Kontsevich Formality, I*

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The story today will take place over k a field of characteristic zero. Here we tell the algebro-geometric story; when k is $\mathbb{R}$ or $\mathbb{C}$ there is also an analytic version of the story, and the proof that we summarize here works in each of these contexts.

Let us recall what a Poisson algebra is (for us Poisson := commutative Poisson).

Definition 0.0.1. A Poisson ($\mathbb{P}_1$)-algebra in $\mathrm{Vect}^{\heartsuit}$ is a commutative algebra P equipped with an additional Lie bracket $\{-,-\}$ such that $\{p,-\}$ is a derivation for the product structure for all $p \in P$.

One place where one gets Poisson algebra is by taking the "$\hbar \to 0$" limit of non-commutative deformations of commutative algebras. For the rest of this subsection, we fix a *commutative* algebra $A \in \mathrm{Vect}^{\heartsuit}$.

Definition 0.0.2. A *formal associative deformation* of A (over $k[\![\hbar]\!]$) is an *associative* algebra $A' \in k[\![\hbar]\!]\text{-mod}^{\heartsuit}$ and an augmentation $A/(\hbar) \simeq A$ as associative algebras. A *gauge equivalence* is a $k[\![\hbar]\!]$-linear automorphism that is compatible with augmentation.

Remark 0.0.1. Subtle point: we are doing the algebraic case here (c.f. [Yek12, A.5]); if doing C^∞-case, should use the word "differential gauge equivalence" everywhere instead.

As we'll see later, it is most natural to consider such deformations up to gauge equivalences. As a mere vector spaces we have $A' \simeq A[\![\hbar]\!]$, so it makes sense to talk about $f \in A$ as an element in A'. It is a quick exercise to show that, up to a gauge transformation, we can make

$$\{f, g\} := \frac{1}{\hbar}(f \star g - g \star f)|_{\hbar=0}$$

(where $\star$ is the associative product on A') anti-symmetric, and in such case it will make $(A, \{-,-\})$ into a Poisson algebra. This gives a map of sets

$$(\text{formal associative deformations of } A)/(\text{gauge equivalences}) \to (\text{Poisson brackets on } A)$$

The *local* question of deformation quantization is to construct a (ideally explicit) section of this map (which in particular means this map is surjective).

Definition 0.0.3. A *formal Poisson deformation* of A (over $k[\![\hbar]\!]$) is a $k[\![\hbar]\!]$-Poisson algebra A' whose underlying commutative structure is $A[\![\hbar]\!]$ and comes equipped with an augmentation $A'/(\hbar) \simeq A$ as commutative algebras. A *gauge equivalence* is same as above.

Theorem 0.0.1 ([Kon97])**.** *When A is a polynomial algebra, the gauge-equivalence classes of formal associative deformations of A has a canonical bijection with gauge-equivalence classes of formal Poisson deformations of A.*

Remark 0.0.2. In fact, this holds for any smooth A, but the proof we present here is only valid for polynomial algebras. See [DTT06] for a proof for all smooth algebras.

This answers the section question raised above, because given a Poisson bracket $\{-,-\}$, there is a canonical Poisson deformation $(A[\![\hbar]\!], \hbar\{-,-\})$. There is also the *global* question, but a satisfying answer takes a bit more word to describe so we push it to the end of this document.

*If you were like me who tried to look for "Deformation quantization of Poisson manifolds, II", know that it doesn't exist.

0.1 Deformations

The two deformation problems described above (formal associative / Poisson deformations) are examples of *deformation functors*. These are the classical shadows of *formal moduli problems*, which can be found in e.g. [Lur11].

Definition 0.1.1. A functor $F : \mathrm{Art}_k \to \mathrm{Set}$ is a *deformation functor* if, for every $B \to A, C \to A$ maps of local Artinian algebras, and $\alpha : F(B \times_A C) \to F(B) \times_{F(A)} F(C)$, α is surjective when $B \to A$ is surjective, α is an isomorphism when $A = k$, and $F(k) = \{*\}$.

Remark 0.1.1. For convergence reasons deformation problems are only to be defined over Artinian rings, but the extension to pro-Artinian rings (in particular $k[\![\hbar]\!]$ is completely formal).

Now we introduce another player: dg Lie algebras. These are, by definition, chain complexes over k equipped with a differential and a bracket that satisfies the graded version of Lie algebra axioms. The guiding philosophy, usually credited to Deligne, is that "deformation problems are controlled by dg Lie algebras." This is, in some very sketchy sense, an extremely beefed-up version of the LieAlg-LieGroup correspondence.

Remark 0.1.2. The use of exponential means naïvely the formulation only works well in char 0. To work in positive characteristic one has to switch to *partition* Lie algebras of [BM19].

The full formalization of this philosophy would have to wait until [Pri07] and [Lur11] because it is inherently homotopical. Nevertheless, one direction can be easily described: to each dg Lie algebra L, we can attach a deformation problem Def_L. Here's the definition:

Definition 0.1.2. For a dg Lie algebra (L, d), the set $\mathrm{MC}(L)$ of Maurer-Cartan elements is defined as the set of elements $a \in L^1$ that satisfies $da + \dfrac{1}{2}[a, a] = 0$. If L^0 is a nilpotent Lie subalgebra, then the exponential of adjoint action ("gauge equivalence") of L^0 preserves solution to the MC equation. We define

$$\mathrm{Def}_L(R) := \mathrm{MC}(L \otimes \mathfrak{m}_R)/\exp(\mathrm{ad}(L^0 \otimes \mathfrak{m}_R)),$$

where $\mathfrak{m}_R$ is the maximal ideal of R. The quotient of gauge equivalence is roughly speaking quotienting out by paths in the Maurer-Cartan space, c.f. [Man05].

Example 0.1.1. To each associative A, we have the *Hochschild center* of A, i.e.

$$\mathrm{HC}^\bullet(A) := \mathrm{Ext}_{(A \otimes A^{\mathrm{op}})\text{-mod}}(A, A).$$

When A is classical, this can be represented by the Hochschild cochain complex, i.e. $(\mathrm{Hom}_k(A^{\otimes n}, A), d_H)$ where d_H is [DTT08, Formula 3.8]. On this complex we can put a *Gerstenhaber* bracket $[]_G$ [DTT08, Formula 3.14], such that $(\mathrm{HC}^\bullet(A)[1], []_G)$ is a dg Lie algebra.

When A is commutative, the corresponding deformation functor valued on R gives the set of gauge equivalence classes of formal associative deformations of A over R. The assignment goes as follows: let mult be the product on $A \otimes \mathfrak{m}_R$, and let $B \in \mathrm{HC}^2(A \otimes \mathfrak{m}_R)$, then $(\mathrm{mult}+B)$ associative iff $[\mathrm{mult}+B, \mathrm{mult}+B]_G = 0$ iff B is an MC element.

Warning 0.1.1. For general associative A, this is *not* the dg Lie algebra controlling deformation of A as an associative algebra; that should be its tangent complex. In fact, the dg Lie algebra T_A is not formal in general (see [DTT08, Section 6]) The difference between these two is A itself as a Lie algebra (this is explained in [Fra11].

Example 0.1.2. To each smooth commutative A, we have the complex of *polyvector fields* of A, i.e.

$$\mathrm{Pol}^\bullet(A) := (\mathrm{Sym}_A(T_A[-1]), 0).$$

We can give it the Schouten-Nijenhuis bracket (i.e. $[v_1, v_2]_{\mathrm{SN}} = [v_1, v_2], [v, f]_{\mathrm{SN}} = v(f), [f, g]_{\mathrm{SN}} = 0$ and uniquely extend by Leibniz rule), which makes $(\mathrm{Pol}(A)[1], []_{\mathrm{SN}})$ again a dg Lie algebra.

This dg Lie algebra controls formal Poisson deformations over R. The assignment is as follows: if $B \in \mathrm{Pol}^2(A \otimes \mathfrak{m}_R)$, then $\{f, g\} := \langle B, df \wedge dg \rangle$ is a Poisson bracket iff $[B, B]_{\mathrm{SN}} = 0$ iff B is a MC element.

It is easy to check that Def_L is preserved under quasi-isomorphism of dg Lie algebras; however, there seem to be no direct quasi-isomorphism between the two dg Lie algebras described above.

0.2 Entering DG World

At this point it is no longer reasonable to stay in the abelian world. We refer readers to [LV12] for terms not defined here.

Definition 0.2.1. A (symmetric) *dg operad* is an associative algebra within the category of symmetric sequences of chain complexes over k.

The definition above is intentionally useless; here's what it entails. A dg operad $\mathcal{P}$ is a family of chain complexes $\mathcal{P}(n)$ for $n \geq 1$, such that $\mathcal{P}(n)$ is equipped with an action of S_n, and we have composition maps

$$\mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \ldots \otimes \mathcal{P}(k_n) \to \mathcal{P}(k_1 + \ldots + k_n)$$

that satisfy various associativity and equivariance conditions.

Definition 0.2.2. For every chain complex V we have an endomorphism dg operad $\mathrm{End}(V)$ given by $\mathrm{End}(V)(n) := \mathrm{Ext}(V^{\otimes n}, V)$. A $\mathcal{P}$-algebra structure on V is a map of dg operads $\mathcal{P} \to \mathrm{End}(V)$.

Example 0.2.1. Lie is the free operad generated by 1 element b in arity 2 and cohomological degree 0, quotient by the relation $b + b \circ \sigma_2$ and $b \circ b_{23} \circ (\mathrm{id} + \zeta + \zeta^2)$, where ζ is the rotating generator of S^3. The dimension of $\mathsf{Lie}(n)$ is $(n-1)!$. Algebras over this operad are precisely Lie operads.

Example 0.2.2. Ger is the free operad generated by two elements m and b in arity 2, where m is in cohomological degree 0 and b is in degree -1, quotient by the relations $b + b \circ \sigma_2$, $b_{12} \circ b_{23} \circ (\mathrm{id} + \zeta + \zeta^2)$, $m \circ \sigma_2 = m$, $b \circ \sigma_2 = b$, $m \circ m_{23} = m \circ m_{12}$, $b \circ m_{12} - m \circ b_{23} = m \circ b_{12} \circ \sigma_{23}$. Algebras over this operad (usually known as *Gerstenhaber* algebras or $\mathbb{P}_2$ algebras) are Poisson algebras whose Lie bracket has cohomological degree -1.

For any dg operad $\mathcal{P}$, the category $\mathcal{P}$-alg is equipped with a model category structure [LV12, B.6.5], whose weak equivalences are quasi-iso and whose fibrations are degree-wise epi. The category Op of dg-operads itself also has a model structure (B.6.3 of *loc.cit.*), whose weak equivalences are arity-wise quasi-iso and whose fibrations are arity-wise, degree-wise epi. Chapter 11 of *loc.cit.* tells us that the homotopy actegory of $\mathcal{P}$-algebras remains the same when we use a cofibrant replacement of $\mathcal{P}$, and it is independent of which cofibrant replacement we choose.

Remark 0.2.1. Many algebras in real life—including the ones we consider here—do not admit a natural dg $\mathcal{P}$-algebra structure, but only an algebra structure for some cofibrant replacement of $\mathcal{P}$. Even though the rectification result mentioned above tells us that said algebra is *quasi-isomorphic* to a dg $\mathcal{P}$-algebra, this quasi-iso is usually completely unwieldy.

Cofibrant replacement is not something mortals are entitled to, but here's a miracle: (see [LV12, Chapter 6 and 7])

Proposition 0.2.1. *Suppose $\mathcal{P}$ is a* Koszul *operad (c.f. [GK07, p. 4.1.3]). Then $\mathcal{P}_\infty := \Omega\mathcal{P}^{\mathsf{i}}$, the cobar of the Koszul dual of $\mathcal{P}$, is a cofibrant replacement of $\mathcal{P}$.*

How does one explicitly access the data of an $\mathcal{P}_\infty$-algebra structure, and more importantly, how to construct an isomorphism of $\mathcal{P}_\infty$-algebras? The main tool is the following:

Proposition 0.2.2 (Rosetta Stone)**.** *A $\mathcal{P}_\infty$-algebra structure on A is equivalently:*

- *A twisted morphism of dg operads from $\mathcal{P}^{\mathsf{i}}$ to* $\mathrm{End}(A)$*;*
- *A map of dg cooperads from $\mathcal{P}^{\mathsf{i}}$ to* $\mathrm{Bar}\,\mathrm{End}(A)$*;*
- *A codifferential on $\mathcal{P}^{\mathsf{i}}(A)$, the $\mathcal{P}$-coBar complex of A.*

For us all we care is the last one. Since the coBar complex is always cofree, it follows from unpacking definition that a $\mathcal{P}_\infty$-algebra structure on A is the *data* of a Vect-morphism $\mathcal{P}^{\mathsf{i}}(A) \to A$ that satisfy *conditions*, and a morphism of $\mathcal{P}_\infty$-algebras $A \to B$ is the *data* of a Vect-morphism $\mathcal{P}^{\mathsf{i}}(A) \to B$ that satisfy *conditions*.

Example 0.2.3. We set $L_\infty := \mathsf{Lie}_\infty$. A L_∞-algebra is a graded vector space V equipped with operations $l_n : \wedge^n V \to V[2-n]$ for $n > 0$, such that if we extend each l_n to a coderivation ∂_n on $\bigwedge V$, then the total coderivation $\partial := \sum_{n>0} \partial_n : \bigwedge V \to \bigwedge V$ satisfies $\partial^2 = 0$. Equivalently, this means that for any $x_1, \ldots, x_n \in L$ we have

$$\sum_{k=1}^{n} (-1)^{n-k} \sigma_{\sigma \in \mathrm{Unsh}(k,n-k)} (-1)^{|\sigma|} l_{n-k+1}(l_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), x_{\sigma(k+1)}, \ldots, x_{\sigma(n)}) = 0.$$

where $\mathrm{Unsh}(k, n-k)$ are $(k, n-k)$-unshuffles i.e. permutations σ such that $\sigma(i) < \sigma(i+1)$ for $i \neq k$. A L_∞-morphism between $(V, \{l_n\})$ and $(W, \{r_n\})$ is a sequence of maps $f_n : \wedge^n V \to W[1-n]$ such that

$$\sum_{p+q=n+1} \sum_{\sigma \in \mathrm{Unsh}(q,n-q)} (-1)^{|\sigma|} f_p(l_q(x_{\sigma(1)}, \ldots, x_{\sigma(q)}), x_{\sigma(q+1)}, \ldots, x_{\sigma(n)}) =$$

$$\sum_{k_1+\cdots+k_j=n}^{k_i \geq 1} \sum_{\sigma \in \mathrm{Unsh}(k_1,\ldots,k_j)} (-1)^{|\sigma|} r_j(f_{k_1}(x_{\sigma(1)}, \ldots, x_{\sigma(k_1)}), \ldots, f_{k_j}(x_{\sigma(n-k_j+1)}, \ldots, x_{\sigma(n)})),$$

where $\mathrm{Unsh}(k_1, \ldots, k_j)$ are unshuffles that preserve each block $[1, k_1 - 1], [k_1 + 1, k_2 - 1]$, etc.

A usual dg Lie algebra is just a L_∞-algebra for which $l_1 = d, l_2 = []$ and $l_n = 0$ for $n > 2$, but note that there might be highly nontrivial L_∞-morphisms between dg Lie algebras.

Example 0.2.4. We set $G_\infty := \mathsf{Ger}_\infty$. It is given by a graded vector space V and operations

$$m^{p_1,\ldots,p_n} : \bigwedge_{i=1}^{n} \wedge^{p_i} V \to V[3 - \sum_i p_i - n]$$

such that ∂ as defined above (where we range over all $n > 0$ and all multi-indices $P = (p_1, \ldots, p_n)$) is a codifferential. The exact formulas are too unwieldy to be written down. Usual Gerstenhaber algebras can be made into G_∞ algebras by setting $m^1 = d, m^{1,1} = [-,-]$ and $m^2 = \mathrm{mult}$.

The key fact here is that Def_L is dependent on L only homotopically; that is,

Theorem 0.2.1. *Every L_∞-isomorphism $L_1 \to L_2$ between dg Lie algebras induces an isomorphism of sets* $\mathrm{Def}_{L_1} \to \mathrm{Def}_{L_2}$.

0.3 Equivalence of Tangents

So, if we had an L_∞-equivalence between $\mathrm{HC}^\bullet(A)[1]$ and $\mathrm{Pol}(A)[1]$, we would be good to go. Recall that there is indeed a HKR map $\mathrm{Pol}(A) \to \mathrm{HC}^\bullet(A)$ that induces isomorphism on cohomologies:

$$\mathrm{HKR}(v_1 \wedge \ldots \wedge v_m)(f_1 \otimes \ldots \otimes f_m) = \frac{1}{m!} \sum_{\sigma \in S_m} v_{\sigma_1}(f_1) \ldots v_{\sigma_m}(f_m)$$

However, this map does not respect the dg Lie algebra structure: indeed, the Gerstenhaber bracket of images of two bivector fields is not necessarily skew-symmetric. (c.f. [Esp14, p. 2.6.2]). Nevertheless, this discrepancy becomes invisible on the level of cohomologies.

There are two solutions to this: that of Kontsevich, and that of Tamarkin. Kontsevich's solution is to directly write down (when A is a polynomial algebra) a completely explicit L_∞ map that 1) has HKR as its linear component, and 2) actually becomes an L_∞-isomorphism. This correction has an interpretation as twisting by the square root of the Todd class (c.f. [CB07b]), akin to what appears in the Duflo isomorphism. This approach has the advantage of being completely explicit, so one can actually write down the resulting star products; see [Kon97] for the polynomial case, and [CFT00] for the global case. It also has a physical explanation [CF99] which hopefully we'll talk about next week.

Tamarkin's solution goes down a different path which we follow. To begin with, note that $\mathrm{Pol}(A)$ has (by definition) a symmetric product, and this product in fact combines with the SN bracket to form a Gerstenhaber algebra. On the other hand, on $\mathrm{HC}^\bullet(A)$ there is a cup product [DTT08, Formula 3.13]. It is homotopy commutative (i.e. $\mathbb{E}_\infty$), but not commutative in the dg sense; nevertheless, the failure of commutativity again disappears in cohomology, and it induces a graded Ger-structure on $\mathbb{H}(\mathrm{HC}^\bullet(A))$ that is isomorphic to $\mathrm{Pol}(A)$, extending the HKR isomorphism above.

Tamarkin's strategy relies on the following fact:

Claim 0.3.1. $\mathrm{HC}^\bullet(A)$*, with the cup product and bracket above, can be upgraded to a* G_∞*-structure.*

This is not at all obvious, and relies on the following two deep facts:

Theorem 0.3.1 (Deligne Conjecture)**.** $\mathrm{HC}^\bullet(A)$ *is an algebra over the* $\mathbb{E}_2$*-operad.*

We take this as an atomic fact today. See [Kon99, Section 2.4] for a history of its proof (basically people all got it wrong initially); note that this claim is a purely homotopical fact, i.e. is true even in the unstable world. The next one, however, is not:

Theorem 0.3.2 ($\mathbb{E}_n$ Formality)**.** *Over a field of char 0, we have a homotopy equivalence of operads* $\mathbb{E}_2 \simeq G_\infty$*.*

This is what we will discuss next time.

Let's see how to use this to get the claim above. Since $\mathrm{HC}^\bullet(A)$ is an $\mathbb{E}_2$-algebra, its cohomology has a graded $\mathbb{H}(\mathbb{E}_2) \simeq \mathsf{Ger}$-algebra structure, which we already observed as above is $\mathrm{Pol}(A)$ equipped with the SN bracket. On the other hand, by formality, $\mathrm{HC}^\bullet(A)$ *itself* has a $\mathbb{E}_2 \simeq G_\infty$-algebra structure. A subtle point is that the latter structure depends on the choie of an associator so is a priori not unique (even homotopically), but it turns out ([DTT06, Theorem 2]), due to essentially degree reasons,

Lemma 0.3.1. *No matter which associator was chosen, the resulting* L_∞ *part of the structure is always the one we described above (with* d_H *and* $[]_G$*).*

So we would win if we can show that these two G_∞-algebra structures coincide. In fact this is true for all regular algebras A (c.f. [DTT06]), but in the case of polynomial ring there is a faster obstruction-theoretic proof:

Proposition 0.3.1 ([Hin00])**.** *The graded* Ger*-algebra* $\mathrm{Pol}(A)$ *is* intrinsically formal*; that is, for any* G_∞*-algebra* B*, any graded* Ger*-equivalence* $\mathbb{H}(B) \simeq \mathrm{Pol}(A)$ *lifts to an* G_∞*-equivalence* $B \simeq \mathrm{Pol}(A)$*.*

The upgrade to G_∞ here is what makes things work: indeed, $\mathrm{Pol}(A)$ is not intrinsically formal as an L_∞-algebra. This follows from a general intrinsic formality criterion for general dg operads, proven in 4.1.3 of *loc.cit.*. The verification of this criterion is done in 5.4 of *loc.cit.* and is a non-trivial computation; the collapse of the spectral sequence requires in particular $H_{\mathrm{dR}}(A) = k$.

Remark 0.3.1. So we see that in Tamarkin's approach, the L_∞-isomorphism that we wanted actually is part of a G_∞-isomorphism. What about Kontsevich's map? Can it be upgraded to an G_∞-isomorphism? This bothered the community for a while, but now has an answer, thanks to [SW09] and [Wil11]. More precisely: the two approaches are homotopic, if we use the Alekseev-Torossian associator in the next lecture.

0.4 The Global Story

Let X be a classical smooth variety. We can attempt to globalize our question in the following way:

Definition 0.4.1. A *formal Poisson deformation* of $\mathcal{O}_X$ is a sheaf $\mathcal{A}$ of flat, $\hbar$-adically complete $k[\![\hbar]\!]$-Poisson algebras on X, whose underlying commutative structure is $\mathcal{O}_X[\![\hbar]\!]$ and comes equipped with an augmentation $\mathcal{A}/(\hbar) \simeq \mathcal{O}_X$ as sheaf of commutative algebras. *Gauge equivalences* and *formal associative deformations* of $\mathcal{O}_X$ are defined analogously.

These are deformation problems. One can then ask, are these the same deformation problems?

Very roughly speaking, the idea of "deformation problem = Lie algebra" explained above takes place locally over X. In today's language, this is saying that a pointed X-formal moduli problem is controlled by an element in $\mathrm{LieAlg}(\mathrm{QCoh}(X))$.

Remark 0.4.1. Traditional accounts of global quantization were done in the language of "formal geometry" of Gelfand-Kazhdan (also called "D-geometry" or "Gelfand-Fuchs trick") involving the jet bundle of X; examples include [Kon97], [Yek03], [BK03] and [Ber06]. The relationship between this and X-formal moduli problems is described (or at least can be extracted) from the last chapter of [GR20].

Thus, granted that our X is smooth, we can break the job into two parts:

- Give an equivalence between the two sheaves of Lie algebras controlling the "localized" deformation problems; and
- Worry about how to patch the "localized" deformation problems together.

The first question has a satisfying answer, with the two controllers being $D_{\mathrm{poly}}(X) := U_{\mathrm{HopfAlgebroid}}(T_X)$ and $T_{\mathrm{poly}}(X) := \mathrm{Sym}_{\mathcal{O}_X}(T_X[-1])$ respectively. By [Cal04] and [CB07a], these are equivalent as Lie algebroids and even as sheaves of G_∞-algebras.

The second part is easier done (e.g. [CFT00]) in the analytical setting, but in algebraic setting there are cohomological obstructions. As observed in [Kon01] (and realized in [Yek09]), the general situation will require the twist of a *gauge gerbe*. In some situations e.g. when X is D-affine [Yek03], the cohomological obstructions vanish and one can obtain global deformation quantization of $\mathcal{O}_X$ as a sheaf of $\mathbb{E}_1$-algebras on X.

Action on Hochschild Chains The (L_∞-algebra, L_∞-module) pair of Hochschild *cochains* acting on Hochschild *chains* also has a formality statement. I'll probably not say anything about this, just leaving some references here: [CRB09], [DTT08] (the keyword to search is "Tsygan formality").

References

[Ber06] Michel Van den Bergh. *On global deformation quantization in the algebraic case*. 2006. eprint: `arXiv:math/0603200`.

[BK03] R. Bezrukavnikov and D. Kaledin. *Fedosov quantization in algebraic context*. 2003. eprint: `arXiv:math/0309290`.

[BM19] Lukas Brantner and Akhil Mathew. *Deformation Theory and Partition Lie Algebras*. 2019. eprint: `arXiv:1904.07352`.

[Cal04] Damien Calaque. "Formality for Lie algebroids". In: (2004). DOI: `10.1007/s00220-005-1350-5`. eprint: `arXiv:math/0404265`.

[CB07a] Damien Calaque and Michel Van den Bergh. "Global formality at the G_∞-level". In: (2007). eprint: `arXiv:0710.4510`.

[CB07b] Damien Calaque and Michel Van den Bergh. "Hochschild cohomology and Atiyah classes". In: (2007). DOI: `10.1016/j.aim.2010.01.012`. eprint: `arXiv:0708.2725`.

[CF99] Alberto S. Cattaneo and Giovanni Felder. "A path integral approach to the Kontsevich quantization formula". In: (1999). DOI: `10.1007/s002200000229`. eprint: `arXiv:math/9902090`.

[CFT00] Alberto S. Cattaneo, Giovanni Felder, and Lorenzo Tomassini. "From local to global deformation quantization of Poisson manifolds". In: (2000). DOI: `10.1215/S0012-7094-02-11524-5`. eprint: `arXiv:math/0012228`.

[CRB09] Damien Calaque, Carlo A. Rossi, and Michel Van den Bergh. "Caldararu's conjecture and Tsygan's formality". In: (2009). DOI: `10.4007/annals.2012.176.2.4`. eprint: `arXiv:0904.4890`.

[DTT06] Vasiliy Dolgushev, Dmitry Tamarkin, and Boris Tsygan. *The homotopy Gerstenhaber algebra of Hochschild cochains of a regular algebra is formal*. 2006. eprint: arXiv:math/0605141.

[DTT08] V. A. Dolgushev, D. E. Tamarkin, and B. L. Tsygan. "Formality theorems for Hochschild complexes and their applications". In: (2008). DOI: 10.1007/s11005-009-0350-3. eprint: arXiv:0901.0069.

[Esp14] Chiara Esposito. *Formality theory: From poisson structures to deformation quantization*. Vol. 2. Springer, 2014.

[Fra11] John Francis. "The tangent complex and Hochschild cohomology of En-rings". In: (2011). DOI: 10.1112/S0010437X12000140. eprint: arXiv:1104.0181.

[GK07] Victor Ginzburg and Mikhail Kapranov. "Koszul duality for Operads". In: (2007). eprint: arXiv:0709.1228.

[GR20] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry: Volume II: deformations, Lie theory and formal geometry*. Vol. 221. American Mathematical Society, 2020.

[Hin00] Vladimir Hinich. *Tamarkin's proof of Kontsevich formality theorem*. 2000. eprint: arXiv:math/0003052.

[Kon01] M. Kontsevich. "Deformation quantization of algebraic varieties". In: (2001). DOI: 10.1023/A:1017957408559. eprint: arXiv:math/0106006.

[Kon97] Maxim Kontsevich. "Deformation quantization of Poisson manifolds, I". In: (1997). DOI: 10.1023/B:MATH.0000027508.00421.bf. eprint: arXiv:q-alg/9709040.

[Kon99] Maxim Kontsevich. "Operads and Motives in Deformation Quantization". In: (1999). DOI: 10.1023/A:1007555725247. eprint: arXiv:math/9904055.

[Lur11] Jacob Lurie. *Derived Algebraic Geometry X: Formal Moduli Problems*. 2011. URL: http://people.math.harvard.edu/~lurie/papers/DAG-X.pdf.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*. Vol. 346. Springer Science & Business Media, 2012.

[Man05] Marco Manetti. "Deformation theory via differential graded Lie algebras". In: (2005). eprint: arXiv:math/0507284.

[Pri07] J. P. Pridham. "Unifying derived deformation theories". In: (2007). eprint: arXiv:0705.0344.

[SW09] Pavol Severa and Thomas Willwacher. "Equivalence of formalities of the little discs operad". In: (2009). DOI: 10.1215/00127094-1443502. eprint: arXiv:0905.1789.

[Wil11] Thomas Willwacher. "The Homotopy Braces Formality Morphism". In: (2011). DOI: 10.1215/00127094-3450644. eprint: arXiv:1109.3520.

[Yek03] Amnon Yekutieli. *Deformation Quantization in Algebraic Geometry*. 2003. eprint: arXiv:math/0310399.

[Yek09] Amnon Yekutieli. *Twisted Deformation Quantization of Algebraic Varieties*. 2009. eprint: arXiv:0905.0488.

[Yek12] Amnon Yekutieli. "Twisted deformation quantization of algebraic varieties (survey)". In: *Contemp. Math* 562 (2012), pp. 279–297.