Talk X: Nekrasov's Ω-deformation

Let k be an $\mathbf{E}_\infty$-ring. In the previous talk, we gave a preliminary definition of a deformation quantization of a graded k-algebra A_0 as a flat graded associative $k[\![t]\!]$-algebra A equipped with an isomorphism $A/t \cong A_0$ of graded k-algebras, where t is placed in homological degree zero and weight 1. We then amended this definition to define a deformation quantization of a graded k-algebra A_0 as a flat graded associative $k[\![\hbar]\!]$-algebra A equipped with an isomorphism $A/t \cong A_0$ of graded k-algebras, where $\hbar$ is placed in homological degree -2 and weight 1. We also defined the notion of *shearing*, which allowed us to construct a deformation quantization of A_0 over $k[\![\hbar]\!]$ from any $k[\![t]\!]$-deformation of A_0. In this talk, we will justify why it is natural to consider the amended notion of deformation quantization, i.e., why it is natural to place the quantization parameter $\hbar$ in homological degree -2. For the remainder of this talk, we will just set $k = \mathbf{C}$.

We will justify this amendment mathematically by appealing to Bezrukavnikov-Finkelberg's (derived) geometric Satake theorem from [BF08]. For simplicity, let G be a connected and simply-connected algebraic group over $\mathbf{C}$, and let $\mathrm{Gr}_G = G(\!(t)\!)/G[\![t]\!]$ denote the affine Grassmannian[1] of G. There is an action of the ind-group scheme $G[\![t]\!]$ on Gr_G. The classical geometric Satake theorem states that if $\mathcal{P}_{G[\![t]\!]}(\mathrm{Gr}_G)$ is the abelian category of $G[\![t]\!]$-equivariant perverse sheaves on Gr_G (defined appropriately), then there is a symmetric monoidal equivalence between $\mathcal{P}_{G[\![t]\!]}(\mathrm{Gr}_G)$ equipped with the convolution monoidal structure[2] and the category $\mathrm{Rep}(G^\vee)$ of representations of the Langlands dual group $G^\vee$. Bezrukavnikov and Finkelberg proved a derived generalization of this result: namely, they showed that if $\mathrm{DMod}_{G[\![t]\!]}(\mathrm{Gr}_G)$ is the differential graded category of $G[\![t]\!]$-equivariant D-modules on Gr_G (again defined appropriately), then there is an $\mathbf{E}_3$-monoidal structure on $\mathrm{DMod}_{G[\![t]\!]}(\mathrm{Gr}_G)$ and a t-exact $\mathbf{E}_3$-monoidal equivalence $\mathrm{DMod}_{G[\![t]\!]}(\mathrm{Gr}_G) \simeq \mathrm{QCoh}(\mathfrak{g}^{\vee,*}[2]/G^\vee)$. The shift of 2 appearing here is unavoidable: one should view $\mathfrak{g}^{\vee,*}[2]/G^\vee$ as the stack $\mathrm{Loc}_{G^\vee}(S^2)$ of $G^\vee$-local systems on the 2-sphere. Note that the underlying classical stack of $\mathfrak{g}^{\vee,*}[2]/G^\vee$ is just $BG^\vee$, so the heart of the canonically-defined t-structure on $\mathrm{QCoh}(\mathfrak{g}^{\vee,*}[2]/G^\vee)$ is $\mathrm{Rep}(G^\vee)$. Since $\mathcal{P}_{G[\![t]\!]}(\mathrm{Gr}_G)$ is the heart of the t-structure on $\mathrm{DMod}_{G[\![t]\!]}(\mathrm{Gr}_G)$, this recovers the underived geometric Satake theory.

There is a canonical $\mathbf{G}_m$-action on Gr_G, given by loop rotation; this arises via an action of $\mathbf{G}_m$ on $G(\!(t)\!)$ and on $G[\![t]\!]$. One can then define a differential graded category $\mathrm{DMod}_{G[\![t]\!]\rtimes\mathbf{G}_m}(\mathrm{Gr}_G)$ of $G[\![t]\!] \rtimes \mathbf{G}_m$-equivariant D-modules on Gr_G; this category lives over $\mathrm{DMod}_{\mathbf{G}_m}(*) \simeq \mathrm{Mod}_{\mathrm{H}^*_{\mathbf{G}_m}(*;\mathbf{C})} \simeq \mathrm{Mod}_{\mathbf{C}[\![\hbar]\!]}$ with $|\hbar| = -2$. The special fiber (i.e., fiber over $\hbar = 0$) of $\mathrm{DMod}_{G[\![t]\!]\rtimes\mathbf{G}_m}(\mathrm{Gr}_G)$ is $\mathrm{DMod}_{G[\![t]\!]}(\mathrm{Gr}_G)$; in this sense, one can regard $\mathrm{DMod}_{G[\![t]\!]\rtimes\mathbf{G}_m}(\mathrm{Gr}_G)$ as a quantization of $\mathrm{DMod}_{G[\![t]\!]}(\mathrm{Gr}_G)$. Bezrukavnikov and Finkelberg also proved a quantization of the derived geometric Satake theorem: they showed that $\mathrm{DMod}_{G[\![t]\!]\rtimes\mathbf{G}_m}(\mathrm{Gr}_G)$ is equivalent as an $\mathbf{E}_1$-monoidal $\mathbf{C}[\![\hbar]\!]$-linear differential graded category to the differential graded category of $G^\vee$-equivariant $U_\hbar(\mathfrak{g}^\vee)$-modules. Here, $U_\hbar(\mathfrak{g}^\vee)$ is the shearing of the universal enveloping algebra; explicitly, it is the quotient of the free associative $\mathbf{C}[\![\hbar]\!]$-algebra generated by $\mathfrak{g}^\vee[-2]$, subject to the relation $xy - yx = \hbar[x,y]$ for $x, y \in \mathfrak{g}^\vee$. When $\hbar \mapsto 0$, the differential graded category of $G^\vee$-equivariant $U_\hbar(\mathfrak{g}^\vee)$-modules degenerates into $\mathrm{QCoh}(\mathfrak{g}^{\vee,*}[2]/G^\vee)$. We therefore see that $\hbar$ naturally appears in homological degree -2 in this story.

Viewing $\hbar$ as a class in homological degree -2 is also extremely natural from the point of view of physics. In fact, the Koszul duality between S^1-actions and deformation quantizations to $\mathbf{C}[\![\hbar]\!]$ was independently observed by physicists in studying integrable systems. Later, we will discuss a more precise version of this relationship; but today, we will discuss the relationship between Nekrasov's Ω-deformation in 3d $\mathcal{N} = 4$ gauge theory and deformation quantization. We will attempt to use minimal physics in our discussion (which will follow [BBB+20]), since explaining all the concepts involved could form an entire seminar by itself.

Let us begin with a brief review of Poisson brackets appearing in supersymmetric topological field theories. Consider a quantum field theory defined on an d-dimensional Riemannian manifold M. For $\delta > 0$ and $x \in M$, let $\mathrm{Op}_\delta(x)$ denote the $\mathbf{C}$-vector space of states defines on the d-sphere $\partial B_\delta(x)$. If the quantum field theory is topological of Schwarz type, then $\mathrm{Op}_\delta(x)$ does not depend on x or on δ; if the quantum field theory is conformal, then $\mathrm{Op}_\delta(x)$ does not depend on δ. Then, the state-operator correspondence tells us that elements of $\mathrm{Op}_\delta(x)$ can be understood as local observables (centered at x, with support in a ball of radius δ). In a supersymmetric field theory, one further

[1] Note that $\mathrm{Gr}_G(\mathbf{C}) \simeq \Omega G(\mathbf{C})$, so Gr_G may be understood as an algebro-geometric analogue of $\Omega G(\mathbf{C})$. In particular, Gr_G admits an algebraic analogue of the $\mathbf{E}_2$-structure on $\Omega G(\mathbf{C})$.

[2] This can be understood as the convolution tensor product associated to the analogue of the multiplication $\Omega G(\mathbf{C}) \times \Omega G(\mathbf{C}) \to \Omega G(\mathbf{C})$.

has a square-zero operator Q (one of the generators of the super-Poincaré algebra). There is a $\mathbf{Z}/2$-grading on $\mathrm{Op}_\delta(x)$, given by the "fermion number", and denoted $F(-)$. The operator Q has fermion number 1. Finally:

Definition 1. The *topological algebra* $\mathcal{A}_\delta(x)$ is the ($\mathbf{Z}/2$-graded) cohomology $\mathrm{H}^*(\mathrm{Op}_\delta(x); Q)$.

In a topological quantum field theory of Witten type[3], $\mathcal{A}_\delta(x)$ does not depend on δ (almost by definition of "Witten type"). We will only ever consider theories of this type; following [BBB$^+$20], we will often just write $\mathcal{A}(x)$ to denote $\mathcal{A}_\delta(x)$. Similarly, we will write $\mathcal{A}$ to denote the assignment $x \mapsto \mathcal{A}(x)$, viewed as a bundle over M.

Example 2. For instance, if X is an oriented compact Riemannian d-manifold, one can consider 1d $\mathcal{N} = 1$ supersymmetric quantum field theory with target X (also known as supersymmetric quantum mechanics; see [Wit82]). If M is a 1-dimensional manifold, then the assignment $M \ni x \mapsto \mathrm{Op}(x)$ assembles into a cosheaf over M, given by the complexified differential forms $\bigoplus_{i=0}^d \Omega_X^i \otimes \mathbf{C}$; the $\mathbf{Z}/2$-grading/fermion number is just given by $i \pmod 2$. The operator Q can be identified with the de Rham differential, so $\mathcal{A}$ can be identified with the de Rham cohomology of X. However, we will consider TQFTs of dimension ≥ 2 below, so this example cannot be used as a toy model.

Example 3 (3d $\mathcal{N} = 4$ σ-model)**.** Let X be a hyperKähler manifold, so X has a $\mathbf{C}P^1$-family of complex structures. Then one can define a 3d $\mathcal{N} = 4$ σ-model with target X, and each complex structure defines a topological twist of this TQFT (known as Rozansky-Witten theory). Fix a complex structure on X (i.e., an element of $\mathbf{C}P^1$). If M is a 3-manifold, the assignment $M \ni x \mapsto \mathrm{Op}(x)$ assembles into a cosheaf over M, given by the antiholomorphic differential forms $\bigoplus_{i=0}^d \Omega_X^{0,i} \otimes \mathbf{C}$; the $\mathbf{Z}/2$-grading/fermion number is just given by $i \pmod 2$. The operator Q can be identified with the Dolbeault differential $\overline{\partial}$, so $\mathcal{A}$ can be identified with the zeroth Dolbeault cohomology $\mathrm{H}^0(X; \Omega_X^{0,\bullet} \otimes \mathbf{C})$, i.e., the holomorphic sections $\mathrm{H}^\bullet(X; \mathcal{O}_X)$.

To begin relating the story involving $\mathcal{A}(x)$ to deformation quantization, we first observe:

Proposition 4. *Let $\mathcal{A}(x)$ be the topological algebra of a Witten-type TQFT. As $x \in M$ varies, the vector spaces $\mathcal{A}(x)$ assemble into a $\mathbf{Z}/2$-graded locally constant factorization algebra over M, which we will denote $\mathcal{A}$.*

Example 5. If $M = \mathbf{R}^d$ with $d \geq 2$, for instance, then this means that $\mathcal{A}$ admits the structure of a $\mathcal{P}_d$-algebra, i.e., a commutative algebra with a Poisson bracket of homological degree $d-1$. If $M = \mathbf{R}$, then this means that $\mathcal{A}$ admits the structure of an associative algebra.

Example 6. Consider the 3d $\mathcal{N} = 4$ σ-model with hyperKähler target X from Example 3, and fix a complex structure on X. Suppose that, in this complex structure, X is in fact an *affine* $\mathbf{C}$-variety, so that $\mathcal{A} = \mathrm{H}^\bullet(X; \mathcal{O}_X)$ is concentrated in degree 0, where it is just the ring $\mathrm{H}^0(X; \mathcal{O}_X)$ of holomorphic functions on X. If we consider the TQFT on $\mathbf{R}^3$, then the Poisson bracket from Proposition 4 on $\mathrm{H}^0(X; \mathcal{O}_X)$ is then determined by the usual formula $\{f, g\} = \omega(X_f, X_g)$, where ω is the holomorphic symplectic form on X determined by the hyperKähler structure, and X_f and X_g are the holomorphic Hamiltonian vector fields on X determined by the holomorphic functions f and g.

Let us describe the product on $\mathcal{A}$. Consider two points $x, y \in M$, and let $\delta < \|x - y\|/2$, so that $B_\delta(x)$ and $B_\delta(y)$ are disjoint. Let B be a larger ball containing $B_\delta(x)$ and $B_\delta(y)$, and let $\mathcal{A}_B$ denote the topological algebra associated to B. Then the product $\mathcal{A}_\delta(x) \otimes \mathcal{A}_\delta(y) \to \mathcal{A}_B$ sends $\mathcal{O}_1(x) \in \mathcal{A}_\delta(x)$ and $\mathcal{O}_2(y) \in \mathcal{A}_\delta(y)$ to $\mathcal{O}_1(x)\mathcal{O}_2(y)$. Note that this product will generally depend on the points $x, y \in M$.

The more interesting structure is the "Poisson bracket" on $\mathcal{A}$. To describe this, recall that one of the conditions in the definition of a Witten-type TQFT is that the energy-momentum tensor $T_{\mu\nu}$ is Q-exact. This means that there is an operator $G_{\mu\nu}$ of fermionic degree -1 such that $T_{\mu\nu} = [Q, G_{\mu\nu}]$. Define the operator Q_μ to be the integral $i \int G_{0\mu} d^{d-1}x$ on the spatial slice of M, so that $P_\mu = \int T_{0\mu} d^{d-1}x$ is equal to $-i[Q, Q_\mu]$. (Often, Q_μ is included as part of the supersymmetry algebra.)

Let $x \in M$, and let $\mathcal{O}(x) \in \mathrm{Op}(x)$. Write $\mathcal{O}_\mu(x) := Q_\mu \mathcal{O}(x)$, and set $\mathcal{O}^{(1)}(x) = \mathcal{O}_\mu(x) dx^\mu$; then, one should view $\mathcal{O}^{(1)}(x)$ as a 1-form observable on M. Similarly, if we write $\mathcal{O}_{\mu\nu}(x) := \frac{1}{2!} Q_\mu Q_\nu \mathcal{O}(x)$, and set $\mathcal{O}^{(2)}(x) = \mathcal{O}_{\mu\nu}(x) dx^\mu \wedge dx^\nu$, then one should view $\mathcal{O}^{(2)}(x)$ as a 2-form observable on M. This process can be iterated, of course:

[3]Meaning that the action of the TQFT is preserved by the supersymmetry, and that the energy-momentum tensor $T_{\mu\nu}$ is Q-exact.

Definition 7. Define an n-form observable on M by

$$\mathcal{O}^{(n)}(x) = \frac{1}{n!} Q_{\mu_1} \cdots Q_{\mu_n} \mathcal{O}(x) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$$

for any $n \geq 0$. The operators $\mathcal{O}^{(n)}(x)$ are called the *topological descents* of $\mathcal{O}(x)$. If $\gamma \subseteq M$ is an n-chain, define $\mathcal{O}(\gamma) = \int_\gamma \mathcal{O}^{(n)}(x)$, so $\mathcal{O}(\gamma)$ is an operator contained in $\mathrm{Op}_\delta(x)$ if $\gamma \subseteq B_\delta(x)$.

In general, one has the *topological descent equation*

$$Q\mathcal{O}^{(n)}(x) = d_{\mathrm{dR}} \mathcal{O}^{(n-1)}(x); \tag{1}$$

for instance,

$$Q\mathcal{O}^{(1)}(x) = QQ_\mu(\mathcal{O}(x))dx^\mu = iP_\mu(\mathcal{O}(x))dx^\mu = d_{\mathrm{dR}}\mathcal{O}(x),$$

where the final equality used the identity $iP_\mu = \partial_{x_\mu}$. Observe that (1) implies

$$Q\mathcal{O}(\gamma) = \int_\gamma Q\mathcal{O}^{(n)}(x) = \int_{\partial\gamma} \mathcal{O}^{(n-1)} x. \tag{2}$$

If γ is an n-cycle, this integral vanishes, and $\mathcal{O}(\gamma)$ defines a class in $\mathcal{A}$.

We can now define the shifted Poisson bracket on $\mathcal{A}$ as follows. Recall that d is the dimension of our TQFT. Let $\mathcal{O}_1 \in \mathrm{Op}(x)$ and $\mathcal{O}_2 \in \mathrm{Op}(y)$ be two topological operators of the TQFT, and let $\mathrm{Conf}_2(M)$ be the ordered configuration space of two points on M. For each $n \geq 0$, define an n-form valued operator $(\mathcal{O}_1 \boxtimes \mathcal{O}_2)^{(n)}$ on $\mathrm{Conf}_2(M)$ by

$$(\mathcal{O}_1 \boxtimes \mathcal{O}_2)^{(n)}(x, y) = \sum_{i=0}^{n} (-1)^{(n-i)F(\mathcal{O}_1)} \mathcal{O}_1^{(i)}(x) \wedge \mathcal{O}_2^{(n-i)}(y).$$

If Γ is an n-cycle on $\mathrm{Conf}_2(M)$, we can then define

$$(\mathcal{O}_1 \boxtimes \mathcal{O}_2)(\Gamma) = \int_\Gamma (\mathcal{O}_1 \boxtimes \mathcal{O}_2)^{(n)}.$$

This is a topological operator on M. Using Equation (1) as in Equation (2), we see that the class of $(\mathcal{O}_1 \boxtimes \mathcal{O}_2)(\Gamma)$ in $\mathcal{A}$ depends only on the classes of $\mathcal{O}_1$ and $\mathcal{O}_2$, and the homology class of Γ in $\mathrm{H}_n(\mathrm{Conf}_2(M); \mathbf{C})$. It follows that there is a map

$$\mathrm{H}_*(\mathrm{Conf}_2(M); \mathbf{C}) \otimes \mathcal{A}^{\otimes 2} \to \mathcal{A}, \ ([\Gamma], [\mathcal{O}_1], [\mathcal{O}_2]) \mapsto [(\mathcal{O}_1 \boxtimes \mathcal{O}_2)(\Gamma)].$$

This extends in an evident way to an action

$$\mathrm{H}_*(\mathrm{Conf}_n(M); \mathbf{C}) \otimes \mathcal{A}^{\otimes n} \to \mathcal{A}.$$

One can check that this action makes $\mathcal{A}$ into an algebra over the operad $\mathrm{H}_*(\mathrm{Conf}_\bullet(M); \mathbf{C})$, which proves Proposition 4.

We can now describe Nekrasov's Ω-deformation. We will consider the story for the 3d $\mathcal{N} = 4$ σ-model with hyperKähler target X from Example 3. Let us specialize to the case when the spacetime M is of the form $\mathbf{R}^2 \times N$ for some 1-manifold N. Then rotation about $0 \in \mathbf{R}^2$ defines an action of S^1 on M; if x and y are coordinates on $\mathbf{R}^2$, then this S^1-action is generated by the vector field $V := x\partial_y - y\partial_x$.

Definition 8. The Ω-*deformation* is a 1-parameter deformation of the TQFT, where the deformation parameter is $\epsilon \in \mathbf{C}$. The deformation of Op_δ is such that the underlying vector space of $\mathrm{Op}_{\delta,\epsilon}$ is independent of ϵ. Moreover, the Ω-deformed theory is invariant under a supercharge Q_ϵ which satisfies $Q_\epsilon^2 = \epsilon V$ and $Q_\epsilon \xrightarrow{\epsilon \to 0} Q$. Furthermore, if we define $Q_V = \frac{1}{\epsilon}(Q_\epsilon - Q)$ for $\epsilon \neq 0$, then $[Q_V, Q_V] = [Q_\mu, Q_V] = 0$.

For $\epsilon \in \mathbf{C}$, define $\mathrm{Op}_\delta^V \subseteq \mathrm{Op}_\delta$ to be the subspace of operators which are V-invariant. Then, $Q_\epsilon^2 = 0$ on Op_δ^V, and we define $\mathcal{A}_\epsilon := \mathrm{H}^*(\mathrm{Op}_\delta^V; Q_\epsilon)$.

Recall that fields of the 3d σ-model are maps $M = \mathbf{R}^2 \times N \to X$, which can be viewed as a 1d σ-model $N \to \mathrm{Map}(\mathbf{R}^2, X)$. The Ω-deformation can be viewed as a one-parameter deformation of this 1d σ-model; from this point of view, $\mathcal{A}_\epsilon$ is the topological algebra associated to the deformation with parameter ϵ. In particular, if we take $N = \mathbf{R}$, then Proposition 4 tells us that $\mathcal{A}_\epsilon$ is a $\mathbf{Z}/2$-graded associative $\mathbf{C}$-algebra. Furthermore, by construction, $\mathcal{A}_{\epsilon=0} = \mathcal{A}$ is a $\mathbf{Z}/2$-graded commutative $\mathbf{C}$-algebra with a Poisson bracket of degree 1. The main conceptual result involving Ω-deformations is the following:

Theorem 9. *Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{A}$ be V-invariant operators in the topological algebra which admit deformations $\widetilde{\mathcal{O}}_1, \widetilde{\mathcal{O}}_2 \in \mathcal{A}_\epsilon$ to Q_ϵ-closed operators. Define a bracket $\{\widetilde{\mathcal{O}}_1, \widetilde{\mathcal{O}}_2\}_\epsilon$ by*

$$\epsilon\{\widetilde{\mathcal{O}}_1, \widetilde{\mathcal{O}}_2\}_\epsilon = \widetilde{\mathcal{O}}_1 \cdot \widetilde{\mathcal{O}}_2 - \widetilde{\mathcal{O}}_2 \cdot \widetilde{\mathcal{O}}_1.$$

Then $\{\widetilde{\mathcal{O}}_1, \widetilde{\mathcal{O}}_2\}_\epsilon$ converges to the bracket $\{-,-\}$ on $\mathcal{A}$ as $\epsilon \to 0$. In other words, $\mathcal{A}_\epsilon$ is a deformation quantization of $\mathcal{A}$.

Remark 10. What degree does ϵ live in? Assume that the fermionic $\mathbf{Z}/2$-grading on $\mathcal{A}$ lifts to a $\mathbf{Z}$-grading, and suppose that $\mathcal{O}_1, \mathcal{O}_2$ live in degrees i and j. Then the commutator $\widetilde{\mathcal{O}}_1 \cdot \widetilde{\mathcal{O}}_2 - \widetilde{\mathcal{O}}_2 \cdot \widetilde{\mathcal{O}}_1$ lives in degree $i+j$. On the other hand, the bracket $\{\mathcal{O}_1, \mathcal{O}_2\}$ lives in degree $i+j+(3-1) = i+j+2$. Therefore, ϵ must live in degree -2; in the $\mathbf{Z}/2$-grading, it lives in degree 0. In fact, as we will see below, ϵ corresponds to the generator of $\mathrm{H}^2_{S^1}(*; \mathbf{Z}) \cong \mathrm{H}^2(\mathbf{C}P^\infty; \mathbf{Z})$, i.e., to $\hbar$.

Remark 11. Recall from Proposition 4 that we may view $\mathcal{A}$ as a locally constant factorization algebra on $M = \mathbf{R}^2 \times N$. Let us assume for simplicity that $M = \mathbf{R}^3$, and the choice of $\mathbf{R}^2 \subseteq \mathbf{R}^3$ is given by some line $\ell = N \subseteq \mathbf{R}^3$. We can then generalize Theorem 9 as follows: suppose $\mathcal{M}$ is an S^1-equivariant locally constant factorization algebra of $\mathbf{C}$-vector spaces on $\mathbf{R}^3$ (for the S^1-action on $\mathbf{R}^3$ given by rotation about ℓ), and let $\mathcal{M}^{hS^1}$ be the homotopy fixed points. Then: $\mathcal{M}^{hS^1}$ is a locally constant factorization algebra of $\mathbf{C}^{hS^1}$-vector spaces on $(\mathbf{R}^3)^{S^1} = \mathbf{R}$. If we write $\mathbf{C}^{hS^1} = \mathbf{C}[\![\hbar]\!]$ with $|\hbar| = -2$, then the parameter $\hbar$ corresponds to the deformation parameter ϵ in Theorem 9. In other words, $\mathcal{M}^{hS^1}$ is an $\mathbf{E}_1$-$\mathbf{C}[\![\hbar]\!]$-algebra whose special fiber is the $\mathbf{E}_3$-$\mathbf{C}$-algebra $\mathcal{M}$ [4]. This generalization of Theorem 9 is a decategorified version of the main result of Talk IX. Extensive discussion of this topic in line with Talk IX is in [But20a, But20b].

Let us now turn to the proof of Theorem 9. For simplicity, we will just suppose $M = \mathbf{R}^3$, and $\ell = N \subseteq \mathbf{R}^3$ is a line. As indicated in Remark 11, it is natural to study $\mathcal{A}_\epsilon$ as ϵ varies, and where ϵ is viewed as a generator of $\mathrm{H}_2^{S^1}(*; \mathbf{C})$. As ϵ varies, the algebras $\mathcal{A}_\epsilon$ assemble into an associative algebra $\widetilde{\mathcal{A}}$ over $\mathbf{C}[\epsilon]$. Indeed, $\widetilde{\mathcal{A}}$ can be viewed as the homotopy S^1-fixed points of $\mathcal{A}$.

Recall that $\mathrm{Conf}_2(\mathbf{R}^3) \simeq S^2$. The Poisson bracket on $\mathcal{A}$ from Proposition 4 arose via the generator $[S^2] \in \mathrm{H}_2(S^2; \mathbf{C})$. Therefore, the philosophy of Remark 11 suggests that Theorem 9 can potentially be proved by studying the S^1-*equivariant* homology $\mathrm{H}_2^{S^1}(\mathrm{Conf}_2(\mathbf{R}^3); \mathbf{C})$, where S^1 acts on $S^2 \simeq \mathrm{Conf}_2(\mathbf{R}^3)$ by rotations.

Let us begin by listing some elements in $\mathrm{H}_*^{S^1}(S^2; \mathbf{C})$. First, we have the fundamental class $[S^2] \in \mathrm{H}_2^{S^1}(S^2; \mathbf{C})$. Next, the S^1-action on S^2 has two fixed points, namely the north and south poles. These therefore define classes $[p_+], [p_-] \in \mathrm{H}_0^{S^1}(S^2; \mathbf{C})$. Finally, we may view $\mathrm{H}_*^{S^1}(S^2; \mathbf{C})$ as a module over $\mathrm{H}^*_{S^1}(*; \mathbf{C}) \cong \mathbf{C}[\![\epsilon]\!]$ with $|\epsilon| = -2$. We then claim that the following relation holds in $\mathrm{H}_0^{S^1}(S^2; \mathbf{C})$:

$$\epsilon[S^2] = [p_+] - [p_-]. \tag{3}$$

The relation (3) implies Theorem 9. Indeed, as we mentioned, $\widetilde{\mathcal{A}}$ is the homotopy S^1-fixed points of $\mathcal{A}$. This makes $\widetilde{\mathcal{A}}$ into an algebra over the operad $\mathrm{H}_*^{S^1}(\mathrm{Conf}_\bullet(\mathbf{R}^3); \mathbf{C})$. In particular, there is a $\mathbf{C}[\epsilon]$-linear action

$$\mathrm{H}_*^{S^1}(\mathrm{Conf}_2(\mathbf{R}^3); \mathbf{C}) \otimes_{\mathbf{C}[\epsilon]} \widetilde{\mathcal{A}} \otimes_{\mathbf{C}[\epsilon]} \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}.$$

If $\widetilde{\mathcal{O}}_1, \widetilde{\mathcal{O}}_2 \in \widetilde{\mathcal{A}}$, the classes $[p_+]$ and $[p_-] \in \mathrm{H}_0^{S^1}(\mathrm{Conf}_2(\mathbf{R}^3); \mathbf{C})$ correspond to the products $\widetilde{\mathcal{O}}_1 \cdot \widetilde{\mathcal{O}}_2$ and $\widetilde{\mathcal{O}}_2 \cdot \widetilde{\mathcal{O}}_1$. Similarly, the class $\epsilon[S^2] \in \mathrm{H}_0^{S^1}(\mathrm{Conf}_2(\mathbf{R}^3); \mathbf{C})$ corresponds to an operation on $\widetilde{\mathcal{O}}_1$ and $\widetilde{\mathcal{O}}_2$ which degenerates to the Poisson bracket $\{\mathcal{O}_1, \mathcal{O}_2\}$ as $\epsilon \to 0$. Therefore, Equation (3) says that the commutator $\widetilde{\mathcal{O}}_1 \cdot \widetilde{\mathcal{O}}_2 - \widetilde{\mathcal{O}}_2 \cdot \widetilde{\mathcal{O}}_1$ degenerates to the Poisson bracket $\{\mathcal{O}_1, \mathcal{O}_2\}$ as $\epsilon \to 0$, as desired.

Let us now prove Equation (3). This is in fact rather straightforward: choose a chain γ on S^2 by a path $p_- \to p_+$. If ∂ denotes the boundary operator in S^1-equivariant chains on S^2, then

$$\partial\gamma = p_+ - p_- - \epsilon S^2,$$

since the S^1-orbit of γ is the entirety of S^2. This implies Equation (3) in $\mathrm{H}_0^{S^1}(S^2; \mathbf{C})$.

For now, this concludes our discussion of Nekrasov's Ω-deformation. There are a few lingering questions:

[4] Note that the deformation of the $\mathbf{E}_3$-algebra is an $\mathbf{E}_1$-algebra, exactly as in the quantization of the derived geometric Satake correspondence. This is not a surprise: in fact, quantization of the derived geometric Satake correspondence can be explained via the Ω-deformation, as explained in [BBB+20].

- It is not clear why S^1 is so special in the above discussion: for instance, since $\mathbf{C}^{h\mathrm{SU}(2)} \simeq \mathrm{H}^*(B\mathrm{SU}(2);\mathbf{C}) \cong \mathbf{C}[\![w]\!]$ is also a polynomial ring in one variable w with $|w| = -4$, why could we not consider SU(2)-actions instead of S^1-actions in relation to deformation quantization? Based on the above discussion, this would correspond to studying a version of Ω-deformation for a 5d TQFT instead. Similarly, can the results of Talk IX be modified to work for SU(2)-actions instead of S^1-actions?
- The original physical motivation for studying the Ω-deformation came from a relationship between gauge theories and integrable systems. How does this relationship manifest in the above setup?

We will discuss the second bullet in detail later, and briefly discuss the first bullet today. We do not have satisfying answers to either of the questions in the first bullet. However, we can make some preliminary comments. First, since the maximal torus of SU(2) is S^1, and the Weyl group $\mathbf{Z}/2$ acts on S^1 by the antipodal action, it seems reasonable to hope that the results of Talk IX can be generalized to SU(2) using $\mathbf{Z}/2$-equivariant homotopy theory. In a later talk, we will relate Frobenius-constant quantizations to cyclotomic spectra. It seems reasonable to hope that a variant of recent work on "Real cyclotomic spectra" (see [QS19]) could give an SU(2)-analogue of these results. Next, a key point in Talk IX was that k^{hS^1} was the $\mathbf{E}_2$-Koszul dual of $k[\![t]\!]$. Interestingly, a similar result is true for $k^{h\mathrm{SU}(2)}$: it is the $\mathbf{E}_2$-Koszul dual of $k[\sigma] = k \otimes \Omega S^3_+$.

Finally, and perhaps most physically relevant, an analogue of Equation (3) *does* hold in $\mathrm{H}_0^{\mathrm{SU}(2)}(\mathrm{Conf}_2(\mathbf{R}^5);\mathbf{C})$. To describe this, let us identify $\mathrm{Conf}_2(\mathbf{R}^5)$ with the unit sphere S^4 in $\mathbf{H}\oplus\mathbf{R}$, where the SU(2)-action on S^4 comes from the quaternions $\mathbf{H}$. First note that $\mathrm{H}_*^{\mathrm{SU}(2)}(S^4;\mathbf{C}) \cong \mathrm{H}_*^{S^1}(S^4;\mathbf{C})^{\mathbf{Z}/2}$, since the Weyl group is $\mathbf{Z}/2$. We will therefore calculate $\mathrm{H}_*^{S^1}(S^4;\mathbf{C})$. Recall that S^4 can be presented as the quotient of $S^2 \times S^2$ by the wedge $S^2 \vee S^2$; this presentation is S^1-equivariant (but not SU(2)-equivariant, since the Weyl group flips the factors of S^2). Let us denote these two S^2 factors by Σ_1 and Σ_2 (and abusively use the same notation for their homology classes), and let p_+, p_- (resp. q_+, q_-) be the poles of Σ_1 (resp. Σ_2). We know that $\mathrm{H}_*^{S^1}(S^2 \times S^2;\mathbf{C})$ is isomorphic to the tensor square of $\mathrm{H}_*^{S^1}(S^2;\mathbf{C})$ over $\mathrm{H}^*_{S^1}(*;\mathbf{C})$, so (3) holds for both the triples (p_+, p_-, Σ_1) and (q_+, q_-, Σ_2).

To get the S^1-equivariant homology of S^4, we have to impose on $\mathrm{H}_*^{S^1}(S^2 \times S^2;\mathbf{C})$ the relations stemming from quotienting by the subspace $S^2 \vee S^2 \subseteq S^2 \times S^2$. These relations are given by $\Sigma_1 + \Sigma_2 = 0$ and $p_+ = q_-$, and they give us $p_- = -q_+$. Therefore, (3) translates to the relations

$$p_+ - p_- = \hbar\Sigma_1, \; p_+ + p_- = \hbar\Sigma_2.$$

Using the fact that $\mathrm{H}_*^{\mathrm{SU}(2)}(S^4;\mathbf{C})$ is the fixed points of the action of the Weyl group $\mathbf{Z}/2$ on $\mathrm{H}_*^{S^1}(S^4;\mathbf{C})$ (which sends $p_\pm$ to $\mp p_\mp$, $\hbar$ to $-\hbar$, and Σ_1 to Σ_2), we conclude that $\mathrm{H}_*^{\mathrm{SU}(2)}(S^4;\mathbf{C})$ is generated by p_+^2 and p_-^2 subject to the following relation analogous to (3):

$$p_+^2 - p_-^2 = (p_+ + p_-)(p_+ - p_-) = \hbar^2([\Sigma_1]\cdot[\Sigma_2]).$$

Of course, $\hbar^2$ is the generator of $\mathrm{H}^*_{\mathrm{SU}(2)}(*;\mathbf{C})$, and $[\Sigma_1]\cdot[\Sigma_2]$ is the equivariant fundamental class of S^4. It would be interesting to explore this purported "SU(2)-Ω-deformation" in a physical example. (The original paper [NS09] does in fact study a "$S^1 \times S^1$-Ω-deformation", which has been used in ways that I don't understand.)

References

[BBB+20] C. Beem, D. Ben-Zvi, M. Bullimore, T. Dimofte, and A. Neitzke. Secondary products in supersymmetric field theory. *Annales Henri Poincare*, 21(4):1235–1310, 2020.

[BF08] R. Bezrukavnikov and M. Finkelberg. Equivariant Satake category and Kostant-Whittaker reduction. *Mosc. Math. J.*, 8(1):39–72, 183, 2008.

[But20a] D. Butson. Equivariant localization in factorization homology and applications in mathematical physics I: Foundations. `https://arxiv.org/abs/2011.14988`, 2020.

[But20b] D. Butson. Equivariant localization in factorization homology and applications in mathematical physics I: Gauge theory applications. `https://arxiv.org/abs/2011.14978`, 2020.

[NS09] N. Nekrasov and S. Shatashvili. Quantization of Integrable Systems and Four Dimensional Gauge Theories. In *16th International Congress on Mathematical Physics*, pages 265–289, 8 2009.

[QS19] J.D. Quigley and J. Shah. On the parametrized Tate construction and two theories of real p-cyclotomic spectra. `https://arxiv.org/abs/1909.03920`, 2019.

[Wit82] E. Witten. Supersymmetry and Morse theory. *J. Differential Geometry*, 17(4):661–692 (1983), 1982.

Email address: `sdevalapurkar@math.harvard.edu, Summer 2021`