QUANTUM NOETHER THEOREM

This will be a two-part talk, so the notes will also be split into two sections. Throughout these talks, we will fix a manifold X over which our quantum/classical field theories will live.

1. Stating the quantum Noether theorem

Before stating the quantum Noether theorem, let us recall the classical Noether theorem. In this talk, I will refer to "L_∞-algebras" simply as a "dg Lie algebra". Let us first recall the finite-dimensional setting, for motivation; temporarily, we will use the phrase "Lie algebra" to refer to the usual notion.

Suppose G is a connected Lie group acting on a symplectic manifold (M, ω) by symplectomorphisms. Let $\mathrm{Symp}(M)$ denote the Lie algebra of symplectic vector fields on M, i.e., vector fields ξ such that the Lie derivative $\mathcal{L}_\xi \omega$ is zero; equivalently, $d\langle \xi, \omega\rangle = 0$. By definition, acting by symplectomorphisms implies that there is a map $\xi : \mathfrak{g} \to \mathrm{Symp}(M)$ of Lie algebras.

There is a canonical map $\mathcal{O}_M \to \mathrm{Symp}(M)$, which sends a function f on M to its associated Hamiltonian vector field $X_f := \omega^{-1}(df)$. We might therefore ask whether $\xi : \mathfrak{g} \to \mathrm{Symp}(M)$ lifts to an action by *Hamiltonian* vector fields. This will not be possible in general, since not all symplectic vector fields are Hamiltonian (unless $\mathrm{H}^1(X; \mathbf{R}) = 0$). Using the fact that ω is nondegenerate, we may identify $\mathrm{Symp}(M)$ with $\Omega^1_{M,\mathrm{cl}}$ (closed 1-forms). Under this identification, the map $\mathcal{O}_M \to \mathrm{Symp}(M)$ is just the exterior derivative $d : \mathcal{O}_M \to \Omega^1_{M,\mathrm{cl}}$. Therefore, there is an exact sequence

$$0 \to \mathrm{H}^0(M; \mathbf{R}) \to \mathcal{O}_M \xrightarrow{f \mapsto X_f} \mathrm{Symp}(M) \cong \Omega^1_{M,\mathrm{cl}} \to \mathrm{H}^1(X; \mathbf{R}) \to 0.$$

Composing the map $\xi : \mathfrak{g} \to \mathrm{Symp}(M)$ with $\mathrm{Symp}(M) \to \mathrm{H}^1(X; \mathbf{R})$ gets us a Lie algebra cocycle, which gives a class in $\mathrm{H}^1(\mathfrak{g}; \mathrm{H}^1(X; \mathbf{R}))$. If this class vanishes (for instance, if $\mathrm{H}^1(X; \mathbf{R}) = 0$), then we can lift ξ to a map $\mathfrak{g} \to \mathcal{O}_M$. *However*, there is an obstruction for this to be a map of Lie algebras, given by the central extension $\widetilde{\mathfrak{g}}$ defined via the following pullback square:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^0(M; \mathbf{R}) & \longrightarrow & \widetilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g} \\ & & \downarrow & & \downarrow \widetilde{\xi} & & \downarrow \xi \\ 0 & \longrightarrow & \mathrm{H}^0(M; \mathbf{R}) & \longrightarrow & \mathcal{O}_M & \xrightarrow{f \mapsto X_f} & \mathrm{Symp}(M) \cong \Omega^1_{M,\mathrm{cl}}. \end{array}$$

This central extension is classified by an element $\alpha \in \mathrm{H}^2(\mathfrak{g}; \mathrm{H}^0(M; \mathbf{R}))$ (recall that in general, n-shifted central extensions are classified by $\mathrm{H}^{k+2}(\mathfrak{g})$), which can be understood as an "anomaly".

As a consequence, there is a map of Lie algebras $\widetilde{\xi} : \widetilde{\mathfrak{g}} \to \mathcal{O}_M$, which gives a map of Lie algebras out of $\mathfrak{g}$ if this obstruction class α vanishes. Since $\mathcal{O}_M$ is a Poisson algebra, it follows that there is a map $\widetilde{\xi} : \mathrm{Sym}(\widetilde{\mathfrak{g}}) \to \mathcal{O}_M$ of Poisson algebras. Suppose for simplicity that M is connected, so that $\mathrm{H}^0(M; \mathbf{R}) \cong \mathbf{R}$. Then $\alpha \in \mathrm{H}^2(\mathfrak{g}; \mathbf{R})$ classifies a central extension of $\mathfrak{g}$ by $\mathbf{R}$, which means we get a map $\mathrm{Sym}(\mathbf{R}) \to \mathrm{Sym}(\widetilde{\mathfrak{g}})$ of Poisson algebras. The algebra $\mathrm{Sym}(\mathbf{R})$ may be identified with $\mathbf{R}[t]$, so we have a composite

$$\mathbf{R}[t] \to \mathrm{Sym}(\widetilde{\mathfrak{g}}) \xrightarrow{\widetilde{\xi}} \mathcal{O}_M.$$

This factors through a map

$$\widetilde{\xi} : \mathrm{Sym}(\widetilde{\mathfrak{g}}) \otimes_{\mathbf{R}[t]} \mathbf{R} \to \mathcal{O}_M,$$

via the augmentation $\mathbf{R}[t] \to \mathbf{R}$ sending $t \mapsto 1$. This motivates a definition.

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra, and let $\alpha \in \mathrm{H}^2(\mathfrak{g}; \mathbf{R})$ classify a central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$. Then $\mathrm{Sym}(\widetilde{\mathfrak{g}}) \otimes_{\mathbf{R}[t]} \mathbf{R}$ will be called a *twisted Poisson algebra*, and will be denoted $U^{\mathrm{Pois}}_\alpha(\mathfrak{g})$. Note that as a vector space, this is just $\mathrm{Sym}(\mathfrak{g})$ itself.

Our discussion can be summarized as follows.

Summary 1.2. In the above setup, there is a canonical moment map $U^{\mathrm{Pois}}_\alpha(\mathfrak{g}) \to \mathcal{O}_M$ of Poisson algebras; this can be interpreted as a map $M \to \mathrm{Spec}\, U^{\mathrm{Pois}}_\alpha(\mathfrak{g})$ of Poisson manifolds. If $\alpha = 0$, then $U^{\mathrm{Pois}}_\alpha(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{g})$, so the moment map goes $\mu : M \to \mathfrak{g}^*$. It follows that each $\xi \in \mathfrak{g}$ (i.e., each infinitesimal G-symmetry) gives a function $\mu^\xi : M \to \mathbf{R}$, which is our desired conserved quantity.

In the Costello-Gwilliam story, one works with 1-shifted Poisson algebras. The numbers appearing above must then be modified as follows: if $\mathfrak{g}$ is a dg Lie algebra, then the 1-shifted Poisson enveloping algebra is $U^{\text{1-sh,Pois}}(\mathfrak{g}) := \mathrm{Sym}(\mathfrak{g}[1])$. Instead of considering central extensions of $\mathfrak{g}$ by $\mathbf{R}$, we now consider (-1)-*shifted* central extensions $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathbf{R}[-1]$, which are classified by $\alpha \in \mathrm{H}^{-1+2}(\mathfrak{g};\mathbf{R}) = \mathrm{H}^1(\mathfrak{g};\mathbf{R})$. Then we get a map $U^{\text{1-sh,Pois}}(\mathbf{R}[-1]) = \mathbf{R}[t] \to U^{\text{1-sh,Pois}}(\widetilde{\mathfrak{g}})$.

Definition 1.3. The preceding discussion implies that we may define the *twisted* 1-*shifted Poisson algebra* $U_\alpha^{\text{1-sh,Pois}}(\mathfrak{g})$ to be $U^{\text{1-sh,Pois}}(\widetilde{\mathfrak{g}}) \otimes_{\mathbf{R}[t]} \mathbf{R}$ (where the augmentation $\mathbf{R}[t] \to \mathbf{R}$ again sends $t \mapsto 1$). This object has the following universal property: if A is a 1-shifted Poisson algebra, then any map $\widetilde{\mathfrak{g}} \to \widetilde{A}[-1]$ of Lie algebras which sends the central element to the unit $1 \in \widetilde{A}[-1]$ extends to a map $U_\alpha^{\text{1-sh,Pois}}(\mathfrak{g}) \to \widetilde{A}$ of 1-shifted Poisson algebras.

This story can be factorized: suppose $\mathcal{L}$ is a local dg Lie algebra acting on a classical field theory $\mathcal{M}$ (which is a local dg Lie algebra equipped with a (-3)-shifted symmetric pairing $\mathcal{M} \cong \mathcal{M}^![-3]$). Leon's talk taught us that this action defines a central extension $\widetilde{\mathcal{L}}_c$ of the pre-cosheaf $\mathcal{L}_c$ of dg Lie algebras, defined by a class $\alpha \in \mathrm{H}^1(C^*_{\text{red,loc}}(\mathcal{L}))$ defining the obstruction to lifting the $\mathcal{L}$-action to an inner action. The classical Noether theorem states:

Theorem 1.4 (Classical Noether theorem)**.** *In the above situation, there is a map* $\widetilde{\mathcal{L}}_c \to \mathrm{Obs}^{\text{cl}}[-1]$ *of pre-cosheaves of dg Lie algebras to the factorization algebra of classical observables, which sends the central element to the unit in* $\mathrm{Obs}^{\text{cl}}[-1]$ *(which lives in homological degree* -1*).*

Remark 1.5. Let us sketch the idea in the finite-dimensional situation. Suppose $\mathfrak{g}$ and $\mathfrak{h}$ are dg Lie algebras, and that $\mathfrak{h}$ is equipped with an invariant pairing of degree -3. Then $C^*(\mathfrak{h})$ is a 1-shifted Poisson algebra, !!!

Let us rephrase Theorem 1.4 in terms of the twisted 1-shifted Poisson enveloping algebra.

Definition 1.6. Let $\mathcal{L}$ be a local dg Lie algebra. Define $\mathcal{U}^{\text{1-sh,Pois}}(\mathcal{L})$ to be the factorization algebra (of 1-shifted Poisson algebras) given by $U \mapsto U^{\text{1-sh,Pois}}(\mathcal{L}_c(U))$; we call this the *enveloping* $\mathbf{P}_0$-*factorization algebra*. Given a local cocycle $\alpha \in \mathrm{H}^1_{\text{loc}}(\mathcal{L})$, the *twisted* enveloping $\mathbf{P}_0$-factorization algebra $\mathcal{U}_\alpha^{\text{1-sh,Pois}}(\mathcal{L}_c)$ is given by $U \mapsto U_\alpha^{\text{1-sh,Pois}}(\mathcal{L}_c(U))$.

By the universal property of these twisted enveloping Poisson algebras, Theorem 1.4 can be rephrased as saying that there is a map $\mathcal{U}_\alpha^{\text{1-sh,Pois}}(\mathcal{L}_c) \to \mathrm{Obs}^{\text{cl}}$ of $\mathbf{P}_0$-factorization algebras. Our goal will be to quantize this latter statement. There are four steps to doing so: understand what it *means* to quantize, quantize the target, quantize the source, and quantize the map. We will only discuss the first and third parts today, state the fourth part, and omit discussion of the second part. Some of this is review of Jae's talk.

As usual, we will return to the finite-dimensional case for inspiration. Recall that if A_0 is a Poisson algebra over $\mathbf{C}$, a *quantization* is an associative algebra A over $\mathbf{C}[\![\hbar]\!]$ which is flat (i.e., $\hbar$-torsionfree) such that $A/\hbar \cong A_0$, and such that if $a, b \in A_0$ have lifts to A which we also denote by a, b, then $[a, b] = \hbar\{a, b\}$. In other words, the Poisson bracket on A_0 is a residue of the failure of the product on A to be commutative. There is a correspondence between quantizations in the above sense and lifts of graded objects to filtered objects, given by the *Rees construction.*

Construction 1.7. Let V be a (finite-dimensional, for simplicity) vector space over $\mathbf{C}$ equipped with an increasing filtration

$$0 = V_0 \subseteq \cdots \subseteq V_n = V.$$

The filtration on V allows us to define a $\mathbf{C}[\hbar]$-module $\mathrm{Rees}_\hbar(V)$ given by $\bigoplus_{i\geq 0} V_i \cdot \hbar^i$. This is an object which satisfies the following properties:

(a) When $\hbar = 1$, it is $\sum_{i\geq 0} V_i$, which is just V itself since $V_i \subseteq \cdots \subseteq V_n = V$. In fact, there is an isomorphism of $\mathbf{C}[\hbar^{\pm 1}]$-modules given by

$$\mathrm{Rees}_\hbar(V) \otimes_{\mathbf{C}[\hbar]} \mathbf{C}[\hbar^{\pm 1}] = \mathrm{Rees}_\hbar(V)[\tfrac{1}{\hbar}] \cong \mathbf{C}[\hbar^{\pm 1}] \otimes_{\mathbf{C}} \sum_{i\geq 0} V_i \cong \mathbf{C}[\hbar^{\pm 1}] \otimes_{\mathbf{C}} V.$$

(b) When $\hbar = 0$, we get the cokernel of $\hbar$-multiplication on $\mathrm{Rees}_\hbar(V)$. Since $\hbar$-multiplication sends $V_i \cdot \hbar^i$ to $V_i \cdot \hbar^{i+1}$, we see that

$$\mathrm{Rees}_\hbar(V)/\hbar \cong \bigoplus_{1\leq i\leq n} V_i/V_{i-1} = \mathrm{gr}(V).$$

Therefore, $\mathrm{Rees}_\hbar(V)$ is a $\mathbf{C}[\hbar]$-module whose generic fiber is V, and whose special fiber is $\mathrm{gr}(V)$ — in other words, it is a quantization of $\mathrm{gr}(V)$. If the filtration on V is *not* finite, then $\mathrm{Rees}_\hbar(V)$ can still be defined using the same symbols and will satisfy the same properties, but it will be a $\mathbf{C}[\![\hbar]\!]$-module instead.

Using this construction, we get a functor $\mathrm{Rees}_\hbar : \mathrm{Vect}_k^{\mathrm{fil}} \to \mathrm{Mod}_{\mathbf{C}[\![\hbar]\!]}$. In fact, one can characterize its essential image. For simplicity, let us restrict to *finite-dimensional* vector spaces. Then, $\mathrm{Rees}_\hbar$ has essential image given by the subcategory of $\hbar$-torsionfree $\mathbf{C}[\hbar]$-modules which have an action of $\mathbf{G}_m$ compatible with the usual $\mathbf{G}_m$-action on $\mathbf{C}[\hbar]$. In more algebro-geometric terms: $\mathrm{Rees}_\hbar$ defines an equivalence between $\mathrm{Vect}_k^{\mathrm{fil}}$ and torsionfree quasicoherent sheaves on $\mathbf{A}^1/\mathbf{G}_m$.

There are several aspects of this construction that are very interesting. For example, the tensor product on $\mathrm{Vect}_k^{\mathrm{fil}}$ (given by taking tensor products of filtered vector spaces) corresponds to the usual tensor product of quasicoherent sheaves on $\mathbf{A}^1/\mathbf{G}_m$. However, $\mathbf{A}^1/\mathbf{G}_m$ is also a commutative group object in stacks, which means that $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$ inherits a symmetric monoidal tensor product by convolution. This corresponds to the *pointwise* tensor product of filtered vector spaces (i.e., the nth filtered piece of $V \otimes W$ is declared to be the tensor product of the nth filtered pieces of V and W).

Summary 1.8. If A_0 is a (0-shifted) Poisson algebra over $\mathbf{C}$ equipped with a grading (such as $\mathrm{Sym}(V)$, where the grading is by degree), then giving a filtered algebra A whose associated graded is A_0 produces a quantization $\mathrm{Rees}_\hbar(A)$ of A_0.

A first example of the Rees construction is the following.

Example 1.9. Let $\mathfrak{g}$ be a Lie algebra. Since $U(\mathfrak{g})$ is a quotient of the tensor algebra $\bigoplus_{i\geq 0} \mathfrak{g}^{\otimes i}$, it admits a (PBW) filtration by declaring the image of $\bigoplus_{0\leq i\leq n} \mathfrak{g}^{\otimes i}$ to be the nth filtered piece. We will denote this filtered object by $\mathrm{F}^\star U(\mathfrak{g})$, so that its associated graded is (by the "PBW theorem") just $\mathrm{Sym}(\mathfrak{g})$. Note that $U(\mathfrak{g})$ is *not* a finite-dimensional $\mathbf{C}$-vector space, and the PBW filtration is not finite. Then (as a good exercise) one can show that $\mathrm{Rees}_\hbar(\mathrm{F}^\star U(\mathfrak{g}))$ is isomorphic to the $\mathbf{C}[\![\hbar]\!]$-algebra generated as an associative $\mathbf{C}[\![\hbar]\!]$-algebra by $\mathfrak{g}$, subject to the relation

$$xy - yx = \hbar[x, y].$$

We will simply denote this by $U_\hbar(\mathfrak{g})$ instead of the more cumbersome $\mathrm{Rees}_\hbar(\mathrm{F}^\star U(\mathfrak{g}))$.

However, the preceding story only produces quantizations for *unshifted* Poisson algebras, but we need a 1-*shifted* analogue of this story. For example, if $\mathfrak{g}$ is a dg Lie algebra, we seek a $\mathbf{C}[\![\hbar]\!]$-algebra such that when $\hbar$ is specialized to 0, we recover $U_\alpha^{\text{1-sh,Pois}}(\mathfrak{g})$.

Construction 1.10. Let $\mathfrak{g}$ be a dg Lie algebra. Recall that $U^{\text{1-sh,Pois}}(\mathfrak{g})$ is the 1-shifted Poisson dg-algebra $\mathrm{Sym}(\mathfrak{g}[1])$, where the differential is given by the internal differential $d_\mathfrak{g}$ of $\mathfrak{g}$. The *quantized* 1-*shifted enveloping algebra*[1] $U_\hbar^{\text{1-sh}}(\mathfrak{g})$ is the dg $\mathbf{R}[\![\hbar]\!]$-algebra whose underlying graded commutative algebra is $\mathrm{Sym}_{\mathbf{R}[\![\hbar]\!]}(\mathfrak{g}[1][\![\hbar]\!]) \cong \mathrm{Sym}(\mathfrak{g}[1])[\![\hbar]\!]$, but whose differential is given by $d_\mathfrak{g} + \hbar d_{\mathrm{CE}}$. This can also be understood from the point of view of filtrations as follows. The $U_\hbar^{\text{1-sh}}(\mathfrak{g})/\hbar = U^{\text{1-sh,Pois}}(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}[1])$, while $U_\hbar^{\text{1-sh}}(\mathfrak{g})[\frac{1}{\hbar}] = C_*(\mathfrak{g})(\!(\hbar)\!)$. Therefore, $U_\hbar^{\text{1-sh}}(\mathfrak{g})$ is the Rees construction on the Postnikov filtration $\tau_{\leq\star} C_*(\mathfrak{g})$. In Costello-Gwilliam, this is denoted $U^{\mathrm{BD}}(\mathfrak{g})$.

Remark 1.11. We can identify the $\hbar$-adic spectral sequence associated to $U_\hbar^{\text{1-sh}}(\mathfrak{g})$, i.e., the associated graded spectral sequence for the filtration of $U_\hbar^{\text{1-sh}}(\mathfrak{g})$ by ideals generated by powers of $\hbar$. It is precisely the spectral sequence

$$E_1^{*,*} = \mathrm{H}_*(\mathfrak{g}, d_\mathfrak{g})[\![\hbar]\!] \Rightarrow \mathrm{H}_* U_\hbar^{\text{1-sh}}(\mathfrak{g}).$$

Remark 1.12. One can rephrase this entire story in terms of *circle actions*: namely, a circle action in characteristic zero is the datum of a differential. We may therefore view d_{CE} on $C_*(\mathfrak{g})$ as defining an S^1-action on the commutative dg-algebra $\mathrm{Sym}(\mathfrak{g}[1])$, and then $U_\hbar^{\text{1-sh}}(\mathfrak{g})$ is its homotopy fixed points. This explanation can be made precise in several ways: the slickest/most modern approach is to use Arpon Raksit's theory of the filtered circle, and note that $C_*(\mathfrak{g})$ is the Hochschild homology of $\mathrm{Rep}(\mathfrak{g})$ viewed as an $\mathbf{R}$-linear ∞-category (which explains the source of this S^1-action). Unfortunately, this perspective is missing from the literature. Another (equivalent) way of getting this circle action is to use a result stating that circle actions are Koszul dual to filtrations (this is a manifestation of the claim that the 1-fold bar construction of $\mathbf{R}[\mathbf{N}] = \mathbf{R}[t]$ is $\mathbf{R}[S^1]$), and noting that $\mathrm{Sym}(\mathfrak{g}[1])$ is the 1-fold bar construction on $\mathrm{Sym}(\mathfrak{g}) = \mathrm{gr}(U(\mathfrak{g}))$.

[1] Also known as the "BD enveloping algebra".

Warning 1.13. It is *not* true that $U_\hbar^{1\text{-sh}}(\mathfrak{g})$ is simply the $\mathbf{R}[\![\hbar]\!]$-linear homological Chevalley-Eilenberg complex! Indeed, although they both have the same underlying graded commutative $\mathbf{R}[\![\hbar]\!]$-algebra (namely, $\mathrm{Sym}_{\mathbf{R}[\![\hbar]\!]}(\mathfrak{g}[1][\![\hbar]\!]) \cong \mathrm{Sym}(\mathfrak{g}[1])[\![\hbar]\!]$), their differentials differ: the differential in $U_\hbar^{1\text{-sh}}(\mathfrak{g})$ is $d_\mathfrak{g} + \hbar d_{\mathrm{CE}}$, while the differential in the Chevalley-Eilenberg complex is $d_\mathfrak{g} + d_{\mathrm{CE}}$.

The quantized 1-shifted enveloping algebra can be twisted just as with the 1-shifted Poisson enveloping algebra. Unfortunately, I am extremely confused by this part of Costello-Gwilliam (they seem to disregard the preceding warning, if I'm reading correctly); so we will spell out an "alternative" approach.

Definition 1.14. Suppose we are given a class $\alpha \in \mathrm{H}^1(\mathfrak{g}[\![\hbar]\!]; \mathbf{R}[\![\hbar]\!])$, where the Lie algebra cohomology is taken over the base ring $\mathbf{R}[\![\hbar]\!]$. Then α defines a $\hbar$-linear central extension

$$0 \to \mathbf{R}[\![\hbar]\!][-1] \to \widehat{\mathfrak{g}_\hbar} \to \mathfrak{g}[\![\hbar]\!] \to 0.$$

Recall how this central extension works: we view α as a class in $\mathrm{H}^2(\mathfrak{g}[\![\hbar]\!]; \mathbf{R}[\![\hbar]\!][-1])$, i.e., as a pairing $\langle -,-\rangle_\alpha : \mathfrak{g}[\![\hbar]\!] \otimes_{\mathbf{R}[\![\hbar]\!]} \mathfrak{g}[\![\hbar]\!] \to \mathbf{R}[\![\hbar]\!][-1]$. Then $\widehat{\mathfrak{g}_\hbar}$ is additively just $\mathfrak{g}[\![\hbar]\!] \oplus \mathbf{R}[\![\hbar]\!][-1]t$ (one might view t as living in homological degree -1), with Lie bracket

$$[x,y]_{\widehat{\mathfrak{g}_\hbar}} = [x,y]_{\mathfrak{g}[\![\hbar]\!]} + t\langle x,y\rangle_\alpha.$$

The map $\mathbf{R}[\![\hbar]\!][-1] \to \widehat{\mathfrak{g}_\hbar}$ of graded $\mathbf{R}[\![\hbar]\!]$-algebras gives a map

$$f : \mathbf{R}[\![\hbar]\!][t] \cong \mathrm{Sym}_{\mathbf{R}[\![\hbar]\!]}(\mathbf{R}[\![\hbar]\!][-1][1]) \to \mathrm{Sym}_{\mathbf{R}[\![\hbar]\!]}(\widehat{\mathfrak{g}_\hbar}[1])$$

of graded commutative $\mathbf{R}[\![\hbar]\!]$-algebras. Note that since $\widehat{\mathfrak{g}_\hbar}$ is additively just $\mathfrak{g}[\![\hbar]\!] \oplus \mathbf{R}[\![\hbar]\!][-1]t$, there are isomorphisms of graded commutative $\mathbf{R}[\![\hbar]\!]$-algebras

$$\mathrm{Sym}_{\mathbf{R}[\![\hbar]\!]}(\widehat{\mathfrak{g}_\hbar}[1]) \cong \mathrm{Sym}(\mathfrak{g}[1])[\![\hbar]\!] \otimes_{\mathbf{R}[\![\hbar]\!]} \mathbf{R}[\![\hbar]\!][t] \cong \mathrm{Sym}(\mathfrak{g}[1])[\![\hbar]\!][t].$$

Therefore, one can equip $\mathrm{Sym}_{\mathbf{R}[\![\hbar]\!]}(\widehat{\mathfrak{g}_\hbar}[1])$ with the differential $d_\mathfrak{g} + \hbar d_{\mathrm{CE}}$ (where d_{CE} is built using the Lie bracket on $\widehat{\mathfrak{g}_\hbar}$). Let us call temporarily call this dg commutative $\mathbf{R}[\![\hbar]\!]$-algebra by $A(\widehat{\mathfrak{g}_\hbar})$; then our map f refines to a map $\mathbf{R}[\![\hbar]\!][t] \to A(\widehat{\mathfrak{g}_\hbar})$ of *dg* commutative $\mathbf{R}[\![\hbar]\!]$-algebras.

Finally, we may define the *twisted quantized* 1*-shifted enveloping algebra* $U_{\hbar,\alpha}^{1\text{-sh}}(\mathfrak{g})$ to be $A(\widehat{\mathfrak{g}_\hbar}) \otimes_{\mathbf{R}[\![\hbar]\!][t]} \mathbf{R}[\![\hbar]\!]$ (where the augmentation $\mathbf{R}[\![\hbar]\!][t] \to \mathbf{R}[\![\hbar]\!]$ again sends $t \mapsto 1$). This is, I believe, what Costello and Gwilliam intend to mean by "$U_\alpha^{\mathrm{BD}}(\mathfrak{g})$".

Running through this construction, one sees that $U_{\hbar,\alpha}^{1\text{-sh}}(\mathfrak{g})/\hbar$ is precisely $U_{\alpha_0}^{1\text{-sh},\mathrm{Pois}}(\mathfrak{g})$ where $\alpha_0 = \alpha \pmod \hbar$ is the resulting class in $\mathrm{H}^1(\mathfrak{g}; \mathbf{R})$. Moreover, $U_{\hbar,\alpha}^{1\text{-sh}}(\mathfrak{g})[\frac{1}{\hbar}]$ is the base-change to $\mathbf{R}(\!(\hbar)\!)$ of an α-twisted version of the Postnikov filtered $\tau_{\leq\star} C_*(\mathfrak{g})$. This twist is precisely given by changing the differential on $C_*(\mathfrak{g})$ to $d_\mathfrak{g} + d_{\mathrm{CE}} - \langle \alpha, -\rangle$, where we view α as specifying a 1-cocycle in $C^*(\mathfrak{g})$. In other words, taking the Rees construction of this filtered commutative dg $\mathbf{R}$-algebra produces $U_{\hbar,\alpha}^{1\text{-sh}}(\mathfrak{g})$.

Let us now return to the factorization setting. In the previous talk, we discussed what it meant for a local dg Lie algebra $\mathcal{L}$ to act on a quantum field theory. We will not recall what this means, since the definition in the Costello-Gwilliam book is quite complicated (and Minta could interject here to tell us details) — the main point is that there is an obstruction to lifting this action to an "inner action", given by a cocycle $\alpha \in \mathrm{H}^1(C^*_{\mathrm{red,loc}}(\mathcal{L}))[\![\hbar]\!]$ (which I believe is just the factorization analogue of what I have been denoting by $\mathrm{H}^1(\mathfrak{g}[\![\hbar]\!]; \mathbf{R}[\![\hbar]\!])$ above).

Theorem 1.15 (Quantum Noether theorem)**.** *In the above setup, define the* 1*-shifted quantized enveloping factorization algebra* $\mathcal{U}_{\hbar,\alpha}^{1\text{-sh}}(\mathcal{L}_c)$ *to be* $U \mapsto U_{\hbar,\alpha}^{1\text{-sh}}(\mathcal{L}_c(U))$*. In the above setup, there is a map* $\mathcal{U}_{\hbar,\alpha}^{1\text{-sh}}(\mathcal{L}_c) \to \mathrm{Obs}^q$ *of* 1*-shifted quantized enveloping factorization algebras.*

The whole point of the above discussion is to make clear that when we set $\hbar = 0$, the source reduces to $\mathcal{U}_\alpha^{1\text{-sh},\mathrm{Pois}}(\mathcal{L}_c)$, the target reduces to $\mathrm{Obs}^{\mathrm{cl}}$, and we get a map $\mathcal{U}_\alpha^{1\text{-sh},\mathrm{Pois}}(\mathcal{L}_c) \to \mathrm{Obs}^{\mathrm{cl}}$ of $\mathbf{P}_0$-factorization algebras. This is nothing but the map of Theorem 1.4.

Having at least stated the main theorem, let us mention how this connects to conserved currents. The constant sheaf $\underline{\mathbf{R}}$ on our manifold X has a resolution by the de Rham complex $\Omega_X^\bullet$ by the Poincaré lemma. Therefore, if $\mathfrak{g}$ is a usual Lie algebra acting on a quantum field theory, then we may replace the constant sheaf $\mathcal{L} = \underline{\mathfrak{g}}$ of Lie algebras in Theorem 1.15 by $\Omega_X^\bullet \otimes \mathfrak{g}$. Then, $\alpha \in \mathrm{H}^1(\Omega_X^\bullet \otimes \mathfrak{g}[1])[\![\hbar]\!]$ defines a central extension of $\Omega_X^\bullet \otimes \mathfrak{g}$, and hence a map $\mathcal{U}_{\hbar,\alpha}^{1\text{-sh}}(\Omega_X^\bullet \otimes \mathfrak{g}) \to \mathrm{Obs}^q$. To understand this in terms of conserved currents, let us unwind what α is; for this, note that we have

$$\mathrm{H}^1(X; \Omega_X^\bullet \otimes \mathfrak{g}[1])[\![\hbar]\!] \cong \bigoplus_{i+j=\dim(X)+1} \mathrm{H}^i(X; \mathbf{R}) \otimes \mathrm{H}^j_{\mathrm{red}}(\mathfrak{g})[\![\hbar]\!].$$

If $X = N \times \mathbf{R}$ with N being compact and oriented, let $\mathrm{pr} : N \times \mathbf{R} \to \mathbf{R}$ denote the projection. We can integrate the $(i,j) = (\dim(X) - 1 = \dim(N), 2)$ component of α along N to get a class $\alpha \in \mathrm{H}^2_{\mathrm{red}}(\mathfrak{g})[\![\hbar]\!]$. This is a $\hbar$-indexed/one-paramter family of (unshifted) central extensions of $\mathfrak{g}$, and so we obtain a 1-parameter family $U_\alpha(\mathfrak{g})$ of twisted enveloping algebras. Moreover, Theorem 1.15 gives a map $U_\alpha(\mathfrak{g}) \to \mathrm{H}^0(\mathrm{pr}_* \mathrm{Obs}^q)$ of factorization algebras on $\mathbf{R}$. this can be interpreted as the map which sends an element of $U_\alpha(\mathfrak{g})$ (which is almost an element of $\mathfrak{g}$ if not for the class α) to an observable on a codimension 1-manifold, which one can understand as a current.

In fact, one can explain this situation in the finite-dimensional story, too. Suppose G is a connected Lie group acting on a connected symplectic manifold (M, ω) by symplectomorphisms, and assume that $\mathrm{H}^1(M; \mathbf{R}) = 0$. Then we get a central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ and a map $\widetilde{\xi} : \widetilde{\mathfrak{g}} \to \mathcal{O}_M$ of Lie algebras, i.e., a map $\mathrm{Sym}(\widetilde{\mathfrak{g}}) \to \mathcal{O}_M$. Let $\mathcal{A}$ be a quantization of M, so that $\mathcal{A}/\hbar = \mathcal{O}_M$ with its Poisson bracket. Then we can ask when $\widetilde{\xi}$ lifts to a map $U_\hbar(\widetilde{\mathfrak{g}}) \to \mathcal{A}$ of associative algebras. This is not possible in general, but one case where it is possible is when $M = T^*N$ and $\mathcal{A} = \mathcal{D}_N$ (the algebra of differential operators): if G acts on N itself, then the obstruction class vanishes, and taking enveloping algebras of the derivative map $\mathfrak{g} \to T_N$ gives our desired quantization.

Therefore, one can think of the existence of the class $\alpha \in \mathrm{H}^1(C^*_{\mathrm{red,loc}}(\mathcal{L}))[\![\hbar]\!]$ lifting $\alpha_0 \in \mathrm{H}^1(C^*_{\mathrm{red,loc}}(\mathcal{L}))$ as a factorization analogue of a lift of the $\mathfrak{g}$-action on M to a $\mathfrak{g}$-action on $\mathcal{A}$. Given such a lift, the same sort of argument as in the beginning of this talk would give our "quantized" moment map $U_\hbar(\widetilde{\mathfrak{g}}) \to \mathcal{A}$; one can interpret Theorem 1.15 as the factorization version of this statement.

2. Examples of the Noether theorems

In this talk, I will give two examples of the Noether theorems.

Example 2.1. Let V be a real vector space equipped with a symmetric bilinear form q, and let $\mathrm{O}(V, q)$ denote the corresponding orthogonal group. Then $\mathrm{O}(V, q)$ acts on $T^*V = V \oplus V^*$ by symplectomorphisms (in fact, by Hamiltonian vector fields), so there is no obstruction cocycle, and we get a moment map[2] $\mu : T^*V \to \mathfrak{o}(V, q)$. One can interpret this as a linear map $\mathfrak{o}(V, q) \to V \oplus V^*$.

Let us unwind this moment map in the case when $V = \mathbf{R}^3$ with the standard bilinear form. Then $\mathfrak{o}(\mathbf{R}^3) = \mathfrak{so}_3$, which can be identified with $\mathbf{R}^3$ via the isomorphism

$$\mathfrak{so}_3 \ni \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \mapsto (a_1 \ a_2 \ a_3) \in \mathbf{R}^3.$$

Under this isomorphism, the Lie bracket on $\mathfrak{so}_3$ is sent to the cross-product on $\mathbf{R}^3$. To compute the moment map, we need to compute the infinitesimal $\mathfrak{so}_3$-action; if $A \in \mathfrak{so}_3$ and $q \in \mathbf{R}^3$, then

$$\rho(A) \cdot q = \tfrac{d}{dt}(e^{tA} \cdot q)\Big|_{t=0} = \tfrac{d}{dt}(tA) \cdot q\big|_{t=0} = A \times q.$$

It follows that if $(p, q) \in T^*\mathbf{R}^3$, then

$$\langle \mu(p, q), A \rangle = p \cdot (A \times q) = A \cdot (q \times p),$$

which implies that $\mu(p, q) = q \times p$.

We can generalize this to the setting of field theories as follows. Let $q, p \in \Omega^*_{\mathbf{R}} \otimes V$, so that the 0-form components are just V-valued functions on $\mathbf{R}$ (corresponding to position and momentum) and the 1-form components are antifields. One should think of q and p as defining a section of $\Omega^*_{\mathbf{R}} \otimes T^*V$, and therefore as describing a σ-model of maps $\mathbf{R} \to T^*V$. Then the action we consider will be given by

$$S[q, p] = \int_{\mathbf{R}} \mathrm{Tr}(p \wedge dq) + \tfrac{1}{2} \mathrm{Tr}(p \wedge p) \wedge dt.$$

Moreover, the equations of motion are $dp = 0$ and $p = dq$.

Clearly, this action functional is invariant under the $\mathrm{O}(V, q)$-action on p, q. The classical Noether theorem therefore tells us that we get a map of factorization Lie algebras from $\mathfrak{so}(V, q)$ (or rather the associated constant sheaf of Lie algebras) to $\mathrm{Obs}^{\mathrm{cl}}[-1]$. One can calculate the effect on H^0, and it turns out that this is given by a map $\mathfrak{so}(V, q) \to \mathrm{Sym}(V \oplus V^*) = \mathcal{O}_{T^*V}$ of Lie algebras. This map is nothing but the moment map discussed earlier. In particular, it refines to a map $\mathrm{Sym}(\mathfrak{so}(V, q)) \to \mathcal{O}_{T^*V}$ of Poisson algebras. (In fact, the map of factorization algebras of 1-shifted Poisson algebras is essentially described by this map.) Note that we did not need to consider the

[2] Recall that the moment map $\mu : X \to \mathfrak{g}^*$ of a G-action on a symplectic manifold (X, ω) is supposed to have the property that if $\xi \in \mathfrak{g}$ and $\rho : \mathfrak{g} \to T_X$ is the derivative of the G-action, then $d\langle \mu, \xi \rangle = \langle \rho(\xi), \omega_X \rangle$.

twisted Poisson enveloping algebra here, since the action of $\mathrm{O}(V,q)$ on T^*V was by Hamiltonian vector fields (because $\mathrm{O}(V,q)$ acts on V itself).

This can be quantized straightforwardly. In the finite-dimensional setting, our moment map went $\mu : T^*V \to \mathfrak{so}(V,q)$, and hence we get a map $\mathrm{Sym}(\mathfrak{so}(V,q)) \to \mathcal{O}_{T^*V}$ of Poisson algebras. The quantization of this moment map is the *quantum* moment map $\mu_\hbar : U_\hbar(\mathfrak{so}(V,q)) \to \mathcal{D}_\hbar(V)$, from the asymptotic enveloping algebra of $\mathfrak{so}(V,q)$ to the asymptotic differential operators on V. To define $\mu_\hbar$, it is useful to view $\mathcal{D}_\hbar(V)$ as the $\mathcal{O}_V$-linear enveloping algebra of the tangent sheaf of V (viewed as an affine space); then $\mu_\hbar$ is the effect of taking enveloping algebras on the derivative map $\mathfrak{so}(V,q) \to T_V$.

The other main example is a little more involved: it arises via chiral conformal field theory, and is called the free $\beta\gamma$-system. We will see that the quantum Noether theorem states the existence of certain maps of factorization algebras; these turn out to recover constructions from the theory of vertex algebras. We will opt to giving an overview of the ideas involved instead of trying to give details.

Example 2.2. Let Σ be a Riemann surface, and let V be a complex vector space. The *free $\beta\gamma$-system* on Σ has fields $\gamma \in \Omega^{0,*}_\Sigma \otimes V$ and $\beta \in \Omega^{1,*}_\Sigma \otimes V^*$. Thinking homotopy-invariantly, these are holomorphic maps $\gamma : \Sigma \to V$ and a holomorphic section of $K_\Sigma \otimes V^*$. The action functional of the theory is given by

$$S[\beta,\gamma] = \int_\Sigma \langle \beta, \overline{\partial}\gamma\rangle,$$

which means that the equations of motion are given by $\overline{\partial}\beta = 0 = \overline{\partial}\gamma$. In other words, the solutions to the equation of motion are *holomorphic* maps $\Sigma \to T^*V$.

The quantum observables of the $\beta\gamma$-system may be described as follows.

Example 2.3. For each open $U \subseteq \Sigma$, define

$$\mathcal{H}(U) = (\Omega^{0,*}_{c,U} \otimes V) \oplus (\Omega^{1,*}_{c,U} \otimes V^*) \oplus \mathbf{C}[-1]\cdot \hbar,$$

where the Lie bracket is defined by

$$[\alpha,\beta] = \hbar \int_U \mathrm{Tr}(\alpha \wedge \beta).$$

One can think of $\mathcal{H}$ as a central extension of the (abelian) Lie algebra of holomorphic forms on U; it can be viewed as a variant on the Heisenberg algebra. This Lie bracket defines a differential Δ on the Chevalley-Eilenberg complex $\mathrm{Sym}(\mathcal{H}(U)[1])$, and the prefactorization algebra of quantum observables of the $\beta\gamma$-system is given by

$$U \mapsto \mathrm{Obs}^q_{\beta\gamma}(U) = (\mathrm{Sym}(\mathcal{H}(U)[1]), \overline{\partial} + \hbar\Delta) = (\mathrm{Sym}(\Omega^{0,*}_{c,U} \otimes V[1]) \oplus (\Omega^{1,*}_{c,U} \otimes V^*[1]))[\hbar], \overline{\partial} + \hbar\Delta).$$

We will not need to know what a vertex algebra is for this talk, even though one of the most interesting component of conformal field theories is the vertex operator expansion (VOA). For the purpose of this talk, it suffices to remark that vertex algebras can be associated to certain factorization algebras over $\mathbf{C}$. In this case, the vertex algebra associated to the factorization algebra $\mathrm{Obs}^q_{\beta\gamma}$ over $\Sigma = \mathbf{C}$ is the so-called Heisenberg algebra (see the end of Chapter 11 and Chapter 12 of Frenkel-Ben-Zvi's book on vertex algebras).

There are two sorts of symmetries of the $\beta\gamma$-system which we will study:

(a) Symmetries of V itself: this gives an action of $\mathrm{GL}(V)$ on the $\beta\gamma$-system.
(b) Symmetries associated to Σ: this will give rise to a relationship with a factorization algebra related to the Virasoro algebra.

Since the first is simpler, we begin with it. Define the action of $\mathrm{GL}(V)$ on β and γ by functoriality: $\mathrm{GL}(V)$ acts on V and V^* (the latter by the transpose, if one identifies $V \cong \mathbf{C}^n \cong V^*$). We would like to understand the meaning of the classical and quantum Noether theorems for this action. For this, we will make a slight digression.

Construction 2.4. Let $\mathfrak{g}$ be a Lie algebra equipped with a nondegenerate Ad-invariant bilinear form $\kappa : \mathfrak{g}\otimes\mathfrak{g} \to \mathbf{C}$. Associated to $\mathfrak{g}$ is the *Kac-Moody algebra* $\hat{\mathfrak{g}}_\kappa$: it is a central extension of $\mathfrak{g}((t))$, so it sits into an exact sequence

$$0 \to \mathbf{C}\cdot c \to \hat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0.$$

The Lie bracket is given by

$$[af(t), bg(t)] = [a,b]f(t)g(t) - \mathrm{Res}_{t=0}(fdg)\kappa(a,b)c.$$

In fact, the space of central extensions $\mathrm{H}^2(\mathfrak{g}((t));\mathbf{C})$ can be calculated to be $\mathrm{H}^4(\mathfrak{g};\mathbf{C})$, i.e., the space of nondegenerate invariant bilinear forms on $\mathfrak{g}$. Therefore, this construction explicitly describes all central extensions of $\mathfrak{g}((t))$.

The Kac-Moody algebra $\hat{\mathfrak{g}}_\kappa$ gives rise to what's called the *Kac-Moody vertex algebra* (whose underlying vector space is $\mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbf{C}\cdot c}^{\hat{\mathfrak{g}}_\kappa}\mathbf{C}$). Recall our remark that vertex algebras can be associated to certain factorization algebras over $\mathbf{C}$. In this case, the factorization algebra is given by the cosheaf on $\Sigma = \mathbf{C}$ sending

$$U \mapsto (\Omega^{0,*}_{c,U}\otimes\mathfrak{g},\overline{\partial})\oplus\mathbf{C}\cdot c,$$

where c is placed in homological degree -1 and the bracket is given by

$$[a\alpha, b\beta] = [a,b]\alpha\wedge\beta - \frac{1}{2\pi i}\left(\int_U \partial\alpha\wedge\beta\right)\kappa(a,b)c.$$

In our case, $\mathfrak{g} = \mathfrak{gl}(V)$, and κ will be the Killing form $(a,b)\mapsto \mathrm{Tr}_V(ab)$. Then the cocycle α defining the above 1-shifted central extension of $(\Omega^{0,*}_{c,\Sigma}\otimes\mathfrak{g},\overline{\partial})$ is given by $\frac{1}{2\pi i}\int_\Sigma \mathrm{Tr}_V(\beta\partial\alpha)$. One computes (using techniques we have not discussed in this seminar) that this is precisely the obstruction cocycle for the action of $\mathcal{L} = \Omega^{0,*}_\Sigma\otimes\mathfrak{gl}(V)$ on the quantized $\beta\gamma$-system.

By the discussion last time, the quantum Noether theorem gives a map $\mathcal{U}^{1\text{-sh}}_{\hbar,\alpha}(\mathcal{L}_c)\to\mathrm{Obs}^q_{\beta\gamma}$ of 1-shifted quantized enveloping factorization algebras. The preceding discussion lets us identify H^0 of $\mathcal{U}^{1\text{-sh}}_{\hbar,\alpha}(\mathcal{L}_c)$ as the enveloping algebra $U(\hat{\mathfrak{g}}_\kappa)\otimes_{\mathbf{C}[c]}\mathbf{C}_{c=1}$, corresponding to $\hat{\mathfrak{g}}_\kappa$-representations on which the central element acts by 1. The map of the quantum Noether theorem in this case descends to an action of the vertex algebra associated to $(\mathfrak{g}=\mathfrak{gl}(V),\kappa=\text{Killing})$ on the Heisenberg vertex algebra. This is a fundamental construction in the theory of representations of affine Lie algebras.

Let us now explore symmetries associated to Σ. In a sense, this story is more interesting, because it exploits features that are specific to 2-*dimensional* field theories (as opposed to the $\mathrm{GL}(V)$-action which exists on any σ-model with target T^*V). Let $\mathcal{T}$ denote the Lie algebra $\Omega^{0,*}(\Sigma;T^{1,0}_\Sigma)$, which is a Dolbeault resolution of the sheaf T^{hol}_Σ of holomorphic vector fields on Σ. Note, for instancee, that if Σ is an annulus, the cohomology of T^{hol}_Σ is concentrated in degree 0, where it is $\mathcal{O}_\Sigma\partial_z$. Inside this is the dense subspace $\mathbf{C}[z^{\pm1}]\partial_z$ of vector fields on the circle.

Definition 2.5. The *Witt algebra* $\mathcal{W}\mathrm{itt}$ is the Lie algebra $\mathbf{C}[z^{\pm1}]\partial_z$ of vector fields on the circle. The standard notation for the generators of $\mathcal{W}\mathrm{itt}$ are $L_n := -z^{n+1}\partial_z$. With this definition, the commutation relations are

$$[L_m, L_n] = (m-n)L_{m+n}.$$

In analogy to the Kac-Moody story above, the Witt algebra is to be understood as analogous to $\mathfrak{g}((t))$. The analogue of the Kac-Moody algebra $\hat{\mathfrak{g}}_\kappa$ is known as the *Virasoro algebra*, and is denoted $\mathcal{V}\mathrm{ir}$. It is a central extension of $\mathcal{W}\mathrm{itt}$, and is therefore described by a class in $\mathrm{H}^2(\mathcal{W}\mathrm{itt};\mathbf{C})$. One can do a somewhat arduous calculation to figure out that $\mathrm{H}^2(\mathcal{W}\mathrm{itt};\mathbf{C})\cong\mathbf{C}$, generated by the *Gelfand-Fuks cocycle*

$$\omega(f(z)\partial_z, g(z)\partial_z) = \frac{1}{12}\mathrm{Res}_{z=0}(g(\partial_z^3 f)dz).$$

In terms of the generators L_n, it is

$$\omega(L_n, L_m) = \frac{n^3-n}{12}\delta_{n,-m} = \frac{1}{12}\mathrm{Res}_{t=0}((m+1)m(m-1)t^{n+1}t^{m-2}dt).$$

Thus the Virasoro algebra sits in an extension

$$0\to\mathbf{C}\cdot c\to\mathcal{V}\mathrm{ir}\to\mathcal{W}\mathrm{itt}\to 0,$$

where the Lie bracket on $\mathcal{V}\mathrm{ir}$ is given by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}c.$$

One can extend this to a (-1)-shifted central extension of $\mathcal{T} = \Omega^{0,*}(\Sigma;T^{1,0}_\Sigma)$ (i.e., T^{hol}_Σ) as follows:

$$\omega(f_0 + f_1 d\overline{z}, g_0 + g_1 d\overline{z}) = \frac{1}{12}\frac{1}{2\pi i}\int((\partial_z^3 f_0)g_1 + (\partial_z^3 f_1)g_0)d^2z.$$

Definition 2.6. The *Virasoro factorization algebra* $\mathcal{V}\mathrm{ir}^{\mathrm{fact}}$ on $\Sigma=\mathbf{C}$ is the enveloping algebra $\mathcal{U}^{1\text{-sh}}_\hbar(\widetilde{\mathcal{T}}_c)$ of the central extension $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ with respect to the above Gelfand-Fuks cocycle. If $\lambda\in\mathbf{C}$, let $\mathcal{V}\mathrm{ir}^{\mathrm{fact}}_{c=\lambda}$ denote the factorization algebra obtained by specializing the central charge to λ.

Remark 2.7. Note that the above formulas do not allow us to define the Virasoro factorization algebra over a general Riemann surface, since we picked the coordinate z on $\Sigma = \mathbf{C}$. It takes some work to generalize the Virasoro vertex algebra to arbitrary Riemann surfaces.

Recall that vertex algebras can be associated to certain factorization algebras over $\mathbf{C}$; Williams showed that the vertex algebra associated to $\mathcal{V}\mathrm{ir}^{\mathrm{fact}}$ is the Virasoro vertex algebra.

Let us now define an action of $\mathcal{W}\mathrm{itt}$ on the $\beta\gamma$-system (over $\Sigma = \mathbf{C}$). The action of a vector field $f(z)\partial_z \in \mathcal{W}\mathrm{itt}$ on a field $\gamma \in \Omega^{0,*}_\Sigma \otimes V$ is given by $f(z)\partial_z\gamma$, and similarly for $\beta \in \Omega^{1,*}_\Sigma \otimes V^*$. One can compute the moment map J (which goes from the phase space to the dual of the Lie algebra) for this action: if γ, β are the fields and $\alpha \in \mathcal{T}_c = \Omega^{0,*}_{c,\Sigma}(T^{1,0}_\Sigma)$, then

$$\langle J(\beta, \gamma), \alpha\rangle = \int_\Sigma \langle \beta, \alpha \cdot \gamma\rangle.$$

One can think of J as defining a map $\mathcal{T}_c \to \mathrm{Obs}^{\mathrm{cl}}[-1]$ of factorization Lie algebras.

Williams calculated that the obstruction cocycle for the action of $\mathcal{L} = \mathcal{T}_c$ on the $\beta\gamma$-system is precisely $\dim(V)\omega$ (where ω is the Gelfand-Fuks cocycle). It follows from the quantum Noether theorem that there is a map $\mathcal{V}\mathrm{ir}^{\mathrm{fact}}_{c=\dim(V)} \to \mathrm{Obs}^q$ of 1-shifted quantized enveloping factorization algebras (over $\mathbf{C}$). This defines an action of the Virasoro vertex vertex algebra on the Heisenberg vertex algebra, which is also a well-known construction (it picks out the conformal vector of the Heisenberg vertex algebra).

References

Email address: sdevalapurkar@math.harvard.edu