

Higher chromatic Thom spectra via unstable homotopy theory

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ABSTRACT. We investigate implications of an old conjecture in unstable homotopy theory related to the Cohen-Moore-Neisendorfer theorem and a conjecture about the \mathbf{E}_2 -topological Hochschild cohomology of certain Thom spectra (denoted A , B , and $T(n)$) related to Ravenel's $X(p^n)$. We show that these conjectures imply that the orientations $M\text{Spin} \rightarrow \text{bo}$ and $M\text{String} \rightarrow \text{tmf}$ admit spectrum-level splittings. This is shown by generalizing a theorem of Hopkins and Mahowald, which constructs \mathbf{HF}_p as a Thom spectrum, to construct $\text{BP}\langle n-1 \rangle$, bo , and tmf as Thom spectra (albeit over $T(n)$, A , and B respectively, and not over the sphere). This interpretation of $\text{BP}\langle n-1 \rangle$, bo , and tmf offers a new perspective on Wood equivalences of the form $\text{bo} \wedge C\eta \simeq \text{bu}$; they are related to the existence of certain EHP sequences in unstable homotopy theory. This construction of $\text{BP}\langle n-1 \rangle$ also provides a different lens on the nilpotence theorem. Finally, we prove a C_2 -equivariant analogue of our construction, describing $\underline{\mathbf{HZ}}$ as a Thom spectrum.

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1. Introduction

1.1. Statement of the main results. One of the goals of this article is to describe a program to prove the following old conjecture (studied, for instance, in [Lau04, LS19], and discussed informally in many places, such as [MR09, Section 7]):

Conjecture 1.1.1. *The Ando-Hopkins-Rezk orientation (see [AHR10]) $MString \rightarrow \text{tmf}$ admits a spectrum-level splitting.*

The key idea in our program is to provide a universal property for mapping out of the spectrum tmf . We give a proof which is conditional on an old conjecture from unstable homotopy theory stemming from the Cohen-Moore-Neisendorfer theorem and a conjecture about the \mathbf{E}_2 -topological Hochschild cohomology of certain Thom spectra (the latter of which simplifies the proof of the nilpotence theorem from [DHS88]). This universal property exhibits tmf as a certain Thom spectrum, similarly to the Hopkins-Mahowald construction of \mathbf{HZ}_p and \mathbf{HF}_p as Thom spectra.

To illustrate the gist of our argument in a simpler case, recall Thom’s classical result from [Tho54]: the unoriented cobordism spectrum MO is a wedge of suspensions of \mathbf{HF}_2 . The simplest way to do so is to show that MO is an \mathbf{HF}_2 -module, which in turn can be done by constructing an \mathbf{E}_2 -map $\mathbf{HF}_2 \rightarrow \text{MO}$. The construction of such a map is supplied by the following theorem of Hopkins and Mahowald:

Theorem (Hopkins-Mahowald; see [Mah79] and [MRS01, Lemma 3.3]). *Let $\mu : \Omega^2 S^3 \rightarrow \text{BO}$ denote the real vector bundle over $\Omega^2 S^3$ induced by extending the map $S^1 \rightarrow \text{BO}$ classifying the Möbius bundle. Then the Thom spectrum of μ is equivalent to \mathbf{HF}_2 as an \mathbf{E}_2 -algebra.*

Remark 1.1.2. The Thomification of the \mathbf{E}_2 -map $\mu : \Omega^2 S^3 \rightarrow \text{BO}$ produces the desired \mathbf{E}_2 -splitting $\mathbf{HF}_2 \rightarrow \text{MO}$.

Our argument for Conjecture 1.1.1 takes this approach: we shall show that an old conjecture from unstable homotopy theory and a conjecture about the \mathbf{E}_2 -topological Hochschild cohomology of certain Thom spectra provide a construction of tmf (as well as bo and $\text{BP}\langle n \rangle$) as a Thom spectrum, and utilize the resulting universal property of tmf to construct an (unstructured) map $\text{tmf} \rightarrow MString$.

Mahowald was the first to consider the question of constructing spectra like bo and tmf as Thom spectra (see [Mah87]). Later work by Rudyak in [Rud98] sharpened Mahowald’s results to show that bo and bu cannot appear as the Thom spectrum of a p -complete spherical fibration. In [AHL09], Angeltveit-Hill-Lawson gave an alternative proof of this fact under the assumption that the p -complete spherical fibration is classified by a map of \mathbf{E}_3 -spaces. Recently, Chatham has shown in [Cha19] that tmf_2^\wedge cannot appear as the Thom spectrum of a structured 2-complete spherical fibration over a loop space. Our goal is to argue that these issues are alleviated if we replace “spherical fibrations” with “bundles of R -lines” for certain well-behaved spectra R .

The first hint of tmf being a generalized Thom spectrum comes from a conjecture of Hopkins and Hahn regarding a construction of the truncated Brown-Peterson spectra $\text{BP}\langle n \rangle$ as Thom spectra. To state this conjecture, we need to recall some definitions. Recall (see [DHS88]) that $X(n)$ denotes the Thom spectrum of the map $\Omega\text{SU}(n) \rightarrow \Omega\text{SU} \simeq \text{BU}$. Upon completion at a prime p , the spectra $X(k)$

for $p^n \leq k \leq p^{n+1} - 1$ split as a direct sum of suspensions of certain homotopy commutative ring spectra $T(n)$, which in turn filter the gap between the p -complete sphere spectrum and BP (in the sense that $T(0) = \mathbb{S}$ and $T(\infty) = \text{BP}$). Then:

Conjecture 1.1.3 (Hahn, Hopkins; unpublished). *There is a map $f : \Omega^2 S^{|v_n|+3} \rightarrow \text{BGL}_1(T(n))$, which detects an indecomposable element $v_n \in \pi_{|v_n|} T(n)$ on the bottom cell of the source, whose Thom spectrum is a form of $\text{BP}\langle n-1 \rangle$.*

The primary obstruction to proving that a map f as in Conjecture 1.1.3 exists stems from the failure of $T(n)$ to be an \mathbf{E}_3 -ring (due to Lawson; [Law20, Example 1.5.31]). If R is an \mathbf{E}_1 - or \mathbf{E}_2 -ring spectrum, let $\mathfrak{Z}_3(R)$ denote the \mathbf{E}_2 -topological Hochschild cohomology of R (see Definition 3.3.2). Hahn suggested that one way to get past the failure of $T(n)$ to be an \mathbf{E}_3 -ring would be via the following conjecture:

Conjecture 1.1.4 (Hahn). *There is an indecomposable element $v_n \in \pi_{|v_n|} T(n)$ which lifts to the \mathbf{E}_2 -topological Hochschild cohomology $\mathfrak{Z}_3(X(p^n))$ of $X(p^n)$.*

We do not know how to prove this conjecture (and have no opinion on whether or not it is true). We shall instead show that Conjecture 1.1.3 is implied by the two conjectures alluded to above. We shall momentarily state these conjectures precisely as Conjectures D and E; let us first state our main results.

We need to introduce some notation. Let $y(n)$ (resp. $yz(n)$) denote the Mahowald-Ravenel-Shick spectrum, constructed as a Thom spectrum over $\Omega J_{p^n-1}(S^2)$ (resp. $\Omega J_{p^n-1}(S^2)\langle 2 \rangle$) introduced in [MRS01] to study the telescope conjecture (resp. in [AQ19] as $z(n)$). Let A denote the \mathbf{E}_1 -quotient \mathbb{S}/ν of the sphere spectrum by $\nu \in \pi_3(\mathbb{S})$; its mod 2 homology is $H_*(A) \cong \mathbf{F}_2[\zeta_1^4]$. The spectrum A has been intensely studied by Mahowald and his coauthors in (for instance) [Mah79, DM81, Mah81b, Mah81a, Mah82, MU77], where it is often denoted X_5 . (See Remark 2.1.8 for motivation for the term “ \mathbf{E}_1 -quotient”.) Let B denote the \mathbf{E}_1 -ring introduced in [Dev19b, Construction 3.1]; it has been briefly studied under the name \overline{X} in [HM02]. It may be constructed as the Thom spectrum of a vector bundle over an \mathbf{E}_1 -space N which sits in a fiber sequence $\Omega S^9 \rightarrow N \rightarrow \Omega S^{13}$. The mod 2 homology of B is $H_*(B) \cong \mathbf{F}_2[\zeta_1^8, \zeta_2^4]$.

We also need to recall some unstable homotopy theory. In [CMN79a, CMN79b, Nei81], Cohen, Moore, and Neisendorfer constructed a map $\phi_n : \Omega^2 S^{2n+1} \rightarrow S^{2n-1}$ whose composite with the double suspension $E^2 : S^{2n-1} \rightarrow \Omega^2 S^{2n+1}$ is the degree p map. (The symbol E stands for “Einhängung”, which is German for “suspension”.) Such a map was also constructed by Gray in [Gra89b, Gra88]. In Section 4.1, we introduce the related notion of a *charming map* (Definition 4.1.1), one example of which is the Cohen-Moore-Neisendorfer map.

Our main result is then:

Theorem A. *Suppose R is a base spectrum of height n as in the second line of Table 1. Let K_{n+1} denote the fiber of a charming map $\Omega^2 S^{2p^{n+1}+1} \rightarrow S^{2p^{n+1}-1}$. Then Conjectures D and E imply that there is a map $\mu : K_{n+1} \rightarrow \text{BGL}_1(R)$ such that the mod p homology of the Thom spectrum K_{n+1}^μ is isomorphic to the mod p homology of the associated designer chromatic spectrum $\Theta(R)$ as a Steenrod comodule¹.*

If R is any base spectrum other than B , the Thom spectrum K_{n+1}^μ is equivalent to $\Theta(R)$ upon p -completion for every prime p . If Conjecture F is true, then the

¹We elected to use the symbol Θ because the first two letters of the English spelling of Θ and of Thom’s name agree.

same is true for B : the Thom spectrum K_{n+1}^μ is equivalent to $\Theta(B) = \mathrm{tmf}$ upon 2-completion.

Height	0	1	2	n	n	n
Base spectrum R	\mathbb{S}_p^\wedge	A	B	$T(n)$	$y(n)$	$yz(n)$
Designer chromatic spectrum $\Theta(R)$	\mathbf{HZ}_p	bo	tmf	$\mathrm{BP}\langle n \rangle$	$k(n)$	$k_{\mathbf{Z}}(n)$

TABLE 1. To go from a base spectrum “of height n ”, say R , in the second line to the third, one takes the Thom spectrum of a bundle of R -lines over K_{n+1} .

Making sense of Theorem A relies on knowing that $T(n)$ admits the structure of an \mathbf{E}_1 -ring; this is proved in [BL21]; see also Warning 3.1.6. Note that the spectra A , B , $y(n)$, and $yz(n)$ all admit \mathbf{E}_1 -structures by construction. In Remark 5.4.7, we sketch how Theorem A relates to the proof of the nilpotence theorem.

Although the form of Theorem A does not resemble Conjecture 1.1.3, we show that Theorem A implies the following result.

Corollary B. *Conjectures D and E imply Conjecture 1.1.3.*

In the case $n = 0$, Corollary B recovers the Hopkins-Mahowald theorem constructing \mathbf{HF}_p . Moreover, Corollary B is true unconditionally when $n = 0, 1$.

Using the resulting universal property of tmf , we obtain a result pertaining to Conjecture 1.1.1.

Theorem C. *Assume that the composite $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \mathrm{MString}$ is an \mathbf{E}_3 -map. Then Conjectures D, E, and F imply that there is a spectrum-level unital splitting of the Ando-Hopkins-Rezk orientation $\mathrm{MString}_{(2)} \rightarrow \mathrm{tmf}_{(2)}$.*

In particular, Conjecture 1.1.1 follows (at least after localizing at $p = 2$; a slight modification of our arguments should work at any prime). We believe that the assumption that the composite $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \mathrm{MString}$ is an \mathbf{E}_3 -map is too strong; we believe that it can be removed using special properties of fibers of charming maps, and we will return to this in future work.

We stress that these splittings are unstructured; it seems unlikely that they can be refined to structured splittings. In [Dev19b], we showed (unconditionally) that the Ando-Hopkins-Rezk orientation $\mathrm{MString} \rightarrow \mathrm{tmf}$ induces a surjection on homotopy, a result which is clearly implied by Theorem C.

We remark that the argument used to prove Theorem C shows that if the composite $\mathfrak{Z}_3(A) \rightarrow A \rightarrow \mathrm{MSpin}$ is an \mathbf{E}_3 -map, then Conjectures D and E imply that there is a spectrum-level unital splitting of the Atiyah-Bott-Shapiro orientation $\mathrm{MSpin} \rightarrow \mathrm{bo}$. This splitting was originally proved unconditionally (i.e., without assuming Conjecture D or Conjecture E) by Anderson-Brown-Peterson in [ABP67] via a calculation with the Adams spectral sequence.

1.2. The statements of Conjectures D, E, and F. We first state Conjecture D. The second part of this conjecture is a compilation of several old conjectures in unstable homotopy theory originally made by Cohen-Moore-Neisendorfer, Gray, and Selick in [CMN79a, CMN79b, Nei81, Gra89b, Gra88, Sel77]. The statement we shall give momentarily differs slightly from the statements made in the literature: for instance, in Conjecture D(b), we demand a \mathcal{Q}_1 -space splitting (Notation 2.2.6), rather than merely an H-space splitting.

Conjecture D. *The following statements are true:*

- (a) *The homotopy fiber of any charming map (Definition 4.1.1) is equivalent as a loop space to the loop space on an Anick space (Definition 4.1.3).*
- (b) *There exists a p -local charming map $f : \Omega^2 S^{2p^n+1} \rightarrow S^{2p^n-1}$ whose homotopy fiber admits a \mathcal{Q}_1 -space retraction off of $\Omega^2(S^{2p^n}/p)$. There are also integrally defined maps $\Omega^2 S^9 \rightarrow S^7$ and $\Omega^2 S^{17} \rightarrow S^{15}$ whose composites with the double suspension on S^7 and S^{15} respectively are the degree 2 maps. Moreover, their homotopy fibers K_2 and K_3 (respectively) admit deloopings, and admit \mathcal{Q}_1 -space retractions off $\Omega^2(S^8/2)$ and $\Omega^2(S^{16}/2)$ (respectively).*

Next, we turn to Conjecture E. This conjecture is concerned with the \mathbf{E}_2 -topological Hochschild cohomology of the Thom spectra $X(p^n - 1)_{(p)}$, A , and B introduced above.

Conjecture E. *Let $n \geq 0$ be an integer. Let R denote $X(p^{n+1} - 1)_{(p)}$, A (in which case $n = 1$), or B (in which case $n = 2$). Then the element $\sigma_n \in \pi_{|v_n|-1}R$ lifts to the \mathbf{E}_2 -topological Hochschild cohomology $\mathfrak{H}_3(R)$ of R , and is p -torsion in $\pi_*\mathfrak{H}_3(R)$ if $R = X(p^{n+1} - 1)_{(p)}$, and is 2-torsion in $\pi_*\mathfrak{H}_3(R)$ if $R = A$ or B .*

Finally, we state Conjecture F. It is inspired by [AP76, AL17]. We believe this conjecture is the most approachable of the conjectures stated here.

Conjecture F. *Suppose X is a spectrum which is bounded below and whose homotopy groups are finitely generated over \mathbf{Z}_p . If there is an isomorphism $H_*(X; \mathbf{F}_p) \cong H_*(\mathrm{tmf}; \mathbf{F}_p)$ of Steenrod comodules, then there is a homotopy equivalence $X_p^\wedge \rightarrow \mathrm{tmf}_p^\wedge$ of spectra.*

After proving Theorem A and Theorem C, we explore relationships between the different spectra appearing on the second line of Table 1 in the remainder of the article. In particular, we prove analogues of Wood’s equivalence $\mathrm{bo} \wedge C\eta \simeq \mathrm{bu}$ (see also [Mat16]) for these spectra. We argue that these are related to the existence of certain EHP sequences.

Finally, we describe a C_2 -equivariant analogue of Corollary B at $n = 1$ as Theorem 7.2.1, independently of a C_2 -equivariant analogue of Conjecture D and Conjecture E. This result constructs \mathbf{HZ} as a Thom spectrum of an equivariant bundle of invertible $T(1)_{\mathbf{R}}$ -modules over $\Omega^\rho S^{2\rho+1}$, where $T(1)_{\mathbf{R}}$ is the free \mathbf{E}_σ -algebra with a nullhomotopy of the equivariant Hopf map $\tilde{\eta} \in \pi_\sigma(\mathbb{S})$, and ρ and σ are the regular and sign representations of C_2 , respectively. This uses results of Behrens-Wilson and Hahn-Wilson from [BW18, HW20]. We believe there is a similar result at odd primes, but we defer discussion of this. We discuss why our methods do not work to yield $\mathrm{BP}\langle n \rangle_{\mathbf{R}}$ for $n \geq 1$ as in Corollary B.

1.3. Outline. Section 2 contains a review some of the theory of Thom spectra from the modern perspective, as well as the proof of the classical Hopkins-Mahowald theorem. The content reviewed in this section will appear in various guises throughout this project, hence its inclusion.

In Section 3, we study certain \mathbf{E}_1 -rings; most of them appear as Thom spectra over the sphere. For instance, we recall some facts about Ravenel’s $X(n)$ spectra, and then define and prove properties about the \mathbf{E}_1 -rings A and B used in the statement of Theorem A. We state Conjecture E, and discuss (Remark 5.4.7) its relation to the nilpotence theorem, in this section.

In Section 4, we recall some unstable homotopy theory, such as the Cohen-Moore-Neisendorfer map and the fiber of the double suspension. These concepts do not show up often in stable homotopy theory, so we hope this section provides useful background to the reader. We state Conjecture D, and then explore properties of Thom spectra of bundles defined over Anick spaces.

In Section 5, we state and prove Theorem A and Corollary B, and state several easy consequences of Theorem A.

In Section 6, we study some applications of Theorem A. For instance, we use it to prove Theorem C, which is concerned with the splitting of certain cobordism spectra. In a previous version of this article, we had two subsections discussing Wood-like equivalences, and topological Hochschild homology of the chromatic Thom spectra of Table 1. However, while making revisions to this article, we decided to split these two sections off into separate articles [Dev22a, Dev22b].

In Section 7, we prove an equivariant analogue of Corollary B at height 1. We construct equivariant analogues of $X(n)$ and A , and describe why our methods fail to produce an equivariant analogue of Corollary B at all heights, even granting an analogue of Conjecture D and Conjecture E.

Finally, in Section 8, we suggest some directions for future research. There are also numerous interesting questions arising from our work, which we have indicated in the body of the article.

Conventions. Unless indicated otherwise, or if it goes against conventional notational choices, a Latin letter with a numerical subscript (such as x_5) denotes an element of degree given by its subscript. If X is a space and R is an \mathbf{E}_1 -ring spectrum, then X^μ will denote the Thom spectrum of some bundle of invertible R -modules determined by a map $\mu : X \rightarrow \mathrm{BGL}_1(R)$. We shall often quietly localize or complete at an implicit prime p . Although we have tried to be careful, *all limits and colimits will be homotopy limits and colimits*; we apologize for any inconvenience this might cause.

We shall denote by $P^k(p)$ the mod p Moore space $S^{k-1} \cup_p e^k$ with top cell in dimension k . The symbols ζ_i and τ_i will denote the *conjugates* of the Milnor generators (commonly written nowadays as ξ_i and τ_i , although, as Haynes Miller pointed out to me, our notation for the conjugates was Milnor's original notation) in degrees $2(p^i - 1)$ and $2p^i - 1$ for $p > 2$ and $2^i - 1$ (for ζ_i) at $p = 2$. Unfortunately, we will use A to denote the \mathbf{E}_1 -ring in appearing in Table 1, and write A_* to denote the dual Steenrod algebra. We hope this does not cause any confusion, since we will always denote the homotopy groups of A by $\pi_* A$ and not A_* .

If \mathcal{O} is an operad, we will simply write \mathcal{O} -ring to denote an \mathcal{O} -algebra object in spectra. A map of \mathcal{O} -rings respecting the \mathcal{O} -algebra structure will often simply be called a \mathcal{O} -map. Unless it is clear that we mean otherwise, all modules over non- \mathbf{E}_∞ -algebras will be left modules.

Hood Chatham pointed out to me that $S^3\langle 4 \rangle$ would be the correct notation for what we denote by $S^3\langle 3 \rangle = \mathrm{fib}(S^3 \rightarrow K(\mathbf{Z}, 3))$. Unfortunately, the literature seems to have chosen $S^3\langle 3 \rangle$ as the preferred notation, so we stick to that in this project.

When we write that Theorem A, Corollary B, or Theorem C implies a statement P, we mean that Conjectures D and Conjecture E (and Conjecture F, if the intended application is to tmf) imply P via Theorem A, Corollary B, or Theorem C.

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2. Background, and some classical positive and negative results

2.1. Background on Thom spectra. In this section, we will recall some facts about Thom spectra and their universal properties; the discussion is motivated by [ABG⁺14].

Definition 2.1.1. Let A be an \mathbf{E}_1 -ring, and let $\mu : X \rightarrow \mathrm{BGL}_1(A)$ be a map of spaces. The Thom A -module X^μ is defined as the homotopy pushout

$$\begin{array}{ccc} \Sigma_+^\infty \mathrm{GL}_1(A) & \longrightarrow & \Sigma_+^\infty \mathrm{fib}(\mu) \\ \downarrow & & \downarrow \\ A & \longrightarrow & X^\mu. \end{array}$$

Remark 2.1.2. Let A be an \mathbf{E}_1 -ring, and let $\mu : X \rightarrow \mathrm{BGL}_1(A)$ be a map of spaces. The Thom A -module X^μ is the homotopy colimit of the functor $X \xrightarrow{\mu} \mathrm{BGL}_1(A) \rightarrow \mathrm{Mod}(A)$, where we have abused notation by identifying X with its associated Kan complex. If A is an \mathbf{E}_1 - R -algebra, then the R -module underlying X can be identified with the homotopy colimit of the composite functor

$$X \xrightarrow{\mu} \mathrm{BGL}_1(A) \rightarrow \mathrm{BAut}_R(A) \rightarrow \mathrm{Mod}(R),$$

where we have identified X with its associated Kan complex. The space $\mathrm{BAut}_R(A)$ can be regarded as the maximal subgroupoid of $\mathrm{Mod}(R)$ spanned by the object A .

The following is immediate from the description of the Thom spectrum as a Kan extension:

Proposition 2.1.3. *Let R and R' be \mathbf{E}_1 -rings with an \mathbf{E}_1 -ring map $R \rightarrow R'$ exhibiting R' as a right R -module. If $f : X \rightarrow \mathrm{BGL}_1(R)$ is a map of spaces, then the Thom spectrum of the composite $X \rightarrow \mathrm{BGL}_1(R) \rightarrow \mathrm{BGL}_1(R')$ is the base-change $X^f \wedge_R R'$ of the (left) R -module Thom spectrum X^f .*

Corollary 2.1.4. *Let R and R' be \mathbf{E}_1 -rings with an \mathbf{E}_1 -ring map $R \rightarrow R'$ exhibiting R' as a right R -module. If $f : X \rightarrow \mathrm{BGL}_1(R)$ is a map of spaces such that the composite $X \rightarrow \mathrm{BGL}_1(R) \rightarrow \mathrm{BGL}_1(R')$ is null, then there is an equivalence $X^f \wedge_R R' \simeq R' \wedge \Sigma_+^\infty X$.*

Moreover (see e.g. [AB19, Corollary 3.2]):

Proposition 2.1.5. *Let X be a k -fold loop space, and let R be an \mathbf{E}_{k+1} -ring. Then the Thom spectrum of an \mathbf{E}_k -map $X \rightarrow \mathrm{BGL}_1(R)$ is an \mathbf{E}_k - R -algebra.*

We will repeatedly use the following classical result, which is again a consequence of the observation that Thom spectra are colimits, as well as the fact that total spaces of fibrations may be expressed as colimits; see also [Bea17, Theorem 1].

Proposition 2.1.6. *Let $X \xrightarrow{i} Y \rightarrow Z$ be a fiber sequence of k -fold loop spaces (where $k \geq 1$), and let R be an \mathbf{E}_m -ring for $m \geq k + 1$. Suppose that $\mu : Y \rightarrow \mathrm{BGL}_1(R)$ is a map of k -fold loop spaces. Then, there is a k -fold loop map $\phi : Z \rightarrow \mathrm{BGL}_1(X^{\mu \circ i})$ whose Thom spectrum is equivalent to Y^μ as \mathbf{E}_{k-1} -rings. Concisely,*

if arrows are labeled by their associated Thom spectra, then there is a diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{i} & Y & \longrightarrow & Z \\
 & \searrow & \downarrow \mu & & \downarrow \phi \\
 & & Y^\mu & & Y^\mu = Z^\phi \\
 & & \downarrow & & \downarrow \\
 & & \text{BGL}_1(R) & \longrightarrow & \text{BGL}_1(X^{\mu \circ i}).
 \end{array}$$

The argument to prove Proposition 2.1.6 also goes through with slight modifications when $k = 0$, and shows:

Proposition. *Let $X \xrightarrow{i} Y \rightarrow Z$ be a fiber sequence of spaces with Z connected, and let R be an \mathbf{E}_m -ring for $m \geq 1$. Suppose that $\mu : Y \rightarrow \text{BGL}_1(R)$ is a map of Kan complexes. Then, there is a map $\phi : Z \rightarrow \text{BAut}_R(X^{\mu \circ i})$ such that the homotopy colimit (i.e., ‘‘Thom spectrum’’) Z^ϕ of the following composite is equivalent to Y^μ as an R -module:*

$$(2.1) \quad Z \xrightarrow{\phi} \text{BAut}_R(X^{\mu \circ i}) \subseteq \text{Mod}_R.$$

We will abusively refer to this result in the sequel also as Proposition 2.1.6.

PROOF OF THE SECOND FORM OF PROPOSITION 2.1.6. It will be convenient to use the model for Thom spectra following [ABG⁺14]. Observe that a fibration $X \rightarrow Y \rightarrow Z$ implies (e.g., by [ABG⁺14, Remark 2.4]) that there is a functor $Z \rightarrow \text{Top}$ whose homotopy colimit is Y , and whose fiber over any vertex of $z \in Z$ is X . Since X is connected, we may write $Y \simeq \text{hocolim}_Z X$. The map $X \rightarrow Y$ is induced by the inclusion $\{z\} \hookrightarrow Z$. Since Y is a Kan complex, the Thom spectrum Y^μ can be identified (by [ABG⁺14, Definition 1.4]) with the homotopy colimit of the composite $Y \xrightarrow{\mu} \text{BGL}_1(R) \simeq R\text{-line} \subseteq \text{Mod}_R$ (which we will temporarily denote by $\mu : Y \rightarrow \text{Mod}_R$). We will write this as $Y^\mu \simeq \text{hocolim}_Y R$. The left Kan extension of the map $Y \rightarrow Z$ along the functor $\mu : Y \rightarrow \text{Mod}_R$ defines a functor $\phi : Z \rightarrow \text{Mod}_R$, which sends $z \in Z$ to $X^{\mu \circ i} \simeq \text{hocolim}(X \rightarrow Y \xrightarrow{\mu} \text{Mod}_R)$. Since Z is connected, this implies that $Y^\mu \simeq \text{hocolim}_Y R$ is the homotopy colimit of the functor (2.1). \square

The following is a slight generalization of [AB19, Theorem 4.10]:

Theorem 2.1.7. *Let R be an \mathbf{E}_{k+1} -ring for $k \geq 0$, and let $\alpha : Y \rightarrow \text{BGL}_1(R)$ be a map from a pointed space Y . For any $0 \leq m \leq k$, let $\tilde{\alpha} : \Omega^m \Sigma^m Y \rightarrow \text{BGL}_1(R)$ denote the extension of α . Then the Thom spectrum $(\Omega^m \Sigma^m Y)^{\tilde{\alpha}}$ is the free \mathbf{E}_m - R -algebra A for which the composite $Y \rightarrow \text{BGL}_1(R) \rightarrow \text{BGL}_1(A)$ is null. More precisely, if A is any \mathbf{E}_m - R -algebra, then*

$$\text{Map}_{\text{Alg}_R^{\mathbf{E}_m}}((\Omega^m \Sigma^m Y)^{\tilde{\alpha}}, A) \simeq \begin{cases} \text{Map}_*(Y, \Omega^\infty A) & \text{if } \alpha : Y \rightarrow \text{BGL}_1(R) \rightarrow \text{BGL}_1(A) \text{ is null,} \\ \emptyset & \text{else.} \end{cases}$$

Remark 2.1.8. Say $Y = S^{n+1}$, so α detects an element $\alpha \in \pi_n R$. Theorem 2.1.7 suggests interpreting the Thom spectrum $(\Omega^m \Sigma^{m+n+1})^{\tilde{\alpha}}$ as an \mathbf{E}_m -quotient; to signify this, we will denote it by $R//_{\mathbf{E}_m} \alpha$. If $m = 1$, then we will simply denote it by $R//\alpha$, while if $m = 0$, then the \mathbf{E}_m -quotient is simply the ordinary quotient R/α . See [AB19, Definition 4.3], where the quotient $R//_{\mathbf{E}_m} \alpha$ is called the *versal R -algebra of characteristic α* .

2.2. The Hopkins-Mahowald theorem. The primary motivation for this project is the following miracle (see [Mah79] for $p = 2$ and [MRS01, Lemma 3.3] for $p > 2$, as well as [AB19, Theorem 5.1] for a proof of the equivalence as one of \mathbf{E}_2 -algebras):

Theorem 2.2.1 (Hopkins-Mahowald). *Let \mathbb{S}_p^\wedge be the p -completion of the sphere at a prime p , and let $f : S^1 \rightarrow \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$ detect the element $1 - p \in \pi_1 \mathrm{BGL}_1(\mathbb{S}_p^\wedge) \simeq \mathbf{Z}_p^\times$. Let $\mu : \Omega^2 S^3 \rightarrow \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$ denote the \mathbf{E}_2 -map extending f ; then there is a p -complete equivalence $(\Omega^2 S^3)^\mu \rightarrow \mathbf{HF}_p$ of \mathbf{E}_2 -ring spectra.*

It is not too hard to deduce the following result from Theorem 2.2.1:

Corollary 2.2.2. *Let $S^3\langle 3 \rangle$ denote the 3-connected cover of S^3 . Then the Thom spectrum of the composite $\Omega^2 S^3\langle 3 \rangle \rightarrow \Omega^2 S^3 \xrightarrow{\mu} \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$ is equivalent to \mathbf{HZ}_p as an \mathbf{E}_2 -ring.*

Remark 2.2.3. Theorem 2.2.1 implies a restrictive version of the nilpotence theorem: if R is an \mathbf{E}_2 -ring spectrum, and $x \in \pi_* R$ is a simple p -torsion element which has trivial \mathbf{HF}_p -Hurewicz image, then x is nilpotent. This is explained in [MNN15, Proposition 4.19]. Indeed, to show that x is nilpotent, it suffices to show that the localization $R[1/x]$ is contractible. Since $px = 0$, the localization $R[1/x]$ is an \mathbf{E}_2 -ring in which $p = 0$, so the universal property of Theorem 2.1.7 implies that there is an \mathbf{E}_2 -map $\mathbf{HF}_p \rightarrow R[1/x]$. It follows that the unit $R \rightarrow R[1/x]$ factors through the Hurewicz map $R \rightarrow R \wedge \mathbf{HF}_p$. In particular, the multiplication by x map on $R[1/x]$ factors as the indicated dotted map:

$$\begin{array}{ccccc} \Sigma^{|x|} R & \xrightarrow{x} & R & \longrightarrow & R[1/x]. \\ \downarrow & & \downarrow & \nearrow \text{dotted} & \\ \mathbf{HF}_p \wedge \Sigma^{|x|} R & \xrightarrow{x} & R \wedge \mathbf{HF}_p & & \end{array}$$

However, the bottom map is null (because x has trivial \mathbf{HF}_p -Hurewicz image), so x must be null in $\pi_* R[1/x]$. This is possible if and only if $R[1/x]$ is contractible, as desired. See Proposition 5.4.1 for the analogous connection between Corollary 2.2.2 and nilpotence.

Since an argument similar to the proof of Theorem 2.2.1 will be necessary later in Step 2 of Section 5.2, we will recall a proof of this theorem. The key non-formal input is the following result of Steinberger's from [BMMS86, Theorems III.2.2 and III.2.3]:

Theorem 2.2.4 (Steinberger). *Let ζ_i denote the conjugate to the Milnor generators ξ_i of the dual Steenrod algebra, and similarly for τ_i at odd primes. Then*

$$(2.2) \quad Q^{p^i} \zeta_i = \zeta_{i+1}, \quad Q^{p^j} \tau_j = \tau_{j+1}$$

for $i, j + 1 > 0$.

PROOF OF THEOREM 2.2.1. By Corollary 2.1.7, the Thom spectrum $(\Omega^2 S^3)^\mu$ is the free \mathbf{E}_2 -ring with a nullhomotopy of p . Since \mathbf{HF}_p is an \mathbf{E}_2 -ring with a nullhomotopy of p , we obtain an \mathbf{E}_2 -map $(\Omega^2 S^3)^\mu \rightarrow \mathbf{HF}_p$. To prove that this map is a p -complete equivalence, it suffices to prove that it induces an isomorphism on mod p homology.

The mod p homology of $(\Omega^2 S^3)^\mu$ can be calculated directly via the Thom isomorphism $\mathbf{HF}_p \wedge (\Omega^2 S^3)^\mu \simeq \mathbf{HF}_p \wedge \Sigma_+^\infty \Omega^2 S^3$. Note that this is *not* an equivalence as $\mathbf{HF}_p \wedge \mathbf{HF}_p$ -comodules: the Thom twisting is highly nontrivial.

For simplicity, we will now specialize to the case $p = 2$, although the same proof works at odd primes. The homology of $\Omega^2 S^3$ is classical: it is a polynomial ring generated by applying \mathbf{E}_2 -Dyer-Lashof operations to a single generator x_1 in degree 1. Theorem 2.2.4 implies that the same is true for the mod 2 Steenrod algebra: it, too, is a polynomial ring generated by applying \mathbf{E}_2 -Dyer-Lashof operations to the single generator $\zeta_1 = \xi_1$ in degree 1. Since the map $(\Omega^2 S^3)^\mu \rightarrow \mathbf{HF}_2$ is an \mathbf{E}_2 -ring map, it preserves \mathbf{E}_2 -Dyer-Lashof operations on mod p homology. By the above discussion, it suffices to show that the generator $x_1 \in H_*(\Omega^2 S^3)^\mu \cong H_*(\Omega^2 S^3)$ in degree 1 is mapped to $\zeta_1 \in H_* \mathbf{HF}_2$.

To prove this, note that x_1 is the image of the generator in degree 1 in homology under the double suspension $S^1 \rightarrow \Omega^2 S^3$, and that ζ_1 is the image of the generator in degree 1 in homology under the canonical map $\mathbb{S}/p \rightarrow \mathbf{HF}_p$. It therefore suffices to show that the Thom spectrum of the spherical fibration $S^1 \rightarrow \mathbf{BGL}_1(\mathbb{S}_p^\wedge)$ detecting $1 - p$ is simply \mathbb{S}/p . This is an easy exercise. \square

Remark 2.2.5. When $p = 2$, one does not need to p -complete in Theorem 2.2.1: the map $S^1 \rightarrow \mathbf{BGL}_1(\mathbb{S}_2^\wedge)$ factors as $S^1 \rightarrow \mathbf{BO} \rightarrow \mathbf{BGL}_1(\mathbb{S})$, where the first map detects the Möbius bundle over S^1 , and the second map is the J -homomorphism.

Notation 2.2.6. Let \mathcal{Q}_1 denote the (operadic nerve of the) *cup-1 operad* from [Law20, Example 1.3.6]: this is the operad whose n th space is empty unless $n = 2$, in which case it is S^1 with the antipodal action of Σ_2 . We will need to slightly modify the definition of \mathcal{Q}_1 when localized at an odd prime p : in this case, it will denote the operad whose n th space is a point if $n < p$, empty if $n > p$, and when $n = p$ is the ordered configuration space $\mathbf{Conf}_p(\mathbf{R}^2)$ with the permutation action of Σ_p . Any homotopy commutative ring admits the structure of a \mathcal{Q}_1 -algebra at $p = 2$, but at other primes it is slightly stronger to be a \mathcal{Q}_1 -algebra than to be a homotopy commutative ring. If $k \geq 2$, any \mathbf{E}_k -algebra structure on a spectrum restricts to a \mathcal{Q}_1 -algebra structure.

Remark 2.2.7. As stated in [Law20, Proposition 1.5.29], the operation Q_1 already exists in the mod 2 homology of any \mathcal{Q}_1 -ring R , where \mathcal{Q}_1 is the cup-1 operad from Notation 2.2.6 — the entire \mathbf{E}_2 -structure is not necessary. With our modification of \mathcal{Q}_1 at odd primes as in Remark 2.2.6, this is also true at odd primes.

Remark 2.2.8. We will again momentarily specialize to $p = 2$ for convenience. Steinberger’s calculation in Theorem 2.2.4 can be rephrased as stating that $Q_1 \zeta_i = \zeta_{i+1}$, where Q_1 is the *lower-indexed* Dyer-Lashof operation. (See [BMMS86, Page 59] for this notation.) As in Remark 2.2.7, the operation Q_1 already exists in the mod p homology of any \mathcal{Q}_1 -ring R . Since homotopy commutative rings are \mathcal{Q}_1 -algebras in spectra, this observation can be used to prove results of Würzler ([Wur86, Theorem 1.1]) and Pazhitnov-Rudyak ([PR84, Theorem in Introduction]).

Remark 2.2.9. The argument with Dyer-Lashof operations and Theorem 2.2.4 used in the proof of Theorem 2.2.1 will be referred to as the *Dyer-Lashof hopping argument*. It will be used again (in the same manner) in the proof of Theorem A.

Remark 2.2.10. Theorem 2.2.1 is *equivalent* to Steinberger’s calculation (Theorem 2.2.4), as well as to Bökstedt’s calculation of $\mathrm{THH}(\mathbf{F}_p)$ (as a ring spectrum, and not just the calculation of its homotopy). Let us sketch an argument. First, Theorem 2.2.4 implies Theorem 2.2.1 (by the proof above). The other direction (i.e., the calculation (2.2)) can be argued by observing that the Thom isomorphism $\mathrm{HF}_p \wedge \mathrm{HF}_p \simeq \mathrm{HF}_p \wedge \Sigma_+^\infty \Omega^2 S^3$ is an equivalence of \mathbf{E}_2 - HF_p -algebras, so that the Dyer-Lashof operations are determined by the operations in $H_*(\Omega^2 S^3; \mathbf{F}_p)$. But the Dyer-Lashof operations are *defined* by classes in $H_*(\Omega^2 S^3; \mathbf{F}_p)$, and Theorem 2.2.4 is a consequence of the fact that the iterates of Q_1 on the generator of $H_1(\Omega^2 S^3; \mathbf{F}_p)$ describe all the polynomial generators $H_*(\Omega^2 S^3; \mathbf{F}_p)$.

It remains to argue that Theorem 2.2.1 is equivalent to the calculation that $\mathrm{THH}(\mathbf{F}_p) \simeq \mathbf{F}_p[\Omega S^3]$ as an \mathbf{E}_1 - \mathbf{F}_p -algebra. This is showed in [KN19, Remark 1.5].

2.3. No-go theorems for higher chromatic heights. In light of Theorem 2.2.1 and Corollary 2.2.2, it is natural to wonder if appropriate higher chromatic analogues of HF_p and HZ , such as $\mathrm{BP}\langle n \rangle$, bo , or tmf can be realized as Thom spectra of spherical fibrations. The answer is known to be negative (see [Mah87, Rud98, Cha19]) in many cases:

Theorem 2.3.1 (Mahowald, Rudyak, Chatham). *There is no space X with a spherical fibration $\mu : X \rightarrow \mathrm{BGL}_1(\mathbb{S})$ (even after completion) such that X^μ is equivalent to $\mathrm{BP}\langle 1 \rangle$ or bo . Moreover, there is no 2-local loop space X' with a spherical fibration determined by an H -map $\mu : X' \rightarrow \mathrm{BGL}_1(\mathbb{S}_2^\wedge)$ such that X'^μ is equivalent to tmf_2^\wedge .*

The proofs rely on calculations in the unstable homotopy groups of spheres.

Remark 2.3.2. Although not written down anywhere, a slight modification of the argument used by Mahowald to show that bu is not the Thom spectrum of a spherical fibration over a loop space classified by an H -map can be used to show that $\mathrm{BP}\langle 2 \rangle$ at $p = 2$ (i.e., $\mathrm{tmf}_1(3)$) is not the Thom spectrum of a spherical fibration over a loop space classified by an H -map. We do not know a proof that $\mathrm{BP}\langle n \rangle$ is not the Thom spectrum of a spherical fibration over a loop space classified by an H -map for all $n \geq 1$ and all primes, but we strongly suspect this to be true.

Remark 2.3.3. A lesser-known no-go result due to Priddy appears in [Lew78, Chapter 2.11], where it is shown that BP cannot appear as the Thom spectrum of a double loop map $\Omega^2 X \rightarrow \mathrm{BGL}_1(\mathbb{S})$. In fact, the argument shows that the same result is true with BP replaced by $\mathrm{BP}\langle n \rangle$ for $n \geq 1$; we had independently come up with this argument for $\mathrm{BP}\langle 1 \rangle$ before learning about Priddy’s argument. Since Lewis’ thesis is not particularly easy to acquire, we give a sketch of Priddy’s argument. By the Thom isomorphism and the calculation (see [LN14, Theorem 4.3], as well as [Wil75, Proposition 1.7] and [AR05, Proposition 5.3]):

$$H_*(\mathrm{BP}\langle n-1 \rangle; \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[\zeta_1^2, \dots, \zeta_{n-1}^2, \zeta_n^2, \zeta_{n+1}, \dots] & p = 2, \\ \mathbf{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_p}(\tau_n, \tau_{n+1}, \dots) & p > 2, \end{cases}$$

we find that the mod p homology of $\Omega^2 X$ would be isomorphic as an algebra to a polynomial ring on infinitely many generators, possibly tensored with an exterior algebra on infinitely many generators. The Eilenberg-Moore spectral sequence then

implies that the mod p cohomology of X is given by

$$H^*(X; \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[b_1, \dots, b_n, c_{n+1}, \dots] & p = 2, \\ \mathbf{F}_p[b_1, b_2, \dots] \otimes \Lambda_{\mathbf{F}_p}(c_{n+1}, \dots) & p > 2, \end{cases}$$

where $|b_i| = 2p^i$ and $|c_i| = 2p^{i-1} + 1$. If p is odd, then since $|b_1| = 2p$, we have $P^p(b_1) = b_1^p$. Liulevicius' formula for P^1 in terms of secondary cohomology operations ([Liu62, Theorem 1]) allows us to write $P^p(b_1)$ as a sum $c_0 \mathcal{R}(b_1) + \sum_{\gamma} c_{0,\gamma} \Gamma_{\gamma}(b_1)$, where $\mathcal{R}(b_1)$ is a coset in $H^{2p+4(p-1)}(X; \mathbf{F}_p)$ and Γ_{γ} is an operation of odd degree, so that $\Gamma_{\gamma}(b_1)$ is in odd degree. We will not need to know what exactly the sum is indexed by, or what any of these operations are. Observe that Γ_{γ} kills b_1 because everything is concentrated in even degrees in the relevant range, and \mathcal{R} also kills b_1 since $|\mathcal{R}(b_1)| = 4(p-1) + 2p^i$ is never a sum of numbers of the form $2p^k$ when $p > 2$. Using this, one can conclude that $b_1^p = 0$, which is a contradiction. A similar calculation works at $p = 2$, using Adams' study of secondary mod 2 cohomology operations in [Ada60].

Remark 2.3.4. Using the calculations of $\mathrm{THH}(\mathrm{bo})$ and $\mathrm{THH}(\mathrm{ku})$ from [AHL10], Angeltveit-Hill-Lawson show in [AHL09] that neither bo nor ku can appear as the Thom spectrum of a double loop map $\Omega^2 X \rightarrow \mathrm{BGL}_1(\mathbb{S})$.

Our primary goal in this project is to argue that the issues in Theorem 2.3.1 are alleviated if we replace $\mathrm{BGL}_1(\mathbb{S})$ with the delooping of the space of units of an appropriate replacement of \mathbb{S} . In the next section, we will construct these replacements of \mathbb{S} .

3. Some Thom spectra

In this section, we introduce certain \mathbf{E}_1 -rings; most of them appear as Thom spectra over the sphere. The following table summarizes the spectra introduced in this section and gives references to their locations in the text. The spectra A and B were introduced in [Dev19b].

Thom spectrum	Definition	“Height”	BP-homology
$T(n)$	Theorem 3.1.5	n	Theorem 3.1.5
$y(n)$ and $y_{\mathbf{Z}}(n)$	Definition 3.2.2	n	Proposition 3.2.3
A	Definition 3.2.8	1	Proposition 3.2.13
B	Definition 3.2.18	2	Proposition 3.2.21

TABLE 2. Certain Thom spectra and their homologies.

3.1. Ravenel’s $X(n)$ spectra. The proof of the nilpotence theorem in [DHS88, HS98] crucially relied upon certain Thom spectra arising from Bott periodicity; these spectra first appeared in Ravenel’s seminal paper [Rav84].

Definition 3.1.1. Let $X(n)$ denote the Thom spectrum of the \mathbf{E}_2 -map $\Omega\mathrm{SU}(n) \subseteq \mathrm{BU} \xrightarrow{J} \mathrm{BGL}_1(\mathbb{S})$, where the first map arises from Bott periodicity.

Example 3.1.2. The \mathbf{E}_2 -ring $X(1)$ is the sphere spectrum, while $X(\infty)$ is MU . Since the map $\Omega\mathrm{SU}(n) \rightarrow \mathrm{BU}$ is an equivalence in dimensions $\leq 2n - 2$, the same is true for the map $X(n) \rightarrow \mathrm{MU}$; the first dimension in which $X(n)$ has an element in its homotopy which is not detected by MU is $2n - 1$.

Remark 3.1.3. The \mathbf{E}_2 -structure on $X(n)$ does *not* extend to an \mathbf{E}_3 -structure (see [Law20, Example 1.5.31]). If $X(n)$ admits such an \mathbf{E}_3 -structure, then the induced map $\mathrm{H}_*(X(n)) \rightarrow \mathrm{H}_*(\mathbf{HF}_p)$ on mod p homology would commute with \mathbf{E}_3 -Dyer-Lashof operations. However, we know that the image of $\mathrm{H}_*(X(n))$ in $\mathrm{H}_*(\mathbf{HF}_p)$ is $\mathbf{F}_p[\zeta_1^2, \dots, \zeta_n^2]$; since Steinberger’s calculation (Theorem 2.2.4) implies that $Q_2(\zeta_i^2) = \zeta_{i+1}^2$ via the relation $Q_2(x^2) = Q_1(x)^2$, we find that the image of $\mathrm{H}_*(X(n))$ in $\mathrm{H}_*(\mathbf{HF}_p)$ cannot be closed under the \mathbf{E}_3 -Dyer-Lashof operation Q_2 .

Remark 3.1.4. The proof of the nilpotence theorem shows that each of the $X(n)$ detects nilpotence. However, it is known (see [Rav84, Theorem 3.1]) that $\langle X(n) \rangle > \langle X(n+1) \rangle$.

After localizing at a prime p , the spectrum MU splits as a wedge of suspensions of BP ; this splitting comes from the Quillen idempotent on MU . The same is true of the $X(n)$ spectra, as explained in [Rav86, Section 6.5]: a multiplicative map $X(n)_{(p)} \rightarrow X(n)_{(p)}$ is determined by a polynomial $f(x) = \sum_{0 \leq i \leq n-1} a_i x^{i+1}$, with $a_0 = 1$ and $a_i \in \pi_{2i}(X(n)_{(p)})$. One can use this to define a truncated form of the Quillen idempotent ϵ_n on $X(n)_{(p)}$ (see [Hop84, Proposition 1.3.7]), and thereby obtain a summand of $X(n)_{(p)}$. We summarize the necessary results in the following theorem.

Theorem 3.1.5. *Let n be such that $p^n \leq k \leq p^{n+1} - 1$. Then $X(k)_{(p)}$ splits as a wedge of suspensions of the spectrum $T(n) = \epsilon_{p^n} \cdot X(p^n)_{(p)}$.*

- The map $T(n) \rightarrow \mathrm{BP}$ is an equivalence in dimensions $\leq |v_{n+1}| - 2$, so there is an indecomposable element $v_i \in \pi_* T(n)$ which maps to an indecomposable element in $\pi_* \mathrm{BP}$ for $0 \leq i \leq n$.

- This map induces the inclusion $\mathrm{BP}_*T(n) = \mathrm{BP}_*[t_1, \dots, t_n] \subseteq \mathrm{BP}_*(\mathrm{BP})$ on BP-homology, and the inclusions $\mathbf{F}_2[\zeta_1^2, \dots, \zeta_n^2] \subseteq \mathbf{F}_2[\zeta_1^2, \zeta_2^2, \dots]$ and $\mathbf{F}_p[\zeta_1, \dots, \zeta_n] \subseteq \mathbf{F}_p[\zeta_1, \zeta_2, \dots]$ on mod 2 and mod p homology.
- $T(n)$ is a homotopy associative and \mathcal{Q}_1 -algebra spectrum.

Remark 3.1.6. It is known that $T(n)$ admits the structure of an \mathbf{E}_1 -ring (see [BL21, Section 7.5]). We will interpret the phrase ‘‘Thom spectrum X^μ of a map $\mu : X \rightarrow \mathrm{BGL}_1(T(n))$ ’’ where μ arises via a map $X \xrightarrow{\mu_0} \mathrm{BGL}_1(X(p^{n+1} - 1))$ to mean the base-change $X^{\mu_0} \wedge_{X(p^{n+1} - 1)} T(n)$.

It is believed that $T(n)$ in fact admits more structure (see [AQ21, Section 6] for some discussion):

Conjecture 3.1.7. *The \mathcal{Q}_1 -ring structure on $T(n)$ extends to an \mathbf{E}_2 -ring structure.*

Remark 3.1.8. This is true at $p = 2$ and $n = 1$. Indeed, in this case $T(1) = X(2)$ is the Thom spectrum of the bundle given by the 2-fold loop map $\Omega S^3 = \Omega^2 \mathrm{BSU}(2) \rightarrow \mathrm{BU}$, induced by the inclusion $\mathrm{BSU}(2) \rightarrow \mathrm{B}^3\mathrm{U} = \mathrm{BSU}$.

Remark 3.1.9. Conjecture 3.1.7 is true at $p = 2$ and $n = 2$. The Stiefel manifold $V_2(\mathbf{H}^2)$ sits in a fiber sequence

$$S^3 \rightarrow V_2(\mathbf{H}^2) \rightarrow S^7.$$

There is an equivalence $V_2(\mathbf{H}^2) \simeq \mathrm{Sp}(2)$, so $\Omega V_2(\mathbf{H}^2)$ admits the structure of a double loop space. There is an \mathbf{E}_2 -map $\mu : \Omega V_2(\mathbf{H}^2) \rightarrow \mathrm{BU}$, given by taking double loops of the composite

$$\mathrm{BSp}(2) \rightarrow \mathrm{BSU}(4) \rightarrow \mathrm{BSU} \simeq \mathrm{B}^3\mathrm{U}.$$

The map μ admits a description as the left vertical map in the following map of fiber sequences:

$$\begin{array}{ccccc} \Omega V_2(\mathbf{H}^2) & \longrightarrow & \Omega S^7 & \longrightarrow & S^3 \\ \downarrow \mu & & \downarrow & & \downarrow \\ \mathrm{BU} & \longrightarrow & * & \longrightarrow & \mathrm{B}^2\mathrm{U}. \end{array}$$

Here, the map $S^3 \rightarrow \mathrm{B}^2\mathrm{U}$ detects the generator of $\pi_2(\mathrm{BU})$ (which maps to $\eta \in \pi_2(\mathrm{BGL}_1(\mathbb{S}))$ under the J-homomorphism). The Thom spectrum $\Omega V_2(\mathbf{H}^2)^\mu$ is equivalent to $T(2)$, and it follows that $T(2)$ admits the structure of an \mathbf{E}_2 -ring. We do not know whether $T(n)$ is the Thom spectrum of a p -complete spherical fibration over some space for $n \geq 3$.

It is possible to construct $X(n+1)$ as an $X(n)$ -algebra (see also [Bea18]):

Construction 3.1.10. There is a fiber sequence

$$\Omega \mathrm{SU}(n) \rightarrow \Omega \mathrm{SU}(n+1) \rightarrow \Omega S^{2n+1}.$$

According to Proposition 2.1.6, the spectrum $X(n+1)$ is the Thom spectrum of an \mathbf{E}_1 -map $\Omega S^{2n+1} \rightarrow \mathrm{BGL}_1(\Omega \mathrm{SU}(n))^\mu = \mathrm{BGL}_1(X(n))$. This \mathbf{E}_1 -map is the extension of a map $S^{2n} \rightarrow \mathrm{BGL}_1(X(n))$ which detects an element $\chi_n \in \pi_{2n-1}X(n)$. This element is equivalently determined by the map $\Sigma_+^\infty \Omega^2 S^{2n+1} \rightarrow X(n)$ given by the Thomification of the nullhomotopic composite

$$\Omega^2 S^{2n+1} \rightarrow \Omega \mathrm{SU}(n) \rightarrow \Omega \mathrm{SU}(n+1) \rightarrow \Omega \mathrm{SU} \simeq \mathrm{BU},$$

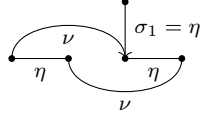


FIGURE 1. $C\eta \wedge C\nu$ shown horizontally, with 0-cell on the left. The element σ_1 is given by the map η on the 4-cell defined by a nullhomotopy of $\eta\nu = 0 \in \pi_4(S^0)$, as indicated in the diagram above.

where the first two maps form a fiber sequence. By Proposition 2.1.6, $X(n+1)$ is the free \mathbf{E}_1 - $X(n)$ -algebra with a nullhomotopy of χ_n .

Remark 3.1.11. Another construction of the map $\chi_n \in \pi_{2n-1}X(n)$ from Construction 3.1.10 is as follows. There is a map $i : \mathbf{C}P^{n-1} \rightarrow \Omega\mathrm{SU}(n)$ given by sending a line $\ell \subseteq \mathbf{C}^n$ to the loop $S^1 \rightarrow \mathrm{SU}(n) = \mathrm{Aut}(\mathbf{C}^n, \langle, \rangle)$ defined as follows: $\theta \in S^1$ is sent to the (appropriate rescaling of the) unitary transformation of \mathbf{C}^n sending a vector to its rotation around the line ℓ by the angle θ . The map i Thomifies to a stable map $\Sigma^{-2}\mathbf{C}P^n \rightarrow X(n)$. The map χ_n is then the composite

$$S^{2n-1} \rightarrow \Sigma^{-2}\mathbf{C}P^n \rightarrow X(n),$$

where the first map is the desuspension of the generalized Hopf map $S^{2n+1} \rightarrow \mathbf{C}P^n$ which attaches the top cell of $\mathbf{C}P^{n+1}$. The fact that this map is indeed χ_n follows immediately from the commutativity of the following diagram:

$$\begin{array}{ccccc} S^{2n-1} & \longrightarrow & \mathbf{C}P^{n-1} & \longrightarrow & \mathbf{C}P^n \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^2 S^{2n+1} & \longrightarrow & \Omega\mathrm{SU}(n) & \longrightarrow & \Omega\mathrm{SU}(n+1), \end{array}$$

where the top row is a cofiber sequence, and the bottom row is a fiber sequence.

An easy consequence of the observation in Construction 3.1.10 is the following lemma.

Lemma 3.1.12. *Let $\sigma_n \in \pi_{|v_{n+1}|-1}T(n)$ denote the element $\chi_{p^{n+1}-1}$. Then the Thom spectrum of the composite $\Omega S^{|v_{n+1}|-1} \rightarrow \mathrm{BGL}_1(X(p^{n+1}-1)) \rightarrow \mathrm{BGL}_1(T(n))$ is equivalent to $T(n+1)$.*

Example 3.1.13. The element $\sigma_0 \in \pi_{|v_1|-1}T(0) = \pi_{2p-3}\mathbb{S}_{(p)}$ is α_1 .

Example 3.1.14. Let us specialize to $p = 2$. Theorem 3.1.5 implies that $H_*T(n) \cong \mathbf{F}_2[\zeta_1^2, \dots, \zeta_n^2]$. Using this, one can observe that the 6-skeleton of $T(1)$ is the smash product $C\eta \wedge C\nu$, and so $\sigma_1 \in \pi_5(C\eta \wedge C\nu)$. This element can be described very explicitly: the cell structure of $C\eta \wedge C\nu$ is shown in Figure 1, and the element σ_1 shown corresponds to the map defined by the relation $\eta\nu = 0$.

Example 3.1.15. The element σ_n in the Adams-Novikov spectral sequence for $T(n)$ is represented by the element $[t_{n+1}]$ in the cobar complex. See [Rav02, Section 1], where σ_{n-1} is denoted by $\alpha(\hat{v}_1/p)$.

A calculation with the Adams-Novikov spectral sequence (as in [Rav02, Theorem 3.17]) proves the following:

Lemma 3.1.16. *The class σ_{n-1} is killed by p in $\pi_{|v_n|-1}X(p^n - 1)$.*

PROOF. The argument is essentially the same as the classical observation that $\alpha_1 \in \pi_{2p-3}(S^0)$ is simple p -torsion. As mentioned in Example 3.1.15, $\sigma_{n-1} = \alpha(\hat{v}_1/p)$ in the notation of [Rav02]. If $\Gamma(n+1) = \text{BP}_*(\text{BP})/(t_1, \dots, t_n)$ denotes the Hopf algebroid of [Rav02] (so that $\text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_*, \text{BP}_*(T(n))) \cong \text{Ext}_{\Gamma(n+1)}(\text{BP}_*, \text{BP}_*)$), then α is the connecting homomorphism in cohomology over $\Gamma(n+1)$ for the short exact sequence

$$0 \rightarrow \text{BP}_* \rightarrow p^{-1}\text{BP}_* \rightarrow p^{-1}\text{BP}_*/\text{BP}_* \rightarrow 0.$$

Since \hat{v}_1/p is of order p in $p^{-1}\text{BP}_*/\text{BP}_*$, we see that $\alpha(\hat{v}_1/p)$ is of order p in the E_2 -page of the Adams-Novikov spectral sequence computing $\pi_*T(n)$. The class $\alpha(\hat{v}_1/p)$ survives to the E_∞ -page; one observes there are no possible additive extensions, so $p\sigma_{n-1} = 0 \in \pi_*T(n)$. \square

In particular, the element $\sigma_{n-1} = \chi_{p^n-1} \in \pi_{|v_n|-1}X(p^n - 1)$ is p -torsion, and the following is a consequence of Example 3.1.15:

Proposition 3.1.17. *The class $\sigma_{n-1} \in \pi_{|v_n|-1}X(p^n - 1)$ is null in $\pi_*X(p^n)$, and the Toda bracket $\langle p, \sigma_{n-1}, 1_{X(p^n)} \rangle$ in $\pi_{|v_n|}X(p^n)$ contains an indecomposable v_n .*

Corollary 3.1.18. *The element $\sigma_{n-1} \in \pi_{|v_n|-1}X(p^n - 1)$ lifts to $\pi_{|v_n|+1}(\mathbf{C}P^{|v_n|/2})$ along the map $\Sigma^{-2}\mathbf{C}P^{|v_n|/2} \rightarrow X(p^n - 1)$.*

PROOF. By Remark 3.1.11, the map $\sigma_{n-1} : S^{|v_n|-1} \rightarrow X(p^n - 1)$ is given by the composite of the generalized Hopf map $S^{|v_n|-1} \rightarrow \Sigma^{-2}\mathbf{C}P^{p^n-1}$ with the map $\Sigma^{-2}\mathbf{C}P^{p^n-1} \rightarrow X(p^n - 1)$. Moreover, this generalized Hopf map is the desuspension of the unstable generalized Hopf map $S^{|v_n|+1} \rightarrow \mathbf{C}P^{p^n-1}$, and so σ_{n-1} lifts to an element of the unstable homotopy group $\pi_{|v_n|+1}(\mathbf{C}P^{|v_n|/2})$. \square

3.2. Related Thom spectra. We now introduce several Thom spectra related to the \mathbf{E}_1 -rings $T(n)$ described in the previous section; some of these were introduced in [Dev19b]. (Relationships to $T(n)$ will be further discussed in Section 6.2.) For the reader's convenience, we have included a table of the spectra introduced below with internal references to their definitions at the beginning of this section.

Remark 3.2.1. Recall (e.g., from [Rav86, Section 4.4]) that under the map

$$\text{BP}_*(\text{BP}) \cong \text{BP}_*[t_1, t_2, \dots] \rightarrow \text{H}_*(\text{BP}; \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[\zeta_1^2, \zeta_2^2, \dots] & p = 2, \\ \mathbf{F}_p[\zeta_1, \zeta_2, \dots] & p > 2, \end{cases}$$

the class t_i is sent to ζ_i^2 (resp. ζ_i) modulo decomposables when $p = 2$ (resp. $p > 2$). Moreover, under the map

$$\text{H}_*(\text{BP}; \mathbf{F}_p) \rightarrow \text{H}_*(\mathbf{H}\mathbf{F}_p; \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[\zeta_1, \zeta_2, \dots] & p = 2, \\ \mathbf{F}_p[\zeta_1, \zeta_2, \dots] \otimes E(\tau_0, \tau_1, \dots) & p > 2, \end{cases}$$

the classes ζ_{i+1} (resp. τ_i) at $p = 2$ (resp. $p > 2$) detect a nullhomotopy of $v_i \in \pi_{2p^i-2}\text{BP}$ in $\mathbf{H}\mathbf{F}_p \otimes \mathbf{H}\mathbf{F}_p$. This implies, for instance, that if X is a spectrum such that $\text{BP}_*(X) \cong \text{BP}_*/(p, \dots, v_{j-1})[t_1, \dots, t_m]$ with $j \leq m$, then $\text{H}_*(X; \mathbf{F}_p) \cong \mathbf{F}_p[\zeta_1, \dots, \zeta_m] \otimes E(\tau_1, \dots, \tau_{j-1})$ for $p > 2$, and $\text{H}_*(X; \mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^2, \dots, \zeta_j^2, \zeta_{j+1}, \dots, \zeta_m]$.

The following Thom spectrum was introduced in [MRS01].

Definition 3.2.2. Let $y(n)$ denote the Thom spectrum of the composite

$$\Omega J_{p^{n-1}}(S^2) \rightarrow \Omega^2 S^3 \xrightarrow{1-p} \mathrm{BGL}_1(\mathbb{S}_p^\wedge).$$

If $J_{p^{n-1}}(S^2)\langle 2 \rangle$ denotes the 2-connected cover of $J_{p^{n-1}}(S^2)$, then let $y_{\mathbf{z}}(n)$ denote the Thom spectrum of the composite

$$\Omega J_{p^{n-1}}(S^2)\langle 2 \rangle \rightarrow \Omega^2 S^3 \langle 3 \rangle \rightarrow \Omega^2 S^3 \xrightarrow{1-p} \mathrm{BGL}_1(\mathbb{S}_p^\wedge),$$

so that both $y(n)$ and $y_{\mathbf{z}}(n)$ admit the structure of \mathbf{E}_1 -rings via [AB19, Corollary 3.2].

Proposition 3.2.3. *As BP_* -comodules, we have*

$$\mathrm{BP}_*(y(n)) \cong \mathrm{BP}_*/I_n[t_1, \dots, t_n], \quad \mathrm{BP}_*(y_{\mathbf{z}}(n)) \cong \mathrm{BP}_*/(v_1, \dots, v_{n-1})[t_1, \dots, t_n],$$

where I_n denotes the invariant ideal (p, v_1, \dots, v_{n-1}) .

PROOF. The claim for $y(n)$ is [MRS01, Equation 2.8]. There is an equivalence $y_{\mathbf{z}}(n)/p \simeq y(n)$, so that $\mathrm{BP}_*(y_{\mathbf{z}}(n))/p \simeq \mathrm{BP}_*(y(n))$. The Bockstein spectral sequence collapses, and the extensions on the E_∞ -page simply place p in filtration 1. This implies the second equivalence. \square

One corollary is the following; this can be deduced from Proposition 3.2.3 using Remark 3.2.1. We also refer to [AQ19, Lemma 2.3] for a direct proof of the following.

Corollary 3.2.4. *As A_* -comodules, we have*

$$\mathrm{H}_*(y(n); \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[\zeta_1, \zeta_2, \dots, \zeta_n] & p = 2 \\ \mathbf{F}_p[\zeta_1, \zeta_2, \dots, \zeta_n] \otimes E(\tau_0, \dots, \tau_{n-1}) & p \geq 3, \end{cases}$$

and

$$\mathrm{H}_*(y_{\mathbf{z}}(n); \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[\zeta_1^2, \zeta_2, \dots, \zeta_n] & p = 2 \\ \mathbf{F}_p[\zeta_1, \zeta_2, \dots, \zeta_n] \otimes E(\tau_1, \dots, \tau_{n-1}) & p \geq 3. \end{cases}$$

We will now relate $y(n)$ and $y_{\mathbf{z}}(n)$ to $T(n)$.

Construction 3.2.5. Let $m \leq n$, and let I_m be the ideal generated by p, v_1, \dots, v_{m-1} , where the v_i are some choices of indecomposables in $\pi_{|v_i|}(T(n))$ which form a regular sequence. Inductively define $T(n)/I_m$ as the cofiber of the map

$$T(n)/I_{m-1} \xrightarrow{v_m \wedge 1} T(n) \wedge T(n)/I_{m-1} \rightarrow T(n)/I_{m-1}.$$

The BP-homology of $T(n)/I_m$ is $\mathrm{BP}_*/I_m[t_1, \dots, t_n]$. The spectrum $T(n)/(v_1, \dots, v_{m-1})$ is defined similarly.

Proposition 3.2.6. *Let $p > 2$. There is an equivalence between $T(n)/I_n$ (resp. $T(n)/(v_1, \dots, v_{n-1})$) and the spectrum $y(n)$ (resp. $y_{\mathbf{z}}(n)$) of Definition 3.2.2.*

PROOF. We will prove the result for $y(n)$; the analogous proof works for $y_{\mathbf{z}}(n)$. By [Gra89a], the space $\Omega J_{p^{n-1}}(S^2)$ is homotopy commutative (since $p > 2$). Moreover, the map $\Omega J_{p^{n-1}}(S^2) \rightarrow \Omega^2 S^3$ is an H-map, so $y(n)$ is a homotopy commutative \mathbf{E}_1 -ring spectrum. It is known (see [Rav86, Section 6.5]) that homotopy

commutative maps $T(n) \rightarrow y(n)$ are equivalent to partial complex orientations of $y(n)$, i.e., factorizations

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & \Sigma^{-2}\mathbf{C}P^{p^n-1} \\ & \searrow 1 & \downarrow \gamma_n \\ & & y(n). \end{array}$$

Such a γ_n indeed exists by obstruction theory: suppose $k < p^n - 1$, and we have a map $\Sigma^{-2}\mathbf{C}P^k \rightarrow y(n)$. Since there is a cofiber sequence

$$S^{2k-1} \rightarrow \Sigma^{-2}\mathbf{C}P^k \rightarrow \Sigma^{-2}\mathbf{C}P^{k+1}$$

of spectra, the obstruction to extending along $\Sigma^{-2}\mathbf{C}P^{k+1}$ is an element of $\pi_{2k-1}y(n)$. However, the homotopy of $y(n)$ is concentrated in even degrees in the appropriate range, so a choice of γ_n does indeed exist. Moreover, this choice can be made such that they fit into a compatible family in the sense that there is a commutative diagram

$$\begin{array}{ccc} \Sigma^{-2}\mathbf{C}P^{p^n-1} & \longrightarrow & \Sigma^{-2}\mathbf{C}P^{p^{n+1}-1} \\ \gamma_n \downarrow & & \downarrow \gamma_{n+1} \\ y(n) & \longrightarrow & y(n+1). \end{array}$$

The formal group law over $\mathbf{H}\mathbf{F}_p$ has infinite height; this forces the elements p, v_1, \dots, v_{n-1} (defined for the “ $(p^n - 1)$ -bud” on $\pi_*y(n)$) to vanish in the homotopy of $y(n)$. It follows that the orientation $T(n) \rightarrow y(n)$ constructed above factors through the quotient $T(n)/I_n$. The induced map $T(n)/I_n \rightarrow y(n)$ can be seen to be an isomorphism on homology (via, for instance, Definition 3.2.2 and Construction 3.2.5). \square

Remark 3.2.7. Since $y(n)$ has a v_n -self-map, we can form the spectrum $y(n)/v_n$; its mod p homology is

$$H_*(y(n)/v_n; \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[\zeta_1, \dots, \zeta_n] \otimes \Lambda_{\mathbf{F}_2}(\zeta_{n+1}) & p = 2 \\ \mathbf{F}_p[\zeta_1, \dots, \zeta_n] \otimes \Lambda_{\mathbf{F}_p}(\tau_0, \dots, \tau_{n-1}, \tau_n) & p \geq 3. \end{cases}$$

It is in fact possible to give a construction of $y(1)/v_1$ as a spherical Thom spectrum. We will work at $p = 2$ for convenience. Define Q to be the fiber of the map $2\eta : S^3 \rightarrow S^2$. There is a map of fiber sequences

$$\begin{array}{ccccc} Q & \longrightarrow & S^3 & \xrightarrow{2\eta} & S^2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{B}\mathbf{G}\mathbf{L}_1(\mathbb{S}) & \longrightarrow & * & \longrightarrow & \mathbf{B}^2\mathbf{G}\mathbf{L}_1(\mathbb{S}). \end{array}$$

By [DM81, Theorem 3.7], the Thom spectrum of the leftmost map is $y(1)/v_1$.

We end this section by recalling the definition of two Thom spectra which, unlike $y(n)$ and $y_{\mathbf{Z}}(n)$, are not indexed by integers (we will see that they are only defined at “heights 1 and 2”). These were both studied in [Dev19b].

Definition 3.2.8. Let $S^4 \rightarrow \mathbf{B}\mathbf{S}\mathbf{p}\mathbf{i}\mathbf{n}$ denote the generator of $\pi_4\mathbf{B}\mathbf{S}\mathbf{p}\mathbf{i}\mathbf{n} \cong \mathbf{Z}$, and let $\Omega S^5 \rightarrow \mathbf{B}\mathbf{S}\mathbf{p}\mathbf{i}\mathbf{n}$ denote the extension of this map, which classifies a real vector bundle of virtual dimension zero over ΩS^5 . Let A denote the Thom spectrum of this bundle.

Remark 3.2.9. As mentioned in the introduction, the spectrum A has been intensely studied by Mahowald and his coauthors in (for instance) [Mah79, DM81, Mah81b, Mah81a, Mah82, MU77], where it is often denoted X_5 .

Remark 3.2.10. The map $\Omega S^5 \rightarrow \mathbf{BSpin}$ is one of \mathbf{E}_1 -spaces, so the Thom spectrum A admits the structure of an \mathbf{E}_1 -ring with an \mathbf{E}_1 -map $A \rightarrow \mathbf{MSpin}$.

Remark 3.2.11. There are multiple equivalent ways to characterize this Thom spectrum. For instance, the J -homomorphism $\mathbf{BSpin} \rightarrow \mathbf{BGL}_1(\mathbb{S})$ sends the generator of $\pi_4 \mathbf{BSpin}$ to $\nu \in \pi_4 \mathbf{BGL}_1(\mathbb{S}) \cong \pi_3 \mathbb{S}$. The universal property of Thom spectra in Theorem 2.1.7 shows that A is the free \mathbf{E}_1 -ring $\mathbb{S} // \nu$ with a nullhomotopy of ν . Note that A is defined integrally, and not just p -locally for some prime p .

Remark 3.2.12. There is a canonical map $A \rightarrow T(1)$ of \mathbf{E}_1 -rings, constructed as follows. By the universal property of A , it suffices to prove that the unit $\mathbb{S} \rightarrow T(1)$ extends along the inclusion $\mathbb{S} \rightarrow C\nu$, i.e., that $\nu = 0 \in \pi_3 T(1)$ up to units. To see this, let us compute $\pi_3 C\eta$ via the exact sequence

$$\pi_3 S^1 \xrightarrow{\eta} \pi_3 S^0 \rightarrow \pi_3 C\eta \rightarrow \pi_2 S^0 \xrightarrow{\eta} \pi_1 S^0.$$

This can be identified with

$$\mathbf{Z}/2\{\eta^2\} \xrightarrow{\eta} \mathbf{Z}/8\{\nu\} \rightarrow \pi_3 C\eta \rightarrow \mathbf{Z}/2\{\eta\} \xrightarrow{\eta} \mathbf{Z}/2\{\eta^2\};$$

the final map is an isomorphism, and the first map sends $\eta^2 \mapsto \eta^3 = 4\nu$. Therefore, $\pi_3 C\eta \cong \mathbf{Z}/4\{\nu\}$. Now, since the class in $\mathbf{H}_4(T(1); \mathbf{F}_2)$ is detected by a nontrivial \mathbf{Sq}^4 , the attaching map of the 4-cell in $T(1)$ must be $\pm\nu$. Therefore, one of $\pm\nu$ must be null in $T(1)$, which implies that there must be a map $C\nu \rightarrow T(1)$ (or $C(-\nu) \rightarrow T(1)$) as claimed.

The following result is [Dev19b, Proposition 2.7]; it is proved there at $p = 2$, but the argument clearly works for $p = 3$ too.

Proposition 3.2.13. *There is an isomorphism $\mathbf{BP}_*(A) \cong \mathbf{BP}_*[y_2]$, where $|y_2| = 4$. There is a map $A_{(p)} \rightarrow \mathbf{BP}$. Under the induced map on \mathbf{BP} -homology, y_2 maps to t_1^2 mod decomposables at $p = 2$, and to t_1 mod decomposables at $p = 3$.*

Remark 3.2.14. For instance, when $p = 2$, we have $\mathbf{BP}_*(A) \cong \mathbf{BP}_*[t_1^2 + v_1 t_1]$.

One corollary (using Remark 3.2.1) is the following.

Corollary 3.2.15. *As A_* -comodules, we have*

$$\mathbf{H}_*(A; \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[\zeta_1^4] & p = 2 \\ \mathbf{F}_3[\zeta_1] & p = 3 \\ \mathbf{F}_p[x_4] & p \geq 5, \end{cases}$$

where x_4 is a polynomial generator in degree 4.

Example 3.2.16. Let us work at $p = 2$ for convenience. Example 3.1.14 showed that σ_1 is the element in $\pi_5(C\eta \wedge C\nu)$ given by the lift of η to the 4-cell (which is attached to the bottom cell by ν) via a nullhomotopy of $\eta\nu$. In particular, σ_1 already lives in $\pi_5(C\nu)$, and as such defines an element of $\mathbb{S} // \nu = A$ (by viewing $C\nu$ as the 4-skeleton of A); note that, by construction, this element is 2-torsion. The image of $\sigma_1 \in \pi_5(A)$ under the canonical map of Remark 3.2.12 is its namesake in $\pi_5(T(1))$. See Figure 2.

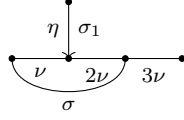


FIGURE 2. 15-skeleton of A at the prime 2 shown horizontally, with 0-cell on the left. The element σ_1 given by the map η on the 4-cell, as indicated in the diagram above.

Remark 3.2.17. The element $\sigma_1 \in \pi_5(A_{(2)})$ defined in Example 3.2.16 in fact lifts to an element of $\pi_5(A)$, because the relation $\eta\nu = 0$ is true integrally, and not just 2-locally. An alternate construction of this map is the following. The Hopf map $\eta_4 : S^5 \rightarrow S^4$ (which lives in the stable range) defines a map $S^5 \rightarrow S^4 \rightarrow \Omega S^5$ whose composite to $B\text{Spin}$ is null (since $\pi_5(B\text{Spin}) = 0$). Upon Thomification of the composite $S^5 \rightarrow \Omega S^5 \rightarrow B\text{Spin}$, one therefore gets a map $S^5 \rightarrow A$ whose composite with $A \rightarrow M\text{Spin}$ is null. The map $S^5 \rightarrow A$ is the element $\sigma_1 \in \pi_5(A)$.

Finally, we have:

Definition 3.2.18. Let BN be the space defined by the homotopy pullback

$$\begin{array}{ccc} BN & \longrightarrow & S^{13} \\ \downarrow & & \downarrow f \\ \text{BO}(9) & \longrightarrow & \text{BO}(10), \end{array}$$

where the map $f : S^{13} \rightarrow \text{BO}(10)$ detects an element of $\pi_{12}\text{O}(10) \cong \mathbf{Z}/12$. There is a fiber sequence

$$S^9 \rightarrow \text{BO}(9) \rightarrow \text{BO}(10),$$

and the image of f under the boundary map in the long exact sequence of homotopy detects $2\nu \in \pi_{12}(S^9) \cong \mathbf{Z}/24$. In particular, there is a fiber sequence

$$S^9 \rightarrow BN \rightarrow S^{13}.$$

If N is defined to be ΩBN , then there is a fiber sequence

$$N \rightarrow \Omega S^{13} \rightarrow S^9.$$

Define a map $N \rightarrow B\text{String}$ via the map of fiber sequences

$$\begin{array}{ccccc} N & \longrightarrow & \Omega S^{13} & \longrightarrow & S^9 \\ \downarrow & & \downarrow & & \downarrow \\ B\text{String} & \longrightarrow & * & \longrightarrow & B^2\text{String}, \end{array}$$

where the map $S^9 \rightarrow B^2\text{String}$ detects a generator of $\pi_8 B\text{String}$. Let B denote the Thom spectrum of the induced bundle over N .

Remark 3.2.19. The map $N \rightarrow B\text{String}$ is in fact one of \mathbf{E}_1 -spaces, so B admits the structure of an \mathbf{E}_1 -ring. To prove this, it suffices to show that there is a map $BN \rightarrow B^2\text{String}$. Recall that $B\text{String} = \tau_{\geq 8}\Omega^\infty \text{KO}$, so the desired map is the same as a class in $\text{KO}^1(BN)$. Using the Serre spectral sequence for the fiber sequence defining BN , one can calculate that there is a class in $\text{KO}^1(BN)$ which lifts the generator of $\text{KO}^1(S^9) \cong \pi_8 \text{KO} \cong \mathbf{Z}$.

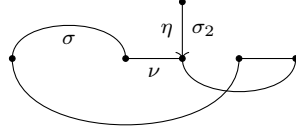


FIGURE 3. Steenrod module structure of the 20-skeleton of B ; the bottom cell (in dimension 0) is on the left; straight lines are Sq^4 , and curved lines correspond to Sq^8 and Sq^{16} , in order of increasing length. The bottom two attaching maps of B are labeled. The map σ_2 is shown.

We introduced the spectrum B and studied its Adams-Novikov spectral sequence in [Dev19b]. The Steenrod module structure of the 20-skeleton of B is shown in [Dev19b, Figure 1], and is reproduced here as Figure 3. As mentioned in the introduction, the spectrum B has been briefly studied under the name \bar{X} in [HM02].

Remark 3.2.20. As with A , there are multiple different ways to characterize B . There is a fiber sequence

$$\Omega S^9 \rightarrow N \rightarrow \Omega S^{13},$$

and the map $\Omega S^9 \rightarrow N \rightarrow \text{BString}$ is an extension of the map $S^8 \rightarrow \text{BString}$ detecting a generator. Under the J -homomorphism $\text{BString} \rightarrow \text{BGL}_1(\mathbb{S})$, this generator maps to $\sigma \in \pi_8 \text{BGL}_1(\mathbb{S}) \cong \pi_7 \mathbb{S}$, so the Thom spectrum of the bundle over ΩS^9 determined by the map $\Omega S^9 \rightarrow \text{BString}$ is the free \mathbf{E}_1 -ring $\mathbb{S} // \sigma$ with a nullhomotopy of ν . Proposition 2.1.6 now implies that N is the Thom spectrum of a map $\Omega S^{13} \rightarrow \text{BGL}_1(\mathbb{S} // \sigma)$. While a direct definition of this map is not obvious, we note that the restriction to the bottom cell S^{12} of the source detects an element $\tilde{\nu}$ of $\pi_{12} \text{BGL}_1(\mathbb{S} // \sigma) \cong \pi_{11} \mathbb{S} // \sigma$. This in turn factors through the 11-skeleton of $\mathbb{S} // \sigma$, which is the same as the 8-skeleton of $\mathbb{S} // \sigma$ (namely, $C\sigma$). This element is precisely a lift of the map $\nu : S^{11} \rightarrow S^8$ to $C\sigma$ determined by a nullhomotopy of $\sigma\nu$ in $\pi_* \mathbb{S}$. Although $\tilde{\nu} \in \pi_{11} C\sigma$ does not come from a class in $\pi_{11} \mathbb{S}$, its representative in the Adams spectral sequence for $C\sigma$ is the image of h_{22} in the Adams spectral sequence for the sphere.

The following result is [Dev19b, Proposition 3.2]; it is proved there at $p = 2$, but the argument clearly works for $p \geq 3$ too.

Proposition 3.2.21. *The BP_* -algebra $\text{BP}_*(B)$ is isomorphic to a polynomial ring $\text{BP}_*[b_4, y_6]$, where $|b_4| = 8$ and $|y_6| = 12$. There is a map $B_{(p)} \rightarrow \text{BP}$. On BP_* -homology, the elements b_4 and y_6 map to t_1^4 and t_2^2 mod decomposables at $p = 2$, and y_6 maps to t_1^3 mod decomposables at $p = 3$.*

One corollary (using Remark 3.2.1) is the following.

Corollary 3.2.22. *As A_* -comodules, we have*

$$H_*(B; \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_2[\zeta_1^8, \zeta_2^4] & p = 2 \\ \mathbf{F}_3[\zeta_1^3, b_4] & p = 3 \\ \mathbf{F}_5[\zeta_1, x_{12}] & p = 5 \\ \mathbf{F}_p[x_8, x_{12}] & p \geq 7, \end{cases}$$

where x_8 and x_{12} are polynomial generators in degree 8 and 12, and b_4 is an element in degree 8.

Example 3.2.23. For simplicity, let us work at $p = 2$. There is a canonical ring map $B \rightarrow T(2)$, and the element $\sigma_2 \in \pi_{13}T(2)$ lifts to B . We can be explicit about this: the 12-skeleton of B is shown in Figure 3, and σ_2 is the element of $\pi_{13}(B)$ existing thanks to the relation $\eta\nu = 0$ and the fact that the Toda bracket $\langle \eta, \nu, \sigma \rangle$ contains 0. This also shows that $\sigma_2 \in \pi_{13}(B)$ is 2-torsion.

Remark 3.2.24. The element $\sigma_2 \in \pi_{13}(B_{(2)})$ defined in Example 3.2.23 in fact lifts to an element of $\pi_{13}(B)$, because the relations $\nu\sigma = 0$, $\eta\nu = 0$, and $0 \in \langle \eta, \nu, \sigma \rangle$ are all true integrally, and not just 2-locally. An alternate construction of this map $S^{13} \rightarrow B$ is the following. The Hopf map $\eta_{12} : S^{13} \rightarrow S^{12}$ (which lives in the stable range) defines a map $S^{13} \rightarrow S^{12} \rightarrow \Omega S^{13}$. Moreover, the composite $S^{13} \rightarrow \Omega S^{13} \rightarrow S^9$ is null, since it detects an element of $\pi_{13}(S^9) = 0$; choosing a nullhomotopy of this composite defines a lift $S^{13} \rightarrow N$. (In fact, this comes from a map $S^{14} \rightarrow BN$.) The composite $S^{13} \rightarrow N \rightarrow \text{BString}$ is null (since $\pi_{13}(\text{BString}) = 0$). Upon Thomification, we obtain a map $S^{13} \rightarrow B$ whose composite with $B \rightarrow \text{MString}$ is null; the map $S^{13} \rightarrow B$ is the element $\sigma_2 \in \pi_{13}(B)$.

The following theorem packages some information contained in this section.

Theorem 3.2.25. *Let R denote any of the spectra in Table 2, and let n denote its “height”. If $R = T(n)$, $y(n)$, or $\mathbf{yZ}(n)$, then there is a map $T(n) \rightarrow R$, and if $R = A$ (resp. B), then there is a map from R to $T(1)$ (resp. $T(2)$). In the first three cases, there is an element $\sigma_n \in \pi_{|v_{n+1}|-1}R$ coming from $\sigma_n \in \pi_{|v_{n+1}|-1}T(n)$, and in the cases $R = A$ and B , there are elements $\sigma_1 \in \pi_5(A)$ and $\sigma_2 \in \pi_{13}(B)$ mapping to the corresponding elements in $T(1)_{(2)}$ and $T(2)_{(2)}$, respectively. Moreover, σ_n is p -torsion in π_*R ; similarly, σ_1 and σ_2 are 2-torsion in $\pi_*A_{(2)}$ and $\pi_*B_{(2)}$.*

PROOF. The existence statement for $T(n)$ is contained in Theorem 3.1.5, while the torsion statement is the content of Lemma 3.1.16. The claims for $y(n)$ and $\mathbf{yZ}(n)$ now follow from Proposition 3.2.6. The existence and torsion statements for A and B are contained in Examples 3.2.16 and 3.2.23. \square

The elements in Theorem 3.2.25 can in fact be extended to infinite families; this is discussed in Section 5.4.

3.3. Centers of Thom spectra. In this section, we review some of the theory of \mathbf{E}_k -centers and state Conjecture E. We begin with the following important result, and refer to [Fra13] and [Lur16, Section 5.5.4] for proofs.

Theorem 3.3.1 ([Lur16, Example 5.5.4.16], [Fra13, Definition 2.5]). *Let \mathcal{C} be a symmetric monoidal presentable ∞ -category, and let A be an \mathbf{E}_k -algebra in \mathcal{C} . Then the category of \mathbf{E}_k - A -modules is equivalent to the category of left modules over the factorization homology $U(A) = \int_{S^{k-1} \times \mathbf{R}} A$ (known as the enveloping algebra of A), which is an \mathbf{E}_1 -algebra in \mathcal{C} .*

Definition 3.3.2. The \mathbf{E}_{k+1} -center $\mathfrak{Z}(A)$ of an \mathbf{E}_k -algebra A in \mathcal{C} is the (\mathbf{E}_{k+1}) -Hochschild cohomology $\text{End}_{U(A)}(A)$, where A is regarded as a left module over its enveloping algebra via Theorem 3.3.1.

Remark 3.3.3. We are using slightly different terminology than the one used in [Lur16, Section 5.3]: our \mathbf{E}_{k+1} -center is his \mathbf{E}_k -center. In other words, Lurie’s

terminology expresses the structure on the input, while our terminology expresses the structure on the output.

The following proposition summarizes some results from [Fra13] and [Lur16, Section 5.3].

Proposition 3.3.4 ([Lur16, Theorem 5.3.2.5], [Fra13, Theorem 1.1]). *The \mathbf{E}_{k+1} -center $\mathfrak{Z}(A)$ of an \mathbf{E}_k -algebra A in a symmetric monoidal presentable ∞ -category \mathcal{C} exists, and satisfies the following properties:*

- (a) $\mathfrak{Z}(A)$ is the universal \mathbf{E}_k -algebra of \mathcal{C} which fits into a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A \otimes \mathfrak{Z}(A) \\ & \searrow & \downarrow \\ & & A \end{array}$$

in $\mathrm{Alg}_{\mathbf{E}_k}(\mathcal{C})$.

- (b) The \mathbf{E}_k -algebra $\mathfrak{Z}(A)$ of \mathcal{C} defined via this universal property in fact admits the structure of an \mathbf{E}_{k+1} -algebra in \mathcal{C} .
 (c) There is a fiber sequence

$$\mathrm{GL}_1(\mathfrak{Z}(A)) \rightarrow \mathrm{GL}_1(A) \rightarrow \Omega^{k-1} \mathrm{End}_{\mathrm{Alg}_{\mathbf{E}_k}(\mathcal{C})}(A)$$

of k -fold loop spaces.

In the sequel, we will need a more general notion:

Definition 3.3.5. Let $m \geq 1$. The \mathbf{E}_{k+m} -center $\mathfrak{Z}_{k+m}(A)$ of an \mathbf{E}_k -algebra A in a presentable symmetric monoidal ∞ -category \mathcal{C} with all limits is defined inductively as the \mathbf{E}_{k+m} -center of the \mathbf{E}_{k+m-1} -center $\mathfrak{Z}_{k+m-1}(A)$. In other words, it is the universal \mathbf{E}_{k+m} -algebra of \mathcal{C} which fits into a commutative diagram

$$\begin{array}{ccc} \mathfrak{Z}_{k+m-1}(A) & \longrightarrow & \mathfrak{Z}_{k+m-1}(A) \otimes \mathfrak{Z}_{k+m}(A) \\ & \searrow & \downarrow \\ & & \mathfrak{Z}_{k+m-1}(A) \end{array}$$

in $\mathrm{Alg}_{\mathbf{E}_{k+m-1}}(\mathcal{C})$.

Proposition 3.3.4 gives:

Corollary 3.3.6. *Let $m \geq 1$. The \mathbf{E}_{k+m-1} -algebra $\mathfrak{Z}_{k+m}(A)$ associated to an \mathbf{E}_k -algebra object A of \mathcal{C} exists, and in fact admits the structure of an \mathbf{E}_{k+m} -algebra in \mathcal{C} .*

We can now finally state Conjecture E:

Conjecture E. *Let $n \geq 0$ be an integer. Let R denote $X(p^{n+1} - 1)_{(p)}$, A (in which case $n = 1$), or B (in which case $n = 2$). Then the element $\sigma_n \in \pi_{|\sigma_n|} R$ lifts to the \mathbf{E}_3 -center $\mathfrak{Z}_3(R)$ of R , and is p -torsion in $\pi_* \mathfrak{Z}_3(R)$ if $R = X(p^{n+1} - 1)_{(p)}$, and is 2-torsion in $\pi_* \mathfrak{Z}_3(R)$ if $R = A$ or B .*

Remark 3.3.7. If R is A or B , then $\mathfrak{Z}_3(R)$ is the \mathbf{E}_3 -center of the \mathbf{E}_2 -center of R . This is a rather unwieldy object, so it would be quite useful to show that the \mathbf{E}_1 -structure on A or B admits an extension to an \mathbf{E}_2 -structure; we do not know if

such extensions exist. Since neither ΩS^5 nor N admit the structure of a double loop space, such an \mathbf{E}_2 -structure would not arise from their structure as Thom spectra. In any case, if such extensions do exist, then $\mathfrak{Z}_3(R)$ in Conjecture E should be interpreted as the \mathbf{E}_3 -center of the \mathbf{E}_2 -ring R . However, we showed in [Dev19a, Theorem 4.2] that $(\mathrm{tmf} \wedge A)[x_2]$ admits an \mathbf{E}_2 -algebra structure, where $|x_2| = 2$.

Remark 3.3.8. In the introduction, we stated Conjecture 1.1.4, which instead asked about whether $v_n \in \pi_{|v_n|} X(p^n)$ lifts to $\pi_* \mathfrak{Z}_3(X(p^n))$. It is natural to ask about the connection between Conjecture E and Conjecture 1.1.4. Proposition 3.1.17 implies that if $\mathfrak{Z}_3(X(p^n))$ admitted an $X(p^n - 1)$ -orientation factoring the canonical $X(p^n - 1)$ -orientation $X(p^n - 1) \rightarrow X(p^n)$, and $\sigma_{n-1} \in \pi_{|v_n|-1} X(p^n - 1)$ was killed by the map $X(p^n - 1) \rightarrow \mathfrak{Z}_3(X(p^n))$, then Conjecture E implies Conjecture 1.1.4. However, we do not believe that either of these statements are true.

Remark 3.3.9. One of the main results of [Kla18] implies that the \mathbf{E}_3 -center of $X(n)$ (which, recall, is the Thom spectrum of a bundle over $\Omega^2 \mathrm{BSU}(n)$) is $\mathrm{Hom}_{\mathrm{SU}(n)_+}(\mathbb{S}, X(n)) \simeq X(n)^{h\mathrm{SU}(n)}$, where $\mathrm{SU}(n)$ acts on $X(n)$ by a Thomification of the conjugation action on $\Omega \mathrm{SU}(n)$.

Remark 3.3.10. Note that the conjugation action of $\mathrm{SU}(n)$ on $X(n)$ can be described very explicitly, via a concrete model for $\Omega \mathrm{SU}(n)$. As explained in [PS86, Zhu17], if G is a reductive linear algebraic group over \mathbf{C} , the loop space $\Omega G(\mathbf{C})$ of its complex points (viewed as a complex Lie group) is equivalent to the homogeneous space $G(\mathbf{C}((t)))/G(\mathbf{C}[[t]])$; this is also commonly studied as the complex points of the affine Grassmannian Gr_G of G . The conjugation action of $G(\mathbf{C})$ on $\Omega G(\mathbf{C})$ arises by restricting the descent (to $G(\mathbf{C}((t)))/G(\mathbf{C}[[t]])$) of the translation action by $G(\mathbf{C}[[t]])$ on $G(\mathbf{C}((t)))$ to the subgroup $G(\mathbf{C}) \subseteq G(\mathbf{C}[[t]])$. Setting $G = \mathrm{SL}_n$ gives a description of the conjugation action of $\mathrm{SU}(n)$ on $\Omega \mathrm{SU}(n)$. In light of its connections to geometric representation theory, we believe that there may be an algebro-geometric approach to proving that χ_n is $\mathrm{SU}(n)$ -trivial in $X(n)$ and in $\Omega \mathrm{SU}(n)$.

Example 3.3.11. The element $\chi_2 \in \pi_3 X(2)$ is central. To see this, note that $\alpha \in \pi_* R$ (where R is an \mathbf{E}_k -ring) is in the \mathbf{E}_{k+1} -center of R if and only if α is in the \mathbf{E}_{k+1} -center of $R_{(p)}$ for all primes $p \geq 0$. It therefore suffices to show that χ_2 is central after p -localizing for all p . First, note that χ_2 is torsion, so it is nullhomotopic (and therefore central) after rationalization. Next, if $p > 2$, then $X(2)_{(p)}$ splits as a wedge of suspensions of spheres. If χ_2 is detected in π_3 of a sphere living in dimension 3, then it could not be torsion, so it must be detected in π_3 of a sphere living in dimension $3 - k$ for some $0 \leq k \leq 2$. If $k = 1$ or 2 , then $\pi_3(S^{3-k})$ is either $\pi_1(S^0)$ or $\pi_2(S^0)$, but both of these groups vanish for $p > 2$. Therefore, χ_2 must be detected in π_3 of the sphere in dimension 0, i.e., in $\pi_3 X(1)$. This group vanishes for $p > 3$, and when $p = 3$, it is isomorphic to $\mathbf{Z}/3$ (generated by α_1). Since $X(1) = S^0$ is an \mathbf{E}_∞ -ring, we conclude that χ_2 is central in $X(2)_{(p)}$ for all $p > 2$.

At $p = 2$, we know the cell structure of $X(2)$ in the bottom few dimensions (see Example 3.1.14; note that σ_1 is *not* χ_2). In dimensions ≤ 3 , it is equivalent to $C\eta$, so $\pi_3 X(2) \cong \pi_3 C\eta$. However, it is easy to see that the canonical map $\pi_3 \mathbb{S} \simeq \mathbf{Z}/8\{\nu\} \rightarrow \pi_3 C\eta$ is surjective and exhibits an isomorphism $\pi_3 C\eta \cong \mathbf{Z}/4\{\nu\}$.

Therefore, χ_2 is in the image of the unit $\mathbb{S} \rightarrow X(2)$, and is therefore vacuously central. We conclude from the above discussion that χ_2 is indeed central in $X(2)$.

4. Review of some unstable homotopy theory

4.1. Charming and Gray maps. A major milestone in unstable homotopy theory was Cohen-Moore-Neisendorfer's result on the p -exponent of unstable homotopy groups of spheres from [CMN79a, CMN79b, Nei81]. They defined for all $p > 2$ and $k \geq 1$ a map $\phi_n : \Omega^2 S^{2n+1} \rightarrow S^{2n-1}$ (the integer k is assumed implicit) such that the composite of ϕ_n with the double suspension $E^2 : S^{2n-1} \rightarrow \Omega^2 S^{2n+1}$ is homotopic to the p^k -th power map. By induction on n , they concluded via a result of Selick's (see [Sel77]) that p^n kills the p -primary component of the homotopy of S^{2n+1} . Such maps will be important in the rest of this article, so we will isolate their desired properties in the definition of a *charming map*, inspired by [ST19]. (Our choice of terminology is non-standard, and admittedly horrible, but it does not seem like the literature has chosen any naming convention for the sort of maps we desire.)

Definition 4.1.1. A p -local map $f : \Omega^2 S^{2np+1} \rightarrow S^{2np-1}$ is called a *Gray map* if the composite of f with the double suspension E^2 is the degree p map, and the composite

$$\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$$

is nullhomotopic. Moreover, a p -local map $f : \Omega^2 S^{2np+1} \rightarrow S^{2np-1}$ is called a *charming map* if the composite of f with the double suspension E^2 is the degree p map, the fiber of f admits the structure of a \mathcal{Q}_1 -space, and if there is a space BK which sits in a fiber sequence

$$S^{2np-1} \rightarrow BK \rightarrow \Omega S^{2np+1}$$

such that the boundary map $\Omega^2 S^{2np+1} \rightarrow S^{2np-1}$ is homotopic to f .

Remark 4.1.2. If f is a charming map, then the fiber of f is a loop space. Indeed, $\text{fib}(f) \simeq \Omega BK$.

Example 4.1.3. Let f denote the Cohen-Moore-Neisendorfer map with $k = 1$. Anick proved (see [Ani93, GT10]) that the fiber of f admits a delooping, i.e., there is a space $T^{2np+1}(p)$ (now known as an *Anick space*) which sits in a fiber sequence

$$S^{2np-1} \rightarrow T^{2np+1}(p) \rightarrow \Omega S^{2np+1}.$$

It follows that f is a charming map.

Remark 4.1.4. We claim that $T^{2p+1}(p) = \Omega S^3 \langle 3 \rangle$, where $S^3 \langle 3 \rangle$ is the 3-connected cover of S^3 . To prove this, we will construct a p -local fiber sequence

$$S^{2p-1} \rightarrow \Omega S^3 \langle 3 \rangle \rightarrow \Omega S^{2p+1}.$$

This fiber sequence was originally constructed by Toda in [Tod62]. To construct this fiber sequence, we first note that there is a p -local fiber sequence

$$S^{2p-1} \rightarrow J_{p-1}(S^2) \rightarrow \mathbf{C}P^\infty,$$

where the first map is the factorization of $\alpha_1 : S^{2p-1} \rightarrow \Omega S^3$ through the $2(p-1)$ -skeleton of ΩS^3 , and the second map is the composite $J_{p-1}(S^2) \rightarrow \Omega S^3 \rightarrow \mathbf{C}P^\infty$. This fiber sequence is simply an odd-primary version of the Hopf fibration $S^3 \rightarrow S^2 \rightarrow \mathbf{C}P^\infty$; the identification of the fiber of the map $J_{p-1}(S^2) \rightarrow \mathbf{C}P^\infty$ is a simple exercise with the Serre spectral sequence. Next, we have the EHP sequence

$$J_{p-1}(S^2) \rightarrow \Omega S^3 \rightarrow \Omega S^{2p+1}.$$

Since $\Omega S^3\langle 3 \rangle$ is the fiber of the map $\Omega S^3 \rightarrow \mathbf{C}P^\infty$, the desired fiber sequence is obtained by taking vertical fibers in the following map of fiber sequences:

$$\begin{array}{ccccc} J_{p-1}(S^2) & \longrightarrow & \Omega S^3 & \longrightarrow & \Omega S^{2p+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{C}P^\infty & \xlongequal{\quad} & \mathbf{C}P^\infty & \longrightarrow & * \end{array}$$

Example 4.1.5. Let W_n denote the fiber of the double suspension $S^{2n-1} \rightarrow \Omega^2 S^{2n+1}$. Gray proved in [Gra89b, Gra88] that W_n admits a delooping BW_n , and that after p -localization, there is a fiber sequence

$$BW_n \rightarrow \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$$

for some map f . As suggested by the naming convention, f is a Gray map.

As proved in [ST19], Gray maps satisfy an important rigidity property:

Proposition 4.1.6 (Selick-Theriault). *The fiber of any Gray map admits an H-space structure, and is H-equivalent to BW_n .*

Remark 4.1.7. It has been conjectured by Cohen-Moore-Neisendorfer and Gray in the papers cited above that there is an equivalence $BW_n \simeq \Omega T^{2np+1}(p)$, and that $\Omega T^{2np+1}(p)$ retracts off of $\Omega^2 P^{2np+1}(p)$ as an H-space, where $P^k(p)$ is the mod p Moore space $S^{k-1} \cup_p e^k$ with top cell in dimension k . For our purposes, we shall require something slightly stronger: namely, the retraction should be one of \mathcal{Q}_1 -spaces. The first part of this conjecture would follow from Proposition 4.1.6 if the Cohen-Moore-Neisendorfer map were a Gray map. In [Ame20], it is shown that the existence of p -primary elements of Kervaire invariant one would imply equivalences of the form $BW_{p^{n-1}} \simeq \Omega T^{2p^n+1}(p)$.

Motivated by Remark 4.1.7 and Proposition 4.1.6, we state the following conjecture; it is slightly weaker than the conjecture mentioned in Remark 4.1.7, and is an amalgamation of slight modifications of conjectures of Cohen, Moore, Neisendorfer, Gray, and Mahowald in unstable homotopy theory, as well as an analogue of Proposition 4.1.6. (For instance, we strengthen having an H-space retraction to having a \mathcal{Q}_1 -space retraction).

Conjecture D. *The following statements are true:*

- (a) *The homotopy fiber of any charming map is equivalent as a loop space to the loop space on an Anick space.*
- (b) *There exists a p -local charming map $f : \Omega^2 S^{2p^n+1} \rightarrow S^{2p^n-1}$ whose homotopy fiber admits a \mathcal{Q}_1 -space retraction off of $\Omega^2 P^{2p^n+1}(p)$. There are also integrally defined maps $\Omega^2 S^9 \rightarrow S^7$ and $\Omega^2 S^{17} \rightarrow S^{15}$ whose composite with the double suspension on S^7 and S^{15} respectively is the degree 2 map, whose homotopy fibers K_2 and K_3 (respectively) admit deloopings, and which admits a \mathcal{Q}_1 -space retraction off $\Omega^2 P^9(2)$ and $\Omega^2 P^{17}(2)$ (respectively).*

Remark 4.1.8. Conjecture D is already not known when $n = 1$. In this case, it asserts that $\Omega^2 S^3\langle 3 \rangle$ retracts off of $\Omega^2 P^{2p+1}(p)$. A theorem of Selick's states that $\Omega^2 S^3\langle 3 \rangle$ retracts off of $\Omega^2 S^{2p+1}\{p\}$ for p odd, where $\Omega^2 S^{2p+1}\{p\}$ is the fiber of the degree p map on $\Omega^2 S^{2p+1}$. This implies that $\Omega^2 S^3\langle 3 \rangle$ retracts off of $\Omega^3 P^{2p+2}(p)$. In

[Coh87, Observation 9.2], the question of whether $\Omega^2 S^3 \langle 3 \rangle$ retracts off of $\Omega^2 P^{2p+1}(p)$ was shown to be equivalent to the question of whether there is a map $\Sigma^2 \Omega^2 S^3 \langle 3 \rangle \rightarrow P^{2p+1}(p)$ which is onto in homology. Some recent results regarding Conjecture D for $n = 1$ can be found in [BT13].

It follows that a retraction of $\Omega^2 S^3 \langle 3 \rangle$ off $\Omega^2 P^{2p+1}(p)$ will be compatible with the canonical map $\Omega^2 S^3 \langle 3 \rangle \rightarrow \Omega^2 S^3$ in the following manner. The p -torsion element $\alpha_1 \in \pi_{2p}(S^3)$ defines a map $P^{2p-1}(p) \rightarrow \Omega^2 S^3$, which extends to an \mathbf{E}_2 -map $\Omega^2 P^{2p+1}(p) \rightarrow \Omega^2 S^3$. We will abusively denote this extension by α_1 . The resulting composite

$$\Omega^2 S^3 \langle 3 \rangle \rightarrow \Omega^2 P^{2p+1}(p) \xrightarrow{\alpha_1} \Omega^2 S^3$$

is homotopic to the canonical map $\Omega^2 S^3 \langle 3 \rangle \rightarrow \Omega^2 S^3$.

The element $\alpha_1 \in \pi_{2p-3}(\mathbb{S}_{(p)})$ defines a map $S^{2p-2} \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)})$, and since it is p -torsion, admits an extension to a map $P^{2p-1}(p) \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)})$. (This extension is in fact unique, because $\pi_{2p-1}(\mathrm{BGL}_1(\mathbb{S}_{(p)})) \cong \pi_{2p-2}(\mathbb{S}_{(p)})$ vanishes.) Since $\mathrm{BGL}_1(\mathbb{S}_{(p)})$ is an infinite loop space, this map further extends to a map $\Omega^2 P^{2p+1}(p) \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)})$. The discussion in the previous paragraph implies that if Conjecture D is true for $n = 1$, then the map $\mu : \Omega^2 S^3 \langle 3 \rangle \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)})$ from Corollary 2.2.2 is homotopic to the composite

$$\Omega^2 S^3 \langle 3 \rangle \rightarrow \Omega^2 P^{2p+1}(p) \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)}).$$

4.2. Fibers of charming maps. We shall need the following proposition.

Proposition 4.2.1. *Let $f : \Omega^2 S^{2p^n+1} \rightarrow S^{2p^n-1}$ be a charming map. Then there are isomorphisms of coalgebras:*

$$\mathrm{H}_*(\mathrm{fib}(f); \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_p[x_{2^{n+1}-1}] \otimes \bigotimes_{k>1} \mathbf{F}_p[x_{2^{n+k}-1}] & p = 2 \\ \bigotimes_{k>0} \mathbf{F}_p[y_{2(p^n+k-1)}] \otimes \bigotimes_{j>0} \Lambda_{\mathbf{F}_p}[x_{2p^n+j-1}] & p > 2. \end{cases}$$

PROOF. This is an easy consequence of the Serre spectral sequence coupled with the well-known coalgebra isomorphisms

$$\mathrm{H}_*(\Omega^2 S^{2n+1}; \mathbf{F}_p) \cong \begin{cases} \bigotimes_{k>0} \mathbf{F}_p[x_{2^k n-1}] & p = 2 \\ \bigotimes_{k>0} \mathbf{F}_p[y_{2(np^k-1)}] \otimes \bigotimes_{j \geq 0} \Lambda_{\mathbf{F}_p}[x_{2np^j-1}] & p > 2, \end{cases}$$

where these classes are generated by the one in dimension $2n-1$ via the single Dyer-Lashof operation (coming already from the cup-1 operad; see Remark 2.2.8). \square

Remark 4.2.2. The Anick spaces $T^{2np+1}(p)$ from Example 4.1.3 sit in fiber sequences

$$S^{2np-1} \rightarrow T^{2np+1}(p) \rightarrow \Omega S^{2np+1},$$

and are homotopy commutative H-spaces. A Serre spectral sequence calculation gives an identification of coalgebras

$$\mathrm{H}_*(T^{2np+1}(p); \mathbf{F}_p) \cong \mathbf{F}_p[a_{2np}] \otimes \Lambda_{\mathbf{F}_p}[b_{2np-1}],$$

with $\beta(a_{2np}) = b_{2np-1}$, where β is the Bockstein homomorphism. An argument with the bar spectral sequence recovers the result of Proposition 4.2.1 in this particular case.

Remark 4.2.3. Suppose that X is a space which sits in a fiber sequence

$$S^{2np-1} \rightarrow X \rightarrow \Omega S^{2np+1}$$

such that the boundary map $\Omega^2 S^{2np+1} \rightarrow S^{2np-1}$ has degree p^j on the bottom cell of the source. The Serre spectral sequence then only has a differential on the E_{2np-1} -page, and:

$$H_i(BK; \mathbf{Z}) \cong \begin{cases} \mathbf{Z} & i = 0 \\ \mathbf{Z}/p^j k & \text{if } i = 2npk - 1 \\ 0 & \text{else.} \end{cases}$$

We conclude this section by investigating Thom spectra of bundles defined over fibers of charming maps. Let R be a p -local \mathbf{E}_1 -ring, and let $\mu : K \rightarrow \mathrm{BGL}_1(R)$ denote a map from the fiber K of a charming map $f : \Omega^2 S^{2np+1} \rightarrow S^{2np-1}$. There is a fiber sequence $\Omega S^{2np-1} \rightarrow K \rightarrow \Omega^2 S^{2np+1}$ of loop spaces, so we obtain a map $\Omega S^{2np-1} \rightarrow \mathrm{BGL}_1(R)$. Such a map gives an element $\alpha \in \pi_{2np-3}R$ via the effect on the bottom cell S^{2np-2} .

Theorem 2.1.7 implies that the Thom spectrum of the map $\Omega S^{2np-1} \rightarrow \mathrm{BGL}_1(R)$ should be thought of as the \mathbf{E}_1 -quotient $R//\alpha$, although this may not make sense if R is not at least \mathbf{E}_2 . However, in many cases (such as the ones we are considering here), the Thom R -module $R//\alpha$ is in fact an \mathbf{E}_1 -ring such that the map $R \rightarrow R//\alpha$ is an \mathbf{E}_1 -map. By Proposition 2.1.6, there is an induced map $\phi : \Omega^2 S^{2np+1} \rightarrow \mathrm{BGL}_1(R//\alpha)$ whose Thom spectrum is equivalent as an \mathbf{E}_1 -ring to K^μ . We would like to determine the element² of $\pi_* R//\alpha$ detected by the restriction to the bottom cell S^{2np-1} of the source of ϕ . First, we note:

Lemma 4.2.4. *The element $\alpha \in \pi_{2np-3}R$ is p -torsion.*

PROOF. Since f is a charming map, the composite $S^{2np-1} \rightarrow \Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1}$ is the degree p map. Therefore, the element $p\alpha \in \pi_{2np-3}R$ is detected by the composite

$$S^{2np-2} \rightarrow \Omega S^{2np-1} \rightarrow \Omega^3 S^{2np-1} \xrightarrow{\Omega f} \Omega S^{2np-1} \rightarrow K \xrightarrow{\mu} \mathrm{BGL}_1(R).$$

But there is a fiber sequence $\Omega^2 S^{2np+1} \xrightarrow{f} S^{2np-1} \rightarrow BK$ by the definition of a charming map, so the composite detecting $p\alpha$ is null, as desired. \square

There is now a square

$$\begin{array}{ccc} S^{2np-2}/p & \longrightarrow & S^{2np-1} \\ \downarrow & & \downarrow \\ K & \longrightarrow & \Omega^2 S^{2np+1} \\ \alpha \downarrow & & \downarrow \\ \mathrm{BGL}_1(R) & \longrightarrow & \mathrm{BGL}_1(R//\alpha), \end{array}$$

and the following result is a consequence of the lemma and the definition of Toda brackets:

Lemma 4.2.5. *The element in $\pi_{2np-2}(R//\alpha)$ detected by the vertical map $S^{2np-1} \rightarrow \mathrm{BGL}_1(R//\alpha)$ lives in the Toda bracket $\langle p, \alpha, 1_{R//\alpha} \rangle$.*

²Technically, this is bad terminology: there are multiple possibilities for the map ϕ , and each gives rise to a map $S^{2np-1} \rightarrow \mathrm{BGL}_1(R//\alpha)$. The elements in $\pi_{2np-2}(R//\alpha)$ determined in this way need not agree, but they are the same modulo the indeterminacy of the Toda bracket $\langle p, \alpha, 1_{R//\alpha} \rangle$.

The upshot of this discussion is the following:

Proposition 4.2.6. *Let R be a p -local \mathbf{E}_1 -ring, and let $\mu : K \rightarrow \mathrm{BGL}_1(R)$ denote a map from the fiber K of a charming map $f : \Omega^2 S^{2np+1} \rightarrow S^{2np-1}$, providing an element $\alpha \in \pi_{2np-3}R$. Assume that the Thom spectrum $R//\alpha$ of the map $\Omega S^{2np-1} \rightarrow \mathrm{BGL}_1(R)$ is an \mathbf{E}_1 - R -algebra. Then there is an element $v \in \langle p, \alpha, 1_{R//\alpha} \rangle$ such that K^μ is equivalent to the Thom spectrum of the map $\Omega^2 S^{2np+1} \xrightarrow{v} \mathrm{BGL}_1(R//\alpha)$.*

Remark 4.2.7. Let R be an \mathbf{E}_1 -ring, and let $\alpha \in \pi_d R$. Then α defines a map $S^{d+1} \rightarrow \mathrm{BGL}_1(R)$, and it is natural to ask when α extends along $S^{d+1} \rightarrow \Omega S^{d+2}$, or at least along $S^{d+1} \rightarrow J_k(S^{d+1})$ for some k . This is automatic if R is an \mathbf{E}_2 -ring, but not necessarily so if R is only an \mathbf{E}_1 -ring. Recall that there is a cofiber sequence

$$S^{(k+1)(d+1)-1} \rightarrow J_k(S^{d+1}) \rightarrow J_{k+1}(S^{d+1}),$$

where the first map is the $(k+1)$ -fold iterated Whitehead product $[\iota_{d+1}, [\dots, [\iota_{d+1}, \iota_{d+1}], \dots]]$. In particular, the map $S^{d+1} \rightarrow \mathrm{BGL}_1(R)$ extends along the map $S^{d+1} \rightarrow J_k(S^{d+1})$ if and only if there are compatible nullhomotopies of the n -fold iterated Whitehead products $[\alpha, [\dots, [\alpha, \alpha]], \dots] \in \pi_* \mathrm{BGL}_1(R)$ for $n \leq k$. These amount to properties of Toda brackets in the homotopy of R . We note, for instance, that the Whitehead bracket $[\alpha, \alpha] \in \pi_{2d+1} \mathrm{BGL}_1(R) \cong \pi_{2d} R$ is the element $2\alpha^2$; therefore, the map $S^{d+1} \rightarrow \mathrm{BGL}_1(R)$ extends to $J_2(S^{d+1})$ if and only if $2\alpha^2 = 0$.

Remark 4.2.8. Let R be a p -local \mathbf{E}_2 -ring, and let $\alpha \in \pi_d(R)$ with d even. Then α defines an element $\alpha \in \pi_{d+2} \mathbf{B}^2 \mathrm{GL}_1(R)$. The p -fold iterated Whitehead product $[\alpha, \dots, \alpha] \in \pi_{p(d+2)-(p-1)} \mathbf{B}^2 \mathrm{GL}_1(R) \cong \pi_{pd+(p-1)} R$ is given by $p!Q_1(\alpha)$ modulo decomposables. This is in fact true more generally. Let R be an \mathbf{E}_n -ring, and suppose $\alpha \in \pi_d(R)$. Let $i < n$, so α defines an element $\alpha \in \pi_{d+i} \mathbf{B}^i \mathrm{GL}_1(R)$. The p -fold iterated Whitehead product $[\alpha, \dots, \alpha] \in \pi_{p(d+i)-(p-1)} \mathbf{B}^i \mathrm{GL}_1(R) \cong \pi_{pd+(i-1)(p-1)} R$ is given by $p!Q_{i-1}(\alpha)$ modulo decomposables.

We will describe this in detail in forthcoming work: the basic idea is to reduce to the universal example of an \mathbf{E}_n -ring, and relate Whitehead products on $\pi_*(S^n)$ to the \mathbf{E}_d -Browder bracket on $\Omega^d S_+^n$ (where $d \geq n$). Recall the isomorphism $\pi_j S^n \cong \pi_{j-d} \Omega^d S^n$. If $\alpha \in \pi_i S^n$ and $\beta \in \pi_j S^n$, then we will show in future work that the stabilization of the Whitehead product $[\alpha, \beta] \in \pi_{i+j-1} S^n \cong \pi_{i+j-d} \Omega^d S^n$ is closely related to the \mathbf{E}_d -Browder bracket $[\alpha, \beta]_{\mathbf{E}_d}$.

5. Chromatic Thom spectra

5.1. Statement of the theorem. To state the main theorem of this section, we set some notation. Fix an integer $n \geq 1$, and work in the p -complete stable category. For each Thom spectrum R of height $n-1$ in Table 1, let $\sigma_{n-1} : S^{|\sigma_{n-1}|} \rightarrow \mathrm{BGL}_1(R)$ denote a map detecting $\sigma_{n-1} \in \pi_{|\sigma_{n-1}|}(R)$ (which exists by Theorem 3.2.25). Let K_n denote the fiber of a p -local charming map $\Omega^2 S^{2p^n+1} \rightarrow S^{2p^n-1}$ satisfying the hypotheses of Conjecture D, and let K_2 (resp. K_3) denote the fiber of an integrally defined charming map $\Omega^2 S^9 \rightarrow S^7$ (resp. $\Omega^2 S^{17} \rightarrow S^{15}$) satisfying the hypotheses of Conjecture D.

Then:

Theorem A. *Let R be a height $n-1$ spectrum as in the second line of Table 1. Then Conjectures D and E imply that there is a map $K_n \rightarrow \mathrm{BGL}_1(R)$ such that the mod p homology of the Thom spectrum K_n^μ is isomorphic to the mod p homology of the associated designer chromatic spectrum $\Theta(R)$ as a Steenrod comodule.*

If R is any base spectrum other than B , the Thom spectrum K_n^μ is equivalent to $\Theta(R)$ upon p -completion for every prime p . If Conjecture F is true, then the same is true for B : the Thom spectrum K_n^μ is equivalent to $\Theta(B) = \mathrm{tmf}$ upon 2-completion.

We emphasize again that naïvely making sense of Theorem A relies on knowing that $T(n)$ admits the structure of an \mathbf{E}_1 -ring; we shall interpret this phrase as in Warning 3.1.6.

Remark 5.1.1. Theorem A is proved independently of the nilpotence theorem. (In fact, it is even independent of Quillen’s identification of $\pi_*\mathrm{MU}$ with the Lazard ring, provided one regards the existence of designer chromatic spectra as being independent of Quillen’s identification.) We shall elaborate on the connection between Theorem A and the nilpotence theorem in future work; a sketch is provided in Remark 5.4.7.

Remark 5.1.2. Theorem A is true unconditionally when $n = 1$, since that case is simply Corollary 2.2.2.

Remark 5.1.3. Note that Table 2 implies that the homology of each of the Thom spectra in Table 1 are given by the Q_0 -Margolis homology of their associated designer chromatic spectra. In particular, the map $R \rightarrow \Theta(R)$ is a rational equivalence.

Before we proceed with the proof of Theorem A, we observe some consequences.

Corollary B. *Conjecture D and Conjecture E imply Conjecture 1.1.3.*

PROOF. This follows from Theorem A, Proposition 4.2.6, and Proposition 3.1.17. \square

Remark 5.1.4. Corollary B is true *unconditionally* when $n = 1$, since Theorem A is true unconditionally in that case by Remark 5.1.2. See also Remark 4.1.4.

Remark 5.1.5. We can attempt to apply Theorem A for $R = A$ in conjunction with Proposition 4.2.6. Theorem A states that Conjecture D and Conjecture E imply that there is a map $K_2 \rightarrow \mathrm{BGL}_1(A)$ whose Thom spectrum is equivalent to bo . There is a fiber sequence

$$\Omega S^7 \rightarrow K_2 \rightarrow \Omega^2 S^9,$$

so we obtain a map $\mu : \Omega S^7 \rightarrow K_2 \rightarrow \mathrm{BGL}_1(A)$. The proof of Theorem A shows that the bottom cell S^6 of the source detects $\sigma_1 \in \pi_5(A)$. A slight variation of the argument used to establish Proposition 4.2.6 supplies a map $\Omega^2 S^9 \rightarrow \mathrm{BAut}((\Omega S^7)^\mu)$ whose Thom spectrum is bo. The spectrum $(\Omega S^7)^\mu$ has mod 2 homology $\mathbf{F}_2[\zeta_1^4, \zeta_2^2]$. However, unlike A , it does not naturally arise an \mathbf{E}_1 -Thom spectrum over the sphere spectrum; this makes it unamenable to study via techniques of unstable homotopy.

More precisely, $(\Omega S^7)^\mu$ is not the Thom spectrum of an \mathbf{E}_1 -map $X \rightarrow \mathrm{BGL}_1(\mathbb{S})$ from a loop space X which sits in a fiber sequence

$$\Omega S^5 \rightarrow X \rightarrow \Omega S^7$$

of loop spaces. Indeed, BX would be a S^5 -bundle over S^7 , which by [Mah87, Lemma 4] implies that X is then equivalent as a loop space to $\Omega S^5 \times \Omega S^7$. The resulting \mathbf{E}_1 -map $\Omega S^7 \rightarrow \mathrm{BGL}_1(\mathbb{S})$ is specified by an element of $\pi_5(\mathbb{S}) \cong 0$, so $(\Omega S^7)^\mu$ must then be equivalent as an \mathbf{E}_1 -ring to $A \wedge \Sigma_+^\infty \Omega S^7$. In particular, $\sigma_1 \in \pi_5(A)$ would map nontrivially to $(\Omega S^7)^\mu$, which is a contradiction.

The proof of Theorem A will also show:

Corollary 5.1.6. *Let R be a height $n - 1$ spectrum as in the second line of Table 1, and assume Conjecture F if $R = B$. Let M be an \mathbf{E}_3 - R -algebra. Conjecture D and Conjecture E imply that if:*

- (a) *the composite $\mathfrak{Z}_3(R) \rightarrow R \rightarrow M$ is an \mathbf{E}_3 -algebra map,*
- (b) *the element σ_{n-1} in $\pi_* M$ is nullhomotopic,*
- (c) *and the bracket $\langle p, \sigma_{n-1}, 1_M \rangle$ contains zero,*

then there is a unital map $\Theta(R) \rightarrow M$.

5.2. The proof of Theorem A. This section is devoted to giving a proof of Theorem A, dependent on Conjecture D and Conjecture E. The proof of Theorem A will be broken down into multiple steps. The result for $y(n)$ and $y_{\mathbf{Z}}(n)$ follow from the result for $T(n)$ by Proposition 3.2.6, so we shall restrict ourselves to the cases of R being $T(n)$, A , and B .

Fix $n \geq 1$. If R is A or B , we will restrict to $p = 2$, and let K_2 and K_3 denote the integrally defined spaces from Conjecture D. By Remarks 3.2.17 and 3.2.24, the elements $\sigma_1 \in \pi_5(A)$ and $\sigma_2 \in \pi_{13}(B)$ are defined integrally. We will write σ_{n-1} to generically denote this element, and will write it as living in degree $|\sigma_{n-1}|$. We shall also write R to denote $X(p^n - 1)$ and *not* $T(n)$; this will be so that we can apply Conjecture D. We apologize for the inconvenience, but hope that this is worth circumventing the task of having to read through essentially the same proofs for these slightly different cases.

Step 1. We begin by constructing a map $\mu : K_n \rightarrow \mathrm{BGL}_1(R)$ as required by the statement of Theorem A; the construction in the case $n = 1$ follows Remark 4.1.8. By Conjecture D, the space K_n splits off $\Omega^2 P^{|\sigma_{n-1}|+4}(p)$ (if $R = T(n)$, then $|\sigma_{n-1}| + 4 = |v_n| + 3$). We are therefore reduced to constructing a map $\Omega^2 P^{|\sigma_{n-1}|+4}(p) \rightarrow \mathrm{BGL}_1(R)$. Theorem 3.2.25 shows that the element $\sigma_{n-1} \in \pi_* R$ is p -torsion, so the map $S^{|\sigma_{n-1}|+1} \rightarrow \mathrm{BGL}_1(R)$ detecting σ_{n-1} extends to a map

$$(5.1) \quad S^{|\sigma_{n-1}|+1}/p = P^{|\sigma_{n-1}|+2}(p) \rightarrow \mathrm{BGL}_1(R).$$

Since $\Omega^2 P^{|\sigma_{n-1}|+4}(p) \simeq \Omega^2 \Sigma^2 P^{|\sigma_{n-1}|+2}(p)$, we would obtain an extension $\tilde{\mu}$ of this map through $\Omega^2 P^{|\sigma_{n-1}|+4}(p)$ if R admits an \mathbf{E}_3 -structure.

Unfortunately, this is not true; but this is where Conjecture E comes in: it says that the element $\sigma_{n-1} \in \pi_{|\sigma_{n-1}|}R$ lifts to the \mathbf{E}_3 -center $\mathfrak{Z}_3(R)$, where it has the same torsion order as in R . (Here, we are abusively writing $\mathfrak{Z}_3(T(n-1))$ to denote the \mathbf{E}_3 -center of $X(p^n - 1)_{(p)}$.) The lifting of σ_{n-1} to $\pi_{|\sigma_{n-1}|}\mathfrak{Z}_3(R)$ provided by Conjecture E gives a factorization of the map from (5.1) as

$$S^{|\sigma_{n-1}|+1}/p = P^{|\sigma_{n-1}|+2}(p) \rightarrow \mathrm{BGL}_1(\mathfrak{Z}_3(R)) \rightarrow \mathrm{BGL}_1(R).$$

Since $\mathfrak{Z}_3(R)$ is an \mathbf{E}_3 -ring, $\mathrm{BGL}_1(\mathfrak{Z}_3(R))$ admits the structure of an \mathbf{E}_2 -space. In particular, the map $P^{|\sigma_{n-1}|+2}(p) \rightarrow \mathrm{BGL}_1(\mathfrak{Z}_3(R))$ factors through $\Omega^2 P^{|\sigma_{n-1}|+4}(p)$, as desired. We let $\tilde{\mu}$ denote the resulting composite

$$\tilde{\mu} : \Omega^2 P^{|\sigma_{n-1}|+4}(p) \rightarrow \mathrm{BGL}_1(\mathfrak{Z}_3(R)) \rightarrow \mathrm{BGL}_1(R).$$

Step 2. Theorem A asserts that there is an identification between the Thom spectrum of the induced map $\mu : K_n \rightarrow \mathrm{BGL}_1(R)$ and the associated designer chromatic spectrum $\Theta(R)$ via Table 1. We shall identify the Steenrod comodule structure on the mod p homology of K_n^μ , and show that it agrees with the mod p homology of $\Theta(R)$.

Designer chromatic spectrum	Mod p homology	
$\mathrm{BP}\langle n-1 \rangle$	$p=2$	$\mathbf{F}_2[\zeta_1^2, \dots, \zeta_{n-1}^2, \zeta_n^2, \zeta_{n+1}, \dots]$
	$p>2$	$\mathbf{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_p}(\tau_n, \tau_{n+1}, \dots)$
$k(n-1)$	$p=2$	$\mathbf{F}_2[\zeta_1, \dots, \zeta_{n-1}, \zeta_n^2, \zeta_{n+1}, \dots]$
	$p>2$	$\mathbf{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_p}(\tau_0, \dots, \tau_{n-2}, \tau_n, \tau_{n+1}, \dots)$
$k_{\mathbf{Z}}(n-1)$	$p=2$	$\mathbf{F}_2[\zeta_1^2, \zeta_2, \dots, \zeta_{n-1}, \zeta_n^2, \zeta_{n+1}, \dots]$
	$p>2$	$\mathbf{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_p}(\tau_1, \dots, \tau_{n-2}, \tau_n, \tau_{n+1}, \dots)$
bo	$p=2$	$\mathbf{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \dots]$
	$p>2$	$\mathbf{F}_p[x_4]/v_1 \otimes \mathbf{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_p}(\tau_2, \tau_3, \dots)$
tmf	$p=2$	$\mathbf{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots]$
	$p=3$	$\Lambda_{\mathbf{F}_3}(b_4) \otimes \mathbf{F}_3[\zeta_1^3, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_3}(\tau_3, \tau_4, \dots)$
	$p \geq 5$	$\mathbf{F}_p[c_4, c_6]/(v_1, v_2) \otimes \mathbf{F}_p[\zeta_1, \zeta_2, \dots] \otimes \Lambda_{\mathbf{F}_p}(\tau_3, \tau_4, \dots)$

TABLE 3. The mod p homology of designer chromatic spectra. See [LN14, Theorem 4.3], as well as [Wil75, Proposition 1.7] and [AR05, Proposition 5.3] for a proof of the statement for $H_*(\mathrm{BP}\langle n-1 \rangle; \mathbf{F}_p)$; this implies the calculations of $H_*(k(n-1); \mathbf{F}_p)$ and $H_*(k_{\mathbf{Z}}(n-1); \mathbf{F}_p)$. See [AR05, Proposition 6.1] for a proof of the statements for $H_*(\mathrm{bo}; \mathbf{F}_2)$ and $H_*(\mathrm{tmf}; \mathbf{F}_2)$, and [Rez07, Theorem 21.5] for $H_*(\mathrm{tmf}; \mathbf{F}_p)$ for any p . For odd p , $\mathrm{bo}_{(p)}$ is a sum of shifts of $\mathrm{BP}\langle 1 \rangle$, which implies the statement about $H_*(\mathrm{bo}; \mathbf{F}_p)$.

In Table 3, we have recorded the mod p homology of the designer chromatic spectra in Table 1 (see [LN14, Theorem 4.3] for $\mathrm{BP}\langle n-1 \rangle$). It follows from Proposition 4.2.1 that there is an isomorphism

$$H_*(K_n^\mu) \cong \begin{cases} H_*(R) \otimes \mathbf{F}_2[x_{2^{n+1}-1}^2] \otimes \bigotimes_{k>1} \mathbf{F}_2[x_{2^{n+k}-1}] & p=2 \\ H_*(R) \otimes \bigotimes_{k>0} \mathbf{F}_p[y_{2(p^{n+k}-1)}] \otimes \bigotimes_{j>0} \Lambda_{\mathbf{F}_p}[x_{2p^{n+j}-1}] & p>2. \end{cases}$$

Combining this isomorphism with Theorem 3.1.5, Proposition 3.2.4, Proposition 3.2.15, and Proposition 3.2.22, we find that there is an *abstract* equivalence between the mod p homology of K_n^μ and the mod p homology of $\Theta(R)$.

We shall now work at $p = 2$ for the remainder of the proof; the same argument goes through with slight modifications at odd primes. We now identify the Steenrod comodule structure on $H_*(K_n^\mu)$. Recall that $\tilde{\mu}$ is the map $\Omega^2 P^{|\sigma_{n-1}|+4}(p) \rightarrow \text{BGL}_1(R)$ from Step 1. By construction, there is a map $K_n^\mu \rightarrow \Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}}$. The map Φ factors through a map $\tilde{\Phi} : \Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}} \rightarrow \Theta(R)$. The Thom spectrum $\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}}$ admits the structure of a \mathcal{Q}_1 -ring. Indeed, it is the smash product $\Omega^2 P^{|\sigma_{n-1}|+4}(p)^\phi \wedge_{\mathfrak{Z}_3(R)} R$, where $\phi : \Omega^2 P^{|\sigma_{n-1}|+4}(p) \rightarrow \text{BGL}_1(\mathfrak{Z}_3(R))$; it therefore suffices to observe that the Thom spectrum $\Omega^2 P^{|\sigma_{n-1}|+4}(p)^\phi$ admits the structure of an $\mathbf{E}_1 \otimes \mathcal{Q}_1$ -ring. (Here, $\mathbf{E}_1 \otimes \mathcal{Q}_1$ denotes the Boardman-Vogt tensor product of the \mathbf{E}_1 - and \mathcal{Q}_1 -operads.) Since there is a map $\mathcal{Q}_1 \rightarrow \mathbf{E}_2$ of ∞ -operads, this is a consequence of the fact that ϕ is a double loop map, and hence an $\mathbf{E}_1 \otimes \mathcal{Q}_1$ -algebra map. Moreover, the image of $H_*(K_n^\mu)$ in $H_*(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}})$ is generated under the single Dyer-Lashof operation (arising from the cup-1 operad; see Remark 2.2.8) by the indecomposables in the image of the map $H_*(R) \rightarrow H_*(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}})$.

The Postnikov truncation map $\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}} \rightarrow \text{H}\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}})$ is one of \mathcal{Q}_1 -rings. Since $\Omega^2 P^{|\sigma_{n-1}|+4}(p)$ is highly connected, $\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}}) \cong \pi_0(R)$. In particular, there is an \mathbf{E}_∞ -map $\text{H}\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}}) \rightarrow \text{HF}_p$. The composite

$$\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}} \rightarrow \text{H}\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}}) \rightarrow \text{HF}_p$$

is therefore a \mathcal{Q}_1 -algebra map. Moreover, the composite

$$R \rightarrow \Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}} \rightarrow \text{H}\pi_0(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}}) \rightarrow \text{HF}_p$$

is simply the Postnikov truncation for R . It follows that the indecomposables in $H_*(\Omega^2 P^{|\sigma_{n-1}|+4}(p)^{\tilde{\mu}})$ which come from the indecomposables in $H_*(R)$ are sent to the indecomposables in $H_*(\text{HF}_p)$. Using the discussion in the previous paragraph, Steinberger's calculation (Theorem 2.2.4), and the Dyer-Lashof hopping argument of Remark 2.2.9, we may conclude that the Steenrod comodule structure on $H_*(K_n^\mu)$ (which, recall, is abstractly isomorphic to $H_*(\Theta(R))$) agrees with the Steenrod comodule structure on $H_*(\Theta(R))$.

Step 3. By Step 2, the mod p homology of the Thom spectrum K_n^μ is isomorphic to the mod p homology of the associated designer chromatic spectrum $\Theta(R)$ as a Steenrod comodule. The main result of [AL17] and [AP76, Theorem 1.1] now imply that unless $R = B$, the Thom spectrum K_n^μ is equivalent to $\Theta(R)$ upon p -completion for every prime p . Finally, if Conjecture F is true, then the same conclusion can be drawn for B : the Thom spectrum K_n^μ is equivalent to $\Theta(B) = \text{tmf}$ upon p -completion for every prime p .

This concludes the proof of Theorem A.

5.3. Remark on the proof. Before proceeding, we note the following consequence of the proof of Theorem A.

Proposition 5.3.1. *Let p be an odd prime. Assume Conjecture D and Conjecture E. Then the composite*

$$g_2 : \Omega^2 S^{|\sigma_{n-1}|+3} \rightarrow \Omega^2 P^{|\sigma_{n-1}|+4}(p) \xrightarrow{\tilde{\mu}} \text{BGL}_1(X(p^n - 1)) \rightarrow \text{BGL}_1(\text{BP}\langle n - 1 \rangle)$$

is null.

PROOF. Let $R = X(p^n - 1)$, and let $\Theta(R) = \mathrm{BP}\langle n - 1 \rangle$. The map g_2 is the composite with $\mathrm{BGL}_1(\mathfrak{Z}_3(R)) \rightarrow \mathrm{BGL}_1(\Theta(R))$ with the extension of the map

$$\sigma_{n-1} : S^{|\sigma_{n-1}|+1} \rightarrow \mathrm{BGL}_1(\mathfrak{Z}_3(R))$$

along the double suspension $S^{|\sigma_{n-1}|+1} \rightarrow \Omega^2 S^{|\sigma_{n-1}|+3}$. Since σ_{n-1} is null in $\pi_*\Theta(R)$, we would be done if g_2 was homotopic to the dotted extension

$$\begin{array}{ccc} S^{|\sigma_{n-1}|+1} & \xrightarrow{\sigma_{n-1}} & \mathrm{BGL}_1(\Theta(R)) \\ \downarrow & \dashrightarrow^{g'_2} & \\ \Omega^2 S^{|\sigma_{n-1}|+3} & & \end{array}$$

The potential failure of these maps to be homotopic stems from the fact that the composite $\mathfrak{Z}_3(R) \rightarrow R \rightarrow \Theta(R)$ need not be a map of \mathbf{E}_3 -rings. It is, however, a map of \mathbf{E}_2 -rings; therefore, the maps

$$g_1 : \Omega S^{|\sigma_{n-1}|+2} \rightarrow \mathrm{BGL}_1(\mathfrak{Z}_3(R)) \rightarrow \mathrm{BGL}_1(\Theta(R))$$

and

$$g'_1 : \Omega S^{|\sigma_{n-1}|+2} \rightarrow \mathrm{BGL}_1(\Theta(R))$$

obtained by extending along the suspension $S^{|\sigma_{n-1}|+1} \rightarrow \Omega S^{|\sigma_{n-1}|+2}$ are homotopic. We now utilize the following result of Serre's:

Proposition 5.3.2 (Serre, [Ser53, p. 281]). *Let p be an odd prime. Then the suspension $S^{2n-1} \rightarrow \Omega S^{2n}$ splits upon p -localization: there is a p -local equivalence*

$$E \times \Omega[\iota_{2n}, \iota_{2n}] : S^{2n-1} \times \Omega S^{4n-1} \rightarrow \Omega S^{2n}.$$

This implies that the suspension map $\Omega S^{|\sigma_{n-1}|+2} \rightarrow \Omega^2 S^{|\sigma_{n-1}|+3}$ admits a splitting as loop spaces. In particular, this implies that the map g_2 is homotopic to the composite

$$\Omega^2 S^{|\sigma_{n-1}|+3} \rightarrow \Omega S^{|\sigma_{n-1}|+2} \xrightarrow{g_1} \mathrm{BGL}_1(\mathfrak{Z}_3(R)) \rightarrow \mathrm{BGL}_1(\Theta(R)),$$

and similarly for g'_2 . Since g_1 and g'_1 are homotopic, and g'_1 (and hence g'_2) is null, we find that g_2 is also null, as desired. \square

5.4. Infinite families and the nilpotence theorem. We now briefly discuss the relationship between Theorem A and the nilpotence theorem. We begin by describing a special case of this connection. Recall from Remark 2.2.3 that Theorem 2.2.1 implies that if R is an \mathbf{E}_2 -ring spectrum, and $x \in \pi_*R$ is a simple p -torsion element which has trivial MU-Hurewicz image, then x is nilpotent. A similar argument implies the following.

Proposition 5.4.1. *Assume Conjecture D when $n = 1$. Then Corollary 2.2.2 (i.e., Theorem A when $n = 1$) implies that if R is a p -local \mathbf{E}_3 -ring spectrum, and $x \in \pi_*R$ is a class with trivial HZ_p -Hurewicz image such that:*

- $\alpha_1 x = 0$ in π_*R ; and
- the Toda bracket $\langle p, \alpha_1, x \rangle$ contains zero;

then x is nilpotent.

PROOF. We claim that the composite

$$(5.2) \quad \Omega^2 S^3 \langle 3 \rangle \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)}) \rightarrow \mathrm{BGL}_1(R[1/x])$$

is null. Remark 4.1.8 implies that Conjecture D for $n = 1$ reduces us to showing that the composite

$$\Omega^2 P^{2p+1}(p) \xrightarrow{\alpha_1} \mathrm{BGL}_1(\mathbb{S}_{(p)}) \rightarrow \mathrm{BGL}_1(R[1/x])$$

is null. Since this composite is one of double loop spaces, it further suffices to show that the composite

$$(5.3) \quad P^{2p-1}(p) \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)}) \rightarrow \mathrm{BGL}_1(R[1/x])$$

is null. The bottom cell S^{2p-2} of $P^{2p-1}(p)$ maps trivially to $\mathrm{BGL}_1(R[1/x])$, because the bottom cell detects α_1 (by Remark 4.1.8), and α_1 is nullhomotopic in $R[1/x]$. Therefore, the map (5.3) factors through the top cell S^{2p-1} of $P^{2p-1}(p)$. The resulting map

$$S^{2p-1} \rightarrow \mathrm{BGL}_1(\mathbb{S}_{(p)}) \rightarrow \mathrm{BGL}_1(R[1/x])$$

detects an element of the Toda bracket $\langle p, \alpha_1, x \rangle$, but this contains zero by hypothesis, so is nullhomotopic.

Since the map (5.2) is null, Corollary 2.2.2 and Theorem 2.1.7 implies that there is a ring map $\mathrm{HZ}_p \rightarrow R[1/x]$. In particular, the composite of the map $x : \Sigma^{|x|}R \rightarrow R$ with the unit $R \rightarrow R[1/x]$ factors as shown:

$$\begin{array}{ccccc} \Sigma^{|x|}R & \xrightarrow{x} & R & \longrightarrow & R[1/x]. \\ \downarrow & & \downarrow & \nearrow & \\ \mathrm{HZ}_p \wedge \Sigma^{|x|}R & \xrightarrow{x} & \mathrm{HZ}_p \wedge R & & \end{array}$$

The bottom map, however, is null, because x has zero HZ_p -Hurewicz image. Therefore, the element $x \in \pi_* R[1/x]$ is null, and hence $R[1/x]$ is contractible. \square

Remark 5.4.2. One can prove by a different argument that Proposition 5.4.1 is true without the assumption that Conjecture D holds when $n = 1$. At $p = 2$, this was shown by Astey in [Ast97, Theorem 1.1].

To discuss the relationship between Theorem A for general n and the nilpotence theorem (which we will expand upon in future work), we embark on a slight digression. The following proposition describes the construction of some infinite families.

Proposition 5.4.3. *Let R be a height $n - 1$ spectrum as in the second line of Table 2, and assume Conjecture E if $R = A$ or B . Then there is an infinite family $\sigma_{n-1,p^k} \in \pi_{p^k|v_n|-1}(R)$. Conjecture E implies that σ_{n-1,p^k} lifts to $\pi_{p^k|v_n|-1}(\mathfrak{J}_3(R))$, where $\mathfrak{J}_3(R)$ abusively denotes the \mathbf{E}_3 -center of $X(p^n - 1)$ if $R = T(n - 1)$.*

PROOF. We construct this family by induction on k . The element $\sigma_{n-1,1}$ is just σ_{n-1} , so assume that we have defined σ_{n-1,p^k} . The element $\sigma_{n-1,p^k} \in \pi_{p^k|v_n|-1}R$ defines a map $\sigma_{n-1,p^k} : S^{p^k|v_n|} \rightarrow \mathrm{BGL}_1(R)$. When $R = T(n - 1)$, Lemma 3.1.12 (and the inductive hypothesis) implies that the map defined by σ_n factors through the map $\mathrm{BGL}_1(X(p^n - 1)) \rightarrow \mathrm{BGL}_1(T(n - 1))$. When $R = A$ or B , Conjecture E (and the inductive hypothesis) implies that the map defined by σ_n factors through the map $\mathrm{BGL}_1(\mathfrak{J}_3(R)) \rightarrow \mathrm{BGL}_1(R)$. This implies that for all R as in the second

line of Table 2, the map $\sigma_{n-1,p^k} : S^{p^k|v_n|} \rightarrow \mathrm{BGL}_1(R)$ factors through an \mathbf{E}_1 -space, which we shall just denote by \mathcal{Z}_R for the purpose of this proof. If we assume Conjecture E, then we may take $\mathcal{Z}_R = \mathrm{BGL}_1(\mathfrak{J}_3(R))$.

Therefore, we get a map $\sigma_{n-1,p^k} : \Omega S^{p^k|v_n|+1} \rightarrow \mathrm{BGL}_1(R)$ via the composite

$$\Omega S^{p^k|v_n|+1} \rightarrow \mathcal{Z}_R \rightarrow \mathrm{BGL}_1(R).$$

Since \mathcal{Z}_R is an \mathbf{E}_1 -space, the map $\Omega S^{p^k|v_n|+1} \rightarrow \mathcal{Z}_R$ is adjoint to a map

$$\bigvee_{j \geq 1} S^{jp^k|v_n|+1} \simeq \Sigma \Omega S^{p^k|v_n|+1} \rightarrow \mathrm{B}\mathcal{Z}_R;$$

the source splits as indicated via the James splitting. These splittings are given by Whitehead products; in particular, the map $S^{p^{k+1}|v_n|+1} = S^{p(p^k|v_n|+1)-(p-1)} \rightarrow \mathrm{B}\mathcal{Z}_R$ is given by the p -fold Whitehead product $[\sigma_{n-1,p^k}, \dots, \sigma_{n-1,p^k}]$. This is divisible by p , so it yields a map $S^{p^{k+1}|v_n|} \rightarrow \mathcal{Z}_R$, and hence a map $S^{p^{k+1}|v_n|} \rightarrow \mathrm{BGL}_1(R)$ given by composing with the map $\mathcal{Z}_R \rightarrow \mathrm{BGL}_1(R)$. This defines the desired element $\sigma_{n-1,p^{k+1}} \in \pi_{p^{k+1}|v_n|-1}(R)$. As the construction makes clear, assuming Conjecture E and taking $\mathcal{Z}_R = \mathrm{BGL}_1(\mathfrak{J}_3(R))$ implies that σ_{n-1,p^k} lifts to $\pi_{p^k|v_n|-1}(\mathfrak{J}_3(R))$. \square

Remark 5.4.4. This infinite family is detected in the 1-line of the ANSS for R by $\delta(v_n^k)$, where δ is the boundary map induced by the map $\Sigma^{-1}R/p \rightarrow R$. This is a consequence of the geometric boundary theorem (see [Rav86, Theorem 2.3.4]) applied to the cofiber sequence $R \xrightarrow{p} R \rightarrow R/p$.

Remark 5.4.5. The element $\sigma_{n-1,1} \in \pi_{2p^n-3}(R)$ is precisely σ_{n-1} .

Remark 5.4.6. When $n = 1$, the ring R is the (p -local) sphere spectrum. The infinite family σ_{n-1,p^k} is the Adams-Toda α -family: namely, $\alpha_{p^k} \in \pi_{2p^k(p-1)-1}(\mathbb{S})$ maps to $\sigma_{0,p^k} \in \pi_{2p^k(p-1)-1}X(p-1)$ under the unit map $\mathbb{S} \rightarrow X(p-1)$.

We now briefly sketch an argument relating Theorem A to the proof of the nilpotence theorem; we shall elaborate on this discussion in forthcoming work.

Remark 5.4.7. The heart of the nilpotence theorem is what is called Step III in [DHS88]; this step amounts to showing that certain self-maps of $T(n-1)$ -module skeleta (denoted G_k in [DHS88]) of $T(n)$ are nilpotent. Let us assume that $p > 2$ for simplicity. Then these self-maps are given by multiplication by the p -fold Toda bracket $b_{n,k} = \langle \sigma_{n-1,p^k}, \dots, \sigma_{n-1,p^k} \rangle$ at an odd prime p ; this lives in degree $p|\sigma_{n-1,p^k}| + p - 2 = 2p^k(p^n - 1) - 2$. (When $p = 2$, the desired element σ_{n-1,p^k} is denoted by h in [Hop87, Theorem 3].) It therefore suffices to establish the nilpotency of the $b_{n,k}$.

This can be proven through Theorem A via induction on k ; we shall assume Conjecture D and Conjecture E for the remainder of this discussion. The motivation for this approach stems from the observation that if R is any $\mathbf{E}_3\text{-}\mathbf{F}_2$ -algebra and $x \in \pi_*(R)$, then there is a relation $Q_1(x)^2 = Q_2(x^2)$ (at odd primes, one has a relation involving the p -fold Toda bracket $\langle Q_1(x), \dots, Q_1(x) \rangle$). In our setting, Proposition 5.4.3 implies that the elements $\sigma_{n-1,k}$ lift to $\pi_*\mathfrak{J}_3(X(p^n - 1))$. At $p = 2$, one can prove (in the same way that the Cartan relation $Q_1(x)^2 = Q_2(x^2)$ is proven) that the construction of this infinite family implies that $\sigma_{n-1,p^{k+1}}^2$ can be described in terms of $Q_2(\sigma_{n-1,p^k}^2)$. At odd primes, there is a similar relation involving the p -fold Toda bracket defining $b_{n,k}$. In particular, induction on k implies

that the $b_{n,k}$ are all nilpotent in $\pi_*\mathfrak{Z}_3(X(p^n - 1))$ if $b_{n,1}$ is nilpotent. Note that $|b_{n,1}| = 2p^{n+1} - 2p - 2$.

To argue that $b_{n,1}$ is nilpotent, one first observes that $\sigma_{n-1}b_{n,1}^p = 0$ in $\pi_*\mathfrak{Z}_3(X(p^n - 1))$; when $n = 0$, this follows from the statement that $\alpha_1\beta_1^p = 0$ in the sphere. To show that $b_{n,1}$ is nilpotent, it suffices to establish that $\mathfrak{Z}_3(X(p^n - 1))[1/b_{n,1}^p]$ is contractible; when $n = 1$, this follows from Proposition 5.4.1. We give a very brief sketch of this nilpotence for general n , by arguing as in Proposition 5.4.1, and with a generous lack of precision which will be remedied in forthcoming work.

For notational convenience, we now denote $d_{n,1} = b_{n,1}^p$, so that $|d_{n,1}| = 2p^{n+2} - 2p^2 - 2p$. It suffices to show that the multiplication-by- $d_{n,1}$ map

$$d_{n,1} : \Sigma^{|d_{n,1}|}\mathfrak{Z}_3(X(p^n - 1)) \rightarrow \mathfrak{Z}_3(X(p^n - 1))[1/d_{n,1}]$$

is nullhomotopic. Since σ_{n-1} kills $d_{n,1}$, we know that σ_{n-1} is nullhomotopic in $\mathfrak{Z}_3(X(p^n - 1))[1/d_{n,1}]$. Moreover, the bracket $\langle p, \sigma_{n-1}, 1_{\mathfrak{Z}_3(X(p^n - 1))[1/d_{n,1}]} \rangle$ contains zero. By arguing as in Proposition 5.4.1, we can conclude that the composite

$$K_n \rightarrow \Omega^2 P^{|\sigma_{n-1}|+4} \xrightarrow{\phi} \mathrm{BGL}_1(\mathfrak{Z}_3(X(p^n - 1))) \rightarrow \mathrm{BGL}_1(\mathfrak{Z}_3(X(p^n - 1))[1/d_{n,1}])$$

is nullhomotopic, where the map ϕ is constructed in Step 1 of the proof of Theorem A. (Recall that the proof of Theorem A shows that the Thom spectrum $(\Omega^2 P^{|\sigma_{n-1}|+4})^\phi$ is an $\mathbf{E}_1 \otimes \mathcal{Q}_1\text{-}\mathfrak{Z}_3(X(p^n - 1))$ -algebra such that $\mathrm{BP}\langle n-1 \rangle$ splits off its base change along the map $\mathfrak{Z}_3(X(p^n - 1)) \rightarrow T(n-1)$.) It follows from Theorem 2.1.7 that the multiplication-by- $d_{n,1}$ map factors as

$$\begin{array}{ccccc} \Sigma^{|d_{n,1}|}\mathfrak{Z}_3(X(p^n - 1)) & \xrightarrow{d_{n,1}} & \mathfrak{Z}_3(X(p^n - 1)) & \longrightarrow & \mathfrak{Z}_3(X(p^n - 1))[1/d_{n,1}]. \\ \downarrow & & \downarrow & \nearrow & \\ \Sigma^{|d_{n,1}|}(\Omega^2 P^{|\sigma_{n-1}|+4})^\phi & \xrightarrow{d_{n,1}} & (\Omega^2 P^{|\sigma_{n-1}|+4})^\phi & & \end{array}$$

To show that the top composite is null, it therefore suffices to show that the self map of K_n^ϕ defined by $d_{n,1}$ is nullhomotopic. This essentially follows from the fact that $(\Omega^2 P^{|\sigma_{n-1}|+4})^\phi$ is an $\mathbf{E}_1 \otimes \mathcal{Q}_1\text{-}\mathfrak{Z}_3(X(p^n - 1))$ -algebra: multiplication by $d_{n,1}$ is therefore null on K_n^ϕ , because $d_{n,1}$ is built from σ_{n-1} (which is null in $(\Omega^2 P^{|\sigma_{n-1}|+4})^\phi$) via \mathbf{E}_1 -power operations.

6. Applications

6.1. Splittings of cobordism spectra. The goal of this section is to prove the following.

Theorem C. *Assume that the composite $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \text{MString}_{(2)}$ is an \mathbf{E}_3 -map. Then Conjectures D, E, and F imply that there is a unital splitting of the Ando-Hopkins-Rezk orientation $\text{MString}_{(2)} \rightarrow \text{tmf}_{(2)}$.*

Remark 6.1.1. We believe that the assumption that the composite $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \text{MString}_{(2)}$ is an \mathbf{E}_3 -map is too strong; we believe that it can be removed using special properties of fibers of charming maps, and we will return to this in future work.

We only construct unstructured splittings; it seems unlikely that they can be refined to structured splittings. A slight modification of our arguments should work at any prime.

Remark 6.1.2. In fact, the same argument used to prove Theorem C shows that if the composite $\mathfrak{Z}_3(A) \rightarrow A \rightarrow \text{MSpin}_{(2)}$ is an \mathbf{E}_3 -map, then Conjecture D and Conjecture E imply that there are unital splittings of the Atiyah-Bott-Shapiro orientation $\text{MSpin}_{(2)} \rightarrow \text{bo}_{(2)}$. This splitting was originally proved unconditionally (i.e., without assuming Conjecture D or Conjecture E) by Anderson-Brown-Peterson in [ABP67] via a calculation with the Adams spectral sequence.

Remark 6.1.3. The inclusion of the cusp on $\overline{\mathcal{M}}_{\text{ell}}$ defines an \mathbf{E}_∞ -map $c : \text{tmf} \rightarrow \text{bo}$ as in [LN14, Theorem 1.2]. The resulting diagram

$$\begin{array}{ccc} \text{MString}_{(2)} & \longrightarrow & \text{MSpin}_{(2)} \\ \downarrow & & \downarrow \\ \text{tmf}_{(2)} & \xrightarrow{c} & \text{bo}_{(2)} \end{array}$$

commutes (see, e.g., [Dev19b, Lemma 6.4]). The splitting $s : \text{tmf}_{(2)} \rightarrow \text{MString}_{(2)}$ of Theorem C defines a composite

$$\text{tmf}_{(2)} \xrightarrow{s} \text{MString}_{(2)} \rightarrow \text{MSpin}_{(2)} \rightarrow \text{bo}_{(2)}$$

which agrees with c .

Remark 6.1.4. The Anderson-Brown-Peterson splitting implies that if X is any compact space, then the Atiyah-Bott-Shapiro \hat{A} -genus (i.e., the index of the Dirac operator in families) $\text{MSpin}^*(X) \rightarrow \text{bo}^*(X)$ is surjective. Similarly, if the composite $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \text{MString}$ is an \mathbf{E}_3 -map, then Conjectures D, E, and F imply that the Ando-Hopkins-Rezk orientation (i.e., the Witten genus in families) $\text{MString}^*(X) \rightarrow \text{tmf}^*(X)$ is also surjective.

Remark 6.1.5. In [Dev19b], we proved (unconditionally) that the map $\pi_* \text{MString} \rightarrow \pi_* \text{tmf}$ is surjective. Our proof proceeds by showing that the map $\pi_* B \rightarrow \pi_* \text{tmf}$ is surjective via arguments with the Adams-Novikov spectral sequence and by exploiting the \mathbf{E}_1 -ring structure on B to lift the powers of Δ living in $\pi_* \text{tmf}$.

The discussion preceding [MR09, Remark 7.3] (in the arXiv version of the document) implies that for a particular model of $\text{tmf}_0(3)$, we have:

Corollary 6.1.6. *Assume that the composite $\mathfrak{Z}_3(B) \rightarrow B_{(2)} \rightarrow \text{MString}_{(2)}$ is an \mathbf{E}_3 -map. Then Conjectures D, E, and F imply that $\Sigma^{16}\text{tmf}_0(3)_2^\wedge$ is a summand of MString_2^\wedge .*

We now turn to the proof of Theorem C.

PROOF OF THEOREM C. First, note that such a splitting exists after rationalization. Indeed, it suffices to check that this is true on rational homotopy; since the orientations under considerations are \mathbf{E}_∞ -ring maps, the induced map on homotopy is one of rings. It therefore suffices to lift the generators.

We now show that the generators of $\pi_*\text{tmf} \otimes \mathbf{Q} \cong \mathbf{Q}[c_4, c_6]$ lift to $\pi_*\text{MString} \otimes \mathbf{Q}$. Although one can argue this by explicitly constructing manifold representatives (as is done for c_4 in [Dev19b, Corollary 6.3]), it is also possible to provide a more homotopy-theoretic proof: the elements c_4 and c_6 live in dimensions 8 and 12 respectively, and the map $\text{MString} \rightarrow \text{tmf}$ is known to be an equivalence in dimensions ≤ 15 by [Hil, Theorem 2.1]. It follows that the same is true rationally, so c_4 and c_6 indeed lift to $\pi_*\text{MString} \otimes \mathbf{Q}$, as desired.

We will now construct a splitting after p -completion where $p = 2$. By Corollary 5.1.6, we obtain a unital map $\text{tmf} \simeq \Theta(B) \rightarrow \text{MString}$ upon p -completion which splits the orientation $\text{MString} \rightarrow \Theta(B)$ because:

- (a) the map $\mathfrak{Z}_3(B) \rightarrow B \rightarrow \text{MString}$ is an \mathbf{E}_3 -ring map (by assumption).
- (b) the element σ_2 vanishes in $\pi_{13}\text{MString}_{(2)}$ (because $\pi_{13}\text{MString}_{(2)} \cong \pi_{13}\text{tmf}_{(2)} \cong 0$),
- (c) and the Toda bracket $\langle 2, \sigma_2, 1_{\text{MString}_{(2)}} \rangle \subseteq \pi_*\text{MString}_{(2)}$ contains zero because $\pi_{14}\text{MString}_{(2)} \cong \pi_{14}\text{tmf}_{(2)}$, and the corresponding bracket $\langle 2, \sigma_2, 1_{\text{MString}_{(2)}} \rangle \subseteq \pi_{14}\text{tmf}_{(2)}$ detects v_3 , hence contains zero.

To obtain a map $\text{tmf}_{(p)} \rightarrow \text{MString}_{(p)}$, we need to show that the induced map $\text{tmf} \otimes \mathbf{Q} \rightarrow \text{tmf}_p^\wedge \otimes \mathbf{Q} \rightarrow \text{MString}_p^\wedge \otimes \mathbf{Q}$ agrees with the rational splitting constructed in the previous paragraph. However, this is immediate from the fact that the splittings $\text{tmf}_p^\wedge \rightarrow \text{MString}_p^\wedge$ are constructed to be equivalences in dimensions ≤ 15 , and the fact that the map out of $\text{tmf} \otimes \mathbf{Q}$ is determined by its effect on the generators c_4 and c_6 . \square

Remark 6.1.7. The proof recalled in Remark 1.1.2 of Thom's splitting of MO proceeded essentially unstably: there is an \mathbf{E}_2 -map $\Omega^2 S^3 \rightarrow \text{BO}$ of spaces over $\text{BGL}_1(\mathbb{S})$, whose Thomification yields the desired \mathbf{E}_2 -map $\text{HF}_2 \rightarrow \text{MO}$. This argument also works for MSO: there is an \mathbf{E}_2 -map $\Omega^2 S^3 \langle 3 \rangle \rightarrow \text{BSO}$ of spaces over $\text{BGL}_1(\mathbb{S})$, whose Thomification yields the desired \mathbf{E}_2 -map $\text{HZ} \rightarrow \text{MSO}$. One might hope for the existence of a similar unstable map which would yield Theorem C. We do not know how to construct such a map. To illustrate the difficulty, let us examine how such a proof would work; we will specialize to the case of MString , but the discussion is the same for MSpin .

According to Theorem A, Conjecture D and Conjecture E imply that there is a map $K_3 \rightarrow \text{BGL}_1(B)$ whose Thom spectrum is equivalent to tmf . There is a map $\text{BN} \rightarrow \text{B}^2\text{String}$, whose fiber we will denote by Q . Then there is a fiber sequence

$$N \rightarrow \text{BString} \rightarrow Q,$$

and so Proposition 2.1.6 implies that there is a map $Q \rightarrow \text{BGL}_1(B)$ whose Thom spectrum is MString . Theorem C would follow if there was a map $f : K_3 \rightarrow Q$ of spaces over $\text{BGL}_1(B)$, since Thomification would produce a map $\text{tmf} \rightarrow \text{MString}$.

Conjecture D reduces the construction of f to the construction of a map $\Omega^2 P^{17}(2) \rightarrow Q$. This map would in particular imply the existence of a map $P^{15}(2) \rightarrow Q$ (and would be equivalent to the existence of such a map if Q was a double loop space), which in turn stems from a 2-torsion element of $\pi_{14}(Q)$. The long exact sequence on homotopy runs

$$\cdots \rightarrow \pi_{14}(\mathbf{BString}) \rightarrow \pi_{14}(Q) \rightarrow \pi_{13}(N) \rightarrow \pi_{13}(\mathbf{BString}) \rightarrow \cdots$$

Bott periodicity states that $\pi_{13}\mathbf{BString} \cong \pi_{14}\mathbf{BString} \cong 0$, so we find that $\pi_{14}(Q) \cong \pi_{13}(N)$. The desired 2-torsion element of $\pi_{14}(Q)$ is precisely the element of $\pi_{13}(N)$ described in Remark 3.2.24. Choosing a particular nullhomotopy of twice this 2-torsion element of $\pi_{14}(Q)$ produces a map $g : P^{15}(2) \rightarrow Q$. To extend this map over the double suspension $P^{15}(2) \rightarrow \Omega^2 P^{17}(2)$, it would suffice to show that there is a double loop space \tilde{Q} with a map $\tilde{Q} \rightarrow Q$ such that g factors through \tilde{Q} .

Unfortunately, we do not know how to prove such a result; this is the unstable analogue of Conjecture E. In fact, such an unstable statement would bypass the need for Conjecture E in Theorem A. (One runs into the same obstruction for \mathbf{MSpin} , except with the fiber of the map $S^5 \rightarrow \mathbf{B}^2\mathbf{Spin}$.) These statements are reminiscent of the conjecture (see Section 4.1) that the fiber $W_n = \text{fib}(S^{2n-1} \rightarrow \Omega^2 S^{2n+1})$ of the double suspension admits the structure of a double loop space.

Remark 6.1.8. The following application of Theorem C was suggested by Mike Hopkins. In [HH92], the Anderson-Brown-Peterson splitting is used to show that the Atiyah-Bott-Shapiro orientation $\mathbf{MSpin} \rightarrow \mathbf{KO}$ induces an isomorphism

$$\mathbf{MSpin}_*(X) \otimes_{\mathbf{MSpin}_*} \mathbf{KO}_* \xrightarrow{\cong} \mathbf{KO}_*(X)$$

of \mathbf{KO}_* -modules for all spectra X . In future work, we shall show that Theorem C can be used to prove the following height 2 analogue of this result: namely, Conjectures D, E, and F imply that the Ando-Hopkins-Rezk orientation $\mathbf{MString} \rightarrow \mathbf{Tmf}$ induces an isomorphism

$$(6.1) \quad \mathbf{MString}_*(X) \otimes_{\mathbf{MString}_*} \mathbf{Tmf}_* \xrightarrow{\cong} \mathbf{Tmf}_*(X)$$

of \mathbf{Tmf}_* -modules for all spectra X . The $K(1)$ -analogue of this isomorphism was obtained by Laures in [Lau04].

6.2. Wood equivalences. The Wood equivalence states that $\text{bo} \wedge C\eta \simeq \text{bu}$. There are generalizations of this equivalence to tmf (see [Mat16]); for instance, there is a 2-local 8-cell complex DA_1 whose cohomology is isomorphic to the double of $A(1)$ as an $A(2)$ -module such that $\text{tmf}_{(2)} \wedge DA_1 \simeq \text{tmf}_1(3) \simeq \mathbf{BP}\langle 2 \rangle$. Similarly, if X_3 denotes the 3-local 3-cell complex $S^0 \cup_{\alpha_1} e^4 \cup_{2\alpha_1} e^8$, then $\text{tmf}_{(3)} \wedge X_3 \simeq \text{tmf}_1(2) \simeq \mathbf{BP}\langle 2 \rangle \vee \Sigma^8 \mathbf{BP}\langle 2 \rangle$. We will use the umbrella term “Wood equivalence” to refer to equivalences of this kind.

Our goal in this section is to revisit these Wood equivalences using the point of view stemming from Theorem A. In particular, we propose that these equivalences are suggested by the existence of certain EHP sequences; we will greatly expand on this in a forthcoming document. We find this to be a rather beautiful connection between stable and unstable homotopy theory.

The first Wood-style result was proved in Proposition 3.2.6. The next result, originally proved in [Mah79, Section 2.5] and [DM81, Theorem 3.7], is the simplest example of a Wood-style equivalence which is related to the existence of certain EHP sequences.

Proposition 6.2.1. *Let $\mathbb{S}\//\eta = X(2)$ (resp. $\mathbb{S}\//2$) denote the \mathbf{E}_1 -quotient of \mathbb{S} by η (resp. 2). If $Y = C\eta \wedge \mathbb{S}/2$ and A_1 is a spectrum whose cohomology is isomorphic to $A(1)$ as a module over the Steenrod algebra, then there are equivalences*

$$A \wedge C\eta \simeq \mathbb{S}\//\eta, \quad A \wedge Y \simeq \mathbb{S}\//2, \quad A \wedge A_1 \simeq y(1)/v_1$$

of A -modules.

Remark 6.2.2. Proposition 6.2.1 implies the Wood equivalence $\text{bo} \wedge C\eta \simeq \text{bu}$. Although this implication is already true before 2-completion, we will work in the 2-complete category for convenience. Recall that Theorem A states that Conjecture D and Conjecture E imply that there is a map $\mu : K_2 \rightarrow \text{BGL}_1(A)$ whose Thom spectrum is equivalent to bo (as left A -modules). Moreover, the Thom spectrum of the composite $K_2 \xrightarrow{\mu} \text{BGL}_1(A) \rightarrow \text{BGL}_1(T(1))$ is equivalent to $\text{BP}\langle 1 \rangle$. Since this Thom spectrum is the base-change $K_2^\mu \wedge_A T(1)$, and Proposition 6.2.1 implies that $T(1) = X(2) \simeq A \wedge C\eta$, we find that

$$\text{BP}\langle 1 \rangle \simeq K_2^\mu \wedge_A (A \wedge C\eta) \simeq K_2^\mu \wedge C\eta \simeq \text{bo} \wedge C\eta,$$

as desired. Similarly, noting that $\mathbb{S}\//2 = y(1)$, we find that Proposition 6.2.1 also proves the equivalence $\text{bo} \wedge Y \simeq k(1)$.

Remark 6.2.3. The argument of Remark 6.2.2 in fact proves that Theorem A for A implies Theorem A for $T(1)$, $y_{\mathbf{Z}}(1)$, and $y(1)$.

PROOF OF PROPOSITION 6.2.1. For the first two equivalences, it suffices to show that $A \wedge C\eta \simeq \mathbb{S}\//\eta$ and that $\mathbb{S}\//\eta \wedge \mathbb{S}/2 \simeq \mathbb{S}\//2$. We will prove the first statement; the proof of the second statement is exactly the same. There is a map $C\eta \rightarrow \mathbb{S}\//\eta$ given by the inclusion of the 2-skeleton. There is also an \mathbf{E}_1 -ring map $A \rightarrow \mathbb{S}\//\eta$ given as follows. The multiplication on $\mathbb{S}\//\eta$ defines a unital map $C\eta \wedge C\eta \rightarrow \mathbb{S}\//\eta$. But since the Toda bracket $\langle \eta, 2, \eta \rangle$ contains ν , there is a unital map $C\nu \rightarrow C\eta \wedge C\eta$. This supplies a unital map $C\nu \rightarrow \mathbb{S}\//\eta$, which, by the universal property of $A = \mathbb{S}\//\nu$ (via Theorem 2.1.7), extends to an \mathbf{E}_1 -ring map $A \rightarrow \mathbb{S}\//\eta$.

For the final equivalence, it suffices to construct a map $A_1 \rightarrow y(1)/v_1$ for which the induced map $A \wedge A_1 \rightarrow y(1)/v_1$ gives an isomorphism on mod 2 homology. Since A_1 may be obtained as the cofiber of a v_1 -self map $\Sigma^2 Y \rightarrow Y$, it suffices to observe that the following diagram commutes; our desired map is the induced map on vertical cofibers:

$$\begin{array}{ccc} \Sigma^2 Y & \longrightarrow & \Sigma^2 y(1) \\ \downarrow v_1 & & \downarrow v_1 \\ Y & \longrightarrow & y(1). \end{array}$$

□

Remark 6.2.4. There are EHP sequences

$$S^1 \rightarrow \Omega S^2 \rightarrow \Omega S^3, \quad S^2 \rightarrow \Omega S^3 \rightarrow \Omega S^5.$$

Recall that $\mathbb{S}/2$, $C\eta$, $\mathbb{S}\//2$, $\mathbb{S}\//\eta = X(2)$, and A are Thom spectra over S^1 , S^2 , ΩS^2 , ΩS^3 , and ΩS^5 respectively. Proposition 2.1.6 therefore implies that there are maps $f : \Omega S^3 \rightarrow \text{BAut}(\mathbb{S}/2)$ and $g : \Omega S^5 \rightarrow \text{BAut}(C\eta)$ whose Thom spectra are equivalent to $\mathbb{S}\//2$ and $\mathbb{S}\//\eta$, respectively. The maps f and g define local systems of spectra over ΩS^3 and ΩS^5 whose fibers are equivalent to $\mathbb{S}/2$ and $C\eta$ (respectively),

and one interpretation of Proposition 6.2.1 is that these local systems in fact factor as

$$\Omega S^3 \xrightarrow{\eta} \mathrm{BGL}_1(\mathbb{S}) \rightarrow \mathrm{BAut}(\mathbb{S}/2), \quad \Omega S^5 \xrightarrow{\nu} \mathrm{BGL}_1(\mathbb{S}) \rightarrow \mathrm{BAut}(C\eta).$$

Proposition 6.2.1 is an immediate consequence of these factorizations. We argue this for the first case in Remark 6.2.5, and for the second in Remark 6.2.6, thereby giving an alternative EHP-based argument for Proposition 6.2.1.

Remark 6.2.5. The first EHP sequence in Remark 6.2.4 splits via the Hopf map $S^3 \rightarrow S^2$. The map $f : \Omega S^3 \rightarrow \mathrm{BAut}(\mathbb{S}/2)$ in fact factors through the dotted map in the following diagram:

$$\begin{array}{ccccc} S^2 & \longrightarrow & \Omega S^2 & \longrightarrow & \Omega S^3 \\ & & \downarrow & \swarrow \text{dotted} & \downarrow \\ & & \mathrm{BGL}_1(\mathbb{S}) & \longrightarrow & \mathrm{BAut}(\mathbb{S}/2). \end{array}$$

Indeed, the composite $\Omega S^3 \rightarrow \Omega S^2 \rightarrow \mathrm{BGL}_1(\mathbb{S})$ is a loop map, and therefore is determined by the composite $\phi : S^3 \rightarrow S^2 \rightarrow B^2\mathrm{GL}_1(\mathbb{S})$. Since the map $S^2 \rightarrow B^2\mathrm{GL}_1(\mathbb{S})$ detects the element $-1 \in \pi_0(\mathbb{S})^\times$, the map ϕ does in fact determine a unit multiple of η . This implies the desired claim.

Remark 6.2.6. The map $g : \Omega S^5 \rightarrow \mathrm{BAut}(C\eta)$ from Remark 6.2.4 factors through $\mathrm{BGL}_1(\mathbb{S})$. To see this, let us begin with the following observation: view BU and BSU as H -spaces via the tensor product of vector bundles. Then the map $\mathrm{BSU} \times \mathbf{C}P^\infty \rightarrow \mathrm{BU}$ classifying $\mathcal{V} \boxtimes \mathcal{L}$, with \mathcal{V} the universal SU -bundle over BSU and \mathcal{L} the universal line bundle over BU , is an equivalence of H -spaces. In particular, there is a fiber sequence

$$\mathbf{C}P^\infty \rightarrow \mathrm{BU} \rightarrow \mathrm{BSU}.$$

The map $\Omega S^3 \rightarrow \mathrm{BGL}_1(\mathbb{S})$ defining $T(1)$ factors as

$$\Omega S^3 \rightarrow \mathrm{BU} \xrightarrow{J} \mathrm{BGL}_1(\mathbb{S});$$

similarly, the map $\Omega S^5 \rightarrow \mathrm{BGL}_1(\mathbb{S})$ defining A factors as

$$\Omega S^5 \rightarrow \mathrm{BSU} \xrightarrow{J} \mathrm{BGL}_1(\mathbb{S}).$$

These factorizations make the following diagram of fiber sequences commute:

$$\begin{array}{ccccc} S^2 & \longrightarrow & \Omega S^3 & \longrightarrow & \Omega S^5 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{C}P^\infty & \longrightarrow & \mathrm{BU} & \longrightarrow & \mathrm{BSU}. \end{array}$$

The map $\Omega S^5 \rightarrow \mathrm{BAut}(C\eta)$ was defined using Proposition 2.1.6. It then follows from the splitting of the bottom fiber sequence in the above diagram that the dotted

map exists in the following diagram:

$$\begin{array}{ccccc}
 S^2 & \longrightarrow & \Omega S^3 & \longrightarrow & \Omega S^5 \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbf{C}P^\infty & \longrightarrow & \mathbf{B}U & \longrightarrow & \mathbf{B}SU \\
 & \searrow & \downarrow J & \swarrow J & \downarrow \\
 & & \mathbf{B}GL_1(\mathbb{S}) & \longrightarrow & \mathbf{B}Aut(C\eta)
 \end{array}$$

The composite

$$\Omega S^5 \rightarrow \mathbf{B}SU \xrightarrow{J} \mathbf{B}GL_1(\mathbb{S}) \rightarrow \mathbf{B}Aut(C\eta)$$

is g , giving our desired factorization.

Next, we have the following result at height 2:

Proposition 6.2.7. *Let DA_1 denote the double of A_1 (see [Mat16]). There are 2-complete equivalences*

$$B \wedge DA_1 \simeq T(2), \quad B \wedge Z \simeq y(2), \quad B \wedge A_2 \simeq y(2)/v_2,$$

where Z is the spectrum “ $\frac{1}{2}A_2$ ” from [MT94, BE16]³, and A_2 is a spectrum whose cohomology is isomorphic to $A(2)$ as a module over the Steenrod algebra.

Remark 6.2.8. Arguing as in Remark 6.2.2 shows that Proposition 6.2.7 and Theorem A imply the Wood equivalences

$$\mathrm{tmf} \wedge DA_1 \simeq \mathrm{tmf}_1(3) = \mathbf{B}P\langle 2 \rangle, \quad \mathrm{tmf} \wedge Z \simeq k(2), \quad \mathrm{tmf} \wedge A_2 \simeq \mathbf{H}\mathbf{F}_2.$$

Remark 6.2.9. Exactly as in Remark 6.2.3, the argument of Remark 6.2.8 in fact proves that Theorem A for B implies Theorem A for $T(2)$, $y\mathbf{z}(2)$, and $y(2)$.

Remark 6.2.10. The telescope conjecture [Rav84, Conjecture 10.5], which we interpret as stating that L_n -localization is the same as L_n^f -localization, is known to be true at height 1. For odd primes, it was proved by Miller in [Mil81], and at $p = 2$ it was proved by Mahowald in [Mah81a, Mah82]. Mahowald’s approach was to calculate the telescopic homotopy of the type 1 spectrum Y . In [MRS01], Mahowald-Ravenel-Shick proposed an approach to *disproving* the telescope conjecture at height 2: they suggest that for $n \geq 2$, the L_n -localization and the v_n -telescopic localization of $y(n)$ have different homotopy groups. They show, however, that the L_1 -localization and the v_1 -telescopic localization of $y(1)$ agree, so this approach (thankfully) does not give a counterexample to the telescope conjecture at height 1.

Motivated by Mahowald’s approach to the telescope conjecture, Behrens-Beaudry-Bhattacharya-Culver-Xu study the v_2 -telescopic homotopy of Z in [BBB⁺19], with inspiration from the Mahowald-Ravenel-Shick approach. Propositions 6.2.1 and 6.2.7 can be used to relate these two (namely, the finite spectrum and the Thom spectrum) approaches to the telescope conjecture. As in Section 6.1, we will let R denote A or B . Moreover, let F denote Y or Z (depending on what R is), and let R' denote $y(1)$ or $y(2)$ (again depending on what R is), so that $R \wedge F = R'$ by Propositions 6.2.1 and 6.2.7. Then:

³In the former source, Z is denoted by \overline{M} .

Corollary 6.2.11. *If the telescope conjecture is true for F (and hence any type 1 or 2 spectrum) or R , then it is true for R' .*

PROOF. Since L_n - and L_n^f -localizations are smashing, we find that if the telescope conjecture is true for F or R , then Propositions 6.2.1 and 6.2.7 yield equivalences

$$L_n^f R' \simeq R \wedge L_n^f F \simeq R \wedge L_n F \simeq L_n R'.$$

□

Finally, we prove Proposition 6.2.7.

PROOF OF PROPOSITION 6.2.7. We first construct maps $B \rightarrow T(2)$ and $DA_1 \rightarrow T(2)$. The top cell of DA_1 is in dimension 12, and the map $T(2) \rightarrow \text{BP}$ is an equivalence in dimensions ≤ 12 . It follows that constructing a map $DA_1 \rightarrow T(2)$ is equivalent to constructing a map $DA_1 \rightarrow \text{BP}$. However, both BP and DA_1 are concentrated in even degrees, so the Atiyah-Hirzebruch spectral sequence collapses, and we find that $\text{BP}^*(DA_1) \cong H^*(DA_1; \text{BP}_*)$. The generator in bidegree $(0, 0)$ produces a map $DA_1 \rightarrow T(2)$; its effect on homology is the additive inclusion $\mathbf{F}_2[\zeta_1^2, \zeta_2^2]/(\zeta_1^8, \zeta_2^4) \rightarrow \mathbf{F}_2[\zeta_1^2, \zeta_2^2]$.

The map $B \rightarrow T(2)$ may be defined via the universal property of Thom spectra from Section 2.1 and Remark 3.2.20. Its effect on homology is the inclusion $\mathbf{F}_2[\zeta_1^8, \zeta_2^4] \rightarrow \mathbf{F}_2[\zeta_1^2, \zeta_2^2]$. We obtain a map $B \wedge DA_1 \rightarrow T(2)$ via the multiplication on $T(2)$, and this induces an isomorphism in mod 2 homology.

For the second equivalence, we argue similarly: the map $B \rightarrow T(2)$ defines a map $B \rightarrow T(2) \rightarrow y(2)$. Next, recall that Z is built through iterated cofiber sequences:

$$\Sigma^2 Y \xrightarrow{v_1} Y \rightarrow A_1, \quad \Sigma^5 A_1 \wedge C\nu \xrightarrow{\sigma_1} A_1 \wedge C\nu \rightarrow Z.$$

As an aside, we note that the element σ_1 is intimately related to the element discussed in Example 3.1.14; namely, it is given by the self-map of $A_1 \wedge C\nu$ given by smashing A_1 with the following diagram:

$$\begin{array}{ccccc} \Sigma^5 C\nu & \longrightarrow & \Sigma^5 A & \xrightarrow{\sigma_1 \wedge \text{id}} & A \wedge A & \longrightarrow & A \\ & \searrow & & \swarrow & & & \uparrow \\ & & & & & & C\nu \end{array}$$

Using these cofiber sequences and Proposition 3.2.6, one obtains a map $Z \rightarrow y(2)$, which induces the additive inclusion $\mathbf{F}_2[\zeta_1, \zeta_2]/(\zeta_1^8, \zeta_2^4) \rightarrow \mathbf{F}_2[\zeta_1, \zeta_2]$ on mod 2 homology. The multiplication on $y(2)$ defines a map $B \wedge Z \rightarrow y(2)$, which induces an isomorphism on mod 2 homology.

For the final equivalence, it suffices to construct a map $A_2 \rightarrow y(2)/v_2$ for which the induced map $B \wedge A_2 \rightarrow y(2)/v_2$ gives an isomorphism on mod 2 homology. Since A_2 may be obtained as the cofiber of a v_2 -self map $\Sigma^6 Z \rightarrow Z$, it suffices to observe that the following diagram commutes; our desired map is the induced map on vertical cofibers:

$$\begin{array}{ccc} \Sigma^6 Z & \longrightarrow & \Sigma^6 y(2) \\ \downarrow v_2 & & \downarrow v_2 \\ Z & \longrightarrow & y(2). \end{array}$$

□

Arguing exactly as in the proof of Proposition 6.2.7 shows the following result at the prime 3:

Proposition 6.2.12. *Let X_3 denote the 8-skeleton of $T(1) = \mathbb{S}/\alpha_1$. There are 3-complete equivalences*

$$B \wedge X_3 \simeq T(2) \vee \Sigma^8 T(2), \quad B \wedge X_3 \wedge \mathbb{S}/(3, v_1) \simeq y(2) \vee \Sigma^8 y(2).$$

In forthcoming work, we will discuss the relation between Proposition 6.2.7 and EHP sequences, along the lines of Remark 6.2.4.

7. C_2 -equivariant analogue of Corollary B

Our goal in this section is to study a C_2 -equivariant analogue of Corollary B at height 1. The odd primary analogue of this result is deferred to the future; it is considerably more subtle.

7.1. C_2 -equivariant analogues of Ravenel's spectra. In this section, we construct the C_2 -equivariant analogue of $T(n)$ for all n . We 2-localize everywhere until mentioned otherwise. There is a C_2 -action on $\Omega\mathrm{SU}(n)$ given by complex conjugation, and the resulting C_2 -space is denoted $\Omega\mathrm{SU}(n)_{\mathbf{R}}$. Real Bott periodicity gives a C_2 -equivariant map $\Omega\mathrm{SU}(n)_{\mathbf{R}} \rightarrow \mathrm{BU}_{\mathbf{R}}$ whose Thom spectrum is the (genuine) C_2 -spectrum $X(n)_{\mathbf{R}}$. This admits the structure of an \mathbf{E}_{ρ} -ring, since it is the Thom spectrum of an \mathbf{E}_{ρ} -map $\Omega^{\rho}\mathrm{B}^{\sigma}\mathrm{SU}(n)_{\mathbf{R}} \rightarrow \Omega^{\rho}\mathrm{B}^{\rho}\mathrm{BU}_{\mathbf{R}} \simeq \Omega^{\rho}\mathrm{BSU}_{\mathbf{R}}$. As in the nonequivariant case, the equivariant Quillen idempotent on $\mathrm{MU}_{\mathbf{R}}$ restricts to one on $X(m)_{\mathbf{R}}$, and therefore defines a summand $T(n)_{\mathbf{R}}$ of $X(m)_{\mathbf{R}}$ for $2^n \leq m \leq 2^{n+1} - 1$. Again, this summand admits the structure of an \mathbf{E}_1 -ring.

Construction 7.1.1. There is an equivariant fiber sequence

$$\Omega\mathrm{SU}(n)_{\mathbf{R}} \rightarrow \Omega\mathrm{SU}(n+1)_{\mathbf{R}} \rightarrow \Omega S^{n\rho+1},$$

where ρ is the regular representation of C_2 ; the equivariant analogue of Proposition 2.1.6 then shows that there is a map $\Omega S^{n\rho+1} \rightarrow \mathrm{BGL}_1(X(n)_{\mathbf{R}})$ (detecting an element $\chi_n \in \pi_{n\rho-1}X(n)_{\mathbf{R}}$) whose Thom spectrum is $X(n+1)_{\mathbf{R}}$. Here, $\mathrm{BGL}_1(X(n)_{\mathbf{R}})$ is the delooping of the \mathbf{E}_{ρ} -space $\mathrm{GL}_1(X(n)_{\mathbf{R}})$, and the C_2 -equivariant notion of Thom spectrum is taken in the sense of [HHK⁺20, Theorem 3.2]. (The constructions from *loc. cit.* can be verified to go through for equivariant maps to $\mathrm{BGL}_1(X(n)_{\mathbf{R}})$; for example, when $n = \infty$, the idea of taking Thom spectra for an equivariant map to $\mathrm{BGL}_1(\mathrm{MU}_{\mathbf{R}})$ was already used in [HS20, Section 3].)

If $\tilde{\sigma}_n$ denotes the image of the element $\chi_{2^{n+1}\rho-1}$ in $\pi_{(2^{n+1}-1)\rho-1}T(n)_{\mathbf{R}}$, then we have a C_2 -equivariant analogue of Lemma 3.1.12:

Lemma 7.1.2. *The Thom spectrum of the map $\Omega S^{(2^{n+1}-1)\rho+1} \rightarrow \mathrm{BGL}_1(X(2^{n+1}-1)_{\mathbf{R}})$ detecting $\tilde{\sigma}_n$ is a direct sum of shifts of $T(n+1)_{\mathbf{R}}$.*

Example 7.1.3. For instance, $T(1)_{\mathbf{R}} = X(2)_{\mathbf{R}}$ is the Thom spectrum of the map $\Omega S^{\rho+1} \rightarrow \mathrm{BU}_{\mathbf{R}}$; upon composing with the equivariant J-homomorphism $\mathrm{BU}_{\mathbf{R}} \rightarrow \mathrm{BGL}_1(\mathbb{S})$, this detects the element $\tilde{\eta} \in \pi_{\sigma}\mathbb{S}$, and the extension of the map $S^{\rho} \rightarrow \mathrm{BGL}_1(\mathbb{S})$ to $\Omega S^{\rho+1}$ uses the \mathbf{E}_1 -structure on $\mathrm{BGL}_1(\mathbb{S})$. The case of $X(2)_{\mathbf{R}}$ exhibits a curious property: $S^{\rho+1}$ is the loop space $\Omega^{\sigma}\mathbf{HP}_{\mathbf{R}}^{\infty}$, and there are equivalences (see [HW20, Proposition 3.4 and Proposition 3.6])

$$\Omega S^{\rho+1} \simeq \Omega^{\sigma+1}\mathbf{HP}_{\mathbf{R}}^{\infty} \simeq \Omega^{\sigma}(\Omega\mathbf{HP}_{\mathbf{R}}^{\infty}).$$

However, $\Omega\mathbf{HP}_{\mathbf{R}}^{\infty} \simeq S^{\rho+\sigma}$, so $\Omega S^{\rho+1} = \Omega^{\sigma}S^{\rho+\sigma}$. The map $\Omega^{\sigma}S^{\rho+\sigma} \rightarrow \mathrm{BGL}_1(\mathbb{S})$ still detects the element $\tilde{\eta} \in \pi_{\sigma}\mathbb{S}$ on the bottom cell, but the extension of the map $S^{\rho} \rightarrow \mathrm{BGL}_1(\mathbb{S})$ to $\Omega^{\sigma}S^{\rho+\sigma}$ is now defined via the \mathbf{E}_{σ} -structure on $\mathrm{BGL}_1(\mathbb{S})$. The upshot of this discussion is that $X(2)_{\mathbf{R}}$ is not only the free \mathbf{E}_1 -ring with a nullhomotopy of $\tilde{\eta}$, but also the free \mathbf{E}_{σ} -algebra with a nullhomotopy of $\tilde{\eta}$.

Warning 7.1.4. Unlike the nonequivariant setting, the element $\tilde{\eta} \in \pi_{\sigma}\mathbb{S}$ is neither torsion nor nilpotent. This is because its geometric fixed points is $\Phi^{C_2}\tilde{\eta} = 2 \in \pi_0\mathbb{S}$; see [DI13, Proposition C.5], although note that the orientations chosen there are the opposite of ours. Briefly, the map $\tilde{\eta}$ is obtained by ρ -desuspending the unstable

equivariant Hopf map $S^{\rho+\sigma} = \mathbf{C}^2 - \{0\} \rightarrow \mathbf{C}P^1 = S^\rho$, whose homotopy fiber is S^σ . In other words, there is a fiber sequence $S^\sigma \rightarrow S^{\rho+\sigma} \xrightarrow{\tilde{\eta}} S^\rho$. On geometric fixed points, this produces the fiber sequence $S^0 = C_2 \rightarrow S^1 \rightarrow S^1$, which forces the map $\Phi^{C_2}\tilde{\eta}$ to have degree 2 (or -2 , depending on the choice of orientation).

Example 7.1.5. Consider the element $\tilde{\sigma}_1 \in \pi_{3\rho-1}T(1)_{\mathbf{R}}$. The underlying nonequivariant element of $\pi_5T(1)_{\mathbf{R}}$ is simply σ_1 . To determine $\Phi^{C_2}\tilde{\sigma}_1 \in \pi_2\Phi^{C_2}T(1)_{\mathbf{R}}$, we first note that $\Phi^{C_2}T(1)_{\mathbf{R}}$ is the Thom spectrum of the map $\Phi^{C_2}\tilde{\eta} : \Phi^{C_2}\Omega S^{\rho+1} \rightarrow \text{BGL}_1(\mathbb{S})$. Since $\Phi^{C_2}\Omega S^{\rho+1} = \Omega S^2$ and $\Phi^{C_2}\tilde{\eta} = 2$, we find that $\Phi^{C_2}T(1)_{\mathbf{R}}$ is the \mathbf{E}_1 -quotient $\mathbb{S}\!/2$. The element $\Phi^{C_2}\tilde{\sigma}_1 \in \pi_2\mathbb{S}\!/2 \cong \pi_2\mathbb{S}/2$ is simply a map $S^2 \rightarrow \mathbb{S}/2$ which is η on the top cell. Such a map exists because $2\eta = 0$.

As an aside, we mention that there is a C_2 -equivariant lift of the spectrum A :

Definition 7.1.6. Let A_{C_2} denote the Thom spectrum of the map $\Omega S^{2\rho+1} \rightarrow \text{BGL}_1(\mathbb{S})$ defined by the extension of the map $S^{2\rho} \rightarrow \text{BGL}_1(\mathbb{S})$ which detects the equivariant Hopf map $\tilde{\nu} \in \pi_{2\rho-1}\mathbb{S}$.

Remark 7.1.7. The underlying spectrum of A_{C_2} is A . To determine the geometric fixed points of A_{C_2} , $\Phi^{C_2}A_{C_2}$ is the Thom spectrum of the map $\Phi^{C_2}\tilde{\nu} : \Phi^{C_2}\Omega S^{2\rho+1} \rightarrow \text{BGL}_1(\mathbb{S})$. We claim that $\tilde{\Phi}^{C_2}\tilde{\nu} = \eta$; indeed, the map $\tilde{\nu}$ is obtained by 2ρ -desuspending the unstable equivariant map $S^{4\rho-1} = \mathbf{H}^2 - \{0\} \rightarrow \mathbf{H}P^1 = S^{2\rho}$. The homotopy fiber of this map is $S^{2\rho-1} = S^{\rho+\sigma}$, so that there is an equivariant fiber sequence $S^{\rho+\sigma} \rightarrow S^{4\rho-1} \rightarrow S^{2\rho}$. On geometric fixed points, we obtain a fiber sequence $S^1 \rightarrow S^3 \rightarrow S^2$, which implies that $\tilde{\Phi}^{C_2}\tilde{\nu}$ be identified with the Hopf fibration $S^3 \rightarrow S^2$. Now, since $\Phi^{C_2}\Omega S^{2\rho+1} = \Omega S^3$, we find that $\Phi^{C_2}A_{C_2} = T(1)$. In particular, A_{C_2} may be thought of as the free C_2 -equivariant \mathbf{E}_1 -ring with a nullhomotopy of $\tilde{\nu}$.

Example 7.1.8. The element $\tilde{\sigma}_1$ lifts to $\pi_{3\rho-1}A_{C_2}$. Indeed, Remark 3.2.17 works equivariantly too: the equivariant Hopf map $S^{3\rho-1} \rightarrow S^{2\rho}$ defines a composite $S^{3\rho-1} \rightarrow S^{2\rho} \rightarrow \Omega S^{2\rho+1}$. The composite $S^{3\rho-1} \rightarrow \Omega S^{2\rho+1} \rightarrow \text{BSU}_{\mathbf{R}}$ is null, since $\pi_{3\rho-1}\text{BSU}_{\mathbf{R}} = 0$. It follows that upon Thomification, the map $S^{3\rho-1} \rightarrow \Omega S^{2\rho+1}$ defines an element $\tilde{\sigma}'_1$ of $\pi_{3\rho-1}A_{C_2}$. In order to show that this element indeed deserves to be called $\tilde{\sigma}_1$, we use Proposition 7.1.9. The map $A_{C_2} \rightarrow T(1)_{\mathbf{R}}$ from the proposition induces a map $\pi_{3\rho-1}A_{C_2} \rightarrow \pi_{3\rho-1}T(1)_{\mathbf{R}}$, and we need to show that the image of $\tilde{\sigma}'_1 \in \pi_{3\rho-1}A_{C_2}$ is in fact $\tilde{\sigma}_1$. By Example 7.1.5, it suffices to observe that the underlying nonequivariant map corresponding to $\tilde{\sigma}'_1 \in \pi_{3\rho-1}T(1)_{\mathbf{R}}$ is σ_1 , and that the geometric fixed points $\Phi_{C_2}\tilde{\sigma}'_1 \in \pi_2\mathbb{S}\!/2$ is the lift of η appearing in Example 7.1.5.

We now prove the proposition used above.

Proposition 7.1.9. *There is a genuine C_2 -equivariant \mathbf{E}_1 -map $A_{C_2} \rightarrow T(1)_{\mathbf{R}}$.*

PROOF. By Remark 7.1.7, it suffices to show that $\tilde{\nu} = 0 \in \pi_{3\rho-1}T(1)_{\mathbf{R}}$. The underlying map is null, because $\nu = 0 \in \pi_5T(1)$. The geometric fixed points are also null, because $\Phi^{C_2}\tilde{\nu} = \eta$ is null in $\pi_2\Phi^{C_2}T(1)_{\mathbf{R}} = \pi_2\mathbb{S}\!/2$. Therefore, $\tilde{\nu}$ is null in $\pi_{3\rho-1}T(1)_{\mathbf{R}}$. \square

In fact, it is easy to prove the following analogue of Proposition 6.2.1:

Proposition 7.1.10. *There is a C_2 -equivariant equivalence $A_{C_2} \wedge C\tilde{\eta} \simeq T(1)_{\mathbf{R}}$.*

PROOF. There are maps $A_{C_2} \rightarrow T(1)_{\mathbf{R}}$ and $C\tilde{\eta} \rightarrow T(1)_{\mathbf{R}}$, which define a map $A_{C_2} \wedge C\tilde{\eta} \rightarrow T(1)_{\mathbf{R}}$ via the multiplication on $T(1)_{\mathbf{R}}$. This map is an equivalence on underlying by Proposition 6.2.1, and on geometric fixed points induces the map $T(1) \wedge \mathbb{S}/2 \rightarrow \mathbb{S}/2$. This was also proved in the course of Proposition 6.2.1. \square

Remark 7.1.11. As in Remark 6.2.2, one might hope that this implies the C_2 -equivariant Wood equivalence $\mathrm{bo}_{C_2} \wedge C\tilde{\eta} \simeq \mathrm{bu}_{\mathbf{R}}$ via some equivariant analogue of Theorem A.

Remark 7.1.12. The equivariant analogue of Remark 6.2.4 remains true: the equivariant Wood equivalence of Proposition 7.1.10 stems from the EHP sequence $S^\rho \rightarrow \Omega S^{\rho+1} \rightarrow \Omega S^{2\rho+1}$. To prove the existence of such a fiber sequence, we use [DH21, Construction 4.26] to get the Hopf map $h : \Omega S^{\rho+1} \rightarrow \Omega S^{2\rho+1}$, as well as a nullhomotopy of the composite $S^\rho \rightarrow \Omega S^{\rho+1} \rightarrow \Omega S^{2\rho+1}$. In particular, if $F = \mathrm{fib}(h)$, there is an equivariant map $S^\rho \rightarrow F$. We claim that this map is an equivalence: it suffices to prove that $S^\rho \rightarrow F$ is an equivalence on underlying and on geometric fixed points, since these functors preserve homotopy limits and colimits, and these functors are jointly conservative. The desired equivalence on underlying spaces follows from the classical EHP sequence $S^2 \rightarrow \Omega S^3 \rightarrow \Omega S^5$, and the equivalence on geometric fixed points follows from the splitting $\Omega S^2 \simeq S^1 \times \Omega S^3$.

7.2. The C_2 -equivariant analogue of Corollary B at $n = 1$. Recall (see [HK01]) that there are indecomposable classes $\bar{v}_n \in \pi_{(2^n-1)\rho} \mathrm{BP}_{\mathbf{R}}$; as in Theorem 3.1.5, these lift to classes in $\pi_* T(m)_{\mathbf{R}}$ if $m \geq n$. The main result of this section is the following:

Theorem 7.2.1. *There is a map $\Omega^\rho S^{2\rho+1} \rightarrow \mathrm{BGL}_1(T(1)_{\mathbf{R}})$ detecting an indecomposable in $\pi_\rho T(1)_{\mathbf{R}}$ on the bottom cell, whose Thom spectrum is $\underline{\mathrm{HZ}}$.*

Note that, as with Corollary B at $n = 1$, this result is *unconditional*. The argument is exactly as in the proof of Corollary B at $n = 1$, with practically no modifications. We need the following analogue of Theorem 2.2.1, originally proved in [BW18, HW20].

Proposition 7.2.2 (Behrens-Wilson, Hahn-Wilson). *Let p be any prime, and let λ denote the 2-dimensional standard representation of C_p on \mathbf{C} . The Thom spectrum of the map $\Omega^\lambda S^{\lambda+1} \rightarrow \mathrm{BGL}_1(S^0)$ extending the map $1 - p : S^1 \rightarrow \mathrm{BGL}_1(S^0)$ is equivalent to $\underline{\mathrm{HF}}_p$ as an \mathbf{E}_λ -ring. Moreover, if $S^{\lambda+1}\langle\lambda+1\rangle$ denotes the $(\lambda+1)$ -connected cover of $S^{\lambda+1}$ (i.e., the fiber of the map $S^{\lambda+1} \rightarrow \Omega^\infty \Sigma^{\lambda+1} \underline{\mathrm{HZ}}$), then the Thom spectrum of the induced map $\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle \rightarrow \mathrm{BGL}_1(S^0)$ is equivalent to $\underline{\mathrm{HZ}}$ as an \mathbf{E}_λ -ring.*

We can now prove Theorem 7.2.1.

PROOF OF THEOREM 7.2.1. In [HW20], the authors prove that there is an equivalence of C_2 -spaces between $\Omega^\lambda S^{\lambda+1}$ and $\Omega^\rho S^{\rho+1}$, and that $\underline{\mathrm{HF}}_2$ is in fact the Thom spectrum of the induced map $\Omega^\rho S^{\rho+1} \rightarrow \mathrm{BGL}_1(S^0)$ detecting -1 . Since both $\Omega^\rho S^{\rho+1}\langle\rho+1\rangle$ and $\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle$ are defined as fibers of maps to S^1 which are degree one on the bottom cell, Hahn and Wilson's equivalence lifts to a C_2 -equivariant equivalence $\Omega^\rho S^{\rho+1}\langle\rho+1\rangle \simeq \Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle$, and we find that $\underline{\mathrm{HZ}}$ is the Thom spectrum of the map $\Omega^\rho S^{\rho+1}\langle\rho+1\rangle \rightarrow \mathrm{BGL}_1(S^0)$.

Since $T(1)_{\mathbf{R}}$ is the Thom spectrum of the composite map $\Omega S^{\rho+1} \rightarrow \Omega^\rho S^{\rho+1}\langle\rho+1\rangle \rightarrow \mathrm{BGL}_1(S^0)$ detecting $\tilde{\eta}$ on the bottom cell of the source, it follows from the

C_2 -equivariant analogue of Proposition 2.1.6 and the above discussion that it is sufficient to define a fiber sequence

$$\Omega S^{\rho+1} \rightarrow \Omega^\rho S^{\rho+1}\langle \rho+1 \rangle \rightarrow \Omega^\rho S^{2\rho+1},$$

and check that the induced map $\Omega^\rho S^{2\rho+1} \rightarrow \mathrm{BGL}_1(T(1)_{\mathbf{R}})$ detects an indecomposable element of $\pi_\rho T(1)_{\mathbf{R}}$. See Remark 4.1.4 for the nonequivariant analogue of this fiber sequence.

Since there is an equivalence $\Omega S^{\rho+1} \simeq \Omega^\sigma S^{\rho+\sigma}$, it suffices to prove that there is a fiber sequence

$$(7.1) \quad S^{\rho+\sigma} \rightarrow \Omega S^{\rho+1}\langle \rho+1 \rangle \rightarrow \Omega S^{2\rho+1};$$

taking σ -loops produces the desired fiber sequence. The fiber sequence (7.1) can be obtained by taking vertical fibers in the following map of fiber sequences

$$\begin{array}{ccccc} S^\rho & \longrightarrow & \Omega S^{\rho+1} & \longrightarrow & \Omega S^{2\rho+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{C}P_{\mathbf{R}}^\infty & \xlongequal{\quad} & \mathbf{C}P_{\mathbf{R}}^\infty & \longrightarrow & * \end{array}$$

Here, the top horizontal fiber sequence is the EHP fiber sequence

$$S^\rho \rightarrow \Omega S^{\rho+1} \rightarrow \Omega S^{2\rho+1}.$$

To identify the fibers, note that there is the Hopf fiber sequence

$$S^{\rho+\sigma} \xrightarrow{\tilde{\eta}} S^\rho \rightarrow \mathbf{C}P_{\mathbf{R}}^\infty.$$

The fiber of the middle vertical map is $\Omega S^{\rho+1}\langle \rho+1 \rangle$ via the definition of $S^{\rho+1}\langle \rho+1 \rangle$ as the homotopy fiber of the map $S^{\rho+1} \rightarrow \mathbf{B}\mathbf{C}P_{\mathbf{R}}^\infty$.

It remains to show that the map $\Omega^\rho S^{2\rho+1} \rightarrow \mathrm{BGL}_1(T(1)_{\mathbf{R}})$ detects an indecomposable element of $\pi_\rho T(1)_{\mathbf{R}}$. Indecomposability in $\pi_\rho T(1)_{\mathbf{R}} \cong \pi_\rho \mathbf{B}P_{\mathbf{R}}$ is the same as not being divisible by 2, so we just need to show that the dotted map in the following diagram does not exist:

$$\begin{array}{ccc} S^{\rho+1} & & \\ E^2 \downarrow & \searrow 2 & \\ \Omega^\rho S^{2\rho+1} & \longrightarrow & S^{\rho+1} \\ \downarrow & \swarrow \text{dotted} & \\ \mathrm{BGL}_1(T(1)_{\mathbf{R}}) & & \end{array}$$

If this factorization existed, there would be an orientation $\underline{\mathbf{H}\mathbf{Z}} \rightarrow T(1)_{\mathbf{R}}$, which is absurd. \square

We now explain why we do not know how to prove the equivariant analogue of Corollary B at higher heights. One could propose an equivariant analogue of Conjecture D, and such a conjecture would obviously be closely tied with the existence of some equivariant analogue of the work of Cohen-Moore-Neisendorfer. We do not know if any such result exists, but it would certainly be extremely interesting.

Suppose that one wanted to prove a result like Corollary B, stating that the equivariant analogues of Conjecture D and Conjecture E imply that there is a map $\Omega^\rho S^{2^n \rho+1} \rightarrow \mathrm{BGL}_1(T(n)_{\mathbf{R}})$ detecting an indecomposable in $\pi_{(2^n-1)\rho} T(n)_{\mathbf{R}}$

on the bottom cell, whose Thom spectrum is $\mathrm{BP}\langle n-1\rangle_{\mathbf{R}}$. One could then try to run the same proof as in the nonequivariant case by constructing a map from the fiber of a charming map $\Omega^\rho S^{2^n \rho+1} \rightarrow S^{(2^n-1)\rho+1}$ to $\mathrm{BGL}_1(T(n-1)_{\mathbf{R}})$, but the issue comes in replicating Step 1 of Section 5.2: there is *no* analogue of Lemma 3.1.16, since the equivariant element $\tilde{\sigma}_n \in \pi_* T(n)$ is neither torsion nor nilpotent. See Warning 7.1.4. This is intimately tied with the failure of an analogue of the nilpotence theorem in the equivariant setting. In future work, we shall describe a related project connecting the $T(n)$ spectra to the Andrews-Gheorghe-Miller w_n -periodicity in \mathbf{C} -motivic homotopy theory (see [AM17, Ghe17, Kra17]).

However, since there is a map $\Omega^\lambda S^{\lambda+1}\langle \lambda+1 \rangle \rightarrow \mathrm{BGL}_1(\mathbb{S})$ as in Proposition 7.2.2, there may nevertheless be a way to construct a suitable map from the fiber of a charming map $\Omega^\rho S^{2^n \rho+1} \rightarrow S^{(2^n-1)\rho+1}$ to $\mathrm{BGL}_1(T(n-1)_{\mathbf{R}})$. Such a construction would presumably provide a more elegant construction of the nonequivariant map used in the proof of Theorem A.

8. Future directions

In this section, we suggest some directions for future investigation. This is certainly not an exhaustive list; there are numerous questions we do not know how to address that are scattered all over this document, but we have tried to condense some of them into the list below. We have tried to order the questions in order of our interest in them. We have partial progress on many of these questions.

- (a) Some obvious avenues for future work are the conjectures studied in this article: Conjectures D, E, F, and 3.1.7. Can the \mathbf{E}_3 -assumption in the statement of Theorem C be removed?
- (b) One of the Main GoalsTM of this project is to rephrase the proof of the nilpotence theorem from [DHS88, HS98]. As mentioned in Remark 2.2.3, the Hopkins-Mahowald theorem for \mathbf{HF}_p immediately implies the nilpotence theorem for simple p -torsion classes in the homotopy of a homotopy commutative ring spectrum (see also [Hop84]). We will expand on the relation between the results of this article and the nilpotence theorem in forthcoming work; see Remark 5.4.7 for a sketch.

From this point of view, Theorem A is very interesting: it connects torsion in the unstable homotopy groups of spheres (via Cohen-Moore-Neisendorfer) to nilpotence in the stable homotopy groups of spheres. We are not sure how to do so, but could the Cohen-Moore-Neisendorfer bound for the exponents of unstable homotopy groups of spheres be used to obtain bounds for the nilpotence exponent of the stable homotopy groups of spheres?

- (c) It is extremely interesting to contemplate the interaction between unstable homotopy theory and chromatic homotopy theory apparent in this article. Connections between unstable homotopy theory and the chromatic picture have appeared elsewhere in the literature (e.g., in [AM99, Aro98, Mah82, MT94]), but their relationship to the content of this project is not clear to me. It would be interesting to have this clarified. One naïve hope is that such connection could stem from a construction of a charming map (such as the Cohen-Moore-Neisendorfer map) via Weiss calculus.
- (d) Let R denote \mathbb{S} or A . The map $R \rightarrow \Theta(R)$ is an equivalence in dimensions $< |\sigma_n|$. Moreover, the $\Theta(R)$ -based Adams-Novikov spectral sequence has a vanishing line of slope $1/|\sigma_n|$ (see [Mah81a] for the case $R = A$). Can another proof of this vanishing line be given using general arguments involving Thom spectra? We have some results in this direction which we shall address in future work.
- (e) The unit maps from each of the Thom spectra on the second line of Table 1 to the corresponding designer spectrum on the third line are surjective on homotopy. In the case of tmf , this requires some computational effort to prove, and has been completed in [Dev19b]. This behavior is rather unexpected: in general, the unit map from a structured ring to some structured quotient will not be surjective on homotopy. Is there a conceptual reason for this surjectivity?
- (f) In [BBB⁺19], the tmf -resolution of a certain type 2 spectrum Z is studied. Mahowald uses the Thom spectrum A to study the bo-resolution of the sphere in [Mah81a], so perhaps the spectrum B could be used to study

the tmf-resolution of Z . This is work in progress. See also Corollary 6.2.11 and the discussion preceding it.

- (g) Is there an equivariant analogue of Theorem A at higher heights and other primes? Currently, we have such an analogue at height 1 and at $p = 2$; see Section 7.
- (h) The Hopkins-Mahowald theorem may be used to define Brown-Gitler spectra. Theorem A produces “relative” Brown-Gitler spectra for $\mathrm{BP}\langle n \rangle$, bo , and tmf . In future work, we will study these spectra and show how they relate to the Davis-Mahowald *non*-splitting of $\mathrm{tmf} \wedge \mathrm{tmf}$ as a wedge of shifts of bo -Brown-Gitler spectra smashed with tmf from [DM10].
- (i) The story outlined in the introduction above could fit into a general framework of “fp-Mahowald spectra” (for “finitely presented Mahowald spectrum”, inspired by [MR99]), of which A , B , $T(n)$, and $y(n)$ would be examples. One might then hope for a generalization of Theorem A which relates fp-Mahowald spectra to fp-spectra. It would also be interesting to prove an analogue of Mahowald-Rezk duality for fp-Mahowald spectra which recovers their duality for fp-spectra upon taking Thom spectra as above.
- (j) One potential approach to the question about surjectivity raised above is as follows. The surjectivity claim at height 0 is the (trivial) statement that the unit map $\mathbb{S} \rightarrow \mathrm{HZ}$ is surjective on homotopy. The Kahn-Priddy theorem, stating that the transfer $\lambda : \Sigma^\infty \mathbf{R}P^\infty \rightarrow \mathbb{S}$ is surjective on $\pi_{* \geq 1}$, can be interpreted as stating that $\pi_* \Sigma^\infty \mathbf{R}P^\infty$ contains those elements of $\pi_* \mathbb{S}$ which are *not* detected by HZ . One is then led to wonder: for each of the Thom spectra R on the second line of Table 1, is there a spectrum P along with a map $\lambda_R : P \rightarrow R$ such that each $x \in \pi_* R$ in the kernel of the map $R \rightarrow \Theta(R)$ lifts along λ_R to $\pi_* P$? (The map $R \rightarrow \Theta(R)$ is an equivalence in dimensions $< |\sigma_n|$ (if R is of height n), so P would have bottom cell in dimension $|\sigma_n|$.)

Since $\Sigma^\infty \mathbf{R}P^\infty \simeq \Sigma^{-1} \mathrm{Sym}^2(\mathbb{S})/\mathbb{S}$, the existence of such a result is very closely tied to an analogue of the Whitehead conjecture (see [Kuh82]; the Whitehead conjecture implies the Kahn-Priddy theorem). In particular, one might expect the answer to the question posed above to admit some interaction with Goodwillie calculus.

- (k) Let $p \geq 5$. Is there a p -primary analogue of B which would provide a Thom spectrum construction (via Table 1) of the conjectural spectrum eo_{p-1} ? Such a spectrum would be the Thom spectrum of a p -complete spherical fibration over a p -local space built via $p-1$ fiber sequences from the loop spaces $\Omega S^{2k(p-1)+1}$ for $2 \leq k \leq p$.
- (l) The spectra $T(n)$ and $y(n)$ have algebro-geometric interpretations: the stack $\mathcal{M}_{T(n)}$ associated (see [DFHH14, Chapter 9]; this stack is well-defined since $T(n)$ is homotopy commutative) to $T(n)$ classifies p -typical formal groups with a coordinate up to degree $p^{n+1} - 1$, while $y(n)$ is the closed substack of $\mathcal{M}_{T(n)}$ defined by the vanishing locus of p, v_1, \dots, v_{n-1} . What are the moduli problems classified by A and B ? We do not know if this question even makes sense at $p = 2$, since A and B are *a priori* only \mathbf{E}_1 -rings. Nonetheless, in [Dev19a], we provide a description of $\mathrm{tmf} \wedge A$ in terms of the Hodge filtration of the universal elliptic curve (even at $p = 2$);

we also showed that $(\mathrm{tmf} \wedge A)[x_2]$ admits an \mathbf{E}_2 -algebra structure, where $|x_2| = 2$.

- (m) Theorem A shows that the Hopkins-Mahowald theorem for HZ_p can be generalized to describe forms of $\mathrm{BP}\langle n \rangle$; at least for small n , these spectra have associated algebro-geometric interpretations (see [DFHH14, Chapter 9]). What is the algebro-geometric interpretation of Theorem A?

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