Talk IX: Quantization and Koszul duality

Our goal in this talk is to begin tying up some of the threads discussed in the previous talks. We will set the subject of cyclotomic spectra aside temporarily, and return to it in a couple of talks when we discuss Frobenius-constant quantizations. To motivate our discussion, let us begin by recalling that the BTT theorem for a Calabi-Yau variety X over a field k of characteristic zero states that the Kodaira-Spencer dg-Lie algebra $\Gamma(X; T_X[1])$ is homotopy abelian. The proof we presented had two parts: first, show that the Hodge-de Rham spectral sequence degenerates at the E_1-page for *any* smooth and proper variety; and second, prove a general result relating degeneration of the Hodge-de Rham spectral sequence when X is Calabi-Yau to the homotopy abelianness of $\Gamma(X; T_X[1])$. Passing to the noncommutative setting, we have seen that the Hodge-de Rham spectral sequence generalizes to the Tate spectral sequence for the S^1-action on Hochschild homology. Furthermore, we have seen that if $\mathcal{C}$ is a smooth and proper k-linear ∞-category, then the Tate spectral sequence $\pi_* \mathrm{HH}(\mathcal{C}/k)(\!(\hbar)\!) \Rightarrow \pi_* \mathrm{HP}(\mathcal{C}/k)$ degenerates at the E_2-page. In Talk VIII, we related degeneration of this Tate spectral sequence to the homotopy abelianness of the Hochschild cohomology $\mathrm{HC}(\mathcal{C}/k)$ when $\mathcal{C}$ is "Calabi-Yau".

In this talk, we will explain *why* deformation quantization is related to S^1-actions. To illustrate an instance of this relationship, consider the following:

Example 1. Over a field k of characteristic zero, we know that an S^1-action on a commutative differential graded k-algebra A is equivalent to providing a "mixed differential" $\Delta : A \to A$ which increases homological degree by 1 and anticommutes with the internal differential on A. An example is given by the derived de Rham complex $L\Omega^\bullet_{X/k}$ of a k-scheme X: the underlying commutative differential graded k-algebra is $\mathrm{Sym}_{\mathcal{O}_X}(L_{X/k}[-1])$, and the mixed differential/S^1-action is specified by the de Rham differential d_{dR}. On the other hand, if X is a smooth k-scheme, then it is well-known that the de Rham complex $L\Omega^\bullet_{X/k}$ may be viewed as the Koszul dual[1] of the sheaf $\mathcal{D}_X$ of (k-linear) differential operators, i.e., as $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X)$ in the differential graded category $\mathrm{DMod}(X)$. The sheaf $\mathcal{D}_X$ was one of the motivating examples of deformation quantization: it is a quantization of $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k})$. Furthermore, the Koszul dual $\mathrm{Hom}_{\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k})}(\mathcal{O}_X, \mathcal{O}_X)$ of $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k})$ is precisely $\mathrm{Sym}_{\mathcal{O}_X}(L_{X/k}[-1])$. In a diagram:

$$\begin{array}{ccc} \mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}) & \xrightarrow{\text{def. quant}} & \mathcal{D}_X \\ {\scriptstyle\text{Koszul dual}}\downarrow & & \downarrow{\scriptstyle\text{Koszul dual}} \\ \mathrm{Sym}_{\mathcal{O}_X}(L_{X/k}[-1]) & \xrightarrow[S^1\text{-action}]{} & L\Omega^\bullet_{X/k}. \end{array}$$

In other words, S^1-actions are Koszul dual to deformation quantization.

The main result of this talk is a general result explaining the final sentence of Example 1. To state this result precisely, let us first categorify the notion of deformation quantization; for the moment, we will only discuss deformation quantization of *commutative* algebras, and discuss deformation quantization of Poisson algebras later.

Definition 2 (Preliminary)**.** Let k be an $\mathbf{E}_\infty$-ring, and let $\mathcal{C}_0$ be a k-linear ∞-category. A *deformation quantization* of $\mathcal{C}_0$ is a $k[\![t]\!]$-linear ∞-category $\mathcal{C}$ and an equivalence $\mathcal{C} \otimes_{k[\![t]\!]} k \simeq \mathcal{C}_0$. This definition may be generalized, of course: if A is an Artinian $\mathbf{E}_2$-algebra over k, then a *deformation* of $\mathcal{C}_0$ to A is an A-linear ∞-category $\mathcal{C}_A$ and an equivalence $\mathcal{C}_A \otimes_A k \simeq \mathcal{C}_0$. Let $\mathrm{Def}_{\mathcal{C}_0} : \mathrm{Alg}_{\mathbf{E}_2}(\mathrm{Mod}_k)^{\mathrm{Art}} \to \mathcal{S}$ denote the functor sending A to the space of deformations of $\mathcal{C}_0$ to A. Let $\mathrm{Quant}'_{\mathcal{C}_0}$ denote the space of deformation quantizations of $\mathcal{C}_0$.

The functor $\mathrm{Def}_{\mathcal{C}_0}$ is not quite an $\mathbf{E}_2$-formal moduli problem in the sense of Talk II, but we have the following:

Theorem 3 ([Lur11])**.** *There is a functor* $\mathrm{Def}^\wedge_{\mathcal{C}_0} : \mathrm{Alg}_{\mathbf{E}_2}(\mathrm{Mod}_k)^{\mathrm{Art}} \to \mathcal{S}$ *which is an* $\mathbf{E}_2$*-formal moduli problem, along with a natural transformation* $\theta : \mathrm{Def}_{\mathcal{C}_0} \to \mathrm{Def}^\wedge_{\mathcal{C}_0}$*. For each Artinian* $\mathbf{E}_2$*-*k*-algebra* A*, the map* $\theta : \mathrm{Def}_{\mathcal{C}_0}(A) \to \mathrm{Def}^\wedge_{\mathcal{C}_0}(A)$ *induces an isomorphism on* π_n *for* $n \geq 2$*, and is injective on* π_1 *(where all the homotopy groups are based at the trivial deformation* $\mathcal{C}_0 \otimes_k A$*).*

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[1]This is not quite correct: $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X)$ is the *Hodge-completion* of $L\Omega^\bullet_{X/k}$. One fix for this is to work in the general setting of filtered k-modules instead, as we will discuss below.

Remark 4. Citing [Lur11] for Theorem 3 is somewhat abusive: in [Lur11], the preceding result is proved only when k is a field. To get the general result (over more general k), one must appeal to the Koszul self-duality of the $\mathbf{E}_n$-operad itself (and not just $C_*(\mathbf{E}_n; k)$). A full proof of this fact was recently given in [CS20].

There is an explicit description of $\mathrm{Def}^{\wedge}_{\mathcal{C}_0}(A)$. To state it, let us recall the following definition.

Definition 5. Let B be an augmented $\mathbf{E}_1$-k-algebra. The bar construction $\mathrm{Bar}(A)$ is defined to be $k \otimes_A k$; note that $\mathrm{Bar}(A)$ is an $\mathbf{E}_1$-k-coalgebra whose diagonal is given by

$$k \otimes_A k \to k \otimes_A A \otimes_A k \xrightarrow{\mathrm{id}\otimes_A \epsilon \otimes_A \mathrm{id}} (k \otimes_A k) \otimes_k (k \otimes_A k),$$

so the k-linear dual of $\mathrm{Bar}(A)$ is an $\mathbf{E}_1$-k-algebra.

Let A be an augmented $\mathbf{E}_2$-k-algebra. The $\mathbf{E}_2$-*Koszul dual* $\mathfrak{D}^{[2]}(A)$ is defined to be the $\mathbf{E}_2$-k-algebra which is the k-linear dual of the 2-fold bar construction $\mathrm{Bar}^{[2]}(A) = \mathrm{Bar}(\mathrm{Bar}(A))$.

Theorem 6. *Let A be an Artinian $\mathbf{E}_2$-k-algebra. Then there is a natural equivalence*

$$\mathrm{Def}^{\wedge}_{\mathcal{C}_0}(A) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_2}(\mathrm{Mod}_k)}(\mathfrak{D}^{[2]}(A), \mathrm{HC}(\mathcal{C}_0/k)). \tag{1}$$

The functor $\mathrm{Def}^{\wedge}_{\mathcal{C}_0}$ naturally extends to pro-Artinian $\mathbf{E}_2$-k-algebras, and (1) *still remains valid.*

One of the categorical properties of Hochschild cohomology is that if B is an $\mathbf{E}_2$-k-algebra, then an $\mathbf{E}_2$-k-algebra map $f : B \to \mathrm{HC}(\mathcal{C}_0/k)$ is equivalent to specifying a k-linear *action* of B on $\mathcal{C}_0$. In other words, the map f provides a lift of $\mathcal{C}_0$ to a B-linear ∞-category. Roughly, this is because the Hochschild cohomology of $\mathcal{C}_0$ is to be viewed as the k-linear *center* of $\mathcal{C}_0$; then, the preceding statement is analogous to the algebraic fact that a map $R \to Z(S)$ from a commutative ring R to the center of an associative ring S is equivalent to promoting S to an R-algebra. This in fact defines an equivalence

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_2}(\mathrm{Mod}_k)}(B, \mathrm{HC}(\mathcal{C}_0/k)) \simeq \mathrm{LinCat}^{\simeq}_B \times_{\mathrm{LinCat}^{\simeq}_k} \{\mathcal{C}_0\}.$$

Combining this with (1), we conclude that if A is a pro-Artinian $\mathbf{E}_2$-k-algebra, then

$$\mathrm{Def}^{\wedge}_{\mathcal{C}_0}(A) \simeq \mathrm{LinCat}^{\simeq}_{\mathfrak{D}^{[2]}(A)} \times_{\mathrm{LinCat}^{\simeq}_k} \{\mathcal{C}_0\}. \tag{2}$$

Example 7. The crucial example relevant for deformation quantization is the case $A = k[\![t]\!]$ (where k is any $\mathbf{E}_\infty$-ring, not necessarily a field). Let us view $k[\![t]\!]$ as the completion of $k[\mathbf{Z}_{\geq 0}]$. Then, $\mathrm{Bar}^{[2]}(k[\![t]\!])$ can be identified with the underlying $\mathbf{E}_2$-coalgebra of the $\mathbf{E}_\infty$-coalgebra $k[B^2\mathbf{Z}_{\geq 0}] \simeq k[\mathbf{C}P^\infty]$; in other words, $\mathfrak{D}^{[2]}(k[\![t]\!]) \simeq k^{\mathbf{C}P^\infty}$. We conclude from (2) that

$$\mathrm{Quant}'_{\mathcal{C}_0} \twoheadrightarrow \mathrm{Def}^{\wedge}_{\mathcal{C}_0}(k[\![t]\!]) \simeq \mathrm{LinCat}^{\simeq}_{k^{\mathbf{C}P^\infty}} \times_{\mathrm{LinCat}^{\simeq}_k} \{\mathcal{C}_0\}.$$

The following result is *incorrect*, because of finiteness issues which we will not address here. If incorporated correctly, though, a version of this result is indeed true.

Proposition 8. *Let k be an $\mathbf{E}_\infty$-ring. Then there is an equivalence* $\mathrm{LinCat}_{k^{\mathbf{C}P^\infty}} \simeq \mathrm{LinCat}_{k[S^1]}$ *of ∞-categories.*

Proof. Recall that $\mathrm{LinCat}_A = \mathrm{LMod}_{\mathrm{LMod}_A}(\mathrm{Pr}^{\mathrm{L,st}})$; therefore, it suffices to show that $\mathrm{LMod}_{k[S^1]} \simeq \mathrm{LMod}_{k^{\mathbf{C}P^\infty}}$ as (symmetric) monoidal ∞-categories, where the left-hand side is given the convolution monoidal structure with respect to the $\mathbf{E}_\infty$-k-algebra structure on $k[S^1]$. First, we claim that if X is a connected and simply-connected *finite CW-complex*, then $\mathrm{LMod}_{k^X} \simeq \mathrm{Fun}(X, \mathrm{Mod}_k)$. Let $\mathrm{pr} : X \to *$ be the crushing functor, and let $\mathrm{pr}_* : \mathrm{Fun}(X, \mathrm{Mod}_k) \to \mathrm{Mod}_k$ be the functor given by pushforward. Explicitly, this takes a bundle of k-modules over X to its cohomology. We claim that pr_* is monadic; for this, we check the conditions of the Barr-Beck-Lurie theorem. First, observe that pr_* is a right adjoint to the pullback pr^*, and therefore preserves all limits. Next, since X is assumed to be a *finite* CW-complex, the functor pr_* also preserves all colimits (in particular, geometric realizations). Finally, we need to show that pr_* is conservative, i.e., that cohomology detects zero objects. This is a consequence of the assumption that X is connected and simply-connected. Of course, the space $\mathbf{C}P^\infty$ does not satisfy the hypothesis that X be a finite CW-complex, so we cannot directly apply the equivalence $\mathrm{LMod}_{k^X} \simeq \mathrm{Fun}(X, \mathrm{Mod}_k)$. However, this can be fixed by working with *filtered* modules over $\mathrm{F}^\star_{\mathbf{C}P} k^{\mathbf{C}P^\infty}$ instead.

We now prove a more general result: if X is a connected space, let us abusively write X for the Kan complex associated to X. Then $\mathrm{LMod}_{k[\Omega X]} \simeq \mathrm{Fun}(X, \mathrm{Mod}_k)$; moreover, if X is an $\mathbf{E}_n$-space, then this equivalence is $\mathbf{E}_n$-monoidal. This result is known as *Koszul duality*: it is a generalization of the classical correspondence between representations of $\pi_1(X)$ and local systems on X. Let

$x : * \to X$ be a point of X; then, $x^* : \mathrm{Fun}(X, \mathrm{Mod}_k) \to \mathrm{Mod}_k$ admits a left adjoint; furthermore, $x_! k$ is a compact generator of $\mathrm{Fun}(X, \mathrm{Mod}_k)$, because $\mathrm{Map}_{\mathrm{Fun}(X,\mathrm{Mod}_k)}(x_! k, M) \simeq \mathrm{Map}_k(k, x^* M)$. If this vanishes, then $x^* M = 0$, which implies that $M = 0$ by the connectedness assumption on X. It follows that $\mathrm{Fun}(X, \mathrm{Mod}_k) \simeq \mathrm{LMod}_A$, where $A = \mathrm{Map}_{\mathrm{Fun}(X,\mathrm{Mod}_k)}(x_! k, x_! k) \simeq \mathrm{Map}_k(k, x^* x_! k)$. But this is just $k[\Omega X]$, as desired. □

Finally, combining with Example 7, we see[2]:

Proto-Theorem 9. *Let k be an $\mathbf{E}_\infty$-ring, and let $\mathcal{C}_0$ be a k-linear ∞-category. Then there is a map*

$$\mathrm{Quant}'_{\mathcal{C}_0} \twoheadrightarrow \mathrm{LinCat}^{\simeq}_{k\mathbf{C}P^\infty} \times_{\mathrm{LinCat}^{\simeq}_k} \{\mathcal{C}_0\} \simeq \mathrm{LinCat}^{\simeq}_{k[S^1]} \times_{\mathrm{LinCat}^{\simeq}_k} \{\mathcal{C}_0\}.$$

In other words, deformation quantizations of $\mathcal{C}_0$ are (almost) equivalent to S^1-actions on $\mathcal{C}_0$. Intuitively, this functor sends a quantization $\mathcal{C}$ of $\mathcal{C}_0$ to the S^1-action on $\mathcal{C}_0$ given by monodromy about the origin in $\mathbf{A}^1 = \mathrm{Spec}\, k[t]$.

Recall that we said that Definition 2 is a preliminary definition: the "correct" definition involves replacing $k[t]$ with the $\mathbf{E}_2$-algebra k^{hS^1}. If k is complex-oriented (e.g., k is a field), we may identify this $\mathbf{E}_2$-algebra with $k[\![\hbar]\!]$, where $\hbar$ is in homological degree -2. Let us now give the "correct" definition of deformation quantization; after stating Proto-Theorem 11, we will discuss why this modification is natural:

Definition 10. Let k be an $\mathbf{E}_\infty$-ring, and let $\mathcal{C}_0$ be a filtered (resp. graded) k-linear ∞-category. Let us reuse the phase *filtered (resp. graded) deformation quantization* of $\mathcal{C}_0$ to mean a $\mathrm{F}^\star_{\mathbf{C}P} k^{hS^1}$-linear (resp. $k[\![\hbar]\!]$-linear) ∞-category $\mathcal{C}$ and an equivalence $\mathcal{C} \otimes_{\mathrm{F}^\star_{\mathbf{C}P} k^{hS^1}} k \simeq \mathcal{C}_0$ of filtered k-linear ∞-categories (similarly in the graded setting). Let $\mathrm{Quant}^{\mathrm{fil}}_{\mathcal{C}_0}$ (not $\mathrm{Quant}'_{\mathcal{C}_0}$) denote the space of filtered deformation quantizations of $\mathcal{C}_0$. Similarly for $\mathrm{Quant}^{\mathrm{gr}}_{\mathcal{C}_0}$.

The same argument as in Proto-Theorem 9 establishes (with the same caveat about finiteness conditions being ignored) the following result; see [Toe14, Theorem 5.1].

Proto-Theorem 11. *Let k be an $\mathbf{E}_\infty$-ring, and let $\mathcal{C}_0$ be a filtered k-linear ∞-category. Then there are equivalences*

$$\mathrm{Quant}^{\mathrm{fil}}_{\mathcal{C}_0} \simeq \mathrm{LinCat}^{\mathrm{fil},\simeq}_{\mathrm{F}^\star k[S^1]} \times_{\mathrm{LinCat}^{\mathrm{fil},\simeq}_k} \{\mathcal{C}_0\},$$
$$\mathrm{Quant}^{\mathrm{gr}}_{\mathcal{C}_0} \simeq \mathrm{LinCat}^{\mathrm{gr},\simeq}_{k[\epsilon]/\epsilon^2} \times_{\mathrm{LinCat}^{\mathrm{gr},\simeq}_k} \{\mathcal{C}_0\},$$

where $\mathrm{F}^\star k[S^1]$ is the filtered k-algebra associated to the filtered space $* \to S^1 \to S^1 \to \cdots$ and $k[\epsilon]/\epsilon^2$ is the graded k-algebra where ϵ is placed in weight 1 and homological degree 1. In other words, deformation quantizations of $\mathcal{C}_0$ are Koszul dual to S^1-actions on $\mathcal{C}_0$. Intuitively, this functor sends a quantization $\mathcal{C}$ of $\mathcal{C}_0$ to the S^1-action on $\mathcal{C}_0$ given by monodromy about the "origin" in $\mathrm{Spec}\, \mathrm{F}^\star_{\mathbf{C}P} k^{hS^1}$.

In Proto-Theorem 11, one has to be careful about the symbol $\mathrm{Spec}\, \mathrm{F}^\star_{\mathbf{C}P} k^{hS^1}$ because, as we proved in Talk VI, $\mathrm{F}^\star_{\mathbf{C}P} k^{hS^1}$ is only a filtered $\mathbf{E}_2$-*algebra*, and this cannot be refined to a filtered $\mathbf{E}_\infty$-algebra structure.

The remainder of this (and the following) talk will be devoted to discussing the following two questions:

- Why are Definition 2 (and Definition 10) concerned with deformation quantizations of commutative algebras viewed as Poisson algebras with trivial bracket? What can be said when the Poisson bracket is nonzero?
- Why is Definition 10 is more natural than the preliminary Definition 2? We will justify this modification through examples, and see that it is also naturally suggested by Nekrasov's Ω-deformation.

Let us begin by discussing the first bullet. The answer to the first question is just that the story of deformation quantizations of commutative algebras viewed as Poisson algebras with trivial bracket is simpler. In a sense, this story is equivalent to the story where the Poisson bracket is nontrivial: recall from the proof of Kontsevich's theorem that a key step was showing that if $(A_0, \{-,-\})$ is a Poisson algebra, then the moduli problems of deforming the multiplication on A_0 and deforming the Poisson bracket on A_0 are equivalent. It is therefore natural to ask if the perspective of S^1-actions as

[2]Again, one should keep in mind the caveat that we are ignoring important finiteness assumptions.

being dual to deformation quantizations also incorporates the case when Poisson bracket is nonzero. The answer to this question is yes, as we will see when discussing Nekrasov's Ω-deformation.

Let us now turn to the second bullet; our discussion will be far more extensive. We begin by observing that if k is an $\mathbf{E}_\infty$-ring with a $\mathbf{E}_\infty$-complex-orientation, then any quantization in the sense of (the graded analogue of) Definition 2 gives rise to a graded quantization in the sense of Definition 10. To describe this construction, we need a definition.

Definition 12. The *shearing* construction $\mathrm{sh} : \mathrm{Mod}_k^{\mathrm{gr}} \to \mathrm{Mod}_k^{\mathrm{gr}}$ sends a graded object $M \in \mathrm{Mod}_k^{\mathrm{gr}}$ to the graded object whose nth piece is $\Sigma^{-2n} M_n$. Clearly, sh is an equivalence, with inverse given by the functor $\mathrm{Mod}_k^{\mathrm{gr}} \to \mathrm{Mod}_k^{\mathrm{gr}}$ sending a graded object $M \in \mathrm{Mod}_k^{\mathrm{gr}}$ to the graded object whose nth piece is $\Sigma^{2n} M_n$.

See [Rak20] for further discussion of the following:

Proposition 13. *Let k be an $\mathbf{E}_\infty$-ring, and let $\mathrm{Mod}_k^{\mathrm{gr}}$ denote the ∞-category of graded k-modules.*

(a) *The functors $\mathrm{sh}, \mathrm{sh}^{-1} : \mathrm{Mod}_k^{\mathrm{gr}} \to \mathrm{Mod}_k^{\mathrm{gr}}$ admit $\mathbf{E}_2$-monoidal structures (for the Day convolution monoidal structures on $\mathrm{Mod}_k^{\mathrm{gr}}$).*

(b) *If the base $\mathbf{E}_\infty$-ring k admits an $\mathbf{E}_\infty$-complex-orientation, then these $\mathbf{E}_2$-monoidal functors admit refinements to symmetric monoidal functors. Moreover, each choice of an $\mathbf{E}_\infty$-complex-orientation defines a particular symmetric monoidal refinement.*

Proof. There is a map $\phi : \mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(\mathbb{S})$ sending $n \mapsto \Sigma^{2n}\mathbb{S}$. Let us begin by showing that it suffices to prove the following two claims: ϕ admits the structure of an $\mathbf{E}_2$-map, and the composite $\mathbf{Z}^{\mathrm{ds}} \xrightarrow{\phi} \mathrm{Pic}(\mathbb{S}) \to \mathrm{Pic}(k)$ admits the structure of an $\mathbf{E}_\infty$-map if k admits an $\mathbf{E}_\infty$-complex orientation. Indeed, observe that defining a lax $\mathbf{E}_2$-monoidal structure on the functor $\mathrm{sh}^{-1} : \mathrm{Mod}_k^{\mathrm{gr}} \to \mathrm{Mod}_k^{\mathrm{gr}}$ is equivalent to a lax $\mathbf{E}_2$-monoidal functor $\mathbf{Z}^{\mathrm{ds}} \times \mathrm{Mod}_k^{\mathrm{gr}} \to \mathrm{Mod}_k^{\mathrm{gr}}$. This is given by the composite of the lax $\mathbf{E}_2$-monoidal functor $\mathbf{Z}^{\mathrm{ds}} \times \mathrm{Mod}_k^{\mathrm{gr}} \to \mathrm{Pic}(\mathbb{S}) \times \mathrm{Mod}_k^{\mathrm{gr}}$ sending $(n, X^\bullet) \mapsto (\phi(n), X^n)$ with the symmetric monoidal functor $\mathrm{Pic}(\mathbb{S}) \times \mathrm{Mod}_k^{\mathrm{gr}}$. Similarly, in the $\mathbf{E}_\infty$-complex-oriented case, the lax symmetric monoidal functor $\mathbf{Z}^{\mathrm{ds}} \times \mathrm{Mod}_k^{\mathrm{gr}} \to \mathrm{Mod}_k^{\mathrm{gr}}$ is given by the composite of the lax symmetric monoidal functor $\mathbf{Z}^{\mathrm{ds}} \times \mathrm{Mod}_k^{\mathrm{gr}} \to \mathrm{Pic}(k) \times \mathrm{Mod}_k^{\mathrm{gr}}$ with the symmetric monoidal functor $\mathrm{Pic}(\mathbb{S}) \times \mathrm{Mod}_k^{\mathrm{gr}}$. Finally, it is easy to check that a lax $\mathbf{E}_2$-monoidal (or lax symmetric monoidal) structure on the functor $\mathrm{sh}^{-1} : \mathrm{Mod}_k^{\mathrm{gr}} \to \mathrm{Mod}_k^{\mathrm{gr}}$ is in fact strictly $\mathbf{E}_2$-monoidal (resp. strictly symmetric monoidal).

Let us now prove the claim about the multiplicative structure on ϕ. The claim that $\phi : \mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(\mathbb{S})$ admits the structure of an $\mathbf{E}_2$-monoidal map is well-known: it can be understood as the composite

$$\mathbf{Z}^{\mathrm{ds}} \simeq \Omega^2 \mathbf{C}P^\infty \to \Omega^2 \mathrm{BU} \xrightarrow{\mathrm{Bott}} \mathrm{BU} \times \mathbf{Z} \xrightarrow{J} \mathrm{Pic}(\mathbb{S}),$$

where the map J is the J-homomorphism. The Bott map admits an $\mathbf{E}_2$-structure (see [Lur15] for a modern proof of this fact), giving the desired $\mathbf{E}_2$-structure on ϕ. We now prove that $\phi : \mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(k)$ admits the structure of an $\mathbf{E}_\infty$-monoidal map if k admits an $\mathbf{E}_\infty$-complex orientation. We may assume that $k = \mathrm{MU}$. Since MU is the Thom spectrum of the $\mathbf{E}_\infty$-map $\mathrm{BU} \xrightarrow{J} \mathrm{Pic}(\mathbb{S})$, it can be understood as the initial $\mathbf{E}_\infty$-ring R equipped with a nullhomotopy of the $\mathbf{E}_\infty$-map $\mathrm{BU} \xrightarrow{J} \mathrm{Pic}(\mathbb{S}) \to \mathrm{Pic}(R)$. In particular, there is a commutative diagram of $\mathbf{E}_\infty$-maps:

$$\begin{array}{ccc} \mathrm{BU} & \xrightarrow{J} & \mathrm{Pic}(\mathbb{S}) \\ \downarrow & & \downarrow \\ \mathbf{Z}^{\mathrm{ds}} & \longrightarrow & \mathrm{Pic}(\mathrm{MU}), \end{array}$$

which proves the desired claim. □

Example 14. The key example is that if k is an $\mathbf{E}_\infty$-ring, and $k[\![t]\!]$ denotes the graded flat polynomial k-algebra where t is placed in homological degree 0 and weight 1, then $\mathrm{sh}(k[\![t]\!]) \simeq k[\![\hbar]\!]$ as $\mathbf{E}_2$-k-algebras. Therefore, if $\mathcal{C}_0$ is a graded k-linear ∞-category, and $\mathcal{C}$ is a graded deformation quantization of $\mathcal{C}_0$ in the sense of Definition 2, then $\mathrm{sh}(\mathcal{C}) := \mathcal{C} \otimes_{\mathrm{Mod}_k^{\mathrm{gr}}} {}^{\mathrm{sh}}\mathrm{Mod}_k^{\mathrm{gr}}$ is a graded deformation quantization of $\mathcal{C}_0$ in the sense of Definition 10.

Concretely, if A_0 is a graded commutative k-algebra, A is a graded deformation quantization of A_0, $\mathcal{C}_0 = \mathrm{Mod}_{A_0}$, and $\mathcal{C} = \mathrm{Mod}_A$, then $\mathrm{sh}(\mathcal{C}) \simeq \mathrm{Mod}_{\mathrm{sh}(A)}$. For instance, in the setting of Example 1, we have $A_0 = \mathrm{Sym}_{\mathcal{O}_X}(T_X)$ with T_X in weight 1 and $A = \mathcal{D}_X$; then, $\mathrm{sh}(\mathcal{D}_X)$ is the quotient of the free associative $\mathcal{O}_X[\![\hbar]\!]$-algebra generated by $T_X[-2]$ subject to the relation $[x, \partial_x] = \hbar$. One often considers a variant of this algebra, where T_X is placed in homological degree 2 (as opposed to homological degree -2); this variant is $\mathrm{sh}^{-1}(\mathcal{D}_X)$, and it is an algebra over $\mathrm{sh}^{-1}(k[\![t]\!]) \simeq k[\sigma]$ instead.

In the next talk, we will describe examples from both mathematics and physics where the homological shift in Definition 10 naturally shows up.

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