E_n formality

Tuesday, June 22, 2021 1:08 PM

Last time: A E_1 algebra $\rightsquigarrow$ $HC(A)$ E_2-alg. controlling deformations of A

$\Rightarrow H^\bullet(HC(A))$ $H^\bullet(E_2)$-algebra

$\overset{\shortparallel}{P_2} \leftarrow$ ass. alg w/ PB of degree 1

$A = \mathcal{O}(X) \rightsquigarrow$

$HC(A) = \Gamma(X, D_{poly}(X))$

$H^\bullet(HC(A)) = \overset{\mathbb{R}}{PV}(X)$

Formality thm: $C_\bullet(E_2) \simeq H_\bullet(E_2) \simeq P_2$

Cor: Intrinsic formality $H^\bullet(HC(A)) \Rightarrow$ formality of $HC(A) \simeq H^\bullet(HC(A))$

$= \Gamma(\operatorname{Spec} A, \Lambda^\bullet T_{\operatorname{Spec} A})$. $

This talk: Proof of formality thm, following [Lambrechts - Volić]

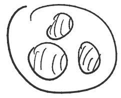
(Ref: Talbot talk Sander Kupers)

Rmk: Nothing special about E_2-algebras. A E_n-alg $\rightsquigarrow$ $HC_{E_n}(A) = E_{n+1}$-alg.

And it's still true that $C_\bullet(E_n) \simeq P_n$.

(In fact, $\boxed{C^\bullet(E_n) \simeq coP_n}$ as operads in coalgs)

Recall: E_2 = operad of little disks, w/ n-ary operation given by embeddings of n disks

e.g. $E_2(3) = \{$  $\} = \operatorname{Emb}(\sqcup_3 D^2, D^2)$.

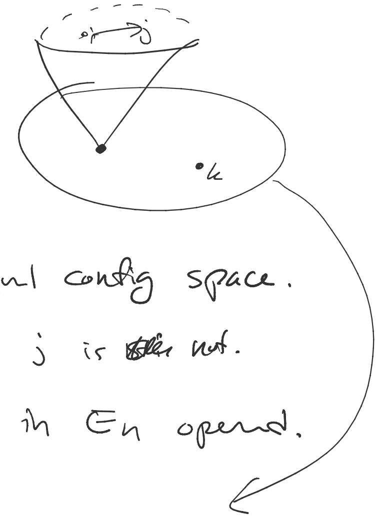
Step 1: Model $E_N \simeq FM_N$ as the Fulton-MacPherson operad of the compactified config. space of $\mathbb{R}^N$. FM_N is defined in 2 steps:

① Start w/ $\operatorname{Emb}([n], \mathbb{R}^N) \Big/ \underset{\text{translation}}{\mathbb{R}^N} \times \underset{\text{scalings}}{\mathbb{R}_+}$ (homotopy equivalent to $C_n(\mathbb{R}^N)$.)

② Embed this into $\prod_{i \neq j \in [n]} S^{N-1} \times \prod_{i \neq j \neq k \in [n]} [0, \infty]$

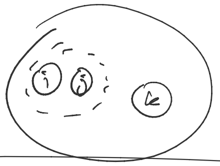
$i \circ \longrightarrow \circ j$ rel. direction

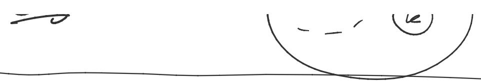
rel. distance $\dfrac{d(i,j)}{d(j,k)}$



The regime when the rel. distances are $\neq 0, \infty$ is the original config space.
When rel. dist. of i,j is 0 wrt k, i,j are "inf. close" but j is not.

$\rightsquigarrow$ get operadic composition matching embeddings of discs in E_n operad.

E.g.: 



Goal: Find a subalg. of $\Omega^\bullet(FM_N)$ which is formal & quasi-iso to all of $\Omega^\bullet(FM_N)$

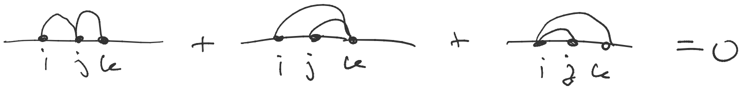
Main example: Arnold's proof of formality of $\mathrm{Conf}_n(\mathbb{C}) = \{(z_1, \dots z_n) \mid z_i \neq z_j \in \mathbb{C}\}$

Pf: For $i \neq j$, define $\omega_{ij} := \dfrac{d(z_i - z_j)}{z_i - z_j}$ ("propagator")

Exercise: These are nonzero in cohomology and they generate all of $H^\bullet(\mathrm{Conf}_n\mathbb{C})$

Check: $\omega_{jk} \wedge \omega_{k\ell} + \omega_{k\ell} \wedge \omega_{\ell j} + \omega_{\ell j} \wedge \omega_{jk} = 0$

Diagrammatically:



$$\rightsquigarrow \Omega^\bullet(\mathrm{Conf}_n\mathbb{C}) \simeq \frac{\Lambda\mathbb{C}\cdot\omega_{jk}}{\text{3-term relation}}$$ ← no differential! formal cdga.

Q: What about $\mathrm{Conf}_n(\mathbb{R}^N)$? Consider $\theta_{ij} : \mathrm{Conf}_n(\mathbb{R}^N) \to S^{N-1}$, define

rel. dirn $i \to j$

$$\omega_{ij} := \theta_{ij}^* \mathrm{Vol}$$

Problem: 3-term relation is only true up to a coboundary term

$$\omega_{ij}\wedge\omega_{jk} + \omega_{k\ell} + \omega\wedge\omega = \boxed{d\beta}$$ ↩ Q: How to find this β?

Consider

$$\begin{tikzcd} FM_N(4) \arrow[d, "p"] \\ FM_N(3) \end{tikzcd}$$

← Fibration that forgets the 4th point

Fiber of this map is $\overline{\mathbb{R}}^N \setminus \bigsqcup_3 D$

$$\partial(\text{Fiber}) = \bigsqcup_3 S^{N-1} \quad \sqcup S^{N-1}$$

↑ when z_4 becomes inf. close to z_1, z_2, z_3

↑ when $z_4 \to \infty$

Study $\Omega^\bullet(FM_N(4))$ by integrating along the fiber

$$\int_F = p_* : \Omega^{3N-3}(FM_N(4)) \to \Omega^{2N-3}(FM_N(3))$$

Stokes thm: $d\left(\int_F \alpha\right) = \cancel{\int_F d\alpha} + \int_{\partial F} \alpha$

$d(\int_F \alpha) = \cancel{\int_F} d\alpha + \boxed{\int_{\partial F} \alpha}$

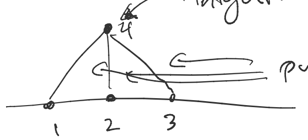
$\Rightarrow \alpha = \omega_{14} \wedge \omega_{24} \wedge \omega_{34}$. $\rightsquigarrow d\alpha = 0$

LHS of 3-term relation

$\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12}$

$\Rightarrow$ can take $\beta = \int_F \omega_{14} \wedge \omega_{24} \wedge \omega_{34}$.

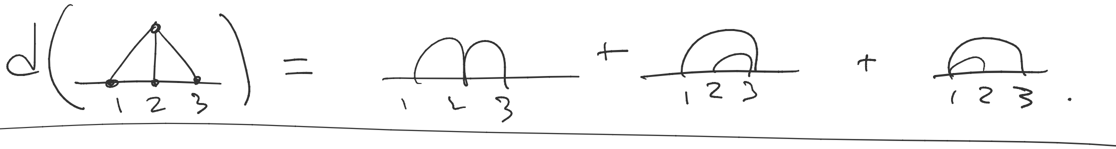
Diagrammatically: β



integrate interior term

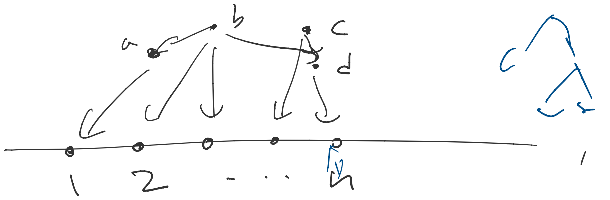
put ω_{ij} on each edge

$\rightsquigarrow$ relation



This strategy will produce an equivalence between $C^\bullet(FM_N)$ and a combinatorial coöperad $\boxed{Feyn_N} \simeq coP_N$

Spanned by admissible graphs Γ, which have n external vertices, and a poset of internal vertices I_Γ, edges E_Γ



with differential

$$d(\Gamma) = \sum_e \pm \Gamma'_e$$

graph obtained by contracting edge e

edges which are not (i) loop
(ii) connecting 2 external vertices
(iii) ending on internal univalent vertex

Feynman integral map

$I : Feyn_N \longrightarrow \Omega^\bullet(FM_N)$

"Put propagators on edges, and integrate over fibres of 'forget points' map."

quasi-iso of operads in cdga's

(Alexeev Torossian associator)

$$\boxed{coP_N \simeq Feyn_N \xrightarrow[\sim]{I} \Omega^\bullet(FM_N) \simeq C^\bullet(FM_N) \simeq C^\bullet(E_N)}$$

Q: For E_2, also the KZ associator ... (?)

Physics: P_n-algebra = functions on an $(n-1)$-symp. mfld X ⇝ target for an n-dim QFT, the theory of maps $\mathrm{Maps}(\Sigma^n, X)$.

(Usually, we linearize this problem by studying formal nbhd of the constant map.)

In this situation, framework for perturbative quantization [Costello]

~~That the~~ The fact that we can compute Feynman integrals via $\overline{C_n(\mathbb{R}^n)}$ relates to the fact that renormalization works nicely for topological theories.