \end{tikzcd}$$

where $I_n = (p, v_1, \cdots, v_{n-1}) \subseteq BP_*$ and $I_n^J = (p^{J_0}, v_1^{J_1}, \cdots, v_{n-1}^{J_{n-1}})$.

Proof. We must first show that the map $R \to R_{v_n}$ factors through $L_{K(n)}R$. It suffices to show that each $v_n^{-1}R/I_n^J$ is $K(n)$-local. The spectrum $v_n^{-1}R/I_n^J$ is built from $v_n^{-1}R/I_n$ by a finite number of cofiber sequence, so it suffices to prove that the spectrum $v_n^{-1}R/I_n$ is $K(n)$-local. This spectrum is a $v_n^{-1}BP/I_n$-module, hence $v_n^{-1}BP/I_n$-local. As $\langle v_n^{-1}BP/I_n\rangle = \langle K(n)\rangle$, it is also $K(n)$-local.

To prove that the map $L_{K(n)}R \to R_{v_n}$ is an equivalence, we must show that $K(n)_*R \xrightarrow{\sim} K(n)_*R_{v_n}$. It suffices to prove this after smashing the map $R \to R_{v_n}$ with a finite complex of type n. Consider the type n complex $X = S/(p^{I_0}, v_1^{I_1}, \cdots, v_{n-1}^{I_{n-1}})$ for some cofinal $(I_0, I_1, \cdots, I_{n-1})$ coming from the Devinatz-Hopkins-Smith nilpotence technology (see [**DHS88, HS98**]); then

$$R_{v_n} \wedge X \simeq \operatorname{holim}_{J\in\mathbf{N}^n}(v_n^{-1}R/I_n^J \wedge X) \simeq v_n^{-1}R/I_n^I.$$

[3] I don't know of a reference for this statement.

Therefore, as $K(n)_*(R \wedge X) \simeq K(n)_*(R/I_n^I)$, we learn that

$$K(n)_*(R \wedge X) \simeq K(n)_*(v_n^{-1} R/I_n^I) \simeq K(n)_*(R_{v_n} \wedge X),$$

as desired. $\square$

Corollary 4.2. *A complex oriented ring spectrum R is $K(n)$-local iff R is I_n-complete and v_n is a unit modulo I_n (in other words, the underlying formal group of the Quillen formal group over $\pi_0 R$ has height at most n).*

Our goal in this section is to give another proof of Corollary 4.2 for $\mathbf{E}_\infty$-rings which des not rely on Devinatz-Hopkins-Smith.

Recall (a standard reference is Paul Goerss' paper [**Goe08**] on quasicoherent sheaves on $\mathscr{M}_{\mathbf{fg}}$):

Definition 4.3. Let $\mathbf{G}_0$ be a formal group over a (classical) $\mathbf{F}_p$-scheme S. Then $\mathbf{G}_0$ has height $\geq n$ if there is a factorization

$$\begin{tikzcd} \mathbf{G}_0 \arrow[r, "\varphi"] \arrow[drrr, "{[p]}"] & \mathbf{G}_0^{(p)} \arrow[r, "\varphi^{(p)}"] & \cdots \arrow[r, "\varphi^{(p^{n-1})}"] & \mathbf{G}_0^{(p^n)} \arrow[d, "T"] \\ & & & \mathbf{G}_0 \end{tikzcd}$$

Construction 4.4. The map T induces a map $T^* : \omega_{\mathbf{G}_0} \to \omega_{\mathbf{G}_0^{(p^n)}} \simeq \omega_{\mathbf{G}_0}^{\otimes p^n}$. As $\omega_{\mathbf{G}_0}$ is a line bundle, this is the same as a map $\mathscr{O}_S \to \omega_{\mathbf{G}_0}^{\otimes(p^n-1)}$. This defines a global section $v_n \in \omega_{\mathbf{G}_0}^{\otimes(p^n-1)}$, called the nth Hasse invariant. Let $\mathscr{M}(n+1)$ be the closed substack of $\mathscr{M}_{\mathbf{fg}}$ defined by the line bundle $\omega_{\mathbf{G}_0}^{\otimes p^n-1}$ and the section v_n.

Definition 4.5. Let $\mathscr{I}_n$ denote the ideal sheaf defining the closed substack $\mathscr{M}(n)$, so that $\mathscr{I}_n$ is the image of the injection $v_n : \omega_{\mathbf{G}_{\mathrm{univ}}}^{\otimes -(p^n-1)} \to \mathscr{O}_{\mathscr{M}_{\mathbf{fg}}}$. If $S = \operatorname{Spec} R$ is a $\mathbf{F}_p$-scheme and $\mathbf{G}_0$ is given by a map $f : S \to \mathscr{M}_{\mathbf{fg}}$, the pullback $f^*\mathscr{I}_n =: I_n^{\mathbf{G}_0}$ defines an ideal of R. This is called the nth Landweber ideal of $\mathbf{G}_0$.

Notation 4.6. If R is an $\mathbf{E}_\infty$-ring and $\mathbf{G}$ is a formal group over R, we set $I_n^{\mathbf{G}} = I_n^{\mathbf{G}_0} \subseteq \pi_0 R$. Let R be an $\mathbf{E}_\infty$-ring, and $\mathbf{G}$ be a formal group over R. Say that $\mathbf{G}$ has height $< n$ if $I_R^{\mathbf{G}} = \pi_0 R$.

Definition 4.7. If R is complex periodic, we set $I_n = I_n^{\widehat{\mathbf{G}}_R^Q}$, with R left implicit; this is the nth Landweber ideal[4] of R.

Let $\widehat{\mathbf{G}}_R^{Q_n}$ denote the base change $\widehat{\mathbf{G}}_R^Q \otimes_{\pi_0 R} \pi_0 R/I_n$. By construction, $\widehat{\mathbf{G}}_R^{Q_n}$ has height $\geq n$. Moreover, it follows from Proposition 2.2 that $\omega_{\widehat{\mathbf{G}}_R^{Q_n}} = \pi_2(R)/I_n$. The section v_n is now an element of $\pi_{2(p^n-1)}(R)/I_n$. Let $\overline{v}_n$ denote any lift of v_n to $\pi_{2(p^n-1)} R$; then I_{n+1} is generated by I_n and $\overline{v}_n \pi_{-2(p^n-1)} R$.

We can now state the generalization of Corollary 4.2. Assume that we have p-localized everywhere.

Theorem 4.8. *Let R be a complex periodic $\mathbf{E}_\infty$-ring and let $n > 0$. The R-module M is $K(n)$-local if and only if the following conditions are satisfied:*

(1) *M is complete with respect to $I_n \subseteq \pi_0 R$.*

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[4]Jacob denotes this by $\mathfrak{I}_n^R$, but again, I do not know how to write a fraktur I.

(2) *multiplication by* $\overline{v}_n$ *induces an equivalence* $\Sigma^{2(p^n-1)}M \to M$.

Proof. Assume that (1) and (2) are satisfied. It suffices to prove the following statement for all $0 \leq m \leq n$: if N is a perfect R-module which is I_m-nilpotent, then $M \otimes_R N$ is $K(n)$-local. Indeed, when $n = 0$, choosing $N = R$ gives us that $M = M \otimes_R R$ is $K(n)$-local.

This statement is proved by descending induction along m. We first prove the statement in the case $m = n$. To prove that $M \otimes_R N$ is $K(n)$-local, we need to show that for any $K(n)$-acyclic[5] spectrum X, the space $\mathrm{Map}_{\mathrm{Sp}}(X, M \otimes_R N) \simeq \mathrm{Map}_{\mathrm{Sp}}(X \otimes N^\vee, M)$ is contractible. As usual, $N^\vee$ denotes the R-linear dual of N. It therefore suffices to prove that $X \otimes N^\vee$ is zero.

The spectrum $MUP \otimes R$ is faithfully flat over R; this is a classical result (e.g., in Adams' blue book) but we have chosen to rephrase it in fancy language. Therefore it suffices to prove that $X \otimes N^\vee \otimes_R MUP \otimes R \simeq X \otimes N^\vee \otimes MUP$ is contractible.

Let $u \in \pi_2 MUP$ be an invertible element. As $v_m \in \pi_{2(p^m-1)} MUP/I_m^{MUP}$, we can choose elements $w_m \in \pi_0 MUP$ such that $w_m = \overline{v}_m u^{-(p^m-1)}$. By construction, $(w_0, \cdots, w_{n-1})$ generate I_n^{MUP}. Clearly I_n^{MUP} and I_n generate the same ideal inside $\pi_0(R \otimes MUP)$, so perfectness and I_n-nilpotence of N implies that $N^\vee \otimes MUP$ is a perfect module over $R \otimes MUP$ which is I_n^{MUP}-nilpotent.

$N^\vee \otimes MUP$ is a retract of $N^\vee \otimes MUP/(w_0^k, \cdots, w_{n-1}^k)$ for $k \gg 0$ by construction, so it suffices to prove that each $X \otimes N^\vee \otimes MUP/(w_0^k, \cdots, w_{n-1}^k)$ vanishes. However, as we can build $MUP/(w_0^k, \cdots, w_{n-1}^k)$ from $MUP/(w_0, \cdots, w_{n-1})$ by a finite number of cofiber sequences, it suffices to show that $X \otimes N^\vee \otimes MUP/(w_0, \cdots, w_{n-1})$ vanishes.

As before, w_n acts invertibly on $N^\vee \otimes MUP$, so it can be regarded as a $R \otimes MUP[w_n^{-1}]$-module. In particular, it suffices to show that $X \otimes N^\vee \otimes MUP/(w_0, \cdots, w_{n-1})[w_n^{-1}]$ vanishes. However, $MUP/(w_0, \cdots, w_{n-1})[w_n^{-1}]$ is $v_n^{-1}BP/I_n$-local, hence $K(n)$-local. As X is $K(n)$-acyclic, we learn that $X \otimes MUP/(w_0, \cdots, w_{n-1})[w_n^{-1}]$ is contractible, as desired.

To prove that (1) and (2) imply that M is $K(n)$-local, it remains to establish the inductive step. Concretely, we need to show that N being a perfect R-module which is I_m-nilpotent implies that $M \otimes_R N$ is $K(n)$-local. Condition (1) says that M is I_n-complete, so perfectness of N implies that $M \otimes_R N$ is also I_n-complete. Therefore

$$M \otimes_R N = \operatorname{holim} M \otimes_R (N/v_m^k).$$

Each N/v_m^k is I_{m+1}-nilpotent, so $M \otimes_R (N/v_m^k)$ is $K(n)$-local by the inductive hypothesis.

It remains to establish that if M is $K(n)$-local, then (1) and (2) are satisfied. To establish (1), we need to show that M is (x)-complete for every $x \in I_n$. In other words, we must show that for every $R[1/x]$-module N, the space $\mathrm{Map}_{\mathrm{Mod}_R}(N, M)$ is contractible. As M is $K(n)$-local, there is an equivalence

$$\mathrm{Map}_{\mathrm{Mod}_R}(N, M) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(L_{K(n)}N, M).$$

It therefore suffices to show that $L_{K(n)}N = 0$, i.e., $K(n) \otimes N = 0$. This is a $K(n) \otimes R[1/x]$-module, so it suffices to show that $K(n) \otimes R[1/x] = 0$. This is easy: the ring $\pi_0(K(n) \otimes N)$ carries two formal group laws, namely the height n formal group law from $K(n)$, and the height $< n$ formal group law from $R[1/t]$. These

[5] There is a typo in Jacob's book.

cannot be isomorphic as they are of different heights, so $K(n) \otimes R[1/x] = 0$, as desired.

To establish (2), we need to show that the map $\Sigma^{2(p^n-1)} M \xrightarrow{\overline{v}_n} M$ is an equivalence. As M is $K(n)$-local, it suffices to show that $\Sigma^{2(p^n-1)} M \otimes K(n) \xrightarrow{\overline{v}_n} M \otimes K(n)$ is an equivalence. As the formal group law over $\pi_0(R \otimes K(n))$ has height n, this map is an isomorphism on homotopy, as desired. □

This recovers a special case of Corollary 4.2:

Corollary 4.9. *Let R be a complex periodic $\mathbf{E}_\infty$-ring and let $n > 0$. Then R is $K(n)$-local if and only if:*

(1) *R is I_n-complete.*
(2) *$I_{n+1} = \pi_0 R$, i.e., $\widehat{\mathbf{G}}_R^Q$ has height $\leq n$.*

Proof. Suppose (1) and (2) are satisfied. As R is I_n-complete, Theorem 4.8 says that R is $K(n)$-local if and only if multiplication by $\overline{v}_n$ induces an equivalence $\Sigma^{2(p^n-1)} R \to R$ of R-modules. In other words, it suffices to establish that $\overline{v}_n$ is invertible in $\pi_* R$. We know that $I_{n+1} = \pi_0 R$ is generated by I_n and the image of $\overline{v}_n : \pi_{-2(p^n-1)} R \to \pi_0 R$. Therefore, $\overline{v}_n$ is invertible modulo I_n. We are now done: the I_n-completeness of $\pi_0 R$ implies that $\overline{v}_n$ is itself invertible.

The proof of the other direction is exactly the same, with the steps reversed. Assume R is $K(n)$-local. Theorem 4.8 implies that R is I_n-complete, so it suffices to establish that $I_{n+1} = \pi_0 R$. Again, I_{n+1} is generated by I_n and the image of $\overline{v}_n : \pi_{-2(p^n-1)} R \to \pi_0 R$ — but condition (2) implies that the latter map is an isomorphism (as $\overline{v}_n$ is invertible in $\pi_* R$ by Theorem 4.8). Therefore $I_{n+1} = \pi_0 R$. □

Remark 4.10. Note that this result is strictly weaker than Corollary 4.2: it requires that R be weakly even periodic and an $\mathbf{E}_\infty$-ring.

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