ROOTS OF UNITY IN $K(n)$-LOCAL RINGS

ABSTRACT. The goal of this paper is to address the following question: if A is an $\mathbf{E}_k$-ring for some $k \geq 1$ and $f: \pi_0 A \to B$ is a map of commutative rings, when can we find an $\mathbf{E}_k$-ring R with an $\mathbf{E}_k$-ring map $g: A \to R$ such that $\pi_0 g = f$? A classical result in the theory of realizing $\mathbf{E}_\infty$-rings, due to Goerss–Hopkins, gives an affirmative answer to this question if f is étale. The goal of this paper is to provide answers to this question when f is ramified. We prove a non-realizability result in the $K(n)$-local setting for every $n \geq 1$ for H_∞-rings containing primitive pth roots of unity. As an application, we give a proof of the folk result that the Lubin–Tate tower from arithmetic geometry does not lift to a tower of H_∞-rings over Morava E-theory.

1. INTRODUCTION

In this paper, we study some rigidity properties of structured ring spectra in $K(n)$-local spectra (see [HS99] for a thorough introduction). The primary question we are concerned with in this paper is the following:

Question 1.1. Let A be an $\mathbf{E}_k$-ring for some $k \geq 1$, and let $f: \pi_0 A \to B$ be a map of commutative rings. Can we find an $\mathbf{E}_k$-ring R with an $\mathbf{E}_k$-ring map $g: A \to R$ such that $\pi_0 g = f$?

As one might expect, Question 1.1 is hard to resolve in general, particularly without any restrictions on the map f. Variants of this question with additional conditions imposed on the map f have been studied quite extensively. In the case $k = \infty$, Question 1.1 was originally raised in [SVW99], where they used topological Hochschild homology to provide a negative answer to the general form of the question: they showed that the extension $\mathbf{Z} \to \mathbf{Z}[i]$ does not lift to an $\mathbf{E}_\infty$-ring extension of the sphere spectrum. The question of adjoining a primitive root of unity to an $\mathbf{E}_\infty$-ring in the unramified case was studied in [SVW99, Theorem 3] and [BR07, Example 2.2.8]. We refer the reader to the lovely paper [Goe09] for further discussion of Question 1.1 from the perspective of chromatic homotopy theory.

One solution to Question 1.1 (which sets up a general framework for answering such questions) is provided by Goerss–Hopkins obstruction theory (see [GH04]); using their work, the following result was deduced in [BR07] (see also [Lur16, Theorem 7.5.0.6]):

Theorem 1.2. *Question 1.1 admits an affirmative answer if f is an étale map.*

For instance, if k is a finite field of characteristic p and $\mathrm{W}(k)$ is the ring of Witt vectors, then the map $\mathbf{Z}_p \to \mathrm{W}(k)$ is étale. As $\pi_0 \mathbf{S}_p = \mathbf{Z}_p$, Theorem 1.2 gives an $\mathbf{E}_\infty$-ring $\mathbf{S}_{\mathrm{W}(k)}$, called the spherical Witt vectors, such that $\pi_0 \mathbf{S}_{\mathrm{W}(k)} = \mathrm{W}(k)$.

The goal of this paper is to address Question 1.1 in the world of $K(n)$-local spectra when f is ramified. Our main result states that f in general cannot be lifted to a map of structured ring spectra, even if one restricts to studying H_∞-rings. A H_∞-ring structure is the analogue of an $\mathbf{E}_\infty$-ring structure in the stable *homotopy* category; in particular, $\mathbf{E}_\infty$-rings provide examples of H_∞-rings. We show:

Theorem 1.3. *Let $n > 0$. There is no nontrivial $K(n)$-local H_∞-ring R at a prime $p > 2$ (resp. at $p = 2$) such that $\pi_0 R$ contains a primitive pth root of unity (resp. a 4th root of unity).*

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Since $\mathbf{E}_\infty$-rings are, in particular, H_∞-rings, Theorem 1.3 shows that no nontrivial $K(n)$-local $\mathbf{E}_\infty$-ring R at a prime $p > 2$ (resp. at $p = 2$) such that $\pi_0 R$ contains a primitive pth root of unity (resp. a 4th root of unity).

One of the primary motivations for desiring a positive answer to Question 1.1 is because it often allows for the construction of sheaves of $\mathbf{E}_\infty$-rings on chromatically interesting moduli stacks M (such as the moduli stack of formal groups M_{FG} and the moduli stack of elliptic curves) which lift the structure sheaf of M. This is the perspective through which Question 1.1 is discussed in [Goe09]. For instance, Theorem 1.2 was used by Goerss–Hopkins–Miller to construct a sheaf of even-periodic $\mathbf{E}_\infty$-rings on the étale site of the moduli stack of elliptic curves; the global sections of this sheaf is the $\mathbf{E}_\infty$-ring TMF. A lot of large-scale phenomena in chromatic homotopy theory are shadows of the hope (discussed in [Goe09]) that there is some Grothendieck topology on the moduli stack M_{FG} of one-dimensional formal groups for which the structure sheaf $\mathcal{O}_{M_{FG}}$ admits a lift to a sheaf of $\mathbf{E}_\infty$-rings; the global sections of this $\mathbf{E}_\infty$-ring would be the sphere spectrum. Theorem 1.3 places severe restrictions on any Grothendieck topology on any moduli stack M (including M_{FG}) for which there is a sheaf of $\mathbf{E}_\infty$-rings on the associated site lifting the structure sheaf of M.

Our proof of Theorem 1.3 spans the whole of Section 2, and relies heavily on power operations obtained from the H_∞-ring structure imposed on R. The proof also relies on a result of Hahn's from [Hah16], which says that a $K(n)$-acyclic H_∞-ring is $K(n+1)$-acyclic.

The following is a natural question motivated by Theorem 1.3:

Question 1.4. Consider the setup of Question 1.1, and let $k \geq 1$ be finite. Is there an analogue of Theorem 1.3 for $K(n)$-local $\mathbf{E}_k$-rings? Namely, is there a $K(n)$-local $\mathbf{E}_k$-ring R such that $\pi_0 R$ contains a primitive pth root of unity?

Remark 1.5. Other notions of ramification in the world of structured ring spectra have been studied. For instance, the map bo $\to$ bu is an example of a ramified extension in the sense of Dundas–Lindenstrauss–Richter [DLR18].

In Section 3, we study an application of Theorem 1.3 to lifting the Lubin–Tate tower from arithmetic geometry into the realm of spectral algebraic geometry. The Lubin–Tate tower, originally introduced in [Dri74], and further studied (for instance) in [RZ96, Chapter 3], plays an important role in the proof of the Jacquet–Langlands correspondence (see [Rog83]), and also carries information about power operations on Morava E-theory by a result of Strickland's (see [Str98]). As the name suggests, it is a tower of moduli problems (which are in fact represented by affine formal schemes) living over the Lubin–Tate moduli space. The nth level of this tower parametrizes deformations of formal groups along with a level $\Gamma_1(p^n)$-structure, i.e., a basis for the p^n-torsion in the deformed formal group. As a consequence of Theorem 1.3, we prove:

Theorem 1.6. *Let $M_k^h = \operatorname{Spf} A_k^h$ denote the Lubin–Tate space at level p^k and height h, and let E_h denote Morava E-theory at height h and the prime p. If $k > 0$, there is no H_∞-ring over E_h whose underlying ring is A_k^h.*

It follows from the main result of [Str98] that there is an isomorphism $A_k^h \cong E_h^0(BC_{p^k})/\operatorname{Tr}$, where Tr is a certain ideal of $E_h^0(BC_{p^k})$, known as the transfer ideal. The ring $E_h^0(BC_{p^k})/\operatorname{Tr}$ parametrizes power operations on height h Morava E-theory. Consequently, Theorem 1.6 says that rings of power operations do not lift to derived algebraic geometry. A different proof of Theorem 1.6 was provided by Niko Naumann in private communication.

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2. Ramification and structured ring spectra

We will use the following terminology.

Definition 2.1. A θ-ring is a ring A with an operator $\theta\colon A \to A$ such that the following identities are satisfied:

$$\begin{aligned} \theta(0) &= 0 \\ \theta(x+y) &= \theta(x) + \theta(y) + \frac{x^p + y^p - (x+y)^p}{p} \\ \theta(xy) &= \theta(x)y^p + \theta(y)x^p + p\theta(x)\theta(y). \end{aligned}$$

These conditions imply that $\psi^p(x) = x^p + p\theta(x)$ is a ring endomorphism of A.

The following result is described in [BMMS86, Chapter VIII], [Bou96], and [Hop14, Section 3] (the latter for $\mathbf{E}_\infty$-rings).

Theorem 2.2. *Let R be a $K(1)$-local H_∞-ring. Then, $\pi_0 R$ naturally has the structure of a θ-ring via power operations, denoted ψ^p and θ.*

Using the existence of these power operations, Mike Hopkins (in unpublished work) gave a proof of the following fact.

Lemma 2.3. *Let R be the spectrum $K_p[\zeta_p]$ obtained by adjoining a primitive pth root of unity to p-adic K-theory. Then R is not an H_∞-ring.*

Proof. The spectrum R is $K(1)$-local. If $\pi_0 R$ contained a primitive pth root of unity ζ, then the power operations would supply an equality

$$\psi^p(\zeta) = 1 + p\theta(\zeta).$$

Since ψ^p is a ring homomorphism, if Φ_p denotes the pth cyclotomic polynomial, we have

$$\Phi_p(\psi^p(\zeta)) = \psi^p(\Phi_p(\zeta)) = \psi^p(0) = 0,$$

so $\psi^p(\zeta)$ is another primitive pth root of unity, i.e., $\psi^p(\zeta) = \zeta^k$ for some $1 \le k \le p-1$. Since $\theta(\zeta) \neq 0$, the prime p must divide $1 - \zeta^k$, which is impossible. $\square$

In an email, Tyler Lawson gave a proof of the following generalization of Lemma 2.3 at the prime 2.

Proposition 2.4. *There is no $K(1)$-local H_∞-ring R such that $\pi_0 R$ contains a primitive 4th root of unity.*

Lawson asked if this result can be generalized to all heights and primes. Theorem 1.3 gives an affirmative answer to this question.

Proof. Denote by i the fourth root of unity in $\pi_0 R$. One can easily check that

$$\theta(xy) = \theta(x)y^2 + \theta(y)x^2 + 2\theta(x)\theta(y);$$

therefore, if $x = y = i$, then

$$-1 = \theta(-1) = -2(\theta(i) - \theta(i)^2).$$

This, however, implies that 2 is invertible, which is impossible by Lemma 2.5. $\square$

In the proof of Proposition 2.4, we utilized the following easy result.

Lemma 2.5. *Let E be a p-local ring spectrum (where p is any prime). If p is a unit in $\pi_0 E$, then E is $K(n)$-locally trivial for any $n \geq 1$.*

Proof. Since p is a unit in $\pi_0 E$, the spectrum E is p-locally rational. This implies that $K(n)_* E$ is zero for any $n \geq 1$. □

Proposition 2.6. *Let $p > 2$ be an odd prime. Suppose A is a θ-ring such that p is not a unit in A. Then A does not contain a primitive pth root of unity ζ.*

Proof. By definition, we have

$$\theta(x+y) = \theta(x) + \theta(y) + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}.$$

By induction, we find that

$$\theta\left(\sum_{i=1}^{k} x_i\right) = \sum_{i=1}^{k} \theta(x_i) + \frac{\sum_{i=1}^{k} x_i^p - \left(\sum_{i=1}^{k} x_i\right)^p}{p} \tag{1}$$

Suppose A is a θ-ring containing a primitive pth root of unity. Since

$$1 + \zeta + \cdots + \zeta^{p-1} = 0, \tag{2}$$

we learn that

$$\theta(1 + \zeta + \cdots + \zeta^{p-1}) = 0.$$

Applying Equation (1) to the left hand side, we get

$$0 = \sum_{i=0}^{p-1} \theta(\zeta^i) + \frac{\sum_{i=0}^{p-1} \zeta^{ip} - \left(\sum_{i=0}^{p-1} \zeta^i\right)^p}{p} \tag{3}$$

Equation (2) implies that

$$\frac{\sum_{i=0}^{p-1} \zeta^{ip} - \left(\sum_{i=0}^{p-1} \zeta^i\right)^p}{p} = \frac{p-0}{p} = 1. \tag{4}$$

It remains to compute $\sum_{i=0}^{p-1} \theta(\zeta^i)$. As ψ^p is multiplicative, we deduce that

$$\theta(x^n) = \frac{(x^p + p\theta(x))^n - x^{np}}{p};$$

this implies that the first term on the right hand side of Equation (3) is

$$\sum_{i=0}^{p-1} \theta(\zeta^i) = \sum_{i=1}^{p-1} \frac{(1+p\theta(\zeta))^i - 1}{p} = \sum_{i=1}^{p-1}\sum_{k=1}^{i} \binom{i}{k} p^{k-1}\theta(\zeta)^k.$$

Splitting up terms, this sum becomes

$$\begin{aligned}\sum_{i=0}^{p-1} \theta(\zeta^i) &= \theta(\zeta)\sum_{i=1}^{p-1} i + \sum_{i=1}^{p-1}\sum_{k=2}^{i} \binom{i}{k} p^{k-1}\theta(\zeta)^k \\ &= p\left(\frac{p-1}{2}\theta(\zeta) + \sum_{i=1}^{p-1}\sum_{k=2}^{i} \binom{i}{k} p^{k-2}\theta(\zeta)^k\right).\end{aligned}$$

Let

$$Z = \frac{p-1}{2}\theta(\zeta) + \sum_{i=1}^{p-1}\sum_{k=2}^{i} \binom{i}{k} p^{k-2}\theta(\zeta)^k.$$

Equation (3) and Equation (4) show that

$$0 = pZ + 1,$$

which implies that p is invertible in A; contradiction. □

Before proceeding, let us recall a theorem of Hahn's from [Hah16]:

Theorem 2.7 (Hahn)**.** *A $K(n)$-acyclic H_∞-ring R is $K(n+1)$-acyclic.*

Proof of Theorem 1.3. Proposition 2.4 proves the result when $p = 2$, so we can restrict to the case when p is odd. Suppose R is a nontrivial $K(1)$-local ring such that $\pi_0 R$ contains a primitive pth root of unity. By Lemma 2.5, p is not a unit in $\pi_* R$. If $n = 1$, then $\pi_0 R$ is a θ-ring. It follows from Proposition 2.6 that $\pi_0 R$ is trivial, so R is contractible.

To prove the corollary at all heights, suppose $n \geq 2$, and let R be a $K(n)$-local $\mathbf{E}_\infty$-ring such that $\pi_0 R$ contains a primitive pth root of unity ζ_p. As $\pi_0 R \xrightarrow{\pi_0 L_{K(1)}} \pi_0 L_{K(1)} R$ is a ring homomorphism, the element $\pi_0 L_{K(1)}(\zeta_p)$ is another primitive pth root of unity in $\pi_0 L_{K(1)} R$. By the arguments laid out above, $L_{K(1)} R \simeq 0$, i.e., R is $K(1)$-acyclic. Theorem 2.7 therefore implies that R is $K(n)$-acyclic for every $n \geq 2$, so $R \simeq L_{K(n)} R \simeq 0$, as desired. □

Remark 2.8. By Theorem 1.3, any $\mathbf{E}_\infty$-ring R such that $p^{1/p} \in \pi_0 R$ must be $K(n)$-acyclic for every $n \geq 1$, since $p^{1/p} + 1$ is, up to units, a primitive pth root of unity. This fact can also be proven directly, by applying $\theta(-)$ to the equation $(p^{1/p})^p - p = 0$ and using the multiplicativity of ψ^p as in the proof above.

3. The Lubin–Tate tower

In arithmetic geometry, the *Lubin–Tate tower* (originally introduced in [Dri74]; see also [RZ96, Chapter 3] and [Wei16, Section 2]) is a tower of finite flat extensions of the Lubin–Tate space, given by the rings representing deformations (of the Honda formal group) along with a chosen Drinfel'd level structure. These level structures play an important role in constructing power operations on Morava E-theory. They are also of immense arithmetic interest, as the Lubin–Tate tower simultaneously admits actions of three groups which form the main object of study in local Langlands.

Let R be a complete local Artinian $\mathrm{W}(\overline{\mathbf{F}_p})$-algebra, and let (G, ι) be a deformation of the Honda formal group of height h to R. Let $\mathfrak{m}$ denote the maximal ideal of R equipped with the group law given by G.

Definition 3.1. A *Drinfel'd level p^k-structure* is a homomorphism $\phi\colon (\mathbf{Z}/p^k)^h \to \mathfrak{m}$ such that there is an inequality of Cartier divisors

$$\sum_{a \in (\mathbf{Z}/p^k)^h[p]} [\phi(a)] \leq G[p].$$

If we pick a coordinate x on G, then this amounts to asking that

$$\left(\prod_{a \in (p^{k-1}\mathbf{Z}/p^k\mathbf{Z})^h} (x - \phi(a)) \right) | [p](x)$$

inside $R[\![x]\!]$.

Drinfel'd proved the following result (see [Dri74, Proposition 4.3]).

Theorem 3.2 (Drinfel'd)**.** *The functor that sends a complete local Artinian $\mathrm{W}(\overline{\mathbf{F}_p})$-algebra $(R, \mathfrak{m})$ to the set of deformations of the Honda formal group (of height h) along with a level p^k-structure is corepresented by an $\pi_0 E_h$-algebra A^h_k.*

Remark 3.3. These rings can be defined inductively, by adjoining the p^k-torsion in G; for instance, if $k > 1$, then $A_k^h = A_{k-1}^h[x_1, ..., x_h]/([p](x_i) - \phi(a_i))$, where $\{a_i\}$ is the canonical basis for $(\mathbf{Z}/p^k)^n$.

Example 3.4. At height $h = 1$, a level p^k-structure on $\mathbb{G}_m$ is a choice of basis for the p^k-torsion; this is exactly the choice of a primitive p^k-torsion point. One might therefore expect that $A_k^1 \cong \mathbf{Z}_p[\zeta_{p^k}]$. This is indeed true: we have $A_1^1 \cong \mathbf{Z}_p[\zeta_p]$ (see, e.g., [Wei16, Section 2.3]; alternatively, this follows immediately from the explicit description given in [Dri74, Lemma in Proposition 4.3]). It follows from this that if $k > 1$, then $A_k^1 \cong \mathbf{Z}_p[\zeta_{p^k}]$. Indeed, recall that $[p](x) = (1+x)^p - 1$. Moreover, note that if $\phi\colon (\mathbf{Z}/p^{k-1})^h \to A_{k-1}^1$ is the universal level p^{k-1}-level structure, then $\phi(1) = \zeta_{p^{k-1}} - 1$. Using Remark 3.3, we find that $A_k^1 \cong A_{k-1}^h[x]/([p](x) - \phi(1)) \cong \mathbf{Z}_p[\zeta_{p^k}]$, as desired.

The main result of this section is:

Theorem 3.5. *Let $M_k^h = \operatorname{Spf} A_k^h$ denote the Lubin–Tate space at level p^k and height h, and let E_h denote Morava E-theory at height h and the prime p. If $k > 0$, there is no H_∞-ring over E_h whose underlying ring is A_k^h.*

Proof. There is a "determinant" map of formal schemes $M_k^h \to M_k^1$ constructed in [Wei16, Section 2.5]. We showed in Example 3.4 that $A_k^1 \cong \mathbf{Z}_p[\zeta_{p^k}]$. The image of ζ_{p^k} under the induced map $\mathbf{Z}_p[\zeta_{p^k}] \to A_k^h$ is a primitive p^kth root of unity in A_k^h. Applying Theorem 1.3, we conclude that A_k^h does not lift to a $K(h)$-local H_∞-ring; in particular, there is no $K(h)$-local E_h-algebra whose homotopy is isomorphic to A_k^h. □

Remark 3.6. The PEL Shimura varieties with Drinfel'd level structures considered in [HT01, §III.1] do not lift to derived stacks: if they did, then the completion at a height h point would give an $\mathbf{E}_\infty$-ring lifting A_k^h, contradicting Theorem 3.5.

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