Talk II: Kontsevich's theorem

This talk is being given by Charles Fu, but these notes were written independently by myself.

Our goal in this and the following talk is to prove the following theorem, mentioned last time:

Theorem 1 (Kontsevich, [Kon03])**.** *The ring $\mathcal{O}_\mathcal{M}$ of smooth functions on any smooth Poisson manifold $\mathcal{M}$ admits a quantization. More precisely, each Poisson bracket $\{-,-\}$ on $\mathcal{M}$ defines an associative product $\star$ on $\mathcal{O}_\mathcal{M}[\![\hbar]\!]$ such that if $f\star g = fg + \sum_{n\geq 1} c_n(f,g)\hbar^n$, then*

$$\{f,g\} = \frac{c_1(f,g) - c_1(g,f)}{\hbar},$$

and each $c_n : \mathcal{O}_\mathcal{M}\otimes\mathcal{O}_\mathcal{M}\to\mathcal{O}_\mathcal{M}$ is a bidifferential operator[1]*.*

In other words, the map from the space of gauge-equivalence classes of smooth quantizations of $\mathcal{O}_\mathcal{M}$ to Poisson brackets on $\mathcal{O}_\mathcal{M}$ is surjective; furthermore, there is an explicit section of this map. We gave a sketch of the proof of Theorem 1 last time; in this talk, we will give some details. Let us begin by stating the following rephrasing of Theorem 1.

Theorem 2. *Let $\mathcal{M}$ be a smooth Poisson manifold, and let $A = \mathcal{O}_\mathcal{M}$. Then:*

(a) *Every Poisson bracket on A admits a lift to a Poisson bracket over $A[\![\hbar]\!]$.*
(b) *There is a bijection between equivalence classes of $\mathbf{R}[\![\hbar]\!]$-linear Poisson brackets over $A[\![\hbar]\!]$ and equivalence classes of quantizations of A.*

Unfortunately, we will not be able to give a proof of (a) in this seminar, because the arguments would take us rather far afield (although we may return to it at the end, if there is time). The proof of (b) will be broken into a few parts:

- Describe the relationship between deformation problems and differential graded Lie algebras, and describe the differential graded Lie algebra associated to quantizations and the differential graded Lie algebra associated to deforming Poisson brackets.
- Show that these differential graded Lie algebras are quasi-isomorphic; this boils down to showing that the $\mathbf{E}_2$-operad is formal in characteristic 0, i.e., that $C_*(\mathbf{E}_2;k)\simeq \mathrm{H}_*(\mathbf{E}_2;k)$ as operads in Mod_k.

The first bullet will be the content of this talk, while the second bullet will be discussed next time.

Let us motivate the relationship between deformation problems and differential graded Lie algebras by the following example.

Example 3. Let k be a field of characteristic zero, and let $f(x)$ be a polynomial (defining some k-scheme $Y = \operatorname{Spec} k[x]/f(x)$). Suppose we have $\lambda\in k$ such that $f(\lambda)=0$, i.e., a k-point of Y. Deformation theory is concerned with studying points of Y which are "close" to our given k-point. In concrete terms, this amounts to understanding the behaviour of f around λ. To do this, we can consider the function $f(x+\lambda)$, whose Taylor expansion is

$$f(x+\lambda) = \sum_{n\geq 1}\frac{x^n}{n!}\left.\frac{d^n f}{dx^n}\right|_{x=\lambda} = xf'(\lambda) + \frac{x^2}{2}f^{(2)}(\lambda)+\cdots$$

Note that this expression only makes sense because k is assumed to of characteristic zero.

To understand the *linear* behaviour of f around λ, we only need to know $f'(\lambda)$. If we write $V = k$, then $f'(\lambda)$ defines a map $\delta : V\to V$ sending $x\mapsto f'(\lambda)x$. For instance, we see that the linear approximation to f does not have any zeros other than λ if $f'(\lambda)\neq 0$, because such zeros would be described by $\delta(x)=0$. This claim is also obvious if $k=\mathbf{R}$.

Let us now try to describe the *quadratic* behaviour of f around λ; for instance, what zeroes near λ can the quadratic approximation of f see? To answer this, we need to know both $f'(\lambda)$ and $f^{(2)}(\lambda)$. We can encode this data in a somewhat funny way: write $t=x^2$, and let $W = k\oplus k$. Then the quadratic behaviour of f is determined by the maps $\delta,\mu : W\to W$ which are given by

$$\delta(x,t) = (f'(\lambda)x, t),\ \ \mu(x,t) = \left(\frac{x^2}{2}f^{(2)}(\lambda), x^2\right).$$

The zeros of the quadratic approximation to f near λ are determined by the equation

$$\delta(w)+\mu(w) = 0. \qquad (1)$$

[1] A product $\star$ where each c_n is a bidifferential operator will be called a *smooth* deformation.

Note that δ is a linear function, while μ is quadratic. In particular, associated to μ is the quadratic form $q : \mathrm{Sym}^2(W) \to W$ given by

$$q(w_1, w_2) = \frac{\mu(w_1 + w_2) - \mu(w_1) - \mu(w_2)}{2}.$$

Let us now consider the differential graded k-module given by

$$\mathfrak{g} := (W[-1] \xrightarrow{\delta} W[-2]);$$

in other words, $\mathfrak{g}$ has underlying graded k-vector space $W[-1] \oplus W[-2]$, and the differential $d_{\mathfrak{g}}$ is given by the map δ from above. The quadratic form map $q : \mathrm{Sym}^2(W) \to W$ naturally endows $\mathfrak{g}$ with the structure of a differential graded Lie algebra in the following way. Recall that if M is a differential graded k-module, then $\mathrm{Sym}^2(M[1]) \simeq (\wedge^2 M)[2]$. Therefore, $\mathrm{Sym}^2(W)[-2] \simeq \wedge^2(W[-1])$. The quadratic form q therefore gives a map

$$\wedge^2(W[-1]) \simeq \mathrm{Sym}^2(W)[-2] \xrightarrow{2q} W[-2].$$

This defines a skew-symmetric map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, which is the claimed Lie bracket. Finally, Equation (1), which describes the zeros of the quadratic approximation to f near λ, translates to the equation

$$d_{\mathfrak{g}}(w) + \frac{[w,w]}{2} = 0 \tag{2}$$

for degree -1 elements $w \in \mathfrak{g}$. Of course, (2) is the Maurer-Cartan equation: we conclude that solutions to the Maurer-Cartan equation for the differential graded Lie algebra $\mathfrak{g}$ describe zeros of the quadratic approximation to f near λ.

One can extend the above discussion to describe zeros of arbitrarily better approximations to f near λ. For instance, if one wishes to understand zeros of the nth order approximation for some $n \geq 2$, then one would naturally be led to a differential graded k-module (which we will also just call $\mathfrak{g}$ for this discussion) equipped with linear maps $\ell_i : \mathfrak{g}^{\otimes i} \to \mathfrak{g}$ for all $i \leq n$ satisfying some compatibility conditions, where $\ell_1 = d_{\mathfrak{g}}$. This is the structure of an *L_∞-algebra*. Then, zeros of the nth order approximation to f near λ would be described by the L_∞-Maurer-Cartan equation

$$d_{\mathfrak{g}}(w) + \sum_{i=2}^{n} \frac{\ell_i(w, \cdots, w)}{i!} = 0 \tag{3}$$

for degree -1 elements $w \in \mathfrak{g}$. Notice that all of this discussion breaks down if k is a field of characteristic $p > 0$ and $n \geq p$; this is why we will stick to k being of characteristic zero.

The philosophy that the preceding example dictates is that deformation theory should be controlled by L_∞-algebras. It turns out that every L_∞-algebra is quasi-isomorphic to a differential graded Lie algebra (that is, a differential graded k-module $\mathfrak{g}$ equipped with a Lie bracket $[-,-]$ satisfying the appropriate version of skew-symmetry and the Leibniz rule $d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$); furthermore, the differential graded Lie algebra can be constructed functorially from the L_∞-algebra. In fact, there is an ∞-category of L_∞-algebras which can be constructed from a model structure on the 1-category of differential graded Lie algebras. We will therefore abusively write Lie_k to denote this ∞-category, and simply refer to objects of this ∞-category as differential graded Lie algebras. To state the main theorem relating differential graded Lie algebras to deformation problems, we must describe the ∞-category of deformation problems.

Definition 4. Let k be a field of characteristic zero. An augmented differential graded k-algebra A is called *small* if A is connective and bounded-above, each $\pi_n(A)$ is a finite-dimensional k-vector space, and $\pi_0(A)$ is a local ring with maximal ideal $\mathfrak{m}_A$ and residue field k. In other words, A is a connective differential graded Artinian k-algebra. Let $\mathrm{CAlg}_k^{\mathrm{sm}}$ denote the ∞-category of small k-algebras.

A functor $F : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ is called a *formal moduli problem* if $F(k) \simeq *$ and F satisfies the following property: if

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

is a pullback diagram of k-algebras where the maps $A \to B$ and $B' \to B$ induce surjections on π_0, then $F(A') \xrightarrow{\sim} F(A) \times_{F(B)} F(B')$. Let Moduli_k denote the ∞-category of formal moduli problems.

Theorem 5 (Lurie, Pridham). *Let k be a field of characteristic zero. There is an equivalence $\Phi : \mathrm{Moduli}_k \xrightarrow{\sim} \mathrm{Lie}_k$ which sends a formal moduli problem F to the differential graded Lie algebra $\mathfrak{g}$ whose zeroth space $\Omega^\infty \mathfrak{g}$ is canonically equivalent to $F(k[\epsilon]/\epsilon^2)$ where $|\epsilon| = 1$. If we write Ψ to denote the inverse of Φ, then the formal moduli problem $\Psi_{\mathfrak{g}}$ associated to a differential graded Lie algebra $\mathfrak{g}$ sends a small k-algebra A to the space of solutions of the Maurer-Cartan equation Equation (2) in $\mathfrak{g} \otimes_k \mathfrak{m}_A$.*

Example 6. Let X be a smooth k-scheme, and let $\mathrm{Def}_X : \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ denote the deformation problem sending a small k-algebra A to the space of lifts of X along $\epsilon : \mathrm{Spec}(k) \to \mathrm{Spec}(A)$. Then the differential graded Lie algebra associated to Def_X is the *Kodaira-Spencer differential graded Lie algebra* $\mathfrak{g}_X = R\Gamma(X; T_X[-1])$, where the Lie bracket is given by the bracket on vector fields. Here, $R\Gamma$ denotes derived global sections (so $\pi_0 \mathfrak{g}_X = \mathrm{H}^1(X; T_X)$). It is a fun exercise to see how this example relates to Example 3.

The next few examples will discuss the formal moduli problems relevant for Theorem 1.

Example 7. Let X be a smooth manifold equipped with a Poisson bracket determined by a Poisson bivector $\pi \in \Gamma(X; \wedge^2 T_X)$, so that $[\pi, \pi] = 0$. Consider the differential graded Lie algebra $\mathcal{T}_X^\pi$ whose degree n piece is $\Gamma(X; \wedge^{n+1} T_X)$, and whose differential d is given by the bracket $[\pi, -]$. Then the formal moduli problem associated to $\mathcal{T}_X^\pi$ classifies deformations of the Poisson structure π on X. To see this, let $A = k[\epsilon]/\epsilon^3$, so that $\mathfrak{m}_A = (\epsilon)$. A deformation of the Poisson structure to A is $\widetilde{\pi} = \pi + \pi'$ such that $[\widetilde{\pi}, \widetilde{\pi}] = 0$. But

$$[\widetilde{\pi}, \widetilde{\pi}] = [\pi + \pi', \pi + \pi'] = 2[\pi, \pi'] + [\pi', \pi'] = 2\left(d\pi' + \frac{[\pi', \pi']}{2}\right),$$

so the condition that $\widetilde{\pi}$ satisfy $[\widetilde{\pi}, \widetilde{\pi}] = 0$ (i.e., that it be a deformation of the Poisson bivector π) is precisely the Maurer-Cartan equation for $\mathcal{T}_X^\pi \otimes_k (\epsilon)$.

Remark 8. If $\pi = 0$ (so X is equipped with the trivial Poisson bracket), then we will just write $\mathcal{T}_X$ instead of $\mathcal{T}_X^\pi$. Note that $\mathcal{T}_X$ has zero differential, and its Poisson bracket is given by the bracket of polyvector fields. Then, solutions to the Maurer-Cartan equation for $\mathcal{T}_X$ are precisely (equivalence classes of) Poisson structures on X.

For the next example, recall that if R is a (discrete) associative k-algebra, then its Hochschild cohomology $\mathrm{HC}(R/k)$ is the complex which is $\mathrm{Hom}(R^{\otimes_k n}, R)$ in degree n, and whose differential sends $f : R^{\otimes_k n-1} \to R$ to

$$(df)(x_1, \cdots, x_n) = x_1 f(x_2, \cdots, x_n) + \sum_{i=1}^{n-1} (-1)^i f(x_1, \cdots, x_i x_{i+1}, \cdots, x_n) + (-1)^n f(x_1, \cdots, x_{n-1}) x_n.$$

Theorem 9. *The following statements are true:*

(a) *The Hochschild cohomology $\mathrm{HC}(R/k)$ admits the structure of an $\mathbf{E}_2$-k-algebra.*
(b) *Let $\mathcal{A}$ be an $\mathbf{E}_2$-k-algebra. Then $\mathcal{A}[1]$ admits the structure of a differential graded Lie algebra over k.*
(c) *The Lie bracket on $\mathrm{HC}(R/k)[1]$ sends $f : R^{\otimes_k n} \to R$ and $g : R^{\otimes_k m} \to R$ to $[f, g] : R^{\otimes_k n+m-1} \to R$ given by*

$$\begin{aligned}[f, g](x_1, \cdots, x_{n+m-1}) = &\sum_{i=1}^{n} (-1)^{im} f(x_1, \cdots, x_{i-1}, g(x_i, \cdots, x_{i+m-1}), x_{i+m}, \cdots, x_{n+m-1}) \\ &+ \sum_{j=1}^{m} (-1)^{jn} g(x_1, \cdots, x_{j-1}, f(x_j, \cdots, x_{j+n-1}), x_{j+n}, \cdots, x_{n+m-1}).\end{aligned}$$

Example 10. Let R be a (discrete) associative k-algebra. Consider the deformation problem of lifting R to an associative A-algebra for some Artinian k-algebra A. For instance, suppose $A = k[\epsilon]/\epsilon^2$, and let $\widetilde{R}$ be a lift of R to A. Let $\star$ denote the multiplication on $\widetilde{R}$, so that $x \star y = xy + \epsilon f(x, y)$ for some $f : R \otimes_k R \to R$. It is easy to check that $\star$ is associative if and only if

$$x f(y, z) - f(xy, z) + f(x, yz) - f(x, y) z = 0. \tag{4}$$

If we only consider lifts $\widetilde{R}$ up to Morita equivalence, then two such multiplications $\star$ and $\star'$ are equivalent if there is an automorphism $\Phi : R[\epsilon]/\epsilon^2 \to R[\epsilon]/\epsilon^2$ such that $\Phi(x) = x + \epsilon g(x)$ for some

k-linear map $g : R \to R$ and such that $\Phi(x) \star \Phi(y) = \Phi(x \star' y)$. Suppose that $f' : R \otimes_k R \to R$ is a linear map such that $x \star' y = xy + \epsilon f'(x, y)$. Then the above condition amounts to asking that

$$f'(x, y) = f(x, y) - g(x)y - xg(y) + g(xy). \tag{5}$$

Therefore, the set of lifts $\widetilde{R}$ up to equivalence are classified by maps $f : R \otimes_k R \to R$ which satisfy the cocycle condition (4), considered modulo the equivalence relation (5) generated by maps $g : R \to R$. This is, by definition, just the Hochschild cohomology $\pi_{-2}\mathrm{HC}(R/k)$.

A similar calculation shows the following. Suppose A is an Artinian k-algebra, and $\star$ is a product on a lift of R to A such that $x \star y = xy + f(x, y)$. Then $\star$ is associative if and only if f is a solution to the Maurer-Cartan equation (2) in $\mathrm{HC}(R/k)[1] \otimes_k \mathfrak{m}_A$. In particular, the differential graded Lie algebra $\mathrm{HC}(R/k)[1]$ classifies the formal moduli problem describing deformations of the k-algebra R.

Example 11. Suppose that in Example 10, the algebra A was $\mathcal{O}_\mathcal{M}$ for some smooth manifold $\mathcal{M}$. To understand *smooth* deformations of $\mathcal{O}_\mathcal{M}$ (as in Theorem 1) following Example 10, we need to restrict to the subcomplex of $\mathrm{HC}(\mathcal{O}_\mathcal{M}/\mathbf{R})[1]$ consisting of those $\mathbf{R}$-linear maps $\mathcal{O}_\mathcal{M}^{\otimes_\mathbf{R} \bullet} \to \mathcal{O}_\mathcal{M}$ which are polydifferential operators. Let us denote this subcomplex by $\mathfrak{D}_\mathcal{M}$.

Let us now return to Theorem 1. Let $\mathcal{M}$ be a smooth manifold. As we discussed in Remark 8, solutions to the Maurer-Cartan equation in $\mathcal{T}_\mathcal{M}$ are in bijection with (equivalence classes of) Poisson brackets on $\mathcal{M}$. Similarly, as we discussed in Example 11, solutions to the Maurer-Cartan equation in $\mathfrak{D}_\mathcal{M}$ are in bijection with (equivalence classes of) smooth deformations of the multiplication on $\mathcal{O}_\mathcal{M}$. Therefore, Theorem 1 will follow once we show:

Theorem 12. *There is a quasi-isomorphism $\mathcal{T}_\mathcal{M} \xrightarrow{\sim} \mathfrak{D}_\mathcal{M}$ of differential graded Lie algebras over $\mathbf{R}$.*

Remark 13. Recall that $\mathcal{T}_\mathcal{M}$ has trivial differential, while $\mathfrak{D}_\mathcal{M}$ has differential given by the differential in Hochschild cohomology. Therefore, Theorem 12 states that $\mathfrak{D}_\mathcal{M}$ is quasi-isomorphic to a differential graded Lie algebra with zero differential; this means that $\mathfrak{D}_\mathcal{M}$ is *formal* as a differential graded Lie algebra. This is why Theorem 12 is known as Kontsevich's formality theorem. Moreover, there is a quasi-isomorphism $h : \mathcal{T}_\mathcal{M} \xrightarrow{\sim} \mathfrak{D}_\mathcal{M}$ of chain complexes of $\mathbf{R}$-modules, but h is *not* a map of differential graded Lie algebras. The quasi-isomorphism h is the HKR isomorphism: it is given by sending

$$X_0 \wedge \cdots X_n \mapsto \left[f_0 \otimes \cdots \otimes f_n \mapsto \frac{\det(X_j(f_i))}{n!} \right].$$

Let us now turn to the proof of Theorem 12. Since $\mathfrak{D}_\mathcal{M}$ is a subcomplex of $\mathrm{HC}(\mathcal{O}_\mathcal{M}/\mathbf{R})$, it suffices to show that $\mathrm{HC}(\mathcal{O}_\mathcal{M}/\mathbf{R})[1]$ is formal as a differential graded Lie algebra over $\mathbf{R}$. Recall from Theorem 9 that the differential graded Lie structure on $\mathrm{HC}(\mathcal{O}_\mathcal{M}/\mathbf{R})[1]$ stems from an action of the $\mathbf{E}_2$-operad on $\mathrm{HC}(\mathcal{O}_\mathcal{M}/\mathbf{R})$. The key step in Tamarkin's approach to Theorem 12 is the following:

Theorem 14. *The $\mathbf{E}_2$-operad is formal in characteristic zero. In particular, $C_*(\mathbf{E}_2; \mathbf{R}) \simeq \mathrm{H}_*(\mathbf{E}_2; \mathbf{R})$ as operads in differential graded $\mathbf{R}$-modules.*

Proof sketch of Theorem 12 given Theorem 14. First, one reduces (by geometric arguments) to the case when $\mathcal{M} = \mathbf{R}^n$; let $V = \mathbf{R}^n$. Then, one shows that the inclusion of $\mathfrak{D}_{\mathbf{R}^n}$ into $\mathrm{HC}(\mathrm{Sym}(V)/\mathbf{R})[1]$ is a quasi-isomorphism, so it suffices to show that $\mathrm{HC}(\mathrm{Sym}(V)/\mathbf{R})[1]$ is a formal differential graded Lie algebra over $\mathbf{R}$. Theorem 9 tells us that $\mathrm{HC}(\mathrm{Sym}(V)/\mathbf{R})$ is an $C_*(\mathbf{E}_2; \mathbf{R})$-algebra in $\mathbf{R}$-modules. A choice of a quasi-isomorphism $C_*(\mathbf{E}_2; \mathbf{R}) \simeq \mathrm{H}_*(\mathbf{E}_2; \mathbf{R})$ as in Theorem 14 then makes $\mathrm{HC}(\mathrm{Sym}(V)/\mathbf{R})$ into a $\mathrm{H}_*(\mathbf{E}_2; \mathbf{R})$-algebra in $\mathrm{Mod}_\mathbf{R}$. Then, one appeals to the following result of Tamarkin's from [Tam03] (which is much easier than Theorem 14; see [Hin03, Section 5.4]): if B is an algebra in $\mathrm{Mod}_\mathbf{R}$ for the operad $\mathrm{H}_*(\mathbf{E}_2; \mathbf{R})$ such that $\pi_* B \cong \pi_* \mathrm{HC}(\mathrm{Sym}(V)/\mathbf{R})$, then B is formal as an algebra over $\mathrm{H}_*(\mathbf{E}_2; \mathbf{R})$. □

In the next talk, we will prove Theorem 14.

References

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