THE RIEMANN-HILBERT CORRESPONDENCE

Our goal in these notes will be to describe the Riemann-Hilbert correspondence. Our primary reference is [HTT08]. I also have slides at `http://www.mit.edu/~sanathd/rh-slides.pdf`.

1. Riemann-Hilbert correspondence

We begin with a simpler case of the Riemann-Hilbert correspondence.

Recollection 1.1. Let X be a connected complex manifold. A local system on X is a sheaf $\mathcal{L}$ of $\mathbf{C}$-vector spaces which is locally constant. Taking the monodromy of $\mathcal{L}$ around a point $x \in X$ defines a (complex) representation of $\pi_1(X, x)$. This defines an equivalence of categories $\mathrm{LocSys}(X) = \mathrm{Rep}(\pi_1(X, x))$.

Recollection 1.2. Let $f : X \to Y$ be a map of complex manifolds, and let $\mathcal{F}$ be a coherent sheaf on X. A connection on $\mathcal{F}$ relative to Y is a map $\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/Y}$ such that if s and f are local sections of $\mathcal{F}$ and $\mathcal{O}_X$ respectively, then

$$\nabla(fs) = df \otimes s + f \cdot \nabla(s).$$

Given a connection ∇ on $\mathcal{F}$, we obtain a map $\nabla : \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^i_{X/Y} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^{i+1}_{X/Y}$ for each $i \geq 0$ by

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^i \omega \wedge \nabla(s).$$

The connection ∇ is said to be integrable if the composite

$$\mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/Y} \xrightarrow{\nabla} \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^2_{X/Y}$$

is zero. (If f is smooth, this amounts to asking that if v and w are vector fields on X relative to Y, then $[\nabla_v, \nabla_w] = \nabla_{[v,w]}$.) In other words, the connection is integrable if it is flat, i.e., if its curvature vanishes. We shall only be interested in the case $Y = *$. Let $\mathrm{Vect}(X)^\nabla$ denote the category of vector bundles on X equipped with an integrable connection.

Example 1.3. Let $\mathcal{L}$ be a local system on X. Then $\mathcal{L}$ defines a vector bundle $\mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_X$ on X. The differential $d : \mathcal{O}_X \to \Omega^1_X$ (which is not $\mathcal{O}_X$-linear, but is $\mathbf{C}$-linear) defines an integrable connection on $\mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_X$ via

$$\mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_X \xrightarrow{1 \otimes d} \mathcal{L} \otimes_{\mathbf{C}} \Omega^1_X.$$

The Riemann-Hilbert correspondence then states that:

Theorem 1.4. *Let X be a complex manifold. There is an equivalence of categories* $\mathrm{LocSys}(X) \xrightarrow{\sim} \mathrm{Vect}(X)^\nabla$*, given by sending a local system $\mathcal{L}$ to $(\mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_X, 1 \otimes d)$ via Example 1.3. The inverse equivalence sends a vector bundle $(\mathcal{F}, \nabla)$ equipped with an integrable connection to the sheaf of horizontal sections* $\ker(\nabla) \subseteq \mathcal{F}$.

Proof. We first need to check that these functors are well-defined. It is clear that $(\mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_X, 1 \otimes d)$ defines a vector bundle with integrable connection (where integrability follows because $d^2 = 0$). Since $\ker(\mathcal{O}_X \to \Omega^1_X)$ consists of the constant functions, it is moreover clear that $\ker(1 \otimes d) \subseteq \mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_X$ is isomorphic to $\mathcal{L}$ itself.

The functor in the other direction is more subtle. There are two things to check: first, that $\ker(\nabla)$ is a local system; second, that there is an isomorphism between $(\mathcal{F}, \nabla)$ and $(\ker(\nabla) \otimes_{\mathbf{C}} \mathcal{O}_X, 1 \otimes d)$. The second fact is easier than the first: one just checks that if ∇ is an integrable connection, then $\mathcal{F} \cong \ker(\nabla) \otimes_{\mathbf{C}} \mathcal{O}_X$. The fact that $\ker(\nabla)$ is a local system boils down to the Frobenius theorem, which in turn is a jazzed up version of the local uniqueness of first-order ODEs with an initial condition. Indeed, here's an informal sketch: the question of being a local system is, well, local. Around any point $x \in X$, we can find a neighbourhood $U \subseteq X$ on which $\nabla = d + A$ for some matrix A of 1-forms. Then $f = (f_1, \cdots, f_n) \in \ker(\nabla)$ if and only if $df_i + \sum_j A_{ij} f_j = 0$. If for every point in $\mathbf{C}^n$ (corresponding to an initial value of f), there is a unique solution to this first-order ODE, then the map sending a function $f \in \ker(\nabla)|_U$ to its initial value defines an isomorphism between $\ker(\nabla)|_U$ and $\mathbf{C}^n$. $\square$

We would like to generalize Theorem 1.4 in two directions: one, we'd like to go to the algebraic setting, rather than only work in the analytic setting; two, we'd like to allow more general D-modules, rather than just vector bundles with a connection. (Recall that D-modules whose underlying $\mathcal{O}_X$-module is locally free are just vector bundle equipped with an integrable connection.)

Roadblock 1.5. In trying to generalize to the algebraic setting, we quickly run into an issue. Let $X = \mathbf{A}^1_{\mathbf{C}}$ with coordinate z. Consider the following two connections on $\mathcal{O}_X$: the first one is $\nabla(f) = df$, and the second one is $\nabla'(f) = df + f\ dz$. These two connections are different algebraically. Upon analytification, these two connections are the same: if we send f to fe^z, then $\nabla(fe^z) = e^z(df + f\ dz) = e^z\nabla'(f)$. In particular, the associated sheaves of horizontal analytic sections are isomorphic, which prevents us from just writing down a naive version of Theorem 1.4.

What is the issue? Consider the differential equation $f' + f = 0$ associated to ∇'. Replacing $t = z^{-1}$, we find that $f'(z) = -t^2 f'(t)$, and so in the coordinate t, the differential equation becomes $-t^2 f'(t) + f(t) = 0$. This differential equation has an irregular singularity at $t = 0$ (i.e., the differential equation $f' + f = 0$ has an irregular singularity at ∞).

Recollection 1.6. Suppose $\sum_{k=0}^n p_k(z) f^{(k)}(z) = 0$ is an ordinary differential equation with each $p_k(z)$ a meromorphic function. If not all p_k are analytic at a point z_0, but the functions $(z - z_0)^{n-k} p_k(x)$ are all analytic at z_0, then z_0 is called a regular singularity. In other words, the differential equation has a regular singularity at z_0 if p_k has a pole of order at most $n - k$ at $z = z_0$.

Roadblock 1.5 suggests that in order to get an algebraic analogue of Theorem 1.4, we have to impose some regularity conditions on the connection on our vector bundle.

The other direction we wished to generalize in was to allow more general D-modules to appear in the Riemann-Hilbert correspondence, rather than just vector bundles with a (regular) connection. This will lead to the notion of a *holonomic D-module*, which is a generalization of the notion of a linear ODE to higher-dimensional complex manifolds. We shall begin by defining the notion of holonomicity.

2. Holonomic D-modules

Let X be a smooth variety over $\mathbf{C}$. Recall:

Recollection 2.1. The sheaf $\mathcal{D}_X$ admits a filtration by the order of a differential operator: if U is an open in X, then

$$F_i\mathcal{D}_X(U) = \left\{ P \in \mathcal{D}_X(U) | P \in \sum_{|\alpha| \leq i} \mathcal{O}_V \partial^\alpha \text{ for any affine open } V \subseteq U \right\}.$$

Recall further that if $\pi : \mathrm{T}^*X \to X$ is the canonical projection from the cotangent bundle, then the associated graded $\mathrm{gr}(\mathcal{D}_X)$ is isomorphic to $\pi_*\mathcal{O}_{\mathrm{T}^*X}$.

Definition 2.2. Let $\mathcal{F}$ be a $\mathcal{D}_X$-module. A filtration $F_i\mathcal{F}$ compatible with the order filtration on $\mathcal{D}_X$ is said to be good if the associated graded $\mathrm{gr}(\mathcal{F})$ is coherent as a $\pi_*\mathcal{O}_{\mathrm{T}^*X}$-module. The characteristic variety/singular support $\mathrm{Ch}(\mathcal{F})$ of a $\mathcal{D}_X$-module $\mathcal{F}$ equipped with a good filtration is the support of the coherent $\mathcal{O}_{\mathrm{T}^*X}$-module $\mathcal{O}_{\mathrm{T}^*X} \otimes_{\pi^{-1}\pi_*\mathcal{O}_{\mathrm{T}^*X}} \pi^{-1}\mathrm{gr}(\mathcal{F})$.

Remark 2.3. It turns out that the characteristic variety is indepedent of the choice of good filtration on $\mathcal{F}$. Moreover, if there is an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, then $\mathrm{Ch}(\mathcal{F}) = \mathrm{Ch}(\mathcal{F}') \cup \mathrm{Ch}(\mathcal{F}'')$.

Definition 2.4. A $\mathcal{D}_X$-module $\mathcal{F}$ is said to be holonomic if either $\mathcal{F} = 0$ or $\dim \mathrm{Ch}(\mathcal{F}) = \dim X$. (A theorem of Bernstein's says that for any nonzero $\mathcal{D}_X$-module $\mathcal{F}$, we have $\dim \mathrm{Ch}(\mathcal{F}) \geq \dim X$.) An object $\mathcal{F} \in \mathrm{D}^b(\mathcal{D}_X)$ in the derived category of bounded $\mathcal{D}_X$-modules is said to be holonomic if each cohomology sheaf is a holonomic $\mathcal{D}_X$-module. Let $\mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ denote the derived category of holonomic $\mathcal{D}_X$-modules.

Example 2.5. Consider a system of differential equations $P_1 f = \cdots = P_k f = 0$ for $P_k \in \mathcal{D}_X$, and consider the associated $\mathcal{D}_X$-module $\mathcal{F} = \mathcal{D}_X/(\mathcal{D}_X P_1 + \cdots + \mathcal{D}_X P_k)$. The characteristic variety $\mathrm{Ch}(\mathcal{F})$ is the set of zeros of the principal symbol $\sigma(Q)$ for each $Q \in \mathcal{D}_X P_1 + \cdots + \mathcal{D}_X P_k$. If the dimension of $\mathcal{F}$ is as small as possible (recall that a theorem of Bernstein's says that its dimension is bounded below by $\dim X$), then this means that the ideal $\mathcal{D}_X P_1 + \cdots + \mathcal{D}_X P_k$ is as large as possible, i.e., that the system of differential equations is overdetermined.

Example 2.6. For a nonzero $\mathcal{D}_X$-module $\mathcal{F}$, the following conditions are equivalent:

- $\mathcal{F}$ defines an integrable connection, i.e., is locally free of finite rank over $\mathcal{O}_X$;
- $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module;
- $\mathrm{Ch}(\mathcal{F}) \cong X$, thought of as sitting inside T^*X via the zero section.

The following is a very important finiteness theorem, which we shall not prove.

Theorem 2.7 ([HTT08, Theorem 3.3.1])**.** *Let $\mathcal{F} \in \mathrm{D}^b_c(\mathcal{D}_X)$ be an object in the constructible bounded derived category of $\mathcal{D}_X$-modules. The following are equivalent.*

(a) *$\mathcal{F}$ is holonomic;*
(b) *for each $i_x : \{x\} \hookrightarrow X$, the cohomology $\mathrm{H}^k(i_x^\dagger \mathcal{F})$ is a finite-dimensional $\mathbf{C}$-vector space;*
(c) *there is a finite sequence $\emptyset = X_{n+1} \subseteq X_n \subseteq \cdots \subseteq X_0 = X$ of closed subsets such that $X_r \setminus X_{r+1}$ is smooth and each $\mathrm{H}^k(i_r^\dagger \mathcal{F})$ is a coherent $\mathcal{O}_X$-module, where $i_r : X_r \setminus X_{r+1} \hookrightarrow X$ is the embedding.*

Theorem 2.7 states that holonomic $\mathcal{D}_X$-modules are precisely those for which there exist a stratification which correspond to integrable connections on each stratum. Comparing to Theorem 1.4, we would expect that a "solution sheaf" on X that might be associated to a holonomic $\mathcal{D}_X$-module $\mathcal{F}$ should be such that there exists a stratification on which the solution sheaf is locally constant. This is precisely a constructible sheaf on X. However, we should not expect the functor from holonomic $\mathcal{D}_X$-modules to constructible sheaves on X to be an equivalence, because of the issue raised in Roadblock 1.5: we need to impose some regularity condition (corresponding to controlling the singularities of a differential equation).

Holonomic D-modules satisfy some nice properties.

Proposition 2.8 ([HTT08, Proposition 3.1.6])**.** *Let $\mathcal{F}$ be a holonomic $\mathcal{D}_X$-module. Then there exists an open dense subset $U \subseteq X$ such that $\mathcal{F}|_U$ is coherent over $\mathcal{O}_U$ (i.e., is a vector bundle over U equipped with an integrable connection).*

Theorem 2.9 ([HTT08, Section 3.2])**.** *The duality functor $\mathbf{D}$ induces an equivalence $\mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}}$. Moreover, the external tensor product induces functors $\boxtimes : \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_X) \times \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_Y) \to \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_{X\times Y})$. If $f : X \to Y$ is a morphism of smooth algebraic varieties, then $\int_f, f^\dagger : \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_X) \to \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_Y)$.*

Remark 2.10. Let $f : X \to Y$ be a morphism of smooth algebraic varieties. We obtain functors $\int_{f^!} = \mathbf{D}_Y \int_f \mathbf{D}_X : \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_X) \to \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_Y)$ (called exceptional direct image) and $f^\star = \mathbf{D}_X f^\dagger \mathbf{D}_Y : \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_Y) \to \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ (called exceptional inverse image) such that $f^\star$ is left adjoint to $\int_f$ and $\int_{f^!}$ is left adjoint to $f^\dagger$. There is also a morphism $\int_{f^!} \to \int_f$ (see [HTT08, Theorem 3.2.16]).

3. Regularity of holonomic D-modules

In this section, we address what it means for a holonomic $\mathcal{D}_X$-module to be regular. To begin with, we recall a theorem of Fuchs (see [HTT08, Theorem 5.1.5]), describing the notion of regularity from Recollection 1.6.

Construction 3.1. Let $\sum_{k=0}^n p_k(z) f^{(k)}(z) = 0$ is an ordinary differential equation with each $p_k(z)$ a meromorphic function. This is equivalent to the differential equation $\left(\sum_{k=0}^n b_k(z)\theta^k\right) f(z) = 0$, where $\theta = z\partial_z$. We then obtain a system of ODEs

$$\frac{d}{dz}\vec{f}(z) = \frac{1}{z}\Gamma(z)\vec{f}(z), \tag{1}$$

where

$$\Gamma(z) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0/b_n & -b_1/b_n & -b_2/b_n & \cdots & -b_{n-1}/b_n \end{pmatrix}.$$

The function f is a solution of our original ODE if and only if the vector $(f, \theta f, \theta^2 f, \cdots, \theta^{n-1} f)$ is a solution to (1).

Theorem 3.2 (Fuchs)**.** *The differential equation $\sum_{k=0}^n p_k(z) f^{(k)}(z) = 0$ is regular if and only if each b_i/b_n is holomorphic.*

This suggests a possible approach to defining regularity.

Definition 3.3. Let D be the open complex disk, and let $\mathring{D}$ denote the punctured disk. Then $\mathcal{O}_D[z^{-1}]$ denotes the sheaf of meromorphic functions on D which are holomorphic on $\mathring{D}$. A meromorphic connection on a vector bundle $\mathcal{F}$ over D is a morphism $\nabla : \mathcal{F} \to \Omega^1_D[z^{-1}] \otimes_{\mathcal{O}_D} \mathcal{F}$ satisfying the Leibniz rule $\nabla(fs) = df \otimes s + f \cdot \nabla(s)$. Motivated by Theorem 3.2, we say that ∇ is regular if there is a choice of local coordinates $e_1, \cdots, e_n$ on $\mathcal{F}$ such that

$$\nabla e_i = \sum_j \frac{b_{ij}(z)}{z} e_j.$$

Equivalently, if we look at the induced $\mathcal{D}_D$-module structure on $\mathcal{F}[z^{-1}]$, then this amounts to saying that there is a $\mathcal{O}_D$-coherent submodule $\mathcal{G} \subseteq \mathcal{F}[z^{-1}]$ with $\mathcal{G}|_{\mathring{D}} = \mathcal{F}[z^{-1}]$ which is stable under $z\nabla$.

This in turn motivates a definition in the algebraic setting.

Definition 3.4. Let $X = \mathbf{A}^1$, and let $U = \mathbf{A}^1 - \{0\}$. Let $i : U \hookrightarrow X$ be the inclusion. Note that $\mathcal{D}_X = \mathbf{C}\langle z, \partial_z\rangle$ and $\mathcal{D}_U = \mathbf{C}\langle z, z^{-1}, \partial_z\rangle$. Let $\theta = z\partial_z$, and let $\mathcal{D}^0_X = \mathbf{C}\langle z, \theta\rangle$. A $\mathcal{O}_U$-coherent $\mathcal{D}_U$-module $\mathcal{F}$ is said to be regular at 0 if $\int_i \mathcal{F}$ is a union of $\mathcal{O}_X$-finitely generated $\mathcal{D}^0_X$-submodules.

Example 3.5. Here are some examples of regular and irregular $\mathcal{D}_X$-modules.

- Consider $\mathcal{F} = \mathcal{D}_U/\mathcal{D}_U \cdot \partial_z = \mathcal{O}_U = \mathbf{C}[z^{\pm 1}]$, so $\int_i \mathcal{F}$ is $\mathbf{C}[z^{\pm 1}]$ as a $\mathbf{C}\langle z, \partial_z\rangle$-module. Then $\theta = z\partial_z$ acts on z^n by nz^n. Since $\mathbf{C}[z^{\pm 1}]$ is $\bigcup_n z^{-n}\mathbf{C}[z]$ and each $z^{-n}\mathbf{C}[z]$ is a $\mathcal{D}^0_X$-submodule with one generator, we find that $\mathcal{F} = \mathcal{O}_U$ is regular at zero.

- More generally, consider $\mathcal{F} = \mathcal{D}_U/\mathcal{D}_U \cdot (z\partial_z - \lambda) = \mathcal{O}_U\{z^\lambda\}$. This is a $\mathbf{C}\langle z, \partial_z\rangle$-module, where $\partial_z(z^\lambda) = \lambda z^{\lambda-1}$. In particular, θ acts on z^λ by λz^λ. Arguing as above, we find that $\mathcal{F}$ is regular at zero.
- Consider $\mathcal{F} = \mathcal{O}_U\{\log(z)\}$, with ∂_z acting on $\log(z)$ by $1/z$. In particular, θ acts on $\log(z)$ by sending it to 1. Since $\mathcal{O}_U\{\log(z)\}$ is $\bigcup_n z^{-n}\mathbf{C}[z]\log(z)$, we find that $\mathcal{F}$ is regular at zero.
- Consider $\mathcal{F} = \mathcal{O}_U\{e^{z^{-n}}\}$ for $n > 1$. Then ∂_z acts on $e^{z^{-n}}$ by $-nz^{-n-1}e^{z^{-n}}$, and so θ acts on $e^{z^{-n}}$ by $-nz^{-n}e^{z^{-n}}$. In particular, if we repeatedly apply θ, we can get all poles, i.e., $\mathbf{C}\langle z, \theta\rangle e^{z^{-n}} = \int_i \mathcal{F}$. Therefore, $\mathcal{F}$ is not regular.

We can now generalize Definition 3.4 to arbitrary smooth curves.

Definition 3.6. Let C be a smooth curve, and let $i : C \hookrightarrow \overline{C}$ denote a smooth compactification of C. Let $Z = \overline{C} \setminus C$, and let $\mathcal{D}_{\overline{C}}^Z$ denote the subsheaf of $\mathcal{D}_{\overline{C}}$ generated by $\mathcal{O}_{\overline{C}}$ and vector fields which vanish at Z. A $\mathcal{O}_C$-coherent $\mathcal{D}_C$-module $\mathcal{F}$ is said to have regular singularities if $\int_i \mathcal{F}$ is a union of $\mathcal{O}_{\overline{C}}$-coherent $\mathcal{D}_{\overline{C}}^Z$-submodules.

Remark 3.7. A theorem of Deligne's says that the definition of having regular singularities does not depend on the choice of compactification.

We can generalize Definition 3.6 to all holonomic $\mathcal{D}_X$-modules, using Proposition 2.8 (which says that holonomic $\mathcal{D}_X$-modules are generically vector bundles with an integrable connection).

Definition 3.8. Let X be a smooth algebraic curve, and let $\mathcal{F}$ be a holonomic $\mathcal{D}_X$-module. Then $\mathcal{F}$ is said to be regular holonomic if there exists a dense open subset $U \subseteq X$ such that $\mathcal{F}|_U$ is a $\mathcal{O}_U$-coherent $\mathcal{D}_U$-module with regular singularities.

We would now like to generalize the notion of regular holonomicity to higher dimensions. This is a little subtle. We need to recall the following theorem. Recall that a nonzero coherent $\mathcal{D}_X$-module $\mathcal{F}$ is said to be simple if there are no nontrivial proper sub-$\mathcal{D}_X$-modules. For any holonomic $\mathcal{D}_X$-module $\mathcal{F}$, there is a Jordan-Hölder filtration $\mathcal{F}_{r+1} = 0 \subseteq \mathcal{F}_r \subseteq \cdots \subseteq \mathcal{F}_0 = \mathcal{F}$ of holonomic sub-$\mathcal{D}_X$-modules such that each quotient $\mathcal{F}_i/\mathcal{F}_{i+1}$ is simple.

Theorem 3.9 ([HTT08, Theorem 3.4.2])**.** *Let X be a smooth algebraic variety. Then:*

(a) *Let $Y \subseteq X$ be a locally closed smooth connected subvariety of X such that the inclusion $i : Y \hookrightarrow X$ is affine. Let $\mathcal{F}$ be a simple holonomic $\mathcal{D}_X$-module. Then the unique simple submodule of $\int_i \mathcal{F}$ is the image $\mathcal{L}(Y, \mathcal{F}) = i_{!*}\mathcal{F}$ of the canonical morphism $\int_{i_!} \mathcal{F} \to \int_i \mathcal{F}$ (called the minimal/Goresky-MacPherson extension).*

(b) *Any simple holonomic $\mathcal{D}_X$-module is isomorphic to the minimal extension $_{!*}\mathcal{F}$, where $Y \subseteq X$ is a locally closed smooth connected subvariety of X such that the inclusion $i : Y \hookrightarrow X$ is affine, and $\mathcal{F}$ is a simple $\mathcal{O}_Y$-coherent $\mathcal{D}_Y$-module (i.e., a vector bundle over Y equipped with an integrable connection).*

Using this theorem and the Jordan-Hölder filtration of any holonomic $\mathcal{D}_X$-module, we may define regularity for holonomic $\mathcal{D}_X$-modules over varieties of any dimension as follows.

Definition 3.10. Let X be a smooth variety over $\mathbf{C}$.

- A $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module $\mathcal{F}$ is said to be regular holonomic if for any smooth curve $C \hookrightarrow X$, the restriction $\mathcal{F}|_C$ is regular holonomic.
- A simple holonomic $\mathcal{D}_X$-module $\mathcal{F}$ is said to be regular holonomic if it is the minimal extension $i_{!*}(\mathcal{G})$ of an embedding $i : Y \hookrightarrow X$ of a locally closed smooth subvariety such that i is affine with $\mathcal{G}$ a regular holonomic $\mathcal{D}_Y$-module (in the sense of (a)).
- A holonomic $\mathcal{D}_X$-module $\mathcal{F}$ is said to be regular holonomic if every simple subquotient of $\mathcal{F}$ is regular holonomic (in the sense of (b)).

- An object $\mathcal{F} \in \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_X)$ is said to be regular holonomic if each cohomology sheaf is regular holonomic.

The most important foundational results involving regular holonomic D-modules are the following, proved in [HTT08, Chapter 6].

Theorem 3.11. *Let X be a smooth algebraic variety. Then $\mathbf{D}$ preserves regular holonomicity, as do direct and exceptional direct image and inverse and exceptional inverse image. Moreover, a holonomic $\mathcal{D}_X$-module $\mathcal{F}$ is regular if and only if $i_C^\dagger \mathcal{F}$ is a regular holonomic $\mathcal{D}_C$-module for every locally closed embedding $i : C \hookrightarrow X$ of a smooth curve X.*

Remark 3.12. In particular, Theorem 3.11 shows that a regular holonomic $\mathcal{D}_X$-module should be thought of as a holonomic $\mathcal{D}_X$-module whose restriction to each curve is generically a vector bundle with integrable connection, whose worst singularities are simple poles (i.e., of order 1).

The following is an important theorem of Deligne's:

Theorem 3.13 (Deligne; [HTT08, Theorem 5.3.8])**.** *Let X be a smooth variety over $\mathbf{C}$. Then the analytification functor defines an equivalence $\mathrm{Vect}^{\mathrm{reg}}(X)^\nabla \xrightarrow{\sim} \mathrm{Vect}(X^{\mathrm{an}})^\nabla$.*

Using Theorem 1.4, we have:

Corollary 3.14. *Let X be a smooth variety over $\mathbf{C}$. Then there is an equivalence $\mathrm{Vect}^{\mathrm{reg}}(X)^\nabla \simeq \mathrm{LocSys}(X^{\mathrm{an}})$.*

Corollary 3.14 is called Deligne's Riemann-Hilbert correspondence.

4. Riemann-Hilbert correspondence

We are now in a position to state the Riemann-Hilbert correspondence. As mentioned in the discussion following Theorem 2.7, one expects an equivalence of (derived) categories between regular holonomic $\mathcal{D}_X$-modules and constructible sheaves on X. In order to define the functor taking a regular holonomic $\mathcal{D}_X$-module to a constructible sheaf on X, we look at Theorem 1.4. The functor from vector bundles equipped with an integrable connection to local systems sent a pair $(\mathcal{F}, \nabla)$ to the local system $\ker(\nabla)$ of horizontal sections. From the perspective of $\mathcal{D}_X$-modules, this is sending the pair $(\mathcal{F}, \nabla)$ regarded as a $\mathcal{D}_X$-module to the solution set of ∇, i.e., $\mathrm{Hom}_{\mathcal{D}_{X^{\mathrm{an}}}}(\mathcal{F}^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$. As usual, the Hom here is taken in the derived sense.

Definition 4.1. Let X be a complex manifold. For $\mathcal{F} \in \mathrm{D}^b(\mathcal{D}_X)$, define the de Rham complex $\mathrm{DR}(\mathcal{F}) \in \mathrm{D}^b(X)$ as the derived tensor product $\omega_X \otimes_{\mathcal{D}_X} \mathcal{F}$ (where we are regarding ω_X as $\Omega_X^{\dim(X)}[\dim(X)]$). Define the solution complex $\mathrm{Sol}(\mathcal{F})$ to be $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{F}, \mathcal{O}_X)$ (where we're taking derived Hom). The de Rham and solution complexes define functors $\mathrm{DR} : \mathrm{D}^b(\mathcal{D}_X) \to \mathrm{D}^b(X)$ and $\mathrm{Sol} : \mathrm{D}^b(\mathcal{D}_X) \to \mathrm{D}^b(X)^{\mathrm{op}}$.

Proposition 4.2 ([HTT08, Proposition 4.2.1])**.** *Let X be a complex manifold. For $\mathcal{F} \in \mathrm{D}^b_c(\mathcal{D}_X)$, we have*

$$\mathrm{DR}(\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{F})[\dim(X)] = \mathrm{Sol}(\mathbf{D}\mathcal{F})[\dim(X)] \cong \mathbf{D}(\mathrm{Sol}(\mathcal{F})[\dim(X)]).$$

This motivates the definition of the functor in the following statement of the Riemann-Hilbert correspondence.

Theorem 4.3 (Riemann-Hilbert correspondence)**.** *Let X be a smooth variety over $\mathbf{C}$. Then the de Rham functor $\mathrm{DR} : \mathrm{D}^b_{\mathrm{rhol}}(\mathcal{D}_X) \to \mathrm{D}^b_c(X^{\mathrm{an}})$ sending $\mathcal{F} \in \mathrm{D}^b_{\mathrm{rhol}}(\mathcal{D}_X)$ to $\omega_{X^{\mathrm{an}}} \otimes_{\mathcal{D}_{X^{\mathrm{an}}}} \mathcal{F}^{\mathrm{an}}$ (where, again, the tensor product is derived).*

Remark 4.4. Some remarks are in order.

- It is not at all clear that the de Rham functor as defined lands in the *constructible* derived category $\mathrm{D}^b_c(X^{\mathrm{an}})$. This is known as Kashiwara's constructibility theorem.
- The equivalence of Theorem 4.3 can be refined further: the category $\mathrm{D}^b_{\mathrm{rhol}}(\mathcal{D}_X)$ admits a t-structure whose heart is the category of regular holonomic $\mathcal{D}_X$-modules, and the equivalence of Theorem 4.3 further says that the de Rham functor is t-exact when $\mathrm{D}^b_c(X)$ is equipped with the perverse t-structure. In particular, taking the heart of the t-exact equivalence of Theorem 4.3 produces an equivalence between regular holonomic $\mathcal{D}_X$-modules and perverse sheaves on X.
- Theorem 4.3 is the *covariant* Riemann-Hilbert correspondence. The solutions functor $\mathrm{DR} : \mathrm{D}^b_{\mathrm{rhol}}(\mathcal{D}_X) \to \mathrm{D}^b_c(X^{\mathrm{an}})^{\mathrm{op}}$ sending $\mathcal{F} \in \mathrm{D}^b_{\mathrm{rhol}}(\mathcal{D}_X)$ to $\mathrm{Hom}_{\mathcal{D}_{X^{\mathrm{an}}}}(\mathcal{F}^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ (where, again, the tensor product is derived) also defines an equivalence (by Proposition 4.2), and is called the *contravariant* Riemann-Hilbert correspondence. We chose to present the covariant Riemann-Hilbert correspondence because that is how it is presented in [HTT08].

Example 4.5. Consider the affine curve $X = \mathbf{A}^1_{\mathbf{C}}$, and the $\mathcal{D}_X$-module $\mathcal{D}_X/\mathcal{D}_X \cdot (z\partial_z - \lambda)$. In Example 3.5, we saw that this was a regular $\mathcal{D}_X$-module. What is the associated constructible sheaf on $X^{\mathrm{an}} = \mathbf{C}$? To describe it, we will use the solutions functor, rather than the de Rham functor (for simplicity). Let $\mathcal{P}$ denote the associated constructible sheaf (in fact, it will be a perverse sheaf, by the previous remark). The perverse sheaf associated to $\mathcal{D}_X/\mathcal{D}_X \cdot (z\partial_z - \lambda)$ under the contravariant Riemann-Hilbert correspondence is $\mathrm{Hom}_{\mathcal{D}_{X^{\mathrm{an}}}}(\mathcal{D}_{X^{\mathrm{an}}}/\mathcal{D}_{X^{\mathrm{an}}} \cdot (z\partial_z - \lambda), \mathcal{O}_{X^{\mathrm{an}}})$, where this means derived Hom. There's an obvious resolution of our $\mathcal{D}_X$-module, and so the derived Hom is given by the complex $\mathbf{C}[z] \xrightarrow{z\partial_z - \lambda} \mathbf{C}[z]$. This is the perverse sheaf associated to our $\mathcal{D}_X$-module.

Let $j : \mathbf{C}^\times \hookrightarrow \mathbf{C}$ denote the inclusion. The restriction of $\mathcal{P}$ to $\mathbf{C}^\times$ consists of solutions to the differential equation $z\partial_z f = \lambda f$, i.e., cz^λ, with $c \in \mathbf{C}$. Note that if λ is not an integer, then these are not algebraic functions — this shows why we get a constructible sheaf in the analytic topology. Therefore, $\mathcal{P}|_{\mathbf{C}^\times}$ is the rank one local system $\mathcal{L}_\lambda$ of functions of the form cz^λ with $c \in \mathbf{C}$. On $\mathbf{C}$ itself, we run into a dichotomy. If $\lambda \notin \mathbf{Z}_{\geq 0}$, then there is no solution to the differential equation $z\partial_z f = \lambda f$ on $\mathbf{C}$, and so $\mathcal{P}$ would be the extension-by-zero of $\mathcal{P}_{\mathbf{C}^\times} = \mathcal{L}_\lambda$, i.e., $\mathcal{P} = j_!(\mathcal{L}_\lambda)$. In other words, the differential equation has nontrivial monodromy around zero. (Note that if $\lambda \in \mathbf{Z}_{\geq 0}$, then we can in fact solve the differential equation $z\partial_z f = \lambda f$ on $\mathbf{C}$, by cz^λ with $c \in \mathbf{C}$.)

Remark 4.6. Let X be a complex manifold. One resolution of ω_X as a right $\mathcal{D}_X$-module is given by

$$0 \to \mathcal{D}_X \to \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \cdots \to \Omega^{n-1}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \omega_X \to 0,$$

where, in local coordinates, we have

$$d(\omega \otimes P) = d\omega \otimes P + \sum_i dx_i \wedge \omega \otimes \partial_{x_i} P.$$

This gives a complex for $\mathrm{DR}(\mathcal{F})$ when $\mathcal{F}$ is a (left) $\mathcal{D}_X$-module:

$$0 \to \mathcal{F} \to \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{F} \to \cdots \to \Omega^{n-1}_X \otimes_{\mathcal{O}_X} \mathcal{F} \to \omega_X \otimes_{\mathcal{O}_X} \mathcal{F};$$

the first map is precisely the connection on $\mathcal{F}$ determined by the $\mathcal{D}_X$-module structure. It follows that the top-dimensional cohomology $\mathrm{H}^{-\dim(X)}(\mathrm{DR}(\mathcal{F}))$ is isomorphic to the sheaf $\ker(\nabla)$ of horizontal sections of $\mathcal{F}$. If $\mathcal{F}$ is a vector bundle equipped with an integrable connection viewed as a $\mathcal{D}_X$-module, then one can show that there are no other cohomology groups of $\mathrm{DR}(\mathcal{F})$, and therefore the functor of Theorem 4.3 agrees with the equivalence of Theorem 1.4 (up to cohomological shift, which goes away if you work with the solutions complex rather than the de Rham complex).

Let us sketch a proof of Theorem 4.3. We begin by stating the following theorem, which ensures that the functor in Theorem 4.3 does in fact land in the constructible derived category.

Theorem 4.7 (Kashiwara's constructibility theorem; [HTT08, Theorem 4.6.3])**.** *Let X be a complex manifold. Let $\mathcal{F}$ be a holonomic $\mathcal{D}_X$-module. Then $\mathrm{DR}(\mathcal{F}) \in \mathrm{D}^b(X)$ is constructible.*

The hard input into Theorem 4.3 is the following theorem, which we shall not prove.

Theorem 4.8 ([HTT08, Section 7.1])**.** *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. The de Rham complex functor $\mathrm{DR} : \mathrm{D}^b_{\mathrm{rhol}}(\mathcal{D}_X) \to \mathrm{D}^b_c(X^{\mathrm{an}})$ commutes with duals, direct and exceptional direct image, inverse image and exceptional inverse image, and products.*

In Theorem 4.8, it is critical that one uses regular $\mathcal{D}_X$-modules. Given Theorem 4.8, the proof of Theorem 4.3 is a calculation.

Proof sketch of Theorem 4.3 assuming Theorem 4.8. The essential surjectivity of DR is rather easy. It suffices to check that the generators of $\mathrm{D}^b_c(X^{\mathrm{an}})$. Since $\mathrm{D}^b_c(X^{\mathrm{an}})$ is generated by $i_*\mathcal{L}$ for a closed embedding $Z \hookrightarrow X$ of a locally closed smooth subvariety and a local system $\mathcal{L}$ on Z^{an} (to check this, just peel off the smooth locus of the support of any constructible sheaf), it suffices to show that there is a $\mathcal{D}_Z$-module $\mathcal{F}$ on Z such that $\mathrm{DR}(\mathcal{F}) = \mathcal{L}$. This follows from Corollary 3.14 (Deligne's Riemann-Hilbert correspondence) and the observation in Remark 4.6 relating the de Rham complex to the classical equivalence of Theorem 1.4.

To show full faithfulness, we just calculate. Let $\Delta : X \to X \times X$ denote the diagonal embedding, and let $p : X \to *$ denote the projection to a point. If $\mathcal{L}, \mathcal{L}' \in \mathrm{D}^b_c(X^{\mathrm{an}})$, then $\mathrm{Hom}(\mathcal{L}, \mathcal{L}') = p_*\Delta^!(\mathbf{D}_X(\mathcal{L}) \boxtimes \mathcal{L}')$. Indeed, simply apply p_* to:

$$\begin{gathered}\mathrm{Hom}(\mathcal{L}, \mathcal{L}') \simeq \mathrm{Hom}(\mathcal{L}, \mathbf{D}^2(\mathcal{L}')) \simeq \mathrm{Hom}(\mathcal{L} \otimes \mathbf{D}(\mathcal{L}'), \omega_X) \\ \simeq \mathbf{D}(\mathcal{L} \otimes \mathbf{D}(\mathcal{L}')) \simeq \mathbf{D}\Delta^{-1}(\mathcal{F} \boxtimes \mathbf{D}(\mathcal{L}')) \simeq \Delta^!(\mathbf{D}(\mathcal{F}) \boxtimes \mathcal{L}').\end{gathered}$$

The same string of identifications (in the setting of D-modules) shows that if $\mathcal{F}, \mathcal{G} \in \mathrm{D}^b_{\mathrm{rhol}}(X)$, then $\int_p \Delta^!(\mathbf{D}_X(\mathcal{F}) \boxtimes \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{F}, \mathcal{G})$. It follows that if $\mathcal{F}, \mathcal{G} \in \mathrm{D}^b_{\mathrm{rhol}}(X)$, then (because DR commutes with everything by Theorem 4.8):

$$\begin{aligned}\mathrm{Hom}(\mathrm{DR}(\mathcal{F}), \mathrm{DR}(\mathcal{G})) &\simeq p_*\Delta^!(\mathbf{D}_X(\mathrm{DR}(\mathcal{F})) \boxtimes \mathrm{DR}(\mathcal{G})) \\ &\simeq \mathrm{DR}_{\mathrm{pt}} \int_p \Delta^!(\mathbf{D}_X(\mathcal{F}) \boxtimes \mathcal{G}) \\ &\simeq \int_p \Delta^!(\mathbf{D}_X(\mathcal{F}) \boxtimes \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{F}, \mathcal{G}).\end{aligned}$$

□

Appendix A. The multitude of functors

There are a lot of functors floating around, so to get our ducks in a row, let us quickly summarize them. Recall that one is supposed to think of left $\mathcal{D}_X$-modules as functions (i.e., things on which differential operators act), and right $\mathcal{D}_X$-modules as duals of functions (think distributions). There is an equivalence of categories between (the derived categories of) left and right $\mathcal{D}_X$-modules, given by sending a left $\mathcal{D}_X$-module $\mathcal{F}$ to $\mathcal{F} \otimes \omega_X$, and the inverse equivalence sends a right $\mathcal{D}_X$-module $\mathcal{F}$ to $\mathcal{F} \otimes \omega_X^{-1}$.

For the remainder of this section, let $f : X \to Y$ be a morphism of smooth algebraic varieties.

Definition A.1. Let $\mathcal{F}$ be a (left) $\mathcal{D}_X$-module. Then the $\mathcal{O}_X$-module $f^*\mathcal{F} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}$ admits the structure of a left $\mathcal{D}_X$-module. There is a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule structure on $f^*\mathcal{D}_Y$, which is denoted $\mathcal{D}_{X\to Y}$.

We will really be concerned with the *derived* category of D-modules, and so the correct thing to be studying is the derived tensor product $\mathcal{O}_X \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}$. We will drop the derived tensor product, and abusively just denote this tensor product by f^*. Notice that

$$\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F} = (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{F} = \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{F}.$$

The shifted inverse image functor $f^\dagger$ is defined by $f^\dagger\mathcal{F} = f^*(\mathcal{F})[\dim(X) - \dim(Y)]$.

Remark A.2. Suppose $\mathcal{F}$ is a right $\mathcal{D}_X$-module. Then $f_*(\mathcal{F} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$ is a right $\mathcal{D}_Y$-module, which one might adopt as the definition of direct images for right D-modules. (To get some intuition for why it is easier to define direct images for right D-modules, recall that we can naturally pushforward forms by integration along fibers, but we can't do something so natural for functions.) This, however, turns out to be poorly behaved homologically. If we imagine that this was the right definition, though, then the equivalence between left and right D-modules discussed above would suggest that the the direct image functor for left D-modules might be defined as follows.

Define a $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-module $\mathcal{D}_{Y\leftarrow X}$ by $\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^{-1}$. Then if $\mathcal{F}$ is a left $\mathcal{D}_X$-module, $f_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{F})$ would be a candidate for the direct image of $\mathcal{F}$ as a left $\mathcal{D}_Y$-module.

Definition A.3. The correct definition of direct images for a left $\mathcal{D}_X$-module $\mathcal{F}$ (rather, an object in the derived category) is the *derived* pushforward $f_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{F})$ of the *derived* tensor product. Again, we will never emphasize the fact that everything is derived. The functor $\mathrm{D}^b(\mathcal{D}_X) \to \mathrm{D}^b(\mathcal{D}_Y)$ is denoted $\int_f$.

It is useful to see this definition worked out in examples.

Example A.4. Consider the case when $i : X \to Y$ is a closed immersion. Around each point $x \in X$, we can choose a coordinate system $\{y_i\}_{1\leq i\leq n}$ on an affine open of Y such that the neighbourhood of x is defined by the vanishing $y_{r+1} = \cdots = y_n = 0$. Then, if we define $\mathcal{D}'_Y = \bigoplus \mathcal{O}_Y \partial_1^{m_1}\cdots\partial_r^{m_r}$, we have $\mathcal{D}_Y = \mathcal{D}'_Y \otimes_{\mathbf{C}} \mathbf{C}[\partial_{y_{r+1}}, \cdots, \partial_{y_n}]$. It is easy to see that $i^*\mathcal{D}'_Y = \mathcal{D}_X$, and so $\mathcal{D}_{X\to Y} = \mathcal{D}_X \otimes_{\mathbf{C}} \mathbf{C}[\partial_{y_{r+1}}, \cdots, \partial_{y_n}]$. Note that this is not of finite type as a $\mathcal{O}_X$-module. Similarly, we can identify $\mathcal{D}_{Y\leftarrow X} = \mathbf{C}[\partial_{y_{r+1}}, \cdots, \partial_{y_n}] \otimes_{\mathbf{C}} \mathcal{D}_X$. In particular, $\int_i \mathbf{C}$ is $i_*(\mathbf{C}[\partial_{y_{r+1}}, \cdots, \partial_{y_n}])$. These are precisely the distributions on Y which are supported on X.

For instance, if $X = \{0\} \hookrightarrow Y = \mathbf{A}^1 = \operatorname{Spec}\mathbf{C}[t]$, then $\int_i \mathbf{C}$ is $i_*(\mathbf{C}[\partial_t])$, which one should think of as the D-module of Dirac delta functions at the origin.

Example A.5. Suppose $Y = *$, so $f : X \to *$ is just the projection. Then $\mathcal{D}_{X\to Y} = f^*\mathcal{D}_Y = f^*\mathcal{O}_Y = \mathcal{O}_X$, so $\mathcal{D}_{Y\leftarrow X} = \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^{-1} = \omega_X$, and therefore $\int_f \mathcal{F}$ is $\omega_X \otimes_{\mathcal{D}_X} \mathcal{F}$. This is precisely the de Rham functor $\mathrm{DR}(\mathcal{F})$; we gave an explicit description via a resolution of ω_X as a right $\mathcal{D}_X$-module in Remark 4.6.

Definition A.6. There is a functor $-\boxtimes- : \mathrm{D}^b(\mathcal{D}_X) \times \mathrm{D}^b(\mathcal{D}_Y) \to \mathrm{D}^b(\mathcal{D}_{X\times Y})$, called the exterior tensor product, defined as follows. Let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to X$; then, for $\mathcal{F} \in \mathrm{D}^b(\mathcal{D}_X)$ and $\mathcal{G} \in \mathrm{D}^b(\mathcal{D}_Y)$, define

$$\mathcal{F} \boxtimes \mathcal{G} = \mathcal{D}_{X\times Y} \otimes_{p_1^{-1}\mathcal{D}_X \otimes_{\mathbf{C}} p_2^{-1}\mathcal{D}_Y} (p_1^{-1}\mathcal{F} \otimes_{\mathbf{C}} p_2^{-1}\mathcal{G}).$$

Because there is an isomorphism

$$\mathcal{D}_{X\times Y} \cong \mathcal{O}_{X\times Y} \otimes_{p_1^{-1}\mathcal{O}_X \otimes_{\mathbf{C}} p_2^{-1}\mathcal{O}_Y} (p_1^{-1}\mathcal{D}_X \otimes_{\mathbf{C}} p_2^{-1}\mathcal{D}_Y),$$

the underlying $\mathcal{O}_{X\times Y}$-module of $\mathcal{F}\boxtimes\mathcal{G}$ is $\mathcal{O}_{X\times Y}\otimes_{p_1^{-1}\mathcal{O}_X\otimes_{\mathbf{C}}p_2^{-1}\mathcal{O}_Y}(p_1^{-1}\mathcal{F}\otimes_{\mathbf{C}}p_2^{-1}\mathcal{G})$. This is the usual exterior tensor product.

If $\mathcal{F},\mathcal{G}\in \mathrm{D}^b(\mathcal{D}_X)$, then the tensor product $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}$ is $\Delta^*(\mathcal{F}\boxtimes\mathcal{G})$, where $\Delta: X\to X\times X$ is the diagonal.

Definition A.7. Let $\mathcal{F}\in\mathrm{D}^b(\mathcal{D}_X)$. The dual $\mathbf{D}(\mathcal{F})\in\mathrm{D}^b(\mathcal{D}_X)$ is defined as $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{F},\mathcal{D}_X)\otimes_{\mathcal{O}_X}\omega_X\cong\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{F},\mathcal{D}_X\otimes_{\mathcal{O}_X}\omega_X)$. (As usual, this means derived Hom.)

Appendix B. A question

I have a possibly silly question that I'd like to get an answer to. Let X be a smooth variety over $\mathbf{C}$. Recall the de Rham space X_{dR}, whose functor of points $\mathrm{CAlg}_{\mathbf{C}}\to\mathrm{Set}$ may be defined by $X_{\mathrm{dR}}(R)=X(R/I)$, where I is the nilradical of R. Then (see the paper by Gaitsgory-Rozenblyum, for instance):

Theorem B.1 (Grothendieck)**.** *There is an equivalence of categories* $\mathrm{QCoh}(X_{\mathrm{dR}})\simeq\mathrm{Mod}(\mathcal{D}_X)$.

Recall how this equivalence goes[1]. A quasicoherent sheaf $\mathcal{F}\in\mathrm{QCoh}(X_{\mathrm{dR}})$ is the data of a quasicoherent sheaf $\mathcal{F}$ on X along with compatible isomorphisms $\mathcal{F}(x)\to\mathcal{F}(y)$ for every pair of "infinitesimally close" R-points $x,y\in X(R)$ (i.e., points whose image under $X(R)\to X(R/I)$ are the same, where I is the nilradical of R). More precisely, if the pair (x,y) is thought of as an R-point of $X\times X$, then x and y are infinitesimally close if and only if they are the same in some thickening of the diagonal $\Delta: X\to X\times X$. Therefore, if $\mathcal{I}$ denotes the ideal sheaf of Δ, then x and y are infinitesimally close if and only if for every $\mathbf{C}$-algebra R, the ideal $(x,y)^*\mathcal{I}^n$ is zero in R for $n\gg 0$, where $(x,y):\mathrm{Spec}(R)\to X\times X$.

Let $X^{(n)}$ denote the closed subscheme of $X\times X$ defined by $\mathcal{I}^{n+1}$. Let p_i denote the projections $(X\times X)^\wedge_X=\mathrm{colim}\,X^{(n)}\to X$, and let $p_i^{(n)}$ denote the induced maps $X^{(n)}\to X$. A quasicoherent sheaf $\mathcal{F}\in\mathrm{QCoh}(X_{\mathrm{dR}})$ is therefore a quasicoherent sheaf $\mathcal{F}$ on X along with the data of compatible isomorphisms $(p_1^{(n)})^*\mathcal{F}\to(p_2^{(n)})^*\mathcal{F}$. This, in turn, is the same as a map $\mathcal{F}\to(p_1^{(n)})_*(p_2^{(n)})^*\mathcal{F}\cong\mathcal{O}_{X^{(n)}}\otimes_{\mathcal{O}_X}\mathcal{F}$.

The key point, now, is that there is a canonical pairing $F_n\mathcal{D}_X\otimes_{\mathcal{O}_X}\mathcal{O}_{X^{(n)}}\to\mathcal{O}_X$. Given a differential operator D and a function $f(x,y)$ defined up to order $n+1$ (i.e., a section of $\mathcal{O}_{X^{(n)}}$), we obtain a function on X by applying D to f (keeping the variable y constant) and evaluating on (x,x) (i.e., $(Df)(x,x)$). When $X=\mathbf{A}^1=\mathrm{Spec}\,\mathbf{C}[t]$, we know that $F_n\mathcal{D}_X$ is the free $\mathbf{C}[t]$-module generated by $\frac{\partial_t^k}{k!}$ for $1\leq k\leq n$, and that $\mathcal{O}_{X^{(n)}}$ is $\mathbf{C}[t,z]/(t-z)^{n+1}$. Applying $\frac{\partial_t^k}{k!}$ to the function t^kz^j in the manner described above produces the function t^j on $\mathbf{A}^1$. In particular, the pairing can be checked to be perfect (and this is true over any smooth variety X). Therefore the maps $\mathcal{F}\to\mathcal{O}_{X^{(n)}}\otimes_{\mathcal{O}_X}\mathcal{F}$ are the same as maps $F_n\mathcal{D}_X\otimes_{\mathcal{O}_X}\mathcal{F}\to\mathcal{F}$, and these assemble into an action of $\mathcal{D}_X$ on $\mathcal{F}$.

In particular, (the derived version of) Theorem B.1 and Theorem 4.3 supply us with the following diagram:

$$\begin{tikzcd}
\mathrm{D}^b_{\mathrm{coh}}(\mathcal{D}_X) \arrow[r, hook] \arrow[d, "\sim"] & \mathrm{D}^b_{\mathrm{rhol}}(\mathcal{D}_X) \arrow[r, hook] \arrow[d, "\sim"] & \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r, hook] & \mathrm{D}^b(\mathcal{D}_X) & \mathrm{D}^b(X_{\mathrm{dR}}), \arrow[l, "\sim"'] \\
\mathrm{LocSys}(X^{\mathrm{an}}) \arrow[r, hook] & \mathrm{D}^b_c(X^{\mathrm{an}}) & & &
\end{tikzcd}$$

where $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{D}_X)$ denotes the subcategory of $\mathcal{D}_X$-modules which are coherent $\mathcal{O}_X$-modules. This leads to the following question:

[1] See these notes by Jacob Lurie: http://people.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/Nov17-19(Crystals).pdf.

Question B.2. What is a natural definition of holonomicity and regularity for a complex of quasicoherent sheaves on X_{dR} which correspond to the same notions under Theorem B.1?

Obviously, one can just transport the definitions along the equivalence of Theorem B.1, but it would be nice to know of "intrinsic" definitions. The property of being a $\mathcal{D}_X$-module which is coherent as a $\mathcal{O}_X$-module is the same on X_{dR}: there is a canonical map $i : X \to X_{\mathrm{dR}}$, and one asks that the pullback along i be coherent as a $\mathcal{O}_X$-module. Theorem 2.7 suggests a definition for holonomicity on the de Rham space.

References

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