1. Summary of six functors on spaces

After I wrote these notes, Peter Haine pointed me to [Vol21], where a similar approach is taken to the six functor formalism: the idea is to use Verdier duality and the $*$-pull/push functors to define !-pull/push functors, and then prove base-change, etc.

Definition 1. Let $\mathcal{X}$ be a small ∞-category equipped with a Grothendieck topology, let $\mathrm{Cov}(\mathcal{X})$ denote the ∞-category of [Lur09, Notation 6.2.2.8], and let $\mathcal{C}$ be an ∞-category which admits small limits. There is a canonical functor $\rho : \mathrm{Cov}(\mathcal{X}) \to \mathcal{X}$, as well a section $s : \mathcal{X} \to \mathrm{Cov}(\mathcal{X})$. Recall that a functor $F : \mathcal{X}^{\mathrm{op}} \to \mathcal{C}$ is called a *sheaf* if the morphism $\rho^* F \to s_* s^* \rho^* F \simeq s_* F$ is an equivalence. Let $\mathrm{Shv}(\mathcal{X}; \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$ spanned by sheaves. If $\mathcal{D}$ is an ∞-category which admits small colimits, let $\mathrm{coShv}(\mathcal{X}; \mathcal{D})$ denote the ∞-category $\mathrm{Shv}(\mathcal{X}; \mathcal{D}^{\mathrm{op}})^{\mathrm{op}}$. This is called the ∞-category of *cosheaves* on $\mathcal{X}$.

Remark 2. The ∞-category $\mathrm{Shv}(\mathcal{X}; \mathcal{S})$ is an ∞-topos by [Lur09, Proposition 6.2.2.7]. Moreover, if $\mathcal{C}$ is a presentable ∞-category, then $\mathrm{Shv}(\mathcal{X}; \mathcal{C}) \simeq \mathrm{Shv}(\mathcal{X}; \mathcal{S}) \otimes \mathcal{C}$. If $\mathcal{C}$ is further assumed to be stable, then $\mathrm{Shv}(\mathcal{X}; \mathcal{C}) \simeq \mathrm{Shv}(\mathcal{X}; \mathrm{Sp}) \otimes \mathcal{C}$. In the rest of this text, we will denote $\mathrm{Shv}(\mathcal{X}; \mathrm{Sp})$ by $\mathrm{Shv}(\mathcal{X})$.

Definition 3. If X is a topological space, let $\mathcal{U}(X)$ denote the poset of open subsets of X ordered by inclusion (viewed as a category). Then $\mathcal{U}(X)$ has a Grothendieck topology, where the covering sieves are given by open covers of X. This defines a Grothendieck topology on $N(\mathcal{U}(X))$. We will denote the ∞-category $\mathrm{Shv}(N(\mathcal{U}(X)); \mathcal{C})$ by $\mathrm{Shv}(X; \mathcal{C})$; this is the ∞-category of $\mathcal{C}$-*valued sheaves on* X.

Theorem 4 (Verdier duality, [Lur16, Theorem 5.5.5.1])**.** *Let X be a locally compact Hausdorff space, and let $\mathcal{C}$ be a stable ∞-category which admits small limits and colimits. Then there is a canonical equivalence of ∞-categories $\mathbf{D} : \mathrm{Shv}(X; \mathcal{C}) \xrightarrow{\sim} \mathrm{coShv}(X; \mathcal{C})$, which sends a sheaf $\mathcal{F} \in \mathrm{Shv}(X; \mathcal{C})$ to the cosheaf $\mathbf{D}(\mathcal{F}) : N(\mathcal{U}(X)) \to \mathcal{C}$ given by*

$$\mathbf{D}(\mathcal{F}) : U \mapsto \Gamma_c(U; \mathcal{F}) := \mathrm{colim}_{K \subseteq U} \mathcal{F}(X) \times_{\mathcal{F}(X-K)} 0_{\mathcal{C}}.$$

Here, the (filtered) colimit is taken over all compact subsets of U.

Remark 5. If $K \subseteq U \subseteq X$ where K is compact and U is an open subset of X, then there is a pullback square

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(X-K) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U-K); \end{array}$$

therefore, we may replace $\mathcal{F}(X) \times_{\mathcal{F}(X-K)} 0_{\mathcal{C}}$ by $\mathcal{F}(U) \times_{\mathcal{F}(U-K)} 0_{\mathcal{C}}$ in the filtered colimit of Theorem 4.

Remark 6. The assumption that X is a locally compact Hausdorff space is relevant for the following reason. Any continuous map $f : X \to Y$ between locally compact Hausdorff spaces factors as a closed immersion (hence proper; i.e., the preimage of any compact subset is compact), an open immersion, and a proper map: namely, f may be identified with the composite

$$X \xrightarrow{\mathrm{graph}} X \times Y \xrightarrow{j} X^c \times Y \xrightarrow{\mathrm{proj}} Y, \tag{1}$$

where X^c is the one-point compactification of X.

We will use Theorem 4 to define the six-functor formalism.

Construction 7. Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces. Then there are canonical functors $f_* : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$ (called *pushforward*) and $f^* : \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$ (called *pullback*), with f^* being left adjoint to f_*. If $\mathcal{C}$ is a presentable stable ∞-category, this defines functors $f_*^{\mathcal{C}} : \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{Shv}(Y; \mathcal{C})$ and $f_{\mathcal{C}}^* : \mathrm{Shv}(Y; \mathcal{C}) \to \mathrm{Shv}(X; \mathcal{C})$ via tensoring up to $\mathcal{C}$.

Construction 8. Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces. Then Construction 7 defines functors $f_*^{\mathcal{C}^{\mathrm{op}}} : \mathrm{Shv}(X; \mathcal{C}^{\mathrm{op}}) \to \mathrm{Shv}(Y; \mathcal{C}^{\mathrm{op}})$ and $f_{\mathcal{C}^{\mathrm{op}}}^* : \mathrm{Shv}(Y; \mathcal{C}^{\mathrm{op}}) \to \mathrm{Shv}(X; \mathcal{C}^{\mathrm{op}})$, and hence functors $f_*^{\mathcal{C}^{\mathrm{op}}} : \mathrm{coShv}(X; \mathcal{C}) \to \mathrm{coShv}(Y; \mathcal{C})$ and $f_{\mathcal{C}^{\mathrm{op}}}^* : \mathrm{coShv}(Y; \mathcal{C}) \to \mathrm{coShv}(X; \mathcal{C})$. Define the functor $f_!^{\mathcal{C}} : \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{Shv}(Y; \mathcal{C})$ as $\mathbf{D}^{-1}(f_*^{\mathcal{C}^{\mathrm{op}}} \circ \mathbf{D})$ of *pushforward with proper support.*

Similarly, define the functor $f^!_{\mathcal{C}} : \mathrm{Shv}(Y;\mathcal{C}) \to \mathrm{Shv}(X;\mathcal{C})$ (called *exceptional pullback*) as $\mathbf{D}^{-1}(f^*_{\mathcal{C}^{\mathrm{op}}} \circ \mathbf{D})$. If $\mathcal{C} = \mathrm{Sp}$, we will drop the super/subscript $\mathcal{C}$ from the notation.

Warning 9. Henceforth, we will assume that $\mathcal{C} = \mathrm{Sp}$ (although for many of the results below, it suffices to assume $\mathcal{C} = \mathcal{S}$). Using the results of [Hai21], all of the results stated below go through by tensoring up to $\mathcal{C}$, if we assume that $\mathcal{C}$ is a presentable stable ∞-category.

Proposition 10. *Let $f : X \to Y$ be a continuous map $f : X \to Y$ between locally compact Hausdorff spaces.*

(a) *There is a natural transformation $f_! \to f_*$ of functors $\mathrm{Shv}(X) \to \mathrm{Shv}(Y)$, which is an equivalence if f is proper.*
(b) *If f is an open immersion, then $f^* \simeq f^!$; therefore, $f_!$ is left adjoint to f^*.*

Proof. It suffices to assume $\mathcal{C} = \mathrm{Sp}$. We first prove (a). The natural transformation $f_! \to f_*$ is specified by a natural transformation $\gamma : f_* \circ \mathbf{D} \to \mathbf{D} \circ f_*$. Let $\mathcal{F} \in \mathrm{Shv}(X)$, and let $U \subseteq Y$ be an open set; then γ is specified by a map

$$\mathrm{colim}_{K \subseteq f^{-1}(U)} \Gamma_K(X;\mathcal{F}) \simeq f_*(\mathbf{D}(\mathcal{F}))(U) \xrightarrow{\gamma_{\mathcal{F}}(U)} \mathbf{D}(f_*(\mathcal{F}))(U) = \Gamma_c(U; f_*\mathcal{F}) \simeq \mathrm{colim}_{K' \subseteq U} \Gamma_{K'}(Y; f_*\mathcal{F})$$

which is natural in $\mathcal{F}$ and U. To define this map, fix a compact subset $K \subseteq f^{-1}(U)$. Because f is continuous, $f(K) \subseteq U$ is compact. Since $K \subseteq f^{-1}(f(K))$, the canonical map $\Gamma(X;\mathcal{F}) \to \Gamma(X - f^{-1}(f(K));\mathcal{F})$ factors as

$$\Gamma(X;\mathcal{F}) \to \Gamma(X - K : \mathcal{F}) \to \Gamma(X - f^{-1}(f(K));\mathcal{F}).$$

This defines a map $\Gamma_K(X;\mathcal{F}) \to \Gamma_{f(K)}(Y; f_*\mathcal{F})$, which gives the desired map $\gamma_{\mathcal{F}}(U)$.

We now show that γ is an equivalence when f is proper by showing that $\gamma_{\mathcal{F}}(U)$ is an equivalence for all $\mathcal{F}$ and U. By the definition of properness, $f^{-1}(f(K))$ is a compact set, and therefore each compact $K \subseteq U$ is contained in the compact $f^{-1}(f(K))$. Since the poset of compact subsets in U is filtered, we conclude that the composite

$$\mathrm{colim}_{K \subseteq f^{-1}(U)} \Gamma_K(X;\mathcal{F}) \xrightarrow{\sim} \mathrm{colim}_{f^{-1}(f(K)) \subseteq f^{-1}(U)} \Gamma_K(X;\mathcal{F}) \xrightarrow{\gamma_{\mathcal{F}}(U)} \mathrm{colim}_{K' \subseteq U} \Gamma_{K'}(Y; f_*\mathcal{F})$$

must be an equivalence.

We now turn to (b): by definition of $f^!$, it suffices to show that $f^*(\mathbf{D}(\mathcal{F})) \simeq \mathbf{D}(f^*(\mathcal{F}))$ for any $\mathcal{F} \in \mathrm{Shv}(X)$. But this is clear by definition of $\mathbf{D}$ and Remark 5. $\square$

Construction 11 (Recollement)**.** Let X be a topological space, let $i : Z \hookrightarrow X$ be a closed immersion, and let $j : U \hookrightarrow X$ be an open immersion. Then for each $\mathcal{F} \in \mathrm{Shv}(X)$, there are canonical cofiber sequences which are functorial in $\mathcal{F}$:

$$j_! j^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}, \; i_! i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}.$$

Note that since i is a closed immersion and j is an open immersion, Proposition 10 implies that $i_! \simeq i_*$ and that $j^! \simeq j^*$.

The following argument is adapted from [Soe89, Section 1.3].

Lemma 12 (Generalized homotopy invariance)**.** *Let X be a topological space equipped with an $\mathbf{C} \setminus \{0\}$-action $\circlearrowleft$: $\mathbf{C} \setminus \{0\} \times X \to X$, and let $\mathrm{pr} : \mathbf{C} \setminus \{0\} \times X \to X$ denote the projection. Suppose that the $\mathbf{C} \setminus \{0\}$-action contracts X to a closed subspace $z : Z \hookrightarrow X$, which by definition means that there is a commutative diagram*

$$\begin{array}{ccccc} X \times \{0\} & \xrightarrow{i} & X \times \mathbf{C} & \xleftarrow{j} & X \times \mathbf{C} \setminus \{0\} \\ \downarrow c & & \downarrow \circlearrowleft_0 & & \downarrow \circlearrowleft \\ Z & \xrightarrow{z} & X & = & X \end{array}$$

such that $cz = \mathrm{id}_Z$. Then the adjunction $c^* c_* \to \mathrm{id}$ gives a natural transformation $c_* \to z^*$ of functors $\mathrm{Shv}(X) \to \mathrm{Shv}(Z)$. Assume that $\mathcal{F} \in \mathrm{Shv}(X)$ is $\mathbf{C} \setminus \{0\}$-monodromic, so that $\mathcal{F}$ admits an equivalence $\circlearrowleft^* \mathcal{F} \simeq \mathrm{pr}^* \mathcal{F}$. Then the map $c_* \mathcal{F} \to z^* \mathcal{F}$ is an equivalence.

Proof. Let $j : U \hookrightarrow X$ denote the complement of Z. Then there is a cofiber sequence

$$j_! j^! \mathcal{F} \to \mathcal{F} \to z_* z^* \mathcal{F}.$$

It is clear that the natural transformation $c_* \to z^*$ is an equivalence on $z_* z^* \mathcal{F}$, since $cz = \mathrm{id}_Z$. Therefore, it suffices to prove that the natural transformation $c_* \to z^*$ is an equivalence on $j_! j^! \mathcal{F}$; but $z^* j_! j^! \mathcal{F} = 0$, so we are reduced to proving the claim in the case when $z^* \mathcal{F} = 0$. In other words, we wish to show that if $z^* \mathcal{F} = 0$, then $c_* \mathcal{F} \simeq 0$.

Consider the following diagram, in which each square is Cartesian:

$$\begin{array}{ccccc} \mathbf{C} \setminus \{0\} \times X & \xrightarrow{j} & \mathbf{C} \times X & \xrightarrow{\mathrm{pr}} & X \\ \downarrow{\scriptstyle \mathrm{id} \times c} & & \downarrow{\scriptstyle q := \mathrm{id} \times c} & & \downarrow{\scriptstyle c} \\ \mathbf{C} \setminus \{0\} \times Z & \xrightarrow[j']{} & \mathbf{C} \times Z & \xrightarrow[\mathrm{pr}']{} & Z. \end{array}$$

Note that we have

$$\mathrm{pr}'_* q_* \mathrm{pr}^* \mathcal{F} \simeq c_* \mathrm{pr}_* \mathrm{pr}^* \mathcal{F} \simeq c_* \mathcal{F}.$$

Let $\theta : \mathbf{C} \times X \to \mathbf{C} \times X$ denote the map sending $(\lambda, x) \mapsto (\lambda, \circlearrowleft (\lambda, x))$. Then the composite

$$X \xrightarrow{i} \mathbf{C} \times X \xrightarrow{\theta} \mathbf{C} \times X \xrightarrow{\mathrm{pr}} X$$

is equivalent to $X \xrightarrow{c} Z \xrightarrow{z} X$.

The map $\mathrm{pr}^* \mathcal{F} \to \theta_* \theta^* \mathrm{pr}^* \mathcal{F}$ induces a map

$$q_* \mathrm{pr}^* \mathcal{F} \to q_* \theta_* \theta^* \mathrm{pr}^* \mathcal{F} \simeq q_* \theta^* \mathrm{pr}^* \mathcal{F}$$

which is an equivalence upon applying j'^*. Let us denote this map by ϕ.

If $z^* \mathcal{F} = 0$, then $i^* \theta^* \mathrm{pr}^* \mathcal{F} = 0$. The recollement cofiber sequence

$$j_! j^* \theta^* \mathrm{pr}^* \mathcal{F} \to \theta^* \mathrm{pr}^* \mathcal{F} \to i_* i^* \theta^* \mathrm{pr}^* \mathcal{F}$$

implies that $j_! j^* \theta^* \mathrm{pr}^* \mathcal{F} \xrightarrow{\sim} \theta^* \mathrm{pr}^* \mathcal{F}$. Since $\mathcal{F}$ is $\mathbf{C} \setminus \{0\}$-monodromic, $j^* \theta^* \mathrm{pr}^* \mathcal{F} \simeq j^* \mathrm{pr}^* \mathcal{F}$, which implies that there is an equivalence

$$j_! j^* \mathrm{pr}^* \mathcal{F} \simeq \theta^* \mathrm{pr}^* \mathcal{F}.$$

This gives a map $\theta^* \mathrm{pr}^* \mathcal{F} \to \mathrm{pr}^* \mathcal{F}$ in $\mathrm{Shv}(\mathbf{C} \times Z)$ which induces an equivalence upon applying j^*, and hence a map $q_* \theta^* \mathrm{pr}^* \mathcal{F} \to q_* \mathrm{pr}^* \mathcal{F}$ which induces an equivalence upon applying j'^*. Let us denote this map by ϕ'.

The maps $\phi : q_* \mathrm{pr}^* \mathcal{F} \to q_* \theta^* \mathrm{pr}^* \mathcal{F}$ and $\phi' : q_* \theta^* \mathrm{pr}^* \mathcal{F} \to q_* \mathrm{pr}^* \mathcal{F}$ induce an endomorphism ψ of $q_* \mathrm{pr}^* \mathcal{F}$, which is an equivalence upon applying j'^*. The contractibility of $\mathbf{C}$ implies that $\mathrm{pr}'_* \psi$ is an automorphism of $\mathrm{pr}'_* q_* \mathrm{pr}^* \mathcal{F} \simeq c_* \mathcal{F}$. We claim that $\mathrm{pr}'_* \psi$ is null, which implies the claim. To see this, it suffices to show that $\mathrm{pr}'_* q_* \theta^* \mathrm{pr}^* \mathcal{F} = 0$. Note that

$$\mathrm{pr}'_* q_* \theta^* \mathrm{pr}^* \mathcal{F} \simeq p_* \mathrm{pr}_* \theta^* \mathrm{pr}^* \mathcal{F} \simeq p_* \mathrm{pr}_* j_! j^* \mathrm{pr}^* \mathcal{F},$$

so that there is a cofiber sequence

$$\mathrm{pr}_* j_! j^* \mathrm{pr}^* \mathcal{F} \to \mathrm{pr}_* \mathrm{pr}^* \mathcal{F} \to \mathrm{pr}_* i_* i^* \mathrm{pr}^* \mathcal{F}.$$

But the latter map is an equivalence since $\mathbf{C}$ is contractible, so that the first term is zero as desired. □

We will need a generalization of Proposition 10 to the case when $f : X \to Y$ is a submersion of smooth manifolds.

Lemma 13. *Let X and Y be locally compact topological spaces. Say that a continuous map $f : X \to Y$ is a submersion if for each $x \in X$, there is an open neighborhood $U \subseteq X$ containing x and a topological space Z such that:*

(a) *Z is locally contractible.*
(b) *$U \cong f(U) \times Z$ as spaces over $f(U) \subseteq Y$.*

Then f^* admits a left adjoint, denoted $f_\#$.

Proof. Because f^* preserves small colimits (being a left adjoint to f_*), it suffices to prove that f^* preserves small limits. Let $\mathcal{I} \to \mathrm{Shv}(Y)$ be a diagram; then there is a canonical map $\eta : f^*(\lim_{i\in\mathcal{I}} \mathcal{F}_i) \to \lim_{i\in\mathcal{I}} f^*(\mathcal{F}_i)$ in $\mathrm{Shv}(X)$. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X. It suffices to check that η is an equivalence after $*$-restriction to U_α for each $\alpha \in A$. By Proposition 10, $*$-restriction to U_α commutes with limits, so we may assume that the map f is of the form $Z \times V \to V$, where V is locally compact and Z is locally contractible. Let $\pi : Z \to *$ denote the projection of Z onto a point. Then f^* is canonically identified with the functor $\mathrm{Shv}(V) \to \mathrm{Shv}(Z \times V) \simeq \mathrm{Shv}(Z) \otimes \mathrm{Shv}(V)$ obtained by tensoring $\pi^* : \mathrm{Sp} \to \mathrm{Shv}(Z)$ with $\mathrm{Shv}(V)$. It therefore suffices to show that $\pi^* : \mathcal{S} \to \mathrm{Shv}(Z; \mathcal{S})$ admits a left adjoint, which by [Lur16, Proposition A.1.8] is equivalent to $\mathrm{Shv}(Z; \mathcal{S})$ being locally of constant shape. But this is equivalent to Z being locally contractible. □

Remark 14. If $f : X \to Y$ is a submersion in the sense of Lemma 13, assume that there is a fixed number n such that for each $x \in X$, we can choose $Z = \mathbf{R}^n$. Then f will be said to be of *relative dimension n*.

Example 15. Any submersion between smooth manifolds is is a submersion in the sense of Lemma 13. Similarly, any vector bundle $\mathcal{E} \to X$ over a topological space X is a submersion in the sense of Lemma 13.

One of the main inputs into relations between the functors defined above is the following:

Proposition 16 (Base-change theorems)**.** *Suppose X, Y, X', and Y' are locally compact Hausdorff topological spaces, and assume that there is a (strict) pullback square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

(a) *There is a natural equivalence $g^* f_! \simeq f'_! g'^*$.*
(b) *If f is a submersion in the sense of Lemma 13, then f' is also a submersion in the sense of Lemma 13, and there is a natural equivalence $g^* f_\# \simeq f'_\# g'^*$ (and hence an equivalence $f^* g_* \simeq g'_* f'^*$ by adjunction).*

Proof. When f is a proper map, part (a) is a consequence of [Lur09, Corollary 7.3.1.18] and [Hai21, Subexample 3.15]. For a general map f, the factorization (1) reduces us to showing the claimed equivalence when f is an open immersion. By Proposition 10(b), the functor $f_!$ is left adjoint to f^*, which produces a natural transformation

$$g'^* \to g'^* f^* f_! \simeq f'^* g^* f_!,$$

and hence a natural transformation $f'_! g'^* \to g^* f_!$. In this case the claim is immediate.

To prove part (b), we first note that the definition of submersion in the sense of Lemma 13 is obviously stable under base-change, so f' is also a submersion. Now we define the natural transformation comparing the two functors: the unit $\mathrm{id} \to f^* f_\#$ defines a map

$$g'^* \to g'^* f^* f_\# \simeq f'^* g^* f_\#,$$

which defines the desired natural transformation $f'_\# g'^* \to g^* f_\#$. This map is obviously an equivalence when $Y = Y' = *$. In the general case, note that the topology on X admits a basis given by open subsets of the form $U \times Z$ where $U \subseteq Y$ is an open subset and Z is locally contractible. We may therefore assume $X = Y \times Z$, in which case $X' = Y' \times Z$. Let $\pi : Z \to *$ denote the projection of Z to a point; then $f = \pi \times \mathrm{id}_Y$ and $f' = \pi \times \mathrm{id}_{Y'}$. Therefore:

$$f'_\# g'^* \simeq (\pi_\# \times \mathrm{id}_{Y',\#}) g'^* \xrightarrow{\sim} \pi_\# \times g^* \simeq g^*(\pi_\# \times \mathrm{id}_{Y,\#}) \simeq g^* f_\#,$$

as desired. □

Proposition 16 has several corollaries.

Corollary 17 (Projection formula)**.** *Let $f : X \to Y$ be a continuous map between locally compact Hausdorff topological spaces, and let $\mathcal{F} \in \mathrm{Shv}(X)$ and $\mathcal{G} \in \mathrm{Shv}(Y)$. Then:*

(a) *There is a canonical equivalence* $f_!(\mathcal{F} \otimes f^*\mathcal{G}) \simeq f_!(\mathcal{F}) \otimes \mathcal{G}$.
(b) *If* f *is a submersion in the sense of Lemma 13, then there is a canonical equivalence* $f_\#(\mathcal{F} \otimes f^*\mathcal{G}) \simeq f_\#(\mathcal{F}) \otimes \mathcal{G}$.

Proof. These equivalences follow by applying Proposition 16 to the strict pullback square

$$\begin{CD} X @>{\mathrm{graph}(f)}>> X \times Y \\ @V{f}VV @VV{f \times \mathrm{id}_Y}V \\ Y @>>{\Delta}> Y \times Y. \end{CD}$$

□

Recollection 18. Let X be a topological space, and let $\mathcal{F} \in \mathrm{Shv}(X)$. Since $\mathrm{Shv}(X)$ is presentably symmetric monoidal, the functor $- \otimes \mathcal{F} : \mathrm{Shv}(X) \to \mathrm{Shv}(X)$ preserves small colimits, and therefore admits a right adjoint $\underline{\mathrm{Hom}}_X(\mathcal{F}, -) : \mathrm{Shv}(X) \to \mathrm{Shv}(X)$. This will be called the *internal Hom*. Let $f : X \to Y$ be a continuous map of topological spaces. Since f^* is symmetric monoidal, one concludes by adjunction that if $\mathcal{F} \in \mathrm{Shv}(Y)$ and $\mathcal{G} \in \mathrm{Shv}(X)$, then $f_*\underline{\mathrm{Hom}}_X(f^*\mathcal{F}, \mathcal{G}) \simeq \underline{\mathrm{Hom}}_Y(\mathcal{F}, f_*\mathcal{G})$.

The tensor-Hom adjunction implies the following by Corollary 17:

Corollary 19. *Let* $f : X \to Y$ *be a continuous map between locally compact Hausdorff topological spaces, and let* $\mathcal{F} \in \mathrm{Shv}(X)$ *and* $\mathcal{G}, \mathcal{G}' \in \mathrm{Shv}(Y)$. *Then:*

(a) *There are canonical equivalences*

$$f_*\underline{\mathrm{Hom}}_X(\mathcal{F}, f^!\mathcal{G}) \simeq \underline{\mathrm{Hom}}_Y(f_!\mathcal{F}, \mathcal{G}), \;\; f^!\underline{\mathrm{Hom}}_Y(\mathcal{G}, \mathcal{G}') \simeq \underline{\mathrm{Hom}}_X(f^*\mathcal{G}, f^!\mathcal{G}').$$

(b) *If* f *is a submersion in the sense of Lemma 13, then there are canonical equivalences*

$$f_*\underline{\mathrm{Hom}}_X(\mathcal{F}, f^*\mathcal{G}) \simeq \underline{\mathrm{Hom}}_Y(f_\#\mathcal{F}, \mathcal{G}), \;\; f^*\underline{\mathrm{Hom}}_Y(\mathcal{G}, \mathcal{G}') \simeq \underline{\mathrm{Hom}}_X(f^*\mathcal{G}, f^*\mathcal{G}').$$

Corollary 20. *Let* $f : X \to Y$ *be a continuous map between locally compact Hausdorff spaces which is a submersion in the sense of Lemma 13. If* $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}(Y)$, *then there is a canonical equivalence* $f^!(\mathcal{F}) \otimes f^*(\mathcal{G}) \xrightarrow{\sim} f^!(\mathcal{F} \otimes \mathcal{G})$.

Proof. We begin by constructing the comparison morphism. This follows from the following sequence of equivalences:

$$\mathrm{Map}_{\mathrm{Shv}(X)}(f^!(\mathcal{F}) \otimes f^*(\mathcal{G}), f^!(\mathcal{F} \otimes \mathcal{G})) \simeq \mathrm{Map}_{\mathrm{Shv}(Y)}(f_!(f^!(\mathcal{F}) \otimes f^*(\mathcal{G})), \mathcal{F} \otimes \mathcal{G}) \simeq \mathrm{Map}(f_!f^!(\mathcal{F}) \otimes \mathcal{G}, \mathcal{F} \otimes \mathcal{G}).$$

The map $f^!(\mathcal{F}) \otimes f^*(\mathcal{G}) \to f^!(\mathcal{F} \otimes \mathcal{G})$ is picked out by the map $f_!f^!(\mathcal{F}) \otimes \mathcal{G} \to \mathcal{F} \otimes \mathcal{G}$ obtained by tensoring $\mathcal{G}$ with the counit $f_!f^!(\mathcal{F}) \to \mathcal{F}$.

To show that the comparison map is an equivalence, it will be convenient to restate the claim after applying Verdier duality. Namely, define a functor $f_{\mathrm{flat}} : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$ as $\mathbf{D}^{-1}(f_\#^{\mathcal{C}^{\mathrm{op}}} \circ \mathbf{D})$, so that f_{flat} is left adjoint to $f^!$ (because $f_\#$ is left adjoint to f^* by Lemma 13). Translating the desired equivalence under Verdier duality, it suffices to prove the following: let $\mathcal{G} \in \mathrm{Shv}(X)$ and $\mathcal{F} \in \mathrm{Shv}(Y)$; then there is a canonical equivalence

$$\underline{\mathrm{Hom}}_Y(\mathcal{F}, f_{\mathrm{flat}}\mathcal{G}) \xrightarrow{\sim} f_*\underline{\mathrm{Hom}}_X(f^!\mathcal{F}, \mathcal{G}).$$

To prove this, let $j : U \hookrightarrow Y$ be an open subset, and let $j' : f^{-1}(U) \to X$ denote its preimage. For notational distinction, let $f' : f^{-1}(U) \to U$ denote the restriction of f to $f^{-1}(U)$, so that there is a pullback square

$$\begin{CD} f^{-1}(U) @>{f'}>> X \\ @V{j'}VV @VV{f}V \\ U @>>{j}> Y. \end{CD}$$

We claim that there is an equivalence $j^* f_{\mathrm{flat}} \simeq f'_{\mathrm{flat}} j'^*$. To see this, note that since j is an open immersion, $j^! = j^*$ by Proposition 10(b), so that $j_{\mathrm{flat}} \simeq j_!$. The claim therefore follows from the equivalence $j^* f_\# \simeq f'_\# j'^*$.

Using this equivalence, we have:

$$\Gamma(U; \underline{\mathrm{Hom}}_Y(\mathcal{F}, f_{\mathrm{flat}}\mathcal{G})) \simeq \mathrm{Hom}_{\mathrm{Shv}(U)}(j^*\mathcal{F}, j^* f_{\mathrm{flat}}\mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{Shv}(U)}(j^*\mathcal{F}, f'_{\mathrm{flat}} j'^*\mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{Shv}(f^{-1}(U))}(f'^! j^*\mathcal{F}, j'^*\mathcal{G}).$$

Since j is an open immersion, $j^! = j^*$ by Proposition 10(b); therefore,

$$f'^! j^* \simeq f'^! j^! \simeq j'^! f^! \simeq j'^* f^!.$$

This implies that

$$\mathrm{Hom}_{\mathrm{Shv}(f^{-1}(U))}(f'^! j^*\mathcal{F}, j'^*\mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{Shv}(f^{-1}(U))}(j'^* f^!\mathcal{F}, j'^*\mathcal{G}) \simeq \Gamma(f^{-1}(U); \underline{\mathrm{Hom}}_X(f^!\mathcal{F}, \mathcal{G})).$$

This in turn can be identified with $\Gamma(U; f_* \underline{\mathrm{Hom}}_X(f^!\mathcal{F}, \mathcal{G}))$, which produces a natural equivalence

$$\Gamma(U; \underline{\mathrm{Hom}}_Y(\mathcal{F}, f_{\mathrm{flat}}\mathcal{G})) \simeq \Gamma(U; f_* \underline{\mathrm{Hom}}_X(f^!\mathcal{F}, \mathcal{G})).$$

This equivalence can be identified with the Verdier dual of the comparison map $f^!(-) \otimes f^*(-) \to f^!(- \otimes -)$ from before, which proves the desired claim. □

Notation 21. If X is a topological space, let $\mathbf{1}_X \in \mathrm{Shv}(X)$ denote the constant sheaf associated to the unit $\mathbf{1} = S^0 \in \mathrm{Sp}$. Concretely, if $\pi : X \to *$ is the projection of X onto a point, then $\mathbf{1}_X = \pi^* \mathbf{1}$.

Corollary 22. *Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces which is a submersion in the sense of Lemma 13. If $\mathcal{F} \in \mathrm{Shv}(Y)$, then there is a canonical equivalence $f^!(\mathbf{1}_Y) \otimes f^*(\mathcal{F}) \xrightarrow{\sim} f^!(\mathcal{F})$. Equivalently, if $\mathcal{G} \in \mathrm{Shv}(X)$, then there is a natural equivalence $f_\# \mathcal{G} \simeq f_!(\mathcal{G} \otimes f^! \mathbf{1}_Y)$.*

Lemma 23. *Let $f : X \to Y$ be a submersion of topological manifolds, and assume that f is of relative dimension n. Then $f^!(\mathbf{1}_Y)$ is an invertible object in $\mathrm{Shv}(X)$: in fact, it is a locally constant sheaf whose stalks are $\mathbf{1}[n]$.*

Proof. In the standard manner, we may reduce to the case when f is a projection $Z \times U \to U$ where U is locally compact and Z is locally contractible. To prove the desired claim, we may further reduce to the case where f is the projection map $Z \to *$, and by working locally on Z, further to the case when f is the projection $\pi : \mathbf{R}^n \to *$. In this case, we claim that $\pi^! \mathbf{1}_* \simeq \mathbf{1}_{\mathbf{R}^n}[n]$. To prove this, let $U \subseteq \mathbf{R}^n$ be an open ball; we claim that the assignment $U \mapsto \mathrm{colim}_{K \subseteq U} \mathbf{1}_{\mathbf{R}^n}(U) \times_{\mathbf{1}_{\mathbf{R}^n}(U-K)} 0$ may be identified with $\mathbf{1}_{\mathbf{R}^n}[n]$. (This implies the desired claim by construction of $\pi^!_{\mathbf{R}^n}$.) Let $\mathcal{K}(U)$ denote the poset of compact subsets $K \subseteq U$, and let $\mathcal{K}'(U)$ denote the sub-poset spanned by the convex compact subsets. The inclusion $\mathcal{K}'(U) \subseteq \mathcal{K}(U)$ is colimit-cofinal (since given a compact subset $K \subseteq U$, one can always find a closed ball in U which contains K), so the desired colimit can be computed as a colimit over $\mathcal{K}'(U)$ instead. But if $K \subseteq U$ is a convex compact subset, then radial projection away from any point $x \in K$ defines a homotopy equivalence $U - K \xrightarrow{\sim} S^{n-1}$. This implies that $\mathbf{1}_{\mathbf{R}^n}(U-K) \simeq S^{n-1}$. Moreover, since U is contractible, $\mathbf{1}_{\mathbf{R}^n}(U) \simeq 0$, so that $\mathbf{1}_{\mathbf{R}^n}(U) \times_{\mathbf{1}_{\mathbf{R}^n}(U-K)} 0 \simeq S^n$. The colimit over $\mathcal{K}'(U)$ is therefore constant, and takes value S^n, as desired. □

Definition 24. Let $f : X \to Y$ be a submersion of topological manifolds. We will call $f^!(\mathbf{1}_Y) \in \mathrm{Shv}(X)$ the *relative dualizing sheaf* of f, and denote it by $\omega_{X/Y}$ (or by ω_f to exhibit the dependence on f). If f is the projection $X \to *$ to a point, we will simply call $f^!(\mathbf{1}_*)$ the *dualizing sheaf of* X and denote it by ω_X.

Corollary 25. *Let $f : X \to Y$ be a submersion of topological manifolds. Then there is an equivalence $\omega_{X/Y} \simeq \omega_X \otimes f^*(\omega_Y^{-1})$.*

Proof. Let $\pi_Y : Y \to *$ denote the projection onto a point, and similarly for π_X. Then

$$\omega_X \simeq \pi_X^!(\mathbf{1}_*) \simeq f^! \pi_Y^!(\mathbf{1}_*) \simeq f^!(\mathbf{1}_Y) \otimes f^*(\omega_Y) \simeq \omega_{X/Y} \otimes f^*(\omega_Y),$$

which gives the desired claim by Lemma 23. □

Lemma 26. *Let X be a topological space, and let $\mathcal{F} \in \mathrm{Shv}(X)$. Then the $\mathbf{1}_X$-linear dual $\mathbf{D}(\mathcal{F})^\vee$ equivalent to $\underline{\mathrm{Hom}}_X(\mathcal{F}, \omega_X)$. If X is locally contractible and $\mathcal{G} \in \mathrm{Shv}(X)$ is dualizable, there is a natural equivalence $\mathbf{D}(\mathcal{F} \otimes \mathbf{D}(\mathcal{G})^\vee)^\vee \simeq \underline{\mathrm{Hom}}_X(\mathcal{F}, \mathcal{G})$.*

Proof. The first sentence is a consequence of Corollary 19(a). For the second claim, note that since $\mathcal{G}$ is assumed to be dualizable, we have

$$\mathbf{D}(\mathcal{G})^\vee \simeq \underline{\mathrm{Hom}}_X(\mathcal{G}, \omega_X) \simeq \mathcal{G}^\vee \otimes \omega_X.$$

This implies the desired claim:

$$\begin{aligned}\mathbf{D}(\mathcal{F} \otimes \mathbf{D}(\mathcal{G})^\vee)^\vee &\simeq \underline{\mathrm{Hom}}_X(\mathcal{F} \otimes \mathbf{D}(\mathcal{G})^\vee, \omega_X) \\ &\simeq \underline{\mathrm{Hom}}_X(\mathcal{F} \otimes \mathcal{G}^\vee \otimes \omega_X, \omega_X) \\ &\simeq \underline{\mathrm{Hom}}_X(\mathcal{F} \otimes \mathcal{G}^\vee, \mathbf{1}_X) \simeq \underline{\mathrm{Hom}}_X(\mathcal{F}, \mathcal{G}).\end{aligned}$$

□

Notation 27. We will denote the functor $\mathrm{Shv}(X)^{\mathrm{op}} \to \mathrm{Shv}(X)$ sending $\mathcal{F} \mapsto \mathbf{D}(\mathcal{F})^\vee$ by $\mathbf{D}^\vee$, and occasionally (abusively) call it Verdier duality.

Construction 28. Let X be a topological space, and let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category. Let $\mathrm{Shv}^!(X; \mathcal{C})$ denote the symmetric monoidal ∞-category whose underlying ∞-category is $\mathrm{Shv}(X; \mathcal{C})$, where the symmetric monoidal structure is inherited from $\mathrm{Shv}(X; \mathcal{C}^{\mathrm{op}})$ via the Verdier duality $\mathbf{D} : \mathrm{Shv}(X; \mathcal{C}) \xrightarrow{\sim} \mathrm{Shv}(X; \mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ of Theorem 4. We will denote the tensor product in $\mathrm{Shv}^!(X; \mathcal{C})$ by $\overset{!}{\otimes}$. Let $f : X \to Y$ be a continuous map of topological spaces. Since $f^*_{\mathcal{C}} : \mathrm{Shv}(Y; \mathcal{C}) \to \mathrm{Shv}(X; \mathcal{C})$ is a symmetric monoidal functor, the same is true of the functor $f^!_{\mathcal{C}} : \mathrm{Shv}^!(Y; \mathcal{C}) \to \mathrm{Shv}^!(X; \mathcal{C})$. Again, we will assume $\mathcal{C} = \mathrm{Sp}$ and drop all mention of $\mathcal{C}$ from the notation.

Remark 29. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}(X)$. By construction, $\mathbf{D}(\mathcal{F}) \overset{!}{\otimes} \mathbf{D}(\mathcal{G}) \simeq \mathbf{D}(\mathcal{F} \otimes \mathcal{G})$. The usual tensor product on $\mathrm{Shv}(X)$ can be understood as follows: let $\Delta : X \to X \times X$ be the diagonal. Then $\mathcal{F} \otimes \mathcal{G} \simeq \Delta^*(\mathcal{F} \boxtimes \mathcal{G})$. Therefore, $\mathbf{D}(\mathcal{F} \otimes \mathcal{G}) \simeq \Delta^! \mathbf{D}(\mathcal{F} \boxtimes \mathcal{G})$. By construction, $\mathbf{D}(\mathcal{F} \boxtimes \mathcal{G})$ is naturally identified with $\mathbf{D}(\mathcal{F}) \boxtimes \mathbf{D}(\mathcal{G})$, so we conclude from the preceding discussion that $\mathbf{D}(\mathcal{F}) \overset{!}{\otimes} \mathbf{D}(\mathcal{G}) \simeq \Delta^!(\mathbf{D}(\mathcal{F}) \boxtimes \mathbf{D}(\mathcal{G}))$. More invariantly, if $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}^!(X)$, then $\mathcal{F} \overset{!}{\otimes} \mathcal{G} \simeq \Delta^!(\mathcal{F} \boxtimes \mathcal{G})$. If X is a topological manifold, it follows from the construction that the unit of the !-tensor product is given by ω_X.

Lemma 30. *For any integer $n \in \mathbf{Z}$, let $\mathrm{Shv}(X)_{\leq n}$ denote the full subcategory of $\mathrm{Shv}(X)$ spanned by those objects $\mathcal{F}$ such that for each open subset $U \subseteq X$, the spectrum $\mathcal{F}(U) \in \mathrm{Sp}_{\leq n}$. This determines a full subcategory $\mathrm{Shv}(X)_{\geq 0}$: an object $\mathcal{G} \in \mathrm{Shv}(X)_{\leq 0}$ if and only if $\mathrm{Hom}_{\mathrm{Shv}(X)}(\mathcal{G}, \mathcal{F}) = 0$ for all $\mathcal{F} \in \mathrm{Shv}(X)_{\leq -1}$. The pair $(\mathrm{Shv}(X)_{\geq 0}, \mathrm{Shv}(X)_{\leq 0})$ determines a compatible t-structure on $\mathrm{Shv}(X)$.*

Proof. This is a consequence of [Lur17, Proposition 1.3.2.7, Remark 1.3.2.6, and Proposition 1.3.4.7]. □

Proposition 31. *Let X be a topological manifold, and let ω_X be its dualizing sheaf in the sense of Definition 24 (i.e., !-pullback of of $\mathbf{1}_*$ along projection to a point). Then the equivalence $\beta : \mathrm{Shv}(X) \to \mathrm{Shv}(X)$ given by tensoring with ω_X is symmetric monoidal for the usual tensor product on the source and the !-tensor product on the target. Furthermore, β is t-exact for the t-structure of Lemma 30.*

Proof. By Lemma 23, tensoring with ω_X defines an equivalence $\mathrm{Shv}(X) \to \mathrm{Shv}(X)$. To prove the symmetric monoidality claim, it suffices to prove that if $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}(X)$, then the functor $\overset{!}{\otimes} : \mathrm{Shv}(X) \times \mathrm{Shv}(X) \to \mathrm{Shv}(X)$ is equivalent to the composite

$$\mathrm{Shv}(X) \otimes \mathrm{Shv}(X) \xrightarrow{\otimes} \mathrm{Shv}(X) \xrightarrow{- \otimes \omega_X^{-1}} \mathrm{Shv}(X).$$

Indeed, then we have

$$\beta(\mathcal{F}) \overset{!}{\otimes} \beta(\mathcal{G}) \simeq \mathcal{F} \otimes \omega_X \otimes \mathcal{G} \otimes \omega_X \otimes \omega_X^{-1} \simeq \mathcal{F} \otimes \mathcal{G} \otimes \omega_X \simeq \beta(\mathcal{F} \otimes \mathcal{G})$$

for any $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}(X)$. To prove the claim about $\overset{!}{\otimes}$, it suffices to prove that $\Delta^!(\mathbf{1}_{X\times X}) \simeq \omega_X^{-1}$. But this is clear by considering !-pullbacks for the composite $X \xrightarrow{\Delta} X \times X \xrightarrow{\pi_{X\times X}} *$ and the observation that $\omega_{X\times X} \simeq \omega_X \boxtimes \omega_X$.

It remains to check that β is t-exact, which is equivalent to ω_X being connective. By Lemma 23, ω_X is a locally constant sheaf on X whose stalks are $\mathbf{1}[\dim(X)]$. It follows that for each open subset $U \subseteq X$, the object $\omega_X(U) \in \mathrm{Sp}_{\geq 0}$; therefore $\omega_X \in \mathrm{Shv}(X)_{\geq 0}$, as desired (in fact, $\omega_X \in \mathrm{Shv}(X)_{\geq \dim(X)}$). □

We will often write the equivalence of Proposition 31 as a symmetric monoidal t-exact equivalence $\beta : \mathrm{Shv}(X) \xrightarrow{\sim} \mathrm{Shv}^!(X)$.

We will now identify $f^!(\mathbf{1}_Y)$ in the case when $f : X \to Y$ is a submersion of smooth manifolds.

Definition 32. Let X be a topological space, and let $q : \mathcal{E} \to X$ be a vector bundle over X. Let $z : X \to \mathcal{E}$ denote the zero section (which is a closed immersion). Then the *Thom spectrum* is defined as $\mathrm{Thom}(X; \mathcal{E}) := q_\# z_!(\mathbf{1}_X) \in \mathrm{Shv}(X)$.

Lemma 33. *Let $j : \mathcal{E} - 0 \to \mathcal{E}$ be the complement of the zero section of q. Then the following composite is a cofiber sequence in* $\mathrm{Shv}(X)$*:*

$$q_\# j_! j^* \mathbf{1}_{\mathcal{E}} \simeq q_\# j_! \mathbf{1}_{\mathcal{E}-0} \to q_\# \mathbf{1}_{\mathcal{E}} \to \mathrm{Thom}(X; \mathcal{E}).$$

Proof. It suffices to show that $\mathrm{cofib}(j_! j^* \mathbf{1}_{\mathcal{E}} \to \mathbf{1}_{\mathcal{E}})$ is equivalent to $z_!(\mathbf{1}_X)$. But this follows from the recollement cofiber sequences of Construction 11 for the diagram $X \xrightarrow{z} \mathcal{E} \leftarrow \mathcal{E} - 0$. □

Lemma 34. *In the above setup,* $\mathrm{Thom}(X; \mathcal{E})$ *is invertible, with inverse* $z^!(\mathbf{1}_{\mathcal{E}})$.

Proof. Let $\pi : X \to *$ denote the projection to a point. By Lemma 33, the preceding definition of $\mathrm{Thom}(X; \mathcal{E})$ agrees with the more classical (∞-categorical) construction presented in [ABG+14]. It follows from the discussion in *loc. cit.* that the Thom spectrum construction is induced by the J-homomorphism $J : \mathrm{Vect}_{\mathbf{R}}^{\simeq} \to \mathrm{Pic}(\mathrm{Sp})$, which upgrades to a functor $J_X : \mathrm{Vect}_{\mathbf{R}}(X) \to \mathrm{Pic}(\mathrm{Shv}(X; \mathrm{Sp}))$ that is natural in spaces X. This implies that $\mathrm{Thom}(X; \mathcal{E}) \in \mathrm{Shv}(X)$ is invertible. Its inverse is therefore its $\mathbf{1}_X$-linear dual, which can compute using Corollary 19:

$$\underline{\mathrm{Hom}}_X(q_\# z_!(\mathbf{1}_X), \mathbf{1}_X) \simeq q_* \underline{\mathrm{Hom}}_{\mathcal{E}}(z_! \mathbf{1}_X, \mathbf{1}_{\mathcal{E}}) \simeq q_* z_* z^! \mathbf{1}_{\mathcal{E}} \simeq z^! \mathbf{1}_{\mathcal{E}}.$$

□

Proposition 35 (Atiyah duality)**.** *Let $f : X \to Y$ be a submersion between smooth manifolds, and let $T_{X/Y}$ denote the relative tangent bundle on X (given by the kernel of the surjective map $T_X \to f^* T_Y$ of bundles on X). Then there is an equivalence $f^! \mathbf{1}_Y \simeq \mathrm{Thom}(X; T_{X/Y})$.*

Proof. Assume the claim is proved for all projection maps $\pi_X : X \to *$; we claim that this implies the general case. Recall from Lemma 34 that $\mathrm{Thom}(X; T_X)$ is invertible. Since $T_{X/Y}$ is the kernel of the derivative map $T_X \to f^* T_Y$, we obtain an equivalence

$$\begin{aligned}\mathrm{Thom}(X; T_{X/Y}) &\simeq \mathrm{Thom}(X; T_X) \otimes \mathrm{Thom}(X; f^* T_Y)^{-1} \\ &\simeq \pi_X^!(\mathbf{1}_*) \otimes (f^* \pi_X^! \mathbf{1}_*)^{-1} \\ &\simeq \pi_X^!(\mathbf{1}_*) \otimes f^!(\mathbf{1}_Y) \otimes (f^! \pi_X^! \mathbf{1}_*)^{-1} \simeq f^!(\mathbf{1}_Y),\end{aligned}$$

as desired.

We now show that $\mathrm{Thom}(X; T_X) \simeq \pi_X^! \mathbf{1}_*$. Let $i : X \hookrightarrow \mathbf{R}^n$ be a closed embedding of X into some Euclidean space. Let $q : N_X \to X$ denote the normal bundle, so that there is an exact sequence $T_X \to i^* T_{\mathbf{R}^n} \to N_X$. Then $\mathrm{Thom}(X; T_X) \simeq \mathrm{Thom}(X; i^* T_{\mathbf{R}^n}) \otimes \mathrm{Thom}(X; N_X)^{-1}$. To determine $\mathrm{Thom}(X; N_X)^{-1}$, recall that the tubular neighborhood theorem says that there is an open subset $j : U \subseteq \mathbf{R}^n$ and a homeomorphism $h : N_X \xrightarrow{\sim} U$ such that i factors as the composite

$$i : X \xrightarrow{z} N_X \xrightarrow{\sim} U \xrightarrow{j} \mathbf{R}^n.$$

By Lemma 34, we know that $\mathrm{Thom}(X; N_X)^{-1} \simeq z^! \mathbf{1}_{N_X}$. Because $\mathbf{1}_{N_X} = h^* j^* \mathbf{1}_{\mathbf{R}^n} \simeq h^* j^! \mathbf{1}_{\mathbf{R}^n}$, we conclude that $\mathrm{Thom}(X; N_X)^{-1} \simeq i^! \mathbf{1}_{\mathbf{R}^n}$. Using Corollary 22, this implies that

$$\mathrm{Thom}(X; T_X) \simeq \mathrm{Thom}(X; i^* T_{\mathbf{R}^n}) \otimes \mathrm{Thom}(X; N_X)^{-1} \simeq i^* \mathrm{Thom}(\mathbf{R}^n; T_{\mathbf{R}^n}) \otimes i^! \mathbf{1}_{\mathbf{R}^n} \simeq i^! \mathrm{Thom}(\mathbf{R}^n; T_{\mathbf{R}^n}).$$

To finish, it suffices to show that $\mathrm{Thom}(\mathbf{R}^n; T_{\mathbf{R}^n}) \simeq \pi^!_{\mathbf{R}^n}\mathbf{1}_*$. By Lemma 33, $\mathrm{Thom}(\mathbf{R}^n; T_{\mathbf{R}^n})$ may be identified with $\mathbf{1}_{\mathbf{R}^n}[n]$, so it remains to show that $\pi^!_{\mathbf{R}^n}\mathbf{1}_* \simeq \mathbf{1}_{\mathbf{R}^n}[n]$. But this follows from (the proof of) Lemma 23. □

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