The nonabelian Hodge correspondence

Sanath Devalapurkar

March 24, 2020


ayo wtf
@YEESSSIIIRRR
future grandson: “hey grandpa want a corona?”
me:

Outline

1. Motivation
2. The proof
3. Consequences
4. An interesting digression whose consequences we won't have time to discuss

The triumvirate...

Let X be a complex manifold. One can then extract the triumvirate:

- Singular cohomology $\mathrm{H}^*(X; \mathbf{C})$;
- de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(X; \mathbf{C})$;
- the Hodge decomposition $\bigoplus_{p+q=n} \mathrm{H}^q(X; \Omega^p_X)$.

These correspond to the topological, smooth, and holomorphic worlds, respectively.

...collapses...

If X is just a smooth manifold, then there is an isomorphism

$$\mathrm{H}^*_{\mathrm{dR}}(X; \mathbf{C}) \xrightarrow{\cong} \mathrm{H}^*(X; \mathbf{C}) \cong \mathsf{Hom}_{\mathbf{C}}(\mathrm{H}_*(X; \mathbf{C}), \mathbf{C}),$$

sending a class $[\omega] \in \mathrm{H}^n_{\mathrm{dR}}(X; \mathbf{C})$ corresponding to an n-form ω to

$$\mathrm{H}_n(X; \mathbf{C}) \ni [M] \mapsto \int_M \omega \in \mathbf{C}.$$

...into one

If X is a complex manifold, then every C^∞-n-form on X can be written as a sum of (p, q)-forms, with $p + q = n$.

If X is also Kähler, then the (p, q)-component of a harmonic n-form is harmonic, and so the space of harmonic n-forms splits as a sum of harmonic (p, q)-forms.

The Hodge theorem now tells us that the space of harmonic n-forms is isomorphic to $\mathrm{H}^n(X; \mathbf{C})$, and so

$$\mathrm{H}^n_{\mathrm{dR}}(X; \mathbf{C}) \cong \bigoplus_{p+q=n} \mathrm{H}^q(X; \Omega^p_X).$$

So, we find that

$$\mathrm{H}^n(X; \mathbf{C}) \cong \mathrm{H}^n_{\mathrm{dR}}(X; \mathbf{C}) \cong \bigoplus_{p+q=n} \mathrm{H}^q(X; \Omega^p_X).$$

Categorification

The de Rham isomorphism $\mathrm{H}^n(X; \mathbf{C}) \cong \mathrm{H}^n_{\mathrm{dR}}(X; \mathbf{C})$ connects the local system $\mathbf{C}$ on X with the vector bundle $\mathcal{O}_X$ equipped with its flat connection $d : \mathcal{O}_X \to \Omega^1_X$.

This is categorified by the Riemann-Hilbert correspondence, a baby case of which says:

Theorem

There is an equivalence:

$$\{\text{Local systems on } X\} \xrightarrow{\sim} \left\{\begin{matrix}\text{Vector bundles on } X + \\ \text{a flat connection}\end{matrix}\right\}.$$

There are many refinements of this, culminating in a correspondence between constructible sheaves and regular holonomic D-modules.

Categorifying the Hodge theorem

We would like to similarly categorify the Hodge theorem. To get some intuition for what to expect, let us look at the Hodge decomposition of $\mathrm{H}^1_{\mathrm{dR}}(X; \mathbf{C})$:

$$\mathrm{H}^1_{\mathrm{dR}}(X; \mathbf{C}) \cong \mathrm{H}^1(X; \mathcal{O}_X) \oplus \mathrm{H}^0(X; \Omega^1_X).$$

Therefore, an element of $\mathrm{H}^1_{\mathrm{dR}}(X; \mathbf{C})$ is a pair (e, ξ) with $e \in \mathrm{H}^1(X; \mathcal{O}_X)$ and $\xi \in \mathrm{H}^0(X; \Omega^1_X)$.

Holomorphic line bundles with vanishing first Chern class give rise to elements of $\mathrm{H}^1(X; \mathcal{O}_X)$, and sections of Ω^1_X are holomorphic 1-forms.

In particular, one might expect the categorification of the Hodge theorem to give a correspondence between:

- Certain vector bundles on X equipped with a flat connection;
- Certain holomorphic bundles on X along with a specified 1-form.

We'd win The Price Is Right

This is, in fact, what happens — and it's called the nonabelian Hodge correspondence.

The category corresponding to the holomorphic side has the following objects:

Higgs bundles

A Higgs bundle is a pair $(\mathcal{F}, \phi)$, with $\mathcal{F}$ a holomorphic bundle on X, and $\phi \in \Gamma(X; \mathrm{End}(\mathcal{F}) \otimes \Omega^1_X)$ which commutes with itself (i.e., $\phi \wedge \phi = 0$).

We will be more precise below, but for now, let's state the impressionists' version of the nonabelian Hodge correspondence:

NAH

There is an equivalence:

$$\left\{\begin{matrix}\text{Vector bundles on } X + \\ \text{a flat connection}\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{Higgs bundles on } X + \\ \text{stability conditions}\end{matrix}\right\}.$$

Getting intuition for the proof

Suppose $(\mathcal{F}, \phi)$ is a Higgs bundle. Then ϕ defines a map

$$\phi : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X,$$

which is $\mathcal{O}_X$-linear: if f is a section of $\mathcal{O}_X$ and s is a section of $\mathcal{F}$, then

$$\phi(fs) = f\phi(s).$$

Compare this to the definition of a connection D on $\mathcal{F}$: this is a map

$$\mathrm{D} : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X,$$

which satisfies the Leibniz rule

$$\mathrm{D}(fs) = s \otimes df + f\mathrm{D}(s).$$

The only difference is the term $s \otimes df$ (which detects whether the map is $\mathcal{O}_X$-linear or not).

Interpol(ation)

To interpolate between Higgs fields and connections, one would therefore like to define some deformation of the notion of a connection, which recovers connections when $\lambda = 1$, and Higgs fields when $\lambda = 0$.

One should think of these intermediaries as analogues of harmonic forms: they interpolate between the smooth world and the holomorphic world.

Here's the definition.

λ-connections

Let $\lambda \in \mathbf{C}$. A λ-connection on a vector bundle $\mathcal{F}$ is a map

$$\mathrm{D}_\lambda : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$$

such that

$$\mathrm{D}_\lambda(fs) = \lambda s \otimes df + f\mathrm{D}_\lambda(s).$$

Fail.

Suppose $(\mathcal{F}, \mathrm{D}_\lambda)$ is a λ-connection. If $\lambda' \in \mathbf{C}$, then $(\mathcal{F}, \lambda' \mathrm{D}_\lambda)$ is a $\lambda\lambda'$-connection.

In particular, there is a $\mathbf{C}^\times$-action on λ-connections.

Because 0-connections are just Higgs bundles, one might hope to obtain the nonabelian Hodge correspondence by starting off with a vector bundle $(\mathcal{F}, \mathrm{D})$, and taking the limit $\lambda \to 0$ under the $\mathbf{C}^\times$-action to get a Higgs bundle.

But this obviously doesn't work: the resulting Higgs field is just zero! We need to work harder (as you might've expected).

The key idea is: allow the holomorphic structure on $\mathcal{F}$ to vary with λ.

Some complex geometry

To understand how we might do this, recall the following beautiful result from complex geometry, known as the Koszul-Malgrange theorem (which in turn is a special case of the Newlander-Nirenberg theorem).

Koszul-Malgrange

The following data on a smooth bundle $\mathcal{F}$ are equivalent:

- A holomorphic structure on $\mathcal{F}$;
- An operator
$$\overline{\partial}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^{0,1}$$
such that
$$\overline{\partial}_{\mathcal{F}}(fs) = s \otimes \overline{\partial} f + f \overline{\partial}_{\mathcal{F}}(s)$$
which satisfies $\overline{\partial}_{\mathcal{F}}^2 = 0$.

The holomorphic sections of $\mathcal{F}$ are then those sections which are killed by $\overline{\partial}_{\mathcal{F}}$.

So what is a Higgs bundle?

Let's see what this means for a Higgs bundle $(\mathcal{F}, \theta)$. Recall that

$$\theta : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^{1,0}.$$

We're now emphasizing that θ lands in $(1,0)$-forms, unlike earlier — this is because we're going to be going between the smooth and holomorphic worlds, and we don't want to confuse notations.

By Koszul-Malgrange, the holomorphic structure on $\mathcal{F}$ is specified by an operator

$$\overline{\partial} : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^{0,1}.$$

The condition that θ be a holomorphic map is encapsulated in the equation

$$\overline{\partial}\theta + \theta\overline{\partial} = 0.$$

So, if we define

$$\mathrm{D}'' = \overline{\partial} + \theta,$$

then $(\mathrm{D}'')^2 = 0$ encapsulates the above condition, the flatness of $\overline{\partial}$, and $\theta \wedge \theta = 0$.

Higgs data

In fact, if we had an operator $\mathrm{D}'' : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$ such that $(\mathrm{D}'')^2 = 0$, then decomposing D'' into its $(1,0)$ and $(0,1)$ components produces:

- a holomorphic structure $\mathrm{D}^{0,1}$ on $\mathcal{F}$; and
- a Higgs structure $\mathrm{D}^{1,0}$ on $\mathcal{F}$.

We'll often just write $(\mathcal{F}, \mathrm{D}'')$ to denote a Higgs bundle.

Flat to Higgs

Suppose $(\mathcal{F}, \mathrm{D})$ is a vector bundle equipped with a flat connection. We'd like to get a Higgs bundle from this.

Write $\mathrm{D} = \mathrm{D}^{1,0} + \mathrm{D}^{0,1}$. Let K be a Hermitian metric on $\mathcal{F}$; then, there are operators

$$\delta^{1,0} : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^{1,0}, \ \delta^{0,1} : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^{0,1}$$

such that $\mathrm{D}^{1,0} + \delta^{0,1}$ and $\mathrm{D}^{0,1} + \delta^{1,0}$ preserve K. In other words, if ∇ denotes either one of these sums, then

$$K(\nabla f, f') + K(f, \nabla f') = dK(f, f').$$

Define the following four operators:

$$\partial_K = \frac{\mathrm{D}^{0,1} + \delta^{1,0}}{2}, \quad \overline{\partial}_K = \frac{\mathrm{D}^{1,0} + \delta^{0,1}}{2}$$
$$\theta_K = \frac{\mathrm{D}^{0,1} - \delta^{1,0}}{2}, \quad \overline{\theta}_K = \frac{\mathrm{D}^{1,0} - \delta^{0,1}}{2}.$$

Flat to Higgs

In particular:

$$\partial_K, \theta_K : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^{1,0},$$

and

$$\overline{\partial}_K, \overline{\theta}_K : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^{0,1}.$$

Further define

$$\mathrm{D}_K' = \partial_K + \overline{\theta}_K,$$
$$\mathrm{D}_K'' = \overline{\partial}_K + \theta_K.$$

It's easy to see that $\mathrm{D}_K' + \mathrm{D}_K'' = \mathrm{D}$.

The pair $(\mathcal{F}, \mathrm{D}_K'')$ looks a lot like the datum we need to specify a Higgs bundle! More precisely:

Observation

If $(\mathrm{D}_K'')^2 = 0$, then $(\mathcal{F}, \mathrm{D}_K'')$ is a Higgs bundle.

Higgs to flat

We can similarly try to produce a vector bundle with flat connection from a Higgs bundle. Suppose that $(\mathcal{F}, \mathrm{D}'') = (\mathcal{F}, \overline{\partial}, \theta)$ is a Higgs bundle. Let K be a Hermitian metric on $\mathcal{F}$.

Again, there is a unique operator

$$\partial_K : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^{1,0}$$

such that $\partial_K + \overline{\partial}$ preserves the metric K.

Define $\overline{\theta}_K : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^{0,1}$ via

$$K(\theta f, f') = K(f, \overline{\theta}_K f').$$

Higgs to flat

Finally, define

$$\mathrm{D}'_K = \partial_K + \overline{\theta}_K,$$
$$\mathrm{D}_K = \mathrm{D}'_K + \mathrm{D}'' = \overline{\partial} + \partial_K + \theta + \overline{\theta}_K.$$

Note that $\mathrm{D}_K - \mathrm{D}'_K = \mathrm{D}''$.

The pair $(\mathcal{F}, \mathrm{D}_K)$ looks a lot like the datum we need to specify a vector bundle with flat connection! More precisely:

Observation

If $(\mathrm{D}_K)^2 = 0$, then $(\mathcal{F}, \mathrm{D}_K)$ is a vector bundle with flat connection.

Harmonic bundles

In both cases, we obtained a tuple $(\mathcal{F}, \partial, \overline{\partial}, \theta, \overline{\theta})$ from the datum of a Higgs bundle/vector bundle with a flat connection equipped with a Hermitian metric.

This is supposed to be reminiscent of the situation in Hodge theory, where one can define harmonic forms after picking a Hermitian metric.

In any case, these considerations motivate the following definition:

Harmonic bundles

A harmonic bundle on X is a tuple $(\mathcal{F}, \mathrm{D}, \mathrm{D}'')$, where $\mathcal{F}$ is a vector bundle, D is a flat connection on $\mathcal{F}$, D'' defines a Higgs structure on $\mathcal{F}$, such that there is a Hermitian metric K on $\mathcal{F}$ for which

$$\mathrm{D}'' = \mathrm{D}''_K,\ \mathrm{D} = \mathrm{D}_K$$

via the above constructions.

Note that the datum of the Hermitian metric K is not included in the definition of a harmonic bundle.

Interpolation, redux

Our discussion implies:

- A vector bundle $(\mathcal{F}, \mathrm{D})$ determines a harmonic bundle if and only if there is a Hermitian metric K on $\mathcal{F}$ such that $(\mathrm{D}''_K)^2 = 0$.
- A Higgs bundle $(\mathcal{F}, \mathrm{D}'')$ determines a harmonic bundle if and only if there is a Hermitian metric K on $\mathcal{F}$ such that $\mathrm{D}_K^2 = 0$.

Here is the key result.

Proposition

Let $(\mathcal{F}, \mathrm{D}, \mathrm{D}'')$ be a harmonic bundle on X. Then there is a family $(\mathcal{F}_\lambda, \mathrm{D}_\lambda)$ of flat λ-connections on X such that

$$(\mathcal{F}_1, \mathrm{D}_1) = (\mathcal{F}, \mathrm{D}),\ (\mathcal{F}_0, \mathrm{D}_0) = (\mathcal{F}, \mathrm{D}'').$$

This is the λ-connection we wanted in the beginning! Given a flat connection $(\mathcal{F}, \mathrm{D})$ which determines a harmonic bundle, just take $\lim_{\lambda\to 0}(\mathcal{F}_\lambda, \mathrm{D}_\lambda)$ to get a Higgs bundle. Similarly for the other direction.

Interpolation, redux

Let $(\mathcal{F}, \mathrm{D}, \mathrm{D}'')$ be our harmonic bundle. Define

$$\mathrm{D}' = \mathrm{D} - \mathrm{D}''.$$

Let us write $\mathrm{D}'' = \overline{\partial} + \theta$.

Because $(\mathcal{F}, \mathrm{D}, \mathrm{D}'')$ is a harmonic bundle, there is a Hermitian metric K on $\mathcal{F}$ such that $\overline{\partial}_K = \overline{\partial}$ and $\theta_K = \theta$. So

$$\mathrm{D}' = \partial_K + \overline{\theta}_K.$$

Define

$$\mathrm{D}_\lambda = \mathrm{D}'' + \lambda \mathrm{D}' = \overline{\partial} + \theta + \lambda\partial_K + \lambda\overline{\theta}_K.$$

Then

$$(\mathrm{D}'_\lambda)^{0,1} = \overline{\partial} + \lambda\overline{\theta},\ (\mathrm{D}'_\lambda)^{1,0} = \partial + \lambda\theta.$$

Because $(\mathcal{F}, \mathrm{D}, \mathrm{D}'')$ is a harmonic bundle, we know that $\mathrm{D}_\lambda^2 = 0$, so these two components commute.

It follows that $\partial + \lambda\theta$ defines a flat λ-connection on $\mathcal{F}$, where the holomorphic structure on $\mathcal{F}$ is determined by $\overline{\partial} + \lambda\overline{\theta}$.

So, how do we get a harmonic bundle?

We observed that if a vector bundle $(\mathcal{F}, \mathrm{D})$ determines a harmonic bundle, then we can get a Higgs bundle. In turn, $(\mathcal{F}, \mathrm{D})$ determines a harmonic bundle if there is a Hermitian metric K on $\mathcal{F}$ such that $(\mathrm{D}''_K)^2 = 0$.

So when does such a metric exist? Using analytic methods, one can prove:

Theorem (Siu, Sampson, Corlette, Deligne)

Let $(\mathcal{F}, \mathrm{D})$ be a vector bundle equipped with a flat connection. Then there exists a Hermitian metric K on $\mathcal{F}$ such that $(\mathrm{D}''_K)^2 = 0$ if and only if $\mathcal{F}$ is semisimple.

Here, semisimplicity means the usual thing; under the Riemann-Hilbert correspondence, it means that the associated representation of $\pi_1(X)$ is semisimple.

So, how do we get a harmonic bundle?

We observed that if a Higgs bundle $(\mathcal{F}, \mathrm{D}'')$ determines a harmonic bundle, then we can get a vector bundle with flat connection. In turn, $(\mathcal{F}, \mathrm{D}'')$ determines a harmonic bundle if there is a Hermitian metric K on $\mathcal{F}$ such that $(\mathrm{D}_K)^2 = 0$.

So when does such a metric exist? Using analytic methods, one can prove:

Theorem (Narasimhan-Seshadri, Donaldson, Uhlenbeck-Yau, Beilinson-Deligne, Hitchin, Simpson)

Let $(\mathcal{F}, \mathrm{D}'')$ be a Higgs bundle. Then there exists a Hermitian metric K on $\mathcal{F}$ such that $\mathrm{D}_K^2 = 0$ if and only if:

- *$\mathcal{F}$ is polystable, meaning that it is a direct sum of stable Higgs bundles of the same slope; and*
- *the first two Chern classes vanish:*

$$c_1(\mathcal{F}) \cdot [\omega]^{\dim(X)-1} = c_2(\mathcal{F}) \cdot [\omega]^{\dim(X)-2} = 0.$$

The theorem

Combining the above results, we find:

NAH

There is an equivalence of categories between:

- Vector bundles equipped with a flat connection which are semisimple;
- Higgs bundles $(\mathcal{F}, \phi)$ on X such that:
  - $(\mathcal{F}, \phi)$ is polystable;
  - the first two Chern classes vanish.

The $\mathbf{C}^\times$-action

Recall from our discussion before that there is a $\mathbf{C}^\times$-action on λ-connections (when you look at all λ in congregate) — this is just given by sending a λ-connection $(\mathcal{F}, \mathrm{D}_\lambda)$ and $\lambda' \in \mathbf{C}^\times$ to the $\lambda\lambda'$-connection $(\mathcal{F}, \lambda'\mathrm{D}_\lambda)$.

In particular, Higgs bundles are sent to Higgs bundles. It turns out that the $\mathbf{C}^\times$-action preserves the conditions imposed in the statement of NAH.

So, we get a $\mathbf{C}^\times$-action on semisimple flat bundles. I haven't seen a description of the action itself, but the fixed points admit a nice description.

Fixed points in Higgs bundles

Let's begin by trying to understand the $\mathbf{C}^\times$-fixed points in Higgs bundles.

Proposition

A Higgs bundle $(\mathcal{F}, \phi)$ is fixed by the $\mathbf{C}^\times$-action if and only if it can be written as $\bigoplus_{i=1}^k \mathcal{F}_k$ satisfying Griffiths transversality:

$$\phi : \mathcal{F}_i \to \mathcal{F}_{i-1} \otimes \Omega^1_X.$$

One might therefore expect that the $\mathbf{C}^\times$-fixed points in Higgs bundles are related to variations of Hodge structures.

Fixed points in Higgs bundles

To see the proposition, let f be an isomorphism $(\mathcal{F}, \phi) \to (\mathcal{F}, t\phi)$ for t not a root of unity. The coefficients of the characteristic polynomial of f are holomorphic functions on X (and therefore are constant).

The decomposition of $\mathcal{F}$ into eigenbundles for f is $\bigoplus_\lambda \mathcal{F}_\lambda$, where $\mathcal{F}_\lambda = \ker((f-\lambda)^n)$ if λ is an eigenvalue of multiplicity n. Because

$$t^n\phi(f-\lambda)^n = (f-t\lambda)^n,$$

we must have

$$\theta : \mathcal{F}_\lambda \to \mathcal{F}_{t\lambda}.$$

Because t is not a root of unity, the set S of eigenvalues of f can be decomposed into strings of the form

$$\lambda, t\lambda, \cdots, t^k\lambda.$$

In particular, $S = \coprod_{i=1}^k S_i$, and one then defines

$$\mathcal{F}_i = \bigoplus_{\lambda \in S_i} \mathcal{F}_\lambda.$$

Variation of Hodge structures

Let X be a smooth projective variety. A complex variation of Hodge structures is the datum of:

- a vector bundle $\mathcal{V} = \bigoplus_{p+q=n} \mathcal{V}^{p,q}$;
- a flat connection D on $\mathcal{V}$ such that

$$\mathrm{D} : \mathcal{V}^{p,q} \to \Omega^{1,0}(\mathcal{V}^{p,q}) \oplus \Omega^{0,1}(\mathcal{V}^{p,q}) \oplus \Omega^{1,0}(\mathcal{V}^{p-1,q+1}) \oplus \Omega^{0,1}(\mathcal{V}^{p+1,q});$$

- a Hermitian form h on $\mathcal{V}$ which makes the decomposition orthogonal, and which is positive (resp. negative) definite on $\mathcal{V}^{p,q}$ if p is even (resp. odd).

Example

The definition is motivated by algebraic geometry.

Suppose $f : Y \to X$ is a smooth projective morphism. Then $\mathcal{V} = \mathrm{R}^n f_*(\mathbf{C}) \otimes_{\mathbf{C}} \mathcal{O}_X$ admits a Hodge decomposition

$$\mathcal{V} \cong \bigoplus_{p+q=n} \mathrm{R}^q f_*(\Omega^p_{Y/X}).$$

The Hermitian form on $\mathcal{V}$ is given by pairing with the Kähler form ω: on each fiber $\mathrm{H}^n(Y_x; \mathbf{C})$, the pairing is defined by

$$\langle \alpha, \beta \rangle = \int_{Y_x} \alpha \wedge \overline{\beta} \wedge \omega^{\dim(Y_x)-n},$$

up to some constant factor.

The Gauss-Manin connection gives the connection D, and the condition required of D comes from Griffiths transversality.

Variation of Hodge structures to Higgs

We shall now describe how to construct a $\mathbf{C}^\times$-fixed point in Higgs bundles from a complex variation of Hodge structures.

Suppose we are given a complex variation of Hodge structures $(\mathcal{V} = \mathcal{V}^{p,q}, \mathrm{D}, h)$, so

$$\mathrm{D} : \mathcal{V}^{p,q} \to \Omega^{1,0}(\mathcal{V}^{p,q}) \oplus \Omega^{0,1}(\mathcal{V}^{p,q}) \oplus \Omega^{1,0}(\mathcal{V}^{p-1,q+1}) \oplus \Omega^{0,1}(\mathcal{V}^{p+1,q}),$$

can be written as

$$\mathrm{D} = \partial \oplus \overline{\partial} \oplus \theta \oplus \overline{\theta}.$$

The operator $\overline{\partial}$ equips $\mathcal{V}^{p,q}$ with a holomorphic structure, and the operator θ equips $\mathcal{V}^{p,q}$ with a map $\mathcal{V}^{p,q} \to \mathcal{V}^{p-1,q+1} \otimes \Omega^1_X$.

Therefore, the bundle $\mathcal{V}$ can be written as a direct sum $\bigoplus_{i=1}^n \mathcal{F}_i$ (where n is the weight of $\mathcal{V}$), with $\mathcal{F}_i = \bigoplus_{p \geq i} \mathcal{V}^{p,q}$.

Since D is assumed to be flat, we find that $\theta \wedge \theta = 0$, so $(\mathcal{V}, \theta)$ is a Higgs bundle. By our proposition, it is a fixed point of the $\mathbf{C}^\times$-action on Higgs bundles.

Example

Consider the complex variation of Hodge structures associated to a morphism $f : Y \to X$, so $\mathcal{V} = \mathrm{R}^n f_*(\mathbf{C})$.

The associated Higgs field sends

$$\mathrm{R}^q f_*(\Omega^p_{Y/X}) \to \mathrm{R}^{q+1} f_*(\Omega^{p-1}_{Y/X}) \otimes \Omega^1_X.$$

On each fiber $x \in X$, this morphism is given by pairing with the Kodaira-Spencer class

$$\eta_x \in \mathrm{Hom}(\mathrm{T}_{X,x}, \mathrm{R}^1 f_*(\mathrm{T}_{Y_x})) \cong \mathrm{R}^1 f_*(\mathrm{T}_{Y_x}) \otimes (\Omega^1_X)_x.$$

Fixed points in flat bundles

It turns out that the mechanism described above (to extract a Higgs bundle from a complex variation of Hodge structures) in fact characterizes the fixed points of the $\mathbf{C}^\times$-action on semisimple flat bundles on X:

Theorem

The fixed points of the $\mathbf{C}^\times$-action on semisimple flat bundles on X are precisely those bundles admitting a complex variation of Hodge structures.

If we have time, there's more that I'd like to say. If not, thanks for listening!

D-modules

If $(\mathcal{F}, \phi)$ is a Higgs bundle, then the $\mathcal{O}_X$-linear coaction of Ω^1_X on $\mathcal{F}$ (defined by ϕ) is equivalent to an action of $\mathrm{Sym}(\mathrm{T}_X) = \mathrm{Sym}((\Omega^1_X)^\vee)$ on $\mathcal{F}$.

In other words, a Higgs bundle is essentially the datum of a coherent sheaf on the cotangent bundle T^*X. There is a similar characterization of vector bundles with flat connection.

Recall that if X is an algebraic variety, then TFAE:

- a vector bundle with a flat connection;
- a $\mathcal{D}_X$-module which is $\mathcal{O}_X$-coherent.

So, we'd like to know if $\mathcal{D}_X$-modules are quasicoherent sheaves on some stack.

The de Rham space

Definition

The de Rham space X_{dR} is the functor $\mathrm{CAlg}_{\mathbf{C}} \to \mathrm{Set}$ defined by $X_{\mathrm{dR}}(R) = X(R/I)$, where I is the nilradical of R. In other words, one identifies "infinitesimally close points" of X.

Then:

Theorem (Grothendieck)

There is an equivalence of categories $\mathrm{QCoh}(X_{\mathrm{dR}}) \simeq \mathrm{Mod}(\mathcal{D}_X)$.

The action of $\mathcal{D}_X$ is roughly given by parallel transport.

λ-connections

We saw that λ-connections interpolate between vector bundles with flat connection and Higgs bundles.

In light of the above discussion, we might hope that there is:

- Some sheaf $\mathcal{D}_X^\lambda$ of algebras which deforms $\mathcal{D}_X$;
- Some stack X_λ which deforms X_{dR}, such that there is an equivalence

$$\mathrm{QCoh}(X_\lambda) \simeq \mathrm{Mod}(\mathcal{D}_X^\lambda).$$

Such objects exist, and admit nice geometric constructions. I will talk about the construction of X_λ.

Presentations

The functor X_{dR} admits a nice presentation: let $\Delta : X \to X \times X$ denote the diagonal; then

$$X_{\mathrm{dR}} \longleftarrow X \leftleftarrows (X \times X)^{\wedge}_{X} \overset{\longleftarrow}{\leftleftarrows} \cdots$$

We may also define a stack X_{Dol}, via the presentation:

$$X_{\mathrm{Dol}} \longleftarrow X \leftleftarrows \mathrm{T}X^{\wedge}_{X} \overset{\longleftarrow}{\leftleftarrows} \cdots$$

where X sits inside $\mathrm{T}X$ via the zero section. Then:

$$\mathrm{QCoh}(X_{\mathrm{Dol}}) \simeq \mathrm{Mod}_{\mathsf{Sym}_{\mathcal{O}_X}(\mathrm{T}X)}(\mathrm{QCoh}(X)) \simeq \mathrm{QCoh}(\mathrm{T}^*X).$$

Therefore, we would like to interpolate between $\mathrm{T}X$ and $X \times X$. This is given by the "deformation to the normal cone" of $\Delta : X \to X \times X$.

The stack X_λ

Let $\widetilde{\mathcal{B}}_\bullet$ be the cosimplicial scheme defined by

$$\widetilde{\mathcal{B}}_\bullet : \Delta \to \mathrm{Aff}_{/\mathbf{A}^1},\ [n] \mapsto \mathrm{Spec}(\mathbf{C}[x, y]/(x^n - y^n)) = \widetilde{\mathcal{B}}_n.$$

There is a canonical map $\widetilde{\mathcal{B}}_n \to \mathbf{A}^1$ detecting the function x, and this morphism is $\mathbb{G}_m$-equivariant for the canonical scaling action on x and y.

The fiber of

$$\mathrm{D}_\bullet := \mathrm{Hom}_{\mathbf{A}^1}(\mathcal{B}_\bullet, X \times \mathbf{A}^1)$$

over $\mathbf{A}^1 - \{0\}$ is simply $X^{\times n} \times (\mathbf{A}^1 - \{0\})$, while the fiber over 0 is $\mathrm{T}X \times_X \cdots \times_X \mathrm{T}X$. In particular, there is a diagonal map $X \times \mathbf{A}^1 \to \mathrm{D}_\bullet$.

Define X_λ to be the geometric realization of the stack given by

$$X_{\lambda,\bullet} = \mathrm{D}_\bullet \times_{(\mathrm{D}_\bullet)_{\mathrm{dR}}} (X \times \mathbf{A}^1)_{\mathrm{dR}}.$$

In other words, $X_{\lambda,\bullet}$ is the formal completion of $\mathrm{D}_\bullet$ along the diagonal $X \times \mathbf{A}^1 \to \mathrm{D}_\bullet$.

The stack X_λ

The $\mathbb{G}_m$-equivariant stack $X_\lambda \to \mathbf{A}^1$ satisfies the properties we described above:

- the fiber over $\mathbf{A}^1 - \{0\}$ is X_{dR};
- the fiber over $\{0\}$ is X_{Dol}.

Let $\underline{\mathrm{Coh}}(X_\lambda)$ denote the stack of coherent sheaves on X_λ.

The proposition we used in the proof of the nonabelian Hodge theorem shows:

Proposition

Any harmonic bundle $(\mathcal{F}, \mathrm{D}, \mathrm{D}'')$ gives rise to a map $\mathbf{A}^1 \to \underline{\mathrm{Coh}}(X_\lambda)$ sending λ to $(\mathcal{F}_\lambda, \mathrm{D}_\lambda)$.

There is a lot more to this story, leading to the Deligne-Hitchin twistor space. But I'm probably way over time, so I'll stop.