Talk XI: Frobenius-constant quantizations

Recall from Talk IX that if k is an $\mathbf{E}_\infty$-ring and $\mathcal{C}_0$ is a graded k-linear ∞-category, then there is an equivalence of ∞-categories between the ∞-category $\mathrm{Quant}^{\mathrm{gr}}_{\mathcal{C}_0}$ of deformation quantizations of $\mathcal{C}_0$ and $\mathrm{LinCat}^{\mathrm{gr},\simeq}_{k[\epsilon]/\epsilon^2} \times_{\mathrm{LinCat}^{\mathrm{gr},\simeq}_k} \{\mathcal{C}_0\}$. In other words, deformation quantizations of $\mathcal{C}_0$ are Koszul dual to S^1-actions on $\mathcal{C}_0$. Our goal in this talk will be to describe the appropriate modification of this result if k is an $\mathbf{E}_\infty$-ring with a Frobenius and "deformation quantization" is upgraded to "Frobenius-constant quantization". Let us begin by recalling the definition of Frobenius-constant quantizations from [BK08], with the modification that the deformation parameter $\hbar$ lives in homological degree -2 and weight 1 (which is a natural modification, as explained in Talks IX and X).

Definition 1. Let k be a field of characteristic $p > 0$, and let A_0 be a commutative graded k-algebra. A *deformation quantization* of A_0 is a pair (A, α) consisting of an associative $\hbar$-torsionfree graded k-algebra A and an isomorphism $\alpha : A \otimes_{k[\![\hbar]\!]} k \xrightarrow{\cong} A_0$. A *Frobenius-constant quantization* of A_0 is a quantization (A, α) along with a *central* $k[\![\hbar]\!]$-linear "splitting" (algebra) map $\varphi : A_0^{(p)} \to A$ which sends graded weight n to graded weight np, such that $\varphi(a) \equiv a^p \pmod{\hbar^{p-1}}$ for each $a \in A_0$. The map φ will be called the *Frobenius-splitting*.

Remark 2. Note that Definition 1 is describing Frobenius-constant quantizations of commutative k-algebras, as opposed to Frobenius-constant quantizations of commutative k-algebras with a possibly nontrivial Poisson bracket. However, we will not discuss this more general case in this talk for the sake of simplicity.

Example 3. Let $\mathfrak{g}$ be a restricted Lie algebra over k, and let $A_0 = \mathrm{Sym}(\mathfrak{g}[-2])$ where $\mathfrak{g}[-2]$ is placed in weight 1. The "asymptotic" enveloping algebra $A = U_\hbar(\mathfrak{g})$ is generated as an associative k-algebra by k and $\mathfrak{g}[-2]$, subject to the relation $x \otimes y - y \otimes x = \hbar[x, y]$. The $k[\![\hbar]\!]$-algebra A admits the structure of a Frobenius-constant quantization of $\mathrm{Sym}(\mathfrak{g}[-2])$. Indeed, it is clear that $U_\hbar(\mathfrak{g}) \otimes_{k[\![\hbar]\!]} k \cong \mathrm{Sym}(\mathfrak{g}[-2])$. The Frobenius-splitting φ of $U_\hbar(\mathfrak{g})$ is defined by the map $\mathrm{Sym}(\mathfrak{g}^{(p)}[-2]) \to U_\hbar(\mathfrak{g})$ which sends $x \mapsto x^p - \hbar^{p-1}\mathcal{Q}_1(x)$. Note that this map sends graded weight n (which implies homological degree $-2n$, since k and $\mathfrak{g}$ are assumed to be discrete, i.e., concentrated in homological degree zero) to graded weight $np - p + 1$ (i.e., homological degree $2(p-1) - 2np$).

Example 4. Let X be a smooth variety over k, and let T_X be the tangent bundle of X. Let $A_0 = \mathrm{Sym}_{\mathcal{O}_X}(T_X[-2])$, where T_X is placed in weight 1 and homological degree -2. Then the "asymptotic" sheaf $\mathcal{D}_X^\hbar$ of differential operators on X is generated as an associative $\mathcal{O}_X$-algebra by $\mathcal{O}_X$ and $T_X[-2]$, subject to the relation $\partial \otimes \partial' - \partial' \otimes \partial = \hbar[\partial, \partial']$. The associative algebra $\mathcal{D}_X^\hbar$ admits the structure of a Frobenius-constant quantization of $\mathrm{Sym}_{\mathcal{O}_X}(T_X[-2])$. Indeed, it is clear that $\mathcal{D}_X^\hbar/\hbar \cong \mathrm{Sym}_{\mathcal{O}_X}(T_X[-2])$. The Frobenius-splitting of $\mathcal{D}_X^\hbar$ is defined by the map $\Psi : \mathrm{Sym}_{\mathcal{O}_{X^{(p)}}}(T_{X^{(p)}}[-2]) \to \mathrm{Frob}_* \mathcal{D}_X^\hbar$ which sends $\partial \mapsto \partial^p - \hbar^{p-1}\partial^{[p]}$. (Note that this map sends graded weight n to graded weight $np - p + 1$, and hence homological degree $-2n$ to homological degree $2(p-1) - 2np$. The failure to preserve homological degree can be remedied by multiplication by a large enough *negative* power of $\hbar$, but this would require inverting $\hbar$.) We will return to this example later when we discuss p-curvature.

Remark 5. The condition that φ be central is extremely important, and is the most nontrivial component of Definition 1. In many cases, the ring $A_0^{(p)}$ is closely related to the center $Z(A)$ of a Frobenius-constant quantization. For instance, in Example 3, the center of $U_\hbar(\mathfrak{g})$ can be identified with $\mathrm{Sym}(\mathfrak{g}^{(p)}[-2]) \otimes_{\mathrm{Sym}(\mathfrak{g}^{(p)}[-2])^G} \mathrm{Sym}(\mathfrak{g}[-2])^G$ (i.e., the ring of functions of $\mathfrak{g}^{*,(p)}[2] \times_{\mathfrak{g}^{*,(p)}[2]/\!\!/G} \mathfrak{g}^*[2]/\!\!/G$), where $\mathrm{Sym}(\mathfrak{g}^{(p)}[-2])$ is the "p-center" of $U_\hbar(\mathfrak{g})$, and $\mathrm{Sym}(\mathfrak{g}[-2])^G$ is the usual (sheared) Harish-Chandra center. The map $\mathfrak{g}^*[2] \to \mathfrak{g}^{*,(p)}[2]$ is induced by the (Artin-Schreier) map $\mathrm{Sym}(\mathfrak{g}^{(p)}[-2]) \to \mathrm{Sym}(\mathfrak{g}[-2])$ sending $x \mapsto x^p - \hbar^{p-1}\mathcal{Q}_1(x)$. Similarly, in Example 4, the center of $\mathrm{Frob}_*\mathcal{D}_X^\hbar$ can be identified (again upon setting $\hbar = 1$) with $\mathrm{Sym}_{\mathcal{O}_{X^{(p)}}}(T_{X^{(p)}}[-2])$.

Warning 6. In the following discussion, we will abusively assume that $\hbar$ lives in degree zero. This is to stick with the convention in [BK08]. One can either view our discussion in this talk as an "approximation" to the sheared notion from Definition 1; or one can 2-periodify the entire discussion below to allow $\hbar$ to move to degree zero. The latter introduces myriad technical difficulties, so we suggest the reader adopt the first point of view.

In fact, both Example 3 and Example 4 fit into the more general paradigm of Frobenius-constant quantizations of *restricted* Poisson algebras. Recall the following definition from Talk I:

Definition 7. Let k be a field of characteristic $p > 0$, and let A be a commutative k-algebra equipped with a Poisson bracket $\{-,-\}$ and a compatible restricted structure φ (i.e., a *restricted Poisson algebra*). Here, φ is said to be *compatible* with $\{-,-\}$ if

$$\begin{aligned} \mathrm{ad}_{\varphi(x)}(y) &= \mathrm{ad}_x^p(y), \\ \varphi(x+y) &= \varphi(x) + \varphi(y) + \mathrm{ad}_x^{p-1}(y), \\ \varphi(xy) &= \varphi(x)y^p + x^p\varphi(y) + \sum_{0\le i,j\le p, i+j\le p} x^i y^j \Gamma_{i,j}(x,y), \end{aligned} \tag{1}$$

for some rather complicated expression $\Gamma_{i,j}(x,y)$ that does not depend on φ. (For instance, $\Gamma_{1,1}(x,y) = \mathrm{ad}_x(y)^{p-1}$, while for $i \neq 0,p$, $\Gamma_{i,p-i}(x,y)$ is the coefficient of t^{i-1} in the expression $\mathrm{ad}_{tx+y}^{p-1}(x)$.) If A is further equipped with an internal differential d, then we require the Leibniz rule $d(\varphi(x)) = \mathrm{ad}_x^{p-1}(dx)$.

A *Frobenius-constant quantization* of A is a quantization $A_\hbar$ of A equipped with a map Φ which satisfies the first two relations in Equation (1) and the following deformation of the third relation:

$$\Phi(xy) = \varphi(x)y^p + x^p\varphi(y) - \hbar^{p-1}x^{[p]}y^{[p]} + \sum_{0\le i,j\le p, i+j\le p} x^i y^j \Gamma_{i,j}(x,y).$$

Remark 8. The relations in Equation (1) are not as general as the relations in [BK08], but they will suffice for our purposes. In particular, these are the relations that the (first) Dyer-Lashof operation $\mathcal{Q}_1$ satisfies with respect to the Browder bracket in the homotopy of an $\mathbf{E}_2$-$\mathbf{F}_p$-algebra.

Our goal in the remainder of this talk is to prove some results from [BK08]. The main results of [BK08] are of two kinds: two results are about the restricted Poisson story, while two results are about the quantization of restricted Poisson structures. In this talk, our primary focus will be on the Poisson side of the story. In the next talk, we will discuss the relationship between Frobenius-constant quantizations and weak cyclotomic structures, and prove an analogue of the main results of [BK08].

We kick off by posing the following questions:

Question 9. Let X be a symplectic scheme over k with symplectic form ω.

(a) When is the subspace of $T_{X/k}$ consisting of Hamiltonian vector fields closed under the restricted Lie operation $\xi \mapsto \xi^{[p]}$?
(b) The symplectic form ω defines a Poisson structure $\{-,-\}$ on $\mathcal{O}_X$. Can one describe the set of restricted structures φ which are compatible with $\{-,-\}$?

Recall that if $f \in \mathcal{O}_X$, then the associated Hamiltonian vector field $X_f \in T_{X/k}$ is defined as the image of $df \in \Omega^1_{X/k}$ under $\omega : T_{X/k} \xrightarrow{\sim} \Omega^1_{X/k}$. In other words, X_f is the derivation on $\mathcal{O}_X$ given by $g \mapsto \{f,g\}$. For notational distinction, we will now write H_f instead of X_f.

Recall that the sheaf $\Omega^i_{X/k,\mathrm{cl}}$ of closed i-forms on X is the quotient $\ker(d : \Omega^i_{X/k} \to \Omega^{i+1}_{X/k})$, and that the *Cartier map* $\mathfrak{C}$ is a Frobenius-linear map $\Gamma(X; \Omega^i_{X/k,\mathrm{cl}}) \to \Gamma(X^{(p)}; \Omega^i_{X^{(p)}/k})$ from closed forms on X to forms on $X^{(p)}$, which kills exact forms. The map $\mathfrak{C}$ induces an isomorphism $\mathcal{H}^i(X; \Omega^\bullet_{X/k}) \xrightarrow{\sim} \Gamma(X^{(p)}; \Omega^i_{X^{(p)}/k})$, whose inverse is multiplicative with respect to the wedge product, which sends $d(f^{(p)})$ to the class $[f^{p-1}df]$, and which is given by the Frobenius in degree zero. The answer to Question 9(a) is then:

Proposition 10. *The symplectic form ω is killed by the Cartier operator if and only if the subspace of $T_{X/k}$ consisting of Hamiltonian vector fields closed under the restricted Lie operation $\xi \mapsto \xi^{[p]}$.*

Proof. Let α be a closed differential form on X, and let $\partial \in T_{X/k}$. Denote by $\partial^{(p)}$ the corresponding vector field on $X^{(p)}$. In the next talk, we will prove that there is an identity

$$\langle \mathfrak{C}(\alpha), \partial^{(p)} \rangle = \mathfrak{C}\left(\langle \alpha, \partial^{[p]} \rangle - \mathcal{L}_\partial^{p-1} \langle \alpha, \partial \rangle \right), \tag{2}$$

where $\mathcal{L}_\partial$ is the Lie derivative. Let $\alpha = \omega$, and assume that ∂ is the Hamiltonian vector field H_f associated to some $f \in \mathcal{O}_X$. Then, $\langle \omega, H_f \rangle = df$ (by definition), and so $\mathcal{L}_{H_f}^{p-1}\langle \omega, H_f \rangle = d(\mathcal{L}_{H_f}^{p-1}(df))$. In particular, this is exact, and hence killed by $\mathfrak{C}$. It follows that Equation (2) states:

$$\langle \mathfrak{C}(\omega), H_f^{(p)} \rangle = \mathfrak{C}\langle \omega, H_f^{[p]} \rangle.$$

The right-hand side is zero if and only if $\langle \omega, H_f^{[p]} \rangle$ is exact, which happens if and only if $H_f^{(p)}$ is a Hamiltonian vector field. Therefore, Hamiltonian vector fields are closed under the restricted Lie operation if and only if $\langle \mathfrak{C}(\omega), H_f^{(p)} \rangle = 0$ for every function $f \in \mathcal{O}_X$. But ω is nondegenerate, so the sub-$\mathcal{O}_X$-module of Hamiltonian vector fields generates all of $T_{X/k}$. In particular, $\langle \mathfrak{C}(\omega), H_f^{(p)} \rangle = 0$ for every function $f \in \mathcal{O}_X$ if and only if $\mathfrak{C}(\omega) = 0$, as desired. □

Remark 11. The condition that $\mathfrak{C}(\omega) = 0$ is equivalent to ω being locally exact (in the Zariski topology).

Let us now turn to Question 9(b). This really has two components: first, does there exist any restricted structure φ compatible with $\{-,-\}$? If so, what is the set of such? We begin with the second part.

Lemma 12. *Suppose that there exists a restricted structure φ compatible with $\{-,-\}$. Then the set of all restricted structures compatible with $\{-,-\}$ is in bijection with Frobenius-derivations on $\mathcal{O}_X$ which land in the Poisson center of $\mathcal{O}_X$, i.e., derivations $\delta : \mathcal{O}_X \to \mathcal{O}_X$ such that*

$$\delta(fg) = \delta(f)g^p + f^p\delta(g) \tag{3}$$

and $\{\delta(f), g\} = 0$ for all $f, g \in \mathcal{O}_X$.

Proof. The bijection is easy to describe: given another restricted structure φ', let $\delta = \varphi' - \varphi$. Then δ is additive and satisfies Equation (3). Observe that

$$\mathrm{ad}_{\delta(x)}(y) = \mathrm{ad}_{\varphi'(x)}(y) - \mathrm{ad}_{\varphi(x)}(y) = \mathrm{ad}_x^p(y) - \mathrm{ad}_x^p(y) = 0,$$

so δ is indeed a Frobenius-derivation which lands in the Poisson center of $\mathcal{O}_X$. □

Remark 13. What is the Poisson center of $\mathcal{O}_X$? A function $f \in \mathcal{O}_X$ is in the Poisson center if and only if $H_f = 0$, which happens if and only if $df = 0$. This in turn happens if and only if f is in the subalgebra of $\mathcal{O}_X$ spanned by pth powers, i.e., in $\mathcal{O}_{X^{(p)}}$.

It follows from Equation (3) that Question 9(b) boils down to asing when there exists a restricted structure φ that is compatible with $\{-,-\}$.

Lemma 14. *Suppose $\xi : \mathcal{O}_X \to \mathcal{O}_X$ is a derivation such that*

$$\xi(\{f,g\}) = \{\xi(f), g\} + \{f, \xi(g)\} - \{f,g\}, \tag{4}$$

and let $\alpha = \langle \omega, \xi \rangle$. Define a map $\varphi : \mathcal{O}_X \to \mathcal{O}_X$ of sets by

$$\varphi(f) = \mathcal{L}_{H_f}^{p-1}\langle H_f, \alpha\rangle - \langle H_f^{[p]}, \alpha\rangle.$$

Then φ is a restricted structure on $\mathcal{O}_X$ compatible with the Poisson bracket.

Proof. To check compatibility, we just compute that Equation (1) holds. To start, recall from the Cartan formula that

$$d\langle \xi, \beta\rangle = \mathcal{L}_\xi \beta - \langle \xi, d\beta\rangle$$

for any differential form β on X. Therefore, $d\alpha$ is $\mathcal{L}_\xi(\omega) - \langle \xi, d\omega\rangle$. However, $d\omega = 0$, so $d\alpha = \mathcal{L}_\xi(\omega)$. We claim that this is equal to ω itself. Indeed, since the sub-$\mathcal{O}_X$-module of Hamiltonian vector fields generates all of $T_{X/k}$, it suffices to check that $\mathcal{L}_\xi(\omega) = \omega$ on Hamiltonian vector fields. If $f, g \in \mathcal{O}_X$, then

$$\begin{aligned} d\alpha(H_f, H_g) &= \mathcal{L}_\xi(\omega)(H_f, H_g) \\ &= \{\xi(f), g\} + \{f, \xi(g)\} - \xi(\{f,g\}) \\ &= \{f,g\} = \omega(H_f, H_g), \end{aligned}$$

as claimed.

Now observe that

$$\begin{aligned} d\varphi(f) &= d\mathcal{L}_{H_f}^{p-1}\langle H_f, \alpha\rangle - d\langle H_f^{[p]}, \alpha\rangle \\ &= \mathcal{L}_{H_f}^{p-1} d\langle H_f, \alpha\rangle - d\langle H_f^{[p]}, \alpha\rangle \\ &= \mathcal{L}_{H_f}^{p-1}(\mathcal{L}_{H_f}(\alpha) - \langle H_f, d\alpha\rangle) - \mathcal{L}_{H_f^{[p]}}(\alpha) + \langle H_f^{[p]}, d\alpha\rangle \\ &= \mathcal{L}_{H_f}^{p}(\alpha) - \mathcal{L}_{H_f^{[p]}}(\alpha) + \langle H_f^{[p]}, \omega\rangle \\ &= \langle H_f^{[p]}, \omega\rangle. \end{aligned}$$

It follows that

$$\{\varphi(f), g\} = H_f^{[p]}(dg) = \mathrm{ad}_f^p(g).$$

The remaining relations in Equation (1) are rather tedious, and we refer to [BK08] for a slicker argument. For instance, let us show the second relation through brute-force calculation. Note that $H_{f+g} = H_f + H_g$, so that

$$H_{f+g}^{[p]} = H_f^{[p]} + H_g^{[p]} + \mathcal{L}_{H_f}^{p-1}(H_g).$$

The second identity in Equation (1) is equivalent to the following claim:

$$\begin{aligned}\mathrm{ad}_f^{p-1}(g) = &((\mathcal{L}_{H_f} + \mathcal{L}_{H_g})^{p-1} - \mathcal{L}_{H_f}^{p-1})\langle H_f, \alpha\rangle + ((\mathcal{L}_{H_f} + \mathcal{L}_{H_g})^{p-1} - \mathcal{L}_{H_g}^{p-1})\langle H_g, \alpha\rangle \\ &- \langle \mathcal{L}_{H_f}^{p-1}(H_g), \alpha\rangle.\end{aligned}$$

This can be proved using the identities

$$\begin{aligned}\mathrm{ad}_f^{p-1}(g) &= \mathrm{ad}_{f+g}^{p-1}(\xi(f) + \xi(g)) - \mathrm{ad}_f^{p-1}(\xi(g)) - \mathrm{ad}_g^{p-1}(\xi(g)) \\ &\quad - \xi(\mathrm{ad}_f^{p-1}(g)), \\ (\mathcal{L}_\xi^n \alpha)(\xi') &= \sum_{i=0}^n (-1)^i \binom{n}{i} \xi^{n-i}\langle \alpha, \mathrm{ad}_\xi^i \xi'\rangle.\end{aligned}$$

□

Note that derivations ξ as in Lemma 14 are in bijection with 1-forms α such that $d\alpha = \omega$, as the proof above shows. We can summarize the above discussion in the following omnibus result, which we will state for affine X:

Theorem 15 (Bezrukavnikov-Kaledin)**.** *Let $X = \mathrm{Spec}(A)$ be a smooth affine scheme over k equipped with a symplectic form ω. Then the following are equivalent:*

(a) *The Poisson algebra A admits a compatible restricted structure.*
(b) *The Lie subalgebra of $T_{X/k}$ consisting of Hamiltonian vector fields closed under the restricted Lie operation $\xi \mapsto \xi^{[p]}$.*
(c) *ω is exact.*
(d) *The Cartier operator $\mathfrak{C}$ kills ω.*

Furthermore, the set of compatible restricted structures is in bijection with the set of 1-forms α such that $d\alpha = \omega$, modulo exact 1-forms.

Proof. The equivalence between (b) and (d) is Proposition 10, and the equivalence between (c) and (d) is Remark 11. The implication (c)⇒(a) is Lemma 14. For the converse, assume that ω is exact; we will then define a bijection between compatible restricted structures and 1-forms α such that $d\alpha = \omega$, modulo exact 1-forms. Since ω is exact, there exists a compatible restricted structure φ, and so the set of such can be identified with Frobenius-derivations on A which land in the Poisson center of A by Lemma 12. Given a Frobenius derivation $\delta : A \to A$ and our chosen compatible restricted structure φ, define α as follows. Since we only need to specify α up to 1-forms, the Cartier isomorphism $\mathfrak{C} : \mathrm{H}^1(\Omega^\bullet_{A/k}) \xrightarrow{\sim} \Omega^1_{A^{(p)}/k}$ allows us to just specify $\mathfrak{C}(\alpha)$. Since the Lie subalgebra of Hamiltonian vector fields generates all of $T_{X/k}$, we will just specify how $\mathfrak{C}(\alpha)$ pairs with Hamiltonian vector fields. If $f \in A$, define:

$$\langle \mathfrak{C}(\alpha), (H_f)^{(p)}\rangle := \mathfrak{C}(\varphi(f) - \delta(f)).$$

We leave checking the well-definedness of α and that $d\alpha = \omega$ to the reader: given the preceding calculations, it is straightforward once one unwinds the notation involved. □

Let us now turn to (special cases of) the results of Bezrukavnikov and Kaledin on Frobenius-constant quantizations. We will not prove any of these results in this talk; instead, in the next talk, we will discuss how they are (almost) consequences of the theory of (weak) cyclotomic structures.

Definition 16. Let A be a k-algebra, and abusively[1] write $\mathfrak{C}^{-1}$ to denote the Frobenius-linear map $\Omega^i_{A/k} \to \Omega^i_{A/k}/d\Omega^{i-1}_{A/k}$ sending df to $f^{p-1}df$. The A-module $\Omega^i_{A/k,\log}$ of *logarithmic i-forms* is defined to be the kernel

$$\Omega^i_{A/k,\log} = \ker(\Omega^i_{A/k} \xrightarrow{1-\mathfrak{C}^{-1}} \Omega^i_{A/k}/d\Omega^{i-1}_{A/k}).$$

This can defines an étale sheaf $\Omega^i_{\log}$ over any smooth k-scheme X. Similarly, let $\widetilde{\Omega}^i_{A/k,\log}$ denote the *cokernel*

$$\widetilde{\Omega}^i_{A/k,\log} = \mathrm{coker}(\Omega^i_{A/k} \xrightarrow{1-\mathfrak{C}^{-1}} \Omega^i_{A/k}/d\Omega^{i-1}_{A/k}).$$

[1]This notation is abusive because we have used $\mathfrak{C}$ above to denote the isomorphism $\mathrm{H}^i(\Omega^\bullet_{A/k}) \xrightarrow{\sim} \Omega^i_{A^{(p)}/k}$.

Since $1-\mathfrak{C}^{-1}$ is surjective locally in the étale topology, $\widetilde{\Omega}^i_{\log}$ vanishes locally in the étale topology. In fact:

$$\mathrm{H}^0_{\mathrm{et}}(\operatorname{Spec} A; \Omega^i_{\log}) = \Omega^i_{A/k,\log},\ \mathrm{H}^1_{\mathrm{et}}(\operatorname{Spec} A; \Omega^i_{\log}) = \widetilde{\Omega}^i_{A/k,\log}.$$

Remark 17. To see why the term "logarithmic" is well-deserved, note that an element in $\Omega^1_{A/k,\log}$ is a 1-form α such that $\mathfrak{C}^{-1}(\alpha)=\alpha$. If $\alpha = gdf$ for some $g\in A$, then $\mathfrak{C}^{-1}(\alpha) = g^p f^{p-1}df$, so we are asking that $gdf = g^p f^{p-1}df$ up to exact forms. In other words, $df = g^{p-1}f^{p-1}df$. This happens if and only if f is a unit in A and $f = g^{-1}$; in other words, $\alpha = \frac{df}{f}$. Observe that if f is a pth power, then $\frac{df}{f}=0$. Using this fact, one can establish an exact sequence

$$0\to (A^{(p)})^\times \to A^\times \xrightarrow{d\log} \Omega^1_{A/k} \xrightarrow{1-\mathfrak{C}^{-1}} \Omega^1_{A/k}/dA \to 0,$$

where the image of $d\log$ is precisely $\Omega^1_{A/k,\log}$. This extends to the following exact sequence of étale sheaves:

$$0\to (\mathbf{G}_m^{[p]})^\times \to \mathbf{G}_m \xrightarrow{d\log} \Omega^1 \xrightarrow{1-\mathfrak{C}^{-1}} \Omega^1/d\mathcal{O}_X \to 0. \tag{5}$$

We will not prove the following theorem in this talk:

Theorem 18. *Let X be a smooth symplectic variety such that $\mathrm{H}^i(X;\mathcal{O}_X)=0$ for $i=1,2,3$. Then:*

(a) *The set of isomorphism classes of Frobenius-constant quantizations of $\mathcal{O}_X$ is in bijection with $\mathrm{H}^1_{\mathrm{et}}(X;\Omega^1_{X/k,\log})$.*
(b) *The image of the class of a Frobenius-constant quantization $\mathcal{O}_\hbar$ along the boundary map $\mathrm{H}^1_{\mathrm{et}}(X;\Omega^1_{X/k,\log}) \to \mathrm{Br}(X/k) = \mathrm{H}^2_{\mathrm{et}}(X;\mathbf{G}_m)$ from Equation (5) is p-torsion. The corresponding class in $\mathrm{Br}(X/k)$ defines a class $\beta\in\mathrm{Br}(X^{(p)}((\hbar)))$.*
(c) *View $\mathcal{O}_\hbar$ as a sheaf of algebras over $X^{(p)}[\![\hbar]\!]$ via the splitting map $\mathcal{O}_{X^{(p)}}\to\mathcal{O}_\hbar$; then, $\mathcal{O}_\hbar[1/\hbar]$ is an Azumaya algebra over $X^{(p)}((\hbar))$ which represents the class β.*

We will see parts (b) and (c) in action in the next talk. Already, the appearance of the Brauer group/Azumaya algebras (as well as the vanishing condition on $\mathrm{H}^i(X;\mathcal{O}_X)$) suggests that the theory of Frobenius-constant quantizations may admit a rephrasing in terms of categories of modules. This is indeed the perspective we will adopt in the next talk: passing to categories of modules allows us to use Koszul duality to view quantizations in terms of S^1-actions. We will also discuss Theorem 18(a) in the next talk. As with parts (b) and (c), the appearance of logarithmic forms in the above discussion is already a suggestion that Frobenius-constant quantizations may be related to cyclotomic structures.

The result of Theorem 18(a) can be explained by relating Frobenius-constant quantizations to restricted analogues of Atiyah algebroids. This notion was formalized in the recent paper [Mun21], whose exposition we will follow.

Definition 19. Let X be a smooth k-scheme.

(a) A *Lie algebroid* over X is a quasicoherent $\mathcal{O}_X$-module $\mathcal{A}$ equipped with a k-linear Lie algebra structure and an *anchor map* $\tau:\mathcal{A}\to T_{X/k}$ such that:
 - τ is a map of $\mathcal{O}_X$-modules and a map of k-linear Lie algebras;
 - If $f\in\mathcal{O}_X$ and $\xi,\xi'\in\mathcal{A}$, we have

$$[\xi, f\cdot\xi'] = f\cdot[\xi,\xi'] + \tau(\xi)(f)\cdot\xi';$$

 in other words, τ measures the failure of the Lie bracket on $\mathcal{A}$ to be $\mathcal{O}_X$-linear.

(b) An *Atiyah algebra* over X is a Lie algebroid $(\mathcal{A},\tau,\varphi)$ over X of the form

$$0\to\mathcal{O}_X\to\mathcal{A}\xrightarrow{\tau} T_{X/k}\to 0, \tag{6}$$

 such that $[\xi,f]=\tau(\xi)(f)$ for all $f\in\mathcal{O}_X$ and $\xi\in T_{X/k}$, and $1\in\mathcal{O}_X\subseteq\mathcal{A}$ is a central element.

(c) A *restricted Lie algebroid* over X is a Lie algebroid $(\mathcal{A},\tau:\mathcal{A}\to T_{X/k})$ along with a restricted operation $x\mapsto\varphi(x)$ such that $\mathcal{A}$ forms a k-linear restricted Lie algebra, and

$$\varphi(f\cdot\xi) = f^p\cdot\varphi(\xi) + \tau(f\cdot\xi)^{p-1}(f)\cdot\xi.$$

(d) A *restricted Atiyah algebra* over X is a restricted Lie algebroid $(\mathcal{A},\tau,\varphi)$ over X whose underlying Lie algebroid is an Atiyah algebra, and such that $\varphi(f)=f^p$ for every local section f of $\mathcal{O}_X$.

Example 20. Let G be an algebraic group, and let $\mathcal{P}$ be a G-bundle over X. Then one has an "Atiyah Lie algebroid" $\mathrm{At}(\mathcal{P})$, which sits in an extension

$$0 \to \mathfrak{g}_{\mathcal{P}} \to \mathrm{At}(\mathcal{P}) \to T_{X/k} \to 0.$$

The Lie algebroid $\mathrm{At}(\mathcal{P})$ is defined as $T_{\mathcal{P}/k}/G$; the above extension is then the quotient by G of the following short exact sequence:

$$0 \to \mathfrak{g} \times \mathcal{P} \to T_{\mathcal{P}/k} \to \pi^* T_{X/k} \to 0,$$

where $\pi : \mathcal{P} \to X$ is the projection, and G acts on $\mathfrak{g} \times \mathcal{P}$ by the diagonal.

If $G = \mathbf{G}_m$, so $\mathcal{P}$ is a line bundle $\mathcal{L}$ over X, then $\mathfrak{g}_{\mathcal{L}} \cong \mathcal{O}_X$. It follows that $\mathrm{At}(\mathcal{L})$ is an extension of $T_{X/k}$ by $\mathcal{O}_X$. In fact, $\mathrm{At}(\mathcal{L})$ can be identified with the sheaf $\mathcal{D}_X^{\mathcal{L},\leq 1}$ of twisted differential operators on $\mathcal{L}$ of order ≤ 1. It may be viewed as a restricted Atiyah algebra, where the restricted operation on a local section of $T_{X/k}$ is just the pth power of the derivation acting on $\mathcal{L}$.

Remark 21. Given an extension as in Equation (6) defining an Atiyah algebra, one can define an associated *enveloping algebra* $U(\mathcal{A})$; this is a filtered sheaf of associative algebras such that there is an isomorphism $\mathrm{gr}(\mathrm{F}^\star U(\mathcal{A})) \cong \mathrm{Sym}_{\mathcal{O}_X}(T_{X/k})$ of Poisson algebras. The restricted structure on $\mathcal{A}$ makes $U(\mathcal{A})$ into a Frobenius-constant quantization of $\mathrm{Sym}_{\mathcal{O}_X}(T_{\mathcal{O}_X/k})$.

Then, one has the following theorem (see [Mun21]):

Theorem 22. *Let X be a smooth scheme over k. Then the set of isomorphism classes of restricted Atiyah algebras over k is in bijection with* $\mathrm{H}^1_{\mathrm{et}}(X; \Omega^1_{X/k,\log})$.

Proof sketch. First, observe that one can identify $\Omega^1_{A/k,\log}$ as

$$\Omega^1_{A/k,\log} = \mathrm{fib}(\Omega^1_{A/k,\mathrm{cl}} \xrightarrow{-^{(p)} - \mathfrak{C}} \Omega^1_{A^{(p)}/k}).$$

To show this, recall from Definition 16 that $\Omega^1_{A/k,\log}$ is the kernel of the map $\Omega^1_{A^{(p)}/k} \xrightarrow{1-\mathfrak{C}^{-1}} \Omega^1_{A/k}/dA$. It therefore suffices to show that the following diagram is Cartesian:

$$\begin{array}{ccc} \Omega^1_{A/k,\mathrm{cl}} & \xrightarrow{-^{(p)} - \mathfrak{C}} & \Omega^1_{A^{(p)}/k} \\ \downarrow \mathfrak{C} & & \downarrow \\ \Omega^1_{A^{(p)}/k} & \xrightarrow{1-\mathfrak{C}^{-1}} & \Omega^1_{A/k}/dA. \end{array}$$

It suffices to show that the kernels of the vertical maps are isomorphic. The Cartier isomorphism tells us that the kernel of the left vertical map is the submodule of exact 1-forms, i.e., dA. But this is also the kernel of the right vertical map; since $\mathfrak{C}$ kills exact 1-forms, the map between the kernels is just $-^{(p)}$, which is an isomorphism.

It follows that a class $\kappa \in \mathrm{H}^1_{\mathrm{et}}(X; \Omega^1_{X/k,\log})$ can be represented in terms of Čech cocycles as follows. Suppose $\{U_i\}$ is a sufficiently refined Zariski open cover of X; then, a class in $\mathrm{H}^1_{\mathrm{et}}(X; \Omega^1_{X/k,\log})$ is represented by a collection $\{\alpha_i, \beta_{ij}\}$ with $\alpha_i \in \mathrm{H}^0(U_i; \Omega^1_{U_i/k,\mathrm{cl}})$ and $\beta_{ij} \in \mathrm{H}^0(U_i^{(p)}; \Omega^1_{U_i^{(p)}/k})$ such that the following Čech cocycle condition is satisfied:

$$\beta_{ij}^{(p)} - \mathfrak{C}(\beta_{ij}) = \alpha_i - \alpha_j.$$

Given such a tuple representing $\kappa \in \mathrm{H}^1_{\mathrm{et}}(X; \Omega^1_{X/k,\log})$, define a restricted Atiyah algebra $\mathcal{A}_\kappa$ on X by gluing the *split* restricted Atiyah algebras $\mathcal{O}_{U_i} \oplus T_{U_i/k}$ with restricted operation

$$\varphi_i : (f, \xi) \mapsto (f^p + \langle \xi, \alpha_i \rangle^p, \xi^{[p]})$$

on the Zariski cover $\{U_i\}$. One can check that the gluing is legal thanks to the Čech cocycle condition. Conversely, given a restricted Atiyah algebra $\mathcal{A}$ on X, one can construct a class in $\mathrm{H}^1_{\mathrm{et}}(X; \Omega^1_{X/k,\log})$ by reversing the above description and using [Mun21, Lemma 4.4], which tells us that the anchor map $\tau : \mathcal{A} \to T_{X/k}$ is Zariski-locally split. □

Remark 23. In the same way, one can prove if k is a field of characteristic zero and X is a smooth k-scheme, then Atiyah algebras on X are classified by closed 1-forms of degree 1, i.e., by $\mathrm{H}^1(X; \Omega^{\geq 1}_{X/k}) = \mathrm{F}^1_{\mathrm{Hdg}} \mathrm{H}^2_{\mathrm{dR}}(X/k)$. Here, $\Omega^{\geq 1}_{X/k}$ denotes the complex

$$\Omega^1_{X/k} \to \Omega^2_{X/k} \to \cdots,$$

where $\Omega^i_{X/k}$ is placed in homological degree[2] $1-i$. Moreover, [BK04, Theorem 1.8] proves that if X is a smooth symplectic variety over a field k of characteristic zero with symplectic form ω, then isomorphism classes of quantizations of X are in bijection with $\mathrm{pr}(\omega) + \hbar \mathrm{H}^1(X; \Omega^{\geq 1}_{X/k})[\![\hbar]\!]$, where $\mathrm{pr}(\omega) \in \mathrm{F}^1_{\mathrm{Hdg}}\mathrm{H}^2_{\mathrm{dR}}(X/k)$ is the projection of ω along a splitting of the inclusion $\mathrm{F}^1_{\mathrm{Hdg}}\mathrm{H}^2_{\mathrm{dR}}(X/k) \subseteq \mathrm{H}^2_{\mathrm{dR}}(X/k)$.

One may understand the classification of Atiyah algebras as follows. There is a cofiber sequence of complexes

$$\Omega^{\geq 2}_{X/k}[-1] \to \Omega^{\geq 1}_{X/k} \to \Omega^1_{X/k},$$

which induces an exact sequence

$$\cdots \to \mathrm{H}^0(X; \Omega^{\geq 2}_{X/k}) \to \mathrm{H}^1(X; \Omega^{\geq 1}_{X/k}) \to \mathrm{H}^1(X; \Omega^1_{X/k}) \to \cdots .$$

If $\alpha \in \mathrm{H}^1(X; \Omega^{\geq 1}_{X/k})$ classifies an Atiyah algebra, then its image in $\mathrm{H}^1(X; \Omega^1_{X/k}) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(T_{X/k}, \mathcal{O}_X)$ classifies the extension (6). Moreover, for a fixed class in $\mathrm{H}^1(X; \Omega^1_{X/k})$ classifying an extension as in (6), the set of Atiyah algebra structures on the extension is classified by the image of the map $\mathrm{H}^0(X; \Omega^{\geq 2}_{X/k}) \to \mathrm{H}^1(X; \Omega^{\geq 1}_{X/k})$. Note that $\mathrm{H}^0(X; \Omega^{\geq 2}_{X/k}) \cong \mathrm{H}^0(X; \Omega^2_{X/k,\mathrm{cl}})$; in particular, if X is *affine*, then Atiyah algebras are classified by $\mathrm{H}^2_{\mathrm{dR}}(X/k)$.

Let us conclude this talk with a brief and unrelated (but fun) observation. Let X be a smooth scheme over k. Then, a (Frobenius-constant) quantization of the commutative $\mathcal{O}_X$-algebra $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k})$ is given by $\mathcal{D}^\hbar_X$. One can consider the quantization $\mathcal{D}^\hbar_{X^{(p)}}$ of $\mathrm{Sym}_{\mathcal{O}_{X^{(p)}}}(T_{X^{(p)}/k})$, too. Recall that the p-curvature map $\Psi : \mathrm{Sym}_{\mathcal{O}_{X^{(p)}}}(T_{X^{(p)}/k}) \to \mathrm{Frob}_*\mathcal{D}_X$ gives an isomorphism between $\mathrm{Sym}_{\mathcal{O}_{X^{(p)}}}(T_{X^{(p)}/k})$ and the center of $\mathcal{D}_X$. It is therefore natural to ask if we can describe the quantization $\mathcal{D}^\hbar_{X^{(p)}}$ (equivalently, the Poisson bracket on $\mathrm{Sym}_{\mathcal{O}_{X^{(p)}}}(T_{X^{(p)}/k})$) in terms of $\mathcal{D}_X$. This is in fact possible, if one uses a quantization of $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k})$ in the *arithmetic* direction: namely, if X lifts to $\widetilde{X}$ over $W_2(k)$, then the deformation $\mathcal{D}^\hbar_{\widetilde{X}}$ of $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k})$ to $W_2(k)[\![\hbar]\!]$ can be used to recover the Poisson bracket on $\mathrm{Sym}_{\mathcal{O}_{X^{(p)}}}(T_{X^{(p)}/k})$.

To explain this, general arguments allow us to reduce to the case $X = \mathbf{A}^1$, with coordinate t. Then, the p-curvature map is $\Psi : k[\partial_t^p, t^p] \to k\langle t, \partial_t\rangle/([t, \partial_t] - 1)$. The Poisson bracket on $k[\partial_t^p, t^p] \cong \mathcal{O}_{T^*(\mathbf{A}^1_k)^{(p)}}$ can then be recovered from the lift $W_2(k)[t]$. This is a consequence of the following lemma:

Lemma 24. *The following identity holds in the Weyl algebra of $W_2(k)[t]$ over $W_2(k)$:*

$$\frac{1}{p}[t^p, (\partial_t)^p] \equiv 1.$$

Proof. Let us calculate $[t^p, (\partial_t)^p]$ in the Weyl algebra of $W(k)[t]$ over $W(k)$. In general, some combinatorics shows that

$$[(\partial_t)^n, t^m] = \sum_{j=1}^n \binom{n}{j}\binom{m}{j} j! t^{m-j} (\partial_t)^{n-j}.$$

When $n = m = p$, we see that

$$\binom{p}{j}\binom{p}{j} j! \equiv \begin{cases} 0 & 1 \leq j \leq p-1 \\ p! \equiv -p & j = p, \end{cases}$$

where the equivalences are taken modulo p^2. This implies the lemma. □

One can prove a Koszul dual version of Lemma 24. Indeed (as we will see next time), the Koszul dual of the isomorphism between $\mathrm{Sym}_{\mathcal{O}_{X^{(p)}}}(T_{X^{(p)}/k})$ and the center of $\mathcal{D}_X$ may be understood as the Cartier isomorphism $\mathfrak{C} : \mathcal{H}^i(X^{(p)}; \mathrm{Frob}_*\Omega^\bullet_{X/k}) \xrightarrow{\sim} \Omega^i_{X^{(p)}/k}$. It is natural to ask whether a lift $\widetilde{X}$ of X to $W_2(k)$ allows one to recover the de Rham differential on $\Omega^*_{X^{(p)}/k}$ under the Cartier isomorphism. The answer is indeed yes; just as with Lemma 24, the claim boils down to the following universal calculation in the case $X = \mathbf{A}^1 = \operatorname{Spec} k[t]$.

[2] Therefore, in "correct" notation, Atiyah algebras are classified by $\pi_{-2}\Gamma(X; \Omega^{\geq 1}_{X/k}) \cong \mathrm{H}^2(X; \Omega^{\geq 1}_{X/k})$, where $\Omega^{\geq 1}_{X/k}$ now denotes the Hodge filtration on the de Rham complex (so $\Omega^i_{X/k}$ is in homological degree $-i$).

Lemma 25. *The de Rham differential on* $k[t^{(p)}] = \mathcal{O}_{(\mathbf{A}^1_k)^{(p)}}$ *is given by the map*

$$(t^{(p)})^n \mapsto -\frac{\partial_t^p}{p}(t^{np})dt^{(p)},$$

where $-\frac{\partial_t^p}{p}(t^{np})$ *is understood modulo* p^2*.*

Proof. Observe that $\partial_t^m(t^n) = \binom{n}{m} m! t^{n-m}$. Therefore,

$$\partial_t^p(t^{np}) = \binom{np}{p} p! t^{(n-1)p} \equiv -npt^{(n-1)p} \pmod{p^2}.$$

For the last equality, note that since $p! \equiv -p \pmod{p^2}$ and $\binom{np}{p} \equiv n \pmod{p}$ for any n by Lucas' theorem, we have $\binom{np}{p} p! \equiv -np \pmod{p^2}$. The lemma follows by identifying $(t^{(p)})^n$ with t^{np} under the Frobenius $\mathbf{A}^1 \to (\mathbf{A}^1)^{(p)}$. □

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