

In the previous two talks, we saw a proof of Kontsevich’s formality theorem for smooth manifolds: namely, that any Poisson structure on a smooth manifold M admits a deformation quantization. The proof proceeded by showing that the formal moduli problem associated to the Hochschild cohomology $\mathrm{HC}(\mathcal{O}_M)$ can always be solved. This was done in two parts: first, one showed that the formal moduli problem associated to deformations of the Poisson bracket is controlled by $\pi_{*-1}\mathrm{HC}(\mathcal{O}_M)$, and that it is always solvable; second, one showed that the \mathbf{E}_2 -algebra $\mathrm{HC}(\mathcal{O}_M)[1]$ is formal, i.e., is equivalent as an \mathbf{E}_2 -algebra to $\pi_{*-1}\mathrm{HC}(\mathcal{O}_M)$.

Our goal in this talk is to prove a *holomorphic* version of this statement, known as the *Bogomolov-Tian-Todorov theorem* (often shortened to “BTT theorem”)¹. In order to state this result, we need to recall the notion of a Calabi-Yau variety. As the definition makes clear, one should think of Calabi-Yau varieties as algebraic analogues of orientable manifolds.

Definition 1. Let k be a commutative ring, and let X be a smooth and proper scheme over k . Say that X is a *Calabi-Yau variety*² if there is an isomorphism $K_X \cong \mathcal{O}_X$, where $K_X = \wedge^{\dim X} \Omega_{X/k}^1$ is the canonical bundle on X . The *datum* of an isomorphism $K_X \cong \mathcal{O}_X$ is called a volume form on X .

Example 2. Suppose X is a smooth and proper curve of genus g , and assume that k is an algebraically closed field. Then the Riemann-Roch formula tells us that $\deg K_X = 2g - 2$, so X is Calabi-Yau if and only if $g = 1$, i.e., X is an elliptic curve. More generally, the adjunction formula and the fact that $K_{\mathbf{P}^n} = \mathcal{O}_{\mathbf{P}^n}(-n - 1)$ implies that any smooth hypersurface of degree $n + 1$ in \mathbf{P}^n is Calabi-Yau.

Theorem 3 (BTT). *Let X be a Calabi-Yau variety over \mathbf{C} of dimension n such that $\mathrm{H}^0(X; T_{X/\mathbf{C}}) = 0$ and $\mathrm{H}^i(X; \mathcal{O}_X) = 0$ for $0 < i < n$. Then the deformation theory of X is unobstructed. More precisely, the functor $\mathrm{Def}_{X/\mathbf{C}} : \mathrm{Art}_{\mathbf{C}} \rightarrow \mathrm{Set}$ is pro-represented by the formal completion of the ring $k[\mathrm{H}^1(X; T_{X/\mathbf{C}})] = k[t_1, \dots, t_{\dim \mathrm{H}^1(X; T_{X/\mathbf{C}})}]$.*

The Calabi-Yau condition on X implies that $T_{X/\mathbf{C}} \cong \Omega_{X/\mathbf{C}}^{n-1}$, so $\mathrm{H}^0(X; T_{X/\mathbf{C}}) \cong \mathrm{H}^0(X; \Omega_{X/\mathbf{C}}^{n-1})$. Hodge theory tells us that $\dim \mathrm{H}^0(X; \Omega_{X/\mathbf{C}}^{n-1}) = \dim \mathrm{H}^1(X; \mathcal{O}_X)$, so the condition that $\mathrm{H}^0(X; T_{X/\mathbf{C}}) = 0$ is satisfied if and only if $\mathrm{H}^1(X; \mathcal{O}_X) = 0$. Note further that $\mathrm{H}^0(X; T_{X/\mathbf{C}})$ vanishing amounts to asking that the deformations of X have no nontrivial automorphisms: this is what allows us to say that $\mathrm{Def}_{X/\mathbf{C}}$ lands in sets, as opposed to groupoids.

The deformation problem $\mathrm{Def}_{X/\mathbf{C}}$ associated to X is controlled by the Kodaira-Spencer differential graded Lie algebra $\mathfrak{g} = \Gamma(X; \wedge^{*+1} T_{X/\mathbf{C}})$ ³. Here (and always), Γ denotes *derived* global sections. Using the general yoga of deformation theory as presented in Talk II, one can then prove Theorem 3 by showing the following:

Theorem 4. *The dg-Lie algebra \mathfrak{g} is homotopy abelian, i.e., \mathfrak{g} is (quasi-)isomorphic to a dg-Lie algebra whose Lie bracket is zero.*

For future discussion, it will be useful to place the argument for this claim in a broader context.

Construction 5. Let \mathcal{A} denote the *graded* dg-algebra $\Gamma(X; \wedge^* T_{X/\mathbf{C}})$, where a polyvector field of degree i is placed in graded weight $-i$ and homological degree i . If $\Sigma^{m,n}$ denotes a shift by homological degree m and graded weight n , then forgetting the grading on $\Sigma^{1,0}\mathcal{A}$ produces \mathfrak{g} . One can construct \mathcal{A} more invariantly using the “Koszul sign rule equivalence” $\mathrm{Sym}_R^n(M[1]) \cong (\wedge^n M)[n]$ for any (graded) commutative ring R and any (graded) R -module M . Then, $\mathcal{A} = \Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(\Sigma^{1,-1} T_{X/\mathbf{C}}))$, where the parenthesis denotes the graded weight. In particular, \mathcal{A} has the structure of a graded commutative \mathbf{C} -algebra.

The Lie bracket on \mathfrak{g} further defines a *1-shifted Lie bracket* on \mathcal{A} : in other words, if a and b are homotopy classes in \mathcal{A} of degrees (m, i) and (n, j) regarded as polyvector fields on X , then the Schouten-Nijenhuis bracket $[a, b]$ is a homotopy class in \mathcal{A} of degree $(m + n + 1, i + j + 1)$.

Placing a grading on \mathcal{A} is not artificial: indeed, the HKR theorem tells us that the Whitehead filtration $\tau_{\geq -*}\mathrm{HC}(X/\mathbf{C})$ on the Hochschild cohomology of X has associated graded given by

$$(1) \quad \Sigma^{0,i} \mathrm{gr}^{-i}(\tau_{\geq -*}\mathrm{HC}(X/\mathbf{C})) = \pi_{-i}\mathrm{HC}(X/\mathbf{C}) \cong \Gamma(X; (\wedge^i T_{X/\mathbf{C}})[-i]).$$

¹Warning to the reader: there are some issues with the grading that I garbled in the first draft of this document. I’ll fix this soon.

²The definition of a Calabi-Yau variety sometimes includes the additional condition that $\mathrm{H}^i(X; \mathcal{O}_X) = 0$ for $0 < i < \dim(X)$, but this will not be necessary for the moment.

³This is a slight lie: the deformation problem $\mathrm{Def}_{X/\mathbf{C}}$ is really controlled by the sub-Lie algebra of \mathfrak{g} given by $\Gamma(X; T_{X/\mathbf{C}})$. However, for the sake of Theorem 4 below, it suffices to work with \mathfrak{g} instead.

This equivalence can be refined in two ways:

- The Deligne conjecture tells us that $\mathrm{HC}(X/\mathbf{C})$ is an \mathbf{E}_2 -algebra over \mathbf{C} . In particular, $\tau_{\geq -*}\mathrm{HC}(X/\mathbf{C})$ is a *filtered* \mathbf{E}_2 -algebra. Then, (1) upgrades to a multiplicative equivalence of graded commutative \mathbf{C} -algebras between the associated graded of $\tau_{\geq -*}\mathrm{HC}(X/\mathbf{C})$ and \mathcal{A} .
- Let B be any \mathbf{E}_2 -algebra over \mathbf{C} . Then π_*B has the structure of a 1-shifted Poisson algebra, where the Poisson bracket is given by the Browder bracket. The Poisson bracket on $\Sigma^{0,\bullet}\mathrm{gr}^{-\bullet}(\tau_{\geq -*}\mathrm{HC}(X/\mathbf{C}))$ then agrees with the 1-shifted Lie bracket on \mathcal{A} .

Observe that $\mathrm{gr}(\tau_{\geq -*}\Sigma B) \cong \Sigma^{-1,-1}\mathrm{gr}(\tau_{\geq -*}B)$. Therefore, we can construct \mathfrak{g} as

$$(2) \quad \mathfrak{g} \simeq \mathrm{gr}(\tau_{\geq -*}\Sigma\mathrm{HC}(X/\mathbf{C})).$$

(This is not quite correct, since the homological degrees do not match up. However, this can be fixed using the *shearing* operation on graded \mathbf{C} -modules (which we will discuss in greater detail later), which sends a graded \mathbf{C} -module M_\bullet to $M_\bullet[2\bullet]$.) Moreover, $\Sigma\mathrm{HC}(X/\mathbf{C})$ is a dg-Lie algebra over \mathbf{C} , and (2) is an equivalence of graded Lie algebras over \mathbf{C} . Therefore, Theorem 4 will follow if we can prove that $\Sigma\mathrm{HC}(X/\mathbf{C})$ is a homotopy abelian dg-Lie algebra over \mathbf{C} . This is essentially the form of the BTT theorem that we will discuss.

If we wish to prove a version of the BTT theorem over fields of positive characteristic (for instance), we must work with the analogue dg-Lie algebra $\Sigma\mathrm{HC}(X/\mathbf{C})$, or rather the \mathbf{E}_2 - \mathbf{C} -algebra $\mathrm{HC}(X/\mathbf{C})$, itself. However, over characteristic zero, we can use the formality of the \mathbf{E}_2 -operad from Talk III to view the \mathbf{E}_2 - \mathbf{C} -algebra $\mathrm{HC}(X/\mathbf{C})$ as the 1-shifted Poisson algebra \mathcal{A} . (We will mainly ignore the grading on \mathcal{A} , because it can be read off from the homological degree.) We will elect to adopt this point of view for the first part of this talk, because \mathcal{A} is a more concrete object than $\mathrm{HC}(X/\mathbf{C})$.

In summary, our goal is to show that if X is a Calabi-Yau variety, then \mathcal{A} is a homotopy abelian dg-Lie algebra over \mathbf{C} . We will do this by upgrading \mathcal{A} into a Batalin-Vilkovisky algebra.

Construction 6. Choose a volume form ω on X . Then ω defines an isomorphism $\Omega_{X/\mathbf{C}}^{n-k} \cong \wedge^k T_{X/\mathbf{C}}$. In particular, the de Rham differential $d : \Omega_{X/\mathbf{C}}^i \rightarrow \Omega_{X/\mathbf{C}}^{i+1}$ induces a map $\Delta : \wedge^{j+1} T_{X/\mathbf{C}} \rightarrow \wedge^j T_{X/\mathbf{C}}$, and hence a map $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ which increases homological degree (and the graded weight) by 1. Since $d^2 = 0$, the operator Δ also squares to zero. Moreover, $d\Delta + \Delta d = 0$.

The crucial analytic input into Theorem 3 (which we will not prove here) is that the Schouten-Nijenhuis bracket can be expressed in terms of Δ :

$$(3) \quad \Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b) = [a, b].$$

This is known as the *Tian-Todorov lemma*: one can interpret (3) as the claim that the bracket on polyvector fields measures the failure of Δ to be a derivation on \mathcal{A} . We can package the algebraic data above into a definition:

Definition 7. A *Batalin-Vilkovisky algebra* (often shortened to \mathcal{BV} -algebra) over a field k is a pair (A, Δ) of a commutative and associative differential graded algebra A (with internal differential d) over k and a square-zero map $\Delta : A \rightarrow A$ of homological degree 1 such that:

- We have $d\Delta + \Delta d = 0$.
- For each $a \in A$, the operator $[a, -] : A \rightarrow A$ defined by Equation (3) is a derivation.

One can compute using Equation (3) that $\Delta[a, b] = [\Delta(a), b] + (-1)^{|a|+1}[a, \Delta(b)]$. Furthermore, one can extend the above notion to the *graded* setting.

Note that (3) tells us that any \mathcal{BV} -algebra over k has an underlying 1-shifted Poisson algebra. Recall that 1-shifted Poisson algebras are algebras for the Poisson operad \mathbf{P}_2 over k . It turns out that \mathcal{BV} -algebras are also algebras for a particular operad over k , which we will denote \mathcal{BV}_2 ; this is true even in the graded setting, where graded \mathcal{BV} -algebras are algebras for a graded k -operad $\mathcal{BV}_2^{\mathrm{gr}}$. Rather than giving a generators-and-relations construction of \mathcal{BV}_2 , we will just cite the following generalization of a theorem of Fred Cohen's:

Theorem 8. Let $\mathbf{E}_2^{\mathrm{fr}}$ denote the framed \mathbf{E}_2 -operad whose n th space is the space of embeddings $\Pi_n D^2 \hookrightarrow D^2$ which on each component is a composite of translations, dilations, and rotations. In general, there is a morphism $\mathcal{BV}_2^{\mathrm{gr}} \rightarrow \mathrm{gr}(\tau_{\geq *}C_*(\mathbf{E}_2^{\mathrm{fr}}; k))$ of graded operads over k ; this map is an equivalence if k is a field of characteristic zero. One also has a formality statement: over a field k of characteristic zero, the operad $C_*(\mathbf{E}_2^{\mathrm{fr}}; k)$ is isomorphic to its homology.

In particular, the k -homology of a framed \mathbf{E}_2 -algebra is a (graded) \mathcal{BV} -algebra. (Later, we will describe the additional structure that one acquires on the \mathcal{BV} -algebra if $\mathbf{F}_p \subseteq k$.) The \mathcal{BV} -structure on \mathcal{A} that we described above in fact comes from a framed \mathbf{E}_2 -structure on $\mathrm{HC}(X/\mathbf{C})$ itself. We will discuss this further later when discussing the positive characteristic story.

We can finally prove Theorem 4 via the following, which appears in [KKP08]:

Proposition 9 (The ‘‘KKP proposition’’). *Let A be a (graded) \mathcal{BV} -algebra over a field k , and let \hbar be a formal variable of homological degree -2 (and graded weight 1). Let $(A[[\hbar]], d + \hbar\Delta)$ denote the differential graded k -module whose underlying k -module is $A[[\hbar]]$ and whose differential is $d + \hbar\Delta$. Suppose that the homology of $(A[[\hbar]], d + \hbar\Delta)$ is a free $k[[\hbar]]$ -module⁴. If $\mathbf{Q} \subseteq k$, then the dg-Lie algebra $\mathfrak{g} = \Sigma A$ is homotopy abelian.*

Proof of Theorem 4 given Proposition 9. According to Proposition 9, we only need to prove that the homology of $(A[[\hbar]], d + \hbar\Delta)$ is a free $\mathbf{C}[[\hbar]]$ -module. Equivalently, it suffices to show that the spectral sequence going from $E_2^{*,*} = (H_*A)[[\hbar]]$ to $H_*(A[[\hbar]], d + \hbar\Delta)$ degenerates at the E_2 -page. This happens if and only if the spectral sequence degenerates after inverting \hbar . We claim that the spectral sequence

$$(4) \quad E_2^{*,*} = (H_*A)((\hbar)) \Rightarrow H_*(A((\hbar)), d + \hbar\Delta)$$

can be identified with a 2-periodified version of the Hodge-de Rham spectral sequence. More precisely, recall that the Calabi-Yau structure on X allows us to identify $\wedge^k T_{X/\mathbf{C}}$ with $\wedge^{n-k} \Omega_{X/\mathbf{C}}^1$. This gives an identification between $\mathcal{A} = \Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(T_{X/\mathbf{C}}[1]))$ and a shift of $\Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(\Omega_{X/\mathbf{C}}^1[1]))$, which is just (up to the shearing operation described above) the underlying graded vector space of the de Rham complex. Under this identification, the differential $\hbar^{-1}(d + \hbar\Delta) = \hbar^{-1}d + \Delta$ on $\mathcal{A}((\hbar))$ translates to the de Rham differential d_{dR} on $\Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(\Omega_{X/\mathbf{C}}^1[1]))((\hbar))$. Therefore, the spectral sequence (4) can be rewritten (up to some shift) as a 2-periodified version of the Hodge-de Rham spectral sequence⁵

$$E_2^{*,*} = H^*(X; \Omega_{X/\mathbf{C}}^*(\hbar)) \Rightarrow H^*(X; \Omega_{X/\mathbf{C}}^\bullet(\hbar)) = H_{\mathrm{dR}}^*(X/\mathbf{C})(\hbar).$$

However, it is a well-known fact (e.g., by Hodge theory) that the Hodge-de Rham spectral sequence degenerates at the first page, proving the desired result. \square

Remark 10. An important observation is that in the above proof, the only place where it was crucial that X lived over \mathbf{C} (or a characteristic zero field) was when we invoked the degeneration of the Hodge-de Rham spectral sequence. More generally, the above argument shows that when X is a Calabi-Yau variety over a field k (of arbitrary characteristic), the \mathcal{BV} -algebra $\Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}[1]))$ satisfies the freeness hypothesis of Proposition 9 if and only if the Hodge-de Rham spectral sequence for X degenerates at the E_1 -page.

Proof of Proposition 9. We will completely ignore the (external) grading on A in this discussion. First, a specialization argument (i.e., degenerating from the complement of the origin in the formal disk $\mathrm{Spf} k[[\hbar]]$ to $\mathrm{Spec} k$) reduces one to showing that the dg-Lie algebra $\Sigma(A((\hbar)), d + \hbar\Delta)$ over $k((\hbar))$ is homotopy abelian. To prove this, let us collapse the $\mathbf{Z} \times \mathbf{Z}/2$ -grading on $A((\hbar))$ (where the \mathbf{Z} -grading is the homological grading on A , and the $\mathbf{Z}/2$ -grading comes from \hbar) to a $\mathbf{Z}/2$ -grading. Let $\delta : A((\hbar)) \rightarrow A((\hbar))$ denote the $\mathbf{Z}/2$ -graded operator on $A((\hbar))$ given by

$$\delta(a) = da + \hbar\Delta(a) + \frac{[a, a]}{2}.$$

In fact, δ is a derivation on $A((\hbar))$. To show that $A((\hbar))$ is homotopy abelian, it suffices to show that there is some $\mathbf{Z}/2$ -graded automorphism Φ of $A((\hbar))$ such that

$$(5) \quad \delta(\Phi(a)) = d\Phi(a) + \hbar\Delta\Phi(a).$$

Define $\Phi : A((\hbar)) \rightarrow A((\hbar))$ to be the $\mathbf{Z}/2$ -graded automorphism given by

$$\Phi(\hbar) = \hbar, \quad \Phi(a) = \hbar(e^{a/\hbar} - 1) = \sum_{n \geq 1} \frac{a^n}{n! \hbar^{n-1}}.$$

⁴Note that this condition would be vacuous if $\Delta = 0$ on the nose. This would then imply the conclusion of Proposition 9, too, via (3).

⁵The classical Hodge-de Rham spectral sequence runs from $E_1^{*,*} = H^*(X; \Omega_{X/\mathbf{C}}^*)$ to $H_{\mathrm{dR}}^*(X/\mathbf{C})$. Note that it starts at the E_1 -page, whereas below we see the E_2 -page of some other spectral sequence. These spectral sequences are related by a *shearing* process, which we will discuss later. For now, let us just mention the renumbering $E_r^{i,j} \mapsto E_{r+1}^{2i+j, -i}$ for the spectral sequence of a bicomplex.

Here, e^x denotes the formal series $\sum_{m \geq 0} \frac{x^m}{m!}$; in particular, if $|x| \equiv 1 \pmod{2}$, then $x^2 = 0$, so $e^x - 1$ just means x . Observe that $|a^n/\hbar^{n-1}| = n|a| - 2(n-1) \equiv n|a| \pmod{2}$, which agrees with $|a|$ if a is in $\mathbf{Z}/2$ -graded weight 0. To show that Φ satisfies Equation (5), we need to compute $\Delta\Phi(a)$. For this, we need a preliminary calculation: an easy induction using $\Delta(x^2) = 2x\Delta(x) + [x, x]$ shows that

$$\Delta(x^n) = nx^{n-1}\Delta(x) + \binom{n}{2}x^{n-2}[x, x].$$

Therefore,

$$\begin{aligned} \Delta\Phi(a) &= \hbar\Delta e^{a/\hbar} = \sum_{n \geq 0} \frac{1}{n!\hbar^{n-1}}\Delta(a^n) \\ &= \left(\Delta(a) + \frac{1}{2} \left[\frac{a}{\hbar}, \frac{a}{\hbar} \right] \right) e^{a/\hbar}. \end{aligned}$$

This implies the desired identity (5):

$$\begin{aligned} (d + \hbar\Delta)\Phi(a) &= \hbar \left(\frac{da}{\hbar} + \hbar\Delta(a) + \frac{\hbar}{2} \left[\frac{a}{\hbar}, \frac{a}{\hbar} \right] \right) e^{a/\hbar} \\ &= \left(da + \hbar\Delta(a) + \frac{[a, a]}{2} \right) e^{a/\hbar} \\ &= \delta(a)e^{a/\hbar} = \delta(e^{a/\hbar}) = \delta\Phi(a). \end{aligned}$$

□

We now turn to the question of generalizing Theorem 4 to Calabi-Yau varieties over fields of characteristic $p > 0$. We will not be able to prove any such statement in this talk, but we will have more things to say in a few weeks. Recall from Remark 10 that if X is a Calabi-Yau variety over a field k (of arbitrary characteristic), then the $\mathcal{B}\mathcal{V}$ -algebra $\Gamma(X; \text{Sym}_{\mathcal{O}_X}(T_{X/k}[1]))$ satisfies the freeness hypothesis of Proposition 9 if and only if the Hodge-de Rham spectral sequence for X degenerates at the E_1 -page. Therefore, we have two tasks set out for us: first, determine when the Hodge-de Rham spectral sequence for X degenerates at the E_1 -page; and second, prove an analogue of Proposition 9.

Unfortunately, it turns out that the correspondence between formal moduli problems and differential graded Lie algebras fails in characteristic $p > 0$: in particular, the Kodaira-Spencer differential graded Lie algebra *no longer* describes the deformation theory of X . However, one remedy for this failure was proposed in [BM19]. Namely, if k is a field of characteristic $p > 0$, there is a correspondence between formal moduli problems (viewed as certain functors from Artinian \mathbf{E}_∞ - k -algebras to ∞ -groupoids) and “partition Lie algebras”. We will not need the precise definition of partition Lie algebras in the sequel, but we will summarize some of its relationships with other, more well-known, notions.

- (a) If k is a field of characteristic zero, then partition Lie algebras are just 1-shifted differential graded Lie algebras over k .
- (b) Partition Lie algebras are algebras in Mod_k for a monad $\text{Lie}_{\mathbf{E}_\infty}^\pi$. This monad preserves limits and sifted colimits. Partition Lie algebras can be viewed roughly as 1-shifted analogues of restricted differential graded Lie algebras. Let \mathfrak{g} be a differential graded Lie algebra over k with internal differential d , and assume that \mathfrak{g} admits a restricted Lie structure $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ (in the sense of Talk I), so that φ sends a class in degree n to a class in degree np , and $d\varphi(x) = \text{ad}_x^{p-1}dx$. Then $\Sigma\mathfrak{g} = \mathfrak{g}[1]$ is a partition Lie algebra: the bracket on $\pi_*\mathfrak{g}[1]$ sends a class in degree n and a class in degree m to a class in degree $n + m - 1$, while the operator φ sends a class in degree n to a class in degree $p(n-1) + 1 = np - (p-1)$.
- (c) If X is a scheme over k , then $\Gamma(X; T_{X/k}[1])$ is a partition Lie algebra over k . If we abusively attempt to describe the partition Lie algebra as a 1-shifted restricted differential graded Lie algebra, then its Lie bracket is given by the usual bracket of vector fields, while the restricted structure is given by sending a vector field to its p th power (which is still a derivation by the Leibniz formula).
- (d) Let A be an \mathbf{E}_2 - k -algebra. Then $\Sigma^2 A$ admits the structure of a partition Lie algebra. The Lie bracket on $\Sigma^2 A$ is given by the Browder bracket on A , while the restricted Lie structure is given by the first Dyer-Lashof operation \mathcal{Q}_1 . This description is abusive in several ways: first, we are attempting describe the partition Lie algebra as a 1-shifted restricted differential graded Lie algebra; second, the Browder bracket and the Dyer-Lashof

operation \mathcal{Q}_1 are both defined on the *homotopy* of A , and are not actual spectrum-level operations on A itself. However, one can make the above description precise: there is a map from the $\mathrm{Lie}_{\mathbf{E}_\infty}^\pi$ -monad to the “double suspension” of the free \mathbf{E}_2 -algebra monad.

- (e) There is a colimit-preserving functor from $\mathrm{Alg}_{\mathrm{Lie}_{\mathbf{E}_\infty}^\pi}(\mathrm{Mod}_k)$ to $\mathrm{Alg}_{\mathbf{E}_2}^{\mathrm{aug}}(\mathrm{Mod}_k)$, which sends a partition Lie algebra \mathfrak{g} over k to its *universal enveloping \mathbf{E}_2 -algebra* $\mathfrak{U}(\mathfrak{g})$; the functor \mathfrak{U} is left adjoint to the forgetful functor $\mathrm{Alg}_{\mathbf{E}_2}^{\mathrm{aug}}(\mathrm{Mod}_k) \rightarrow \mathrm{Alg}_{\mathrm{Lie}_{\mathbf{E}_\infty}^\pi}(\mathrm{Mod}_k)$ from the previous bullet. This construction can be refined to a functor $\mathfrak{U} : \mathrm{Alg}_{\mathrm{Lie}_{\mathbf{E}_\infty}^\pi}(\mathrm{Mod}_k) \rightarrow \mathrm{coCAlg}(\mathrm{Alg}_{\mathbf{E}_2}^{\mathrm{aug}}(\mathrm{Mod}_k))$. Moreover, if V is a k -module and \mathfrak{g} is the free partition Lie algebra on V , then $\mathfrak{U}(\mathfrak{g})$ is the free augmented \mathbf{E}_2 - k -algebra on V .
- (f) There is a Koszul duality functor $\mathfrak{D} : \mathrm{CAlg}_k^{\mathrm{aug}} \rightarrow \mathrm{Alg}_{\mathrm{Lie}_{\mathbf{E}_\infty}^\pi}(\mathrm{Mod}_k)$ which sends an augmented k -algebra R to the partition Lie algebra $L_{k/R}^\vee[1] \simeq (L_{R/k} \otimes_R k)^\vee$. If \mathfrak{g} is a partition Lie algebra, then the associated formal moduli problem $\Psi_{\mathfrak{g}} : \mathrm{CAlg}_k^{\mathrm{Art}} \rightarrow \mathcal{S}$ is given by

$$\Psi_{\mathfrak{g}}(R) = \mathrm{Map}_{\mathrm{Alg}_{\mathrm{Lie}_{\mathbf{E}_\infty}^\pi}(\mathrm{Mod}_k)}(\mathfrak{D}(R), \mathfrak{g}).$$

Our goal, therefore, is to prove an analogue of Theorem 4 in the case when \mathfrak{g} is replaced by the partition Lie algebra $\Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}[1]))$. Since any deformation of X defines a deformation of $\mathrm{QCoh}(X)$ as a symmetric monoidal ∞ -category, a simpler question is to consider symmetric monoidal deformations of the k -linear (∞ -)category $\mathrm{QCoh}(X)$ instead. It is even easier to consider deformations of $\mathrm{QCoh}(X)$ as an *unstructured* k -linear category (we will return to partition Lie algebras and deformations of X itself at the end of this seminar). One of our goals in future talks will be to run the argument for Theorem 4 in this setting. Namely, we will:

- Recall a proof that the noncommutative Hodge-de Rham spectral sequence for the Hochschild homology of $\mathrm{QCoh}(X)$ degenerates at the E_2 -page, with some assumptions on $\mathrm{QCoh}(X)$ if $\mathbf{F}_p \subseteq k$.
- Show that an analogue of Proposition 9 holds even in characteristic $p > 0$ (using partition Lie algebras instead), at least with some assumptions on A which allow us to make sense of expressions like the exponential. For this, we will give a reproof of Proposition 9, using *operadic* methods instead.

Together, these results will allow us to prove that if X is a Calabi-Yau variety over a field k of characteristic $p > 0$ such that the Hodge-de Rham spectral sequence of X degenerates and $\dim(X) < p$, then the deformation theory of $\mathrm{QCoh}(X)$ is unobstructed. However, it will take us a few more talks before arriving at the point where we can prove such a result.

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