STABLE SPLITTINGS OF CLASSIFYING SPACES OF COMPACT LIE GROUPS

1. Introduction

The goal of this talk is to provide a proof of the following result, following [Sna79].

Theorem 1.1. *Let G_n denote one of the groups $U(n), Sp(n)$, or $O(2n)$. Then:*

1. *There are stable equivalences $\nu_n : \Sigma^\infty_+ BG_n \to \bigvee_{t \le n} \Sigma^\infty BG_t/BG_{t-1}$ which are compatible as n varies.*
2. *There is a stable equivalence $\Sigma^\infty_+ BG_\infty \simeq \bigvee_t \Sigma^\infty BG_t/BG_{t-1}$.*

Let us briefly mention how this is relevant to the immersion conjecture. In the previous lecture, Jeremy constructed the space BO/I_n. The immersion conjecture is equivalent to the existence of a homotopy lift in the following diagram:

$$\begin{tikzcd} & BO(n-\alpha(n)) \arrow[d] \\ BO/I_n \arrow[r] \arrow[ur, dashed] & BO. \end{tikzcd}$$

While we will not go into details about the use of Theorem 1.1 in the proof of this result, we shall prove the following proposition which will be relevant for future lectures.

Proposition 1.2. *There is a stable map $\Sigma^\infty BO/I_n \to \Sigma^\infty BO(n-\alpha(n))$ lifting the map $\Sigma^\infty BO/I_n \to \Sigma^\infty BO$.*

Proof. Let us first prove that for every integer m, the space $BO/BO(m)$ has the $(2m+1)$-dimensional homotopy type of a product of mod 2 Eilenberg-MacLane spaces $K(V,t)$ for $t > m$ and V an $\mathbf{F}_2$-vector space. Because $BO/BO(m)$ is m-connected, this is a claim about the stable range. We know that $BO(m+1)/BO(m)$ is homotopy equivalent to $\Sigma^{m+1}MO$ through dimension $2m+1$. It follows that we get a splitting of $\Sigma^\infty BO/BO(m)$ as $\Sigma^{m+1}MO \vee \Sigma^\infty BO/BO(m+1)$ through dimension $2m+1$. The claim now follows by recalling that MO is a wedge of Eilenberg-MacLane spectra and continuing to split $\Sigma^\infty BO/BO(m+1)$.

We now return to the proof of the proposition. The obstruction constructing such a lift is the composite $\Sigma^\infty BO/I_n \to \Sigma^\infty BO \to \Sigma^\infty BO/BO(n-\alpha(n))$. We know that $BO/BO(n-\alpha(n))$ has the $(2(n-\alpha(n))+1)$-dimensional homotopy type of a product of mod 2 Eilenberg-MacLane spaces. Since BO/I_n has the homotopy type of an n-dimensional complex, and $n \le 2(n-\alpha(n))+1$, this lifting problem is entirely a cohomological obstruction. We know, however, that all cohomological obstructions vanish because of Massey's result stating that the normal Stiefel-Whitney classes $\widetilde{w}_i(M^n)$ of a compact n-manifold M^n vanish for $i > n-\alpha(n)$. □

2. Preliminaries: the transfer

Before proceeding, we discuss the construction of the stable transfer map, which is originally due to Becker and Gottlieb. The following description of the transfer map is discussed in [Mil89, Lecture 23].

Construction 2.1. Let X be a locally compact Hausdorff space, and let $U \subseteq X$ be an open subset. Then, there is a Pontryagin-Thom map $X_+ \to X_+/(X_+ - U) \simeq U_+$.

Suppose now that $F \to E \xrightarrow{p} B$ is a smooth fiber bundle of compact manifolds. Then, the Whitney embedding theorem allows us to embed $i : E \hookrightarrow \mathbf{R}^n$ for some $n \geq 0$, so we obtain an inclusion $E \to B \times \mathbf{R}^n$. Since this inclusion is not necessarily open, we will consider the normal bundle ν_E of this inclusion. Then, we have a tubular neighborhood N of E inside $B \times \mathbf{R}^n$, which, by definition, is open. It follows from the discussion in the previous paragraph that there is a map $(B \times \mathbf{R}^n)_+ \to N_+$. But we have identifications $N_+ \cong \overline{N}/\partial N = E^{\nu_E}$ and $(B \times \mathbf{R}^n)_+ \cong B^{n\epsilon} = \Sigma^n B_+$ of Thom spaces, so the Pontryagin-Thom construction yields a map $\Sigma^n B_+ \to E^{\nu_E}$.

We claim that $\Sigma^\infty_+ E^{\nu_E} = \Sigma^\infty_+ E^{-\tau(p)}$, where $\tau(p)$ is the bundle over E of tangent vectors along the fibers. To prove this, we note that

$$\nu_E + p^*\tau_B + \tau(p) = \nu_E + \tau_E = i^*\tau_{B\times\mathbf{R}^n} = p^*\tau_B + n\epsilon_E.$$

It follows that $\nu_E + \tau(p) = n\epsilon_E$, i.e., $\nu_E = n\epsilon_E - \tau(p)$, which proves the desired result. Stably, we therefore obtain a map $\Sigma^\infty_+ B \to \Sigma^\infty_+ E^{-\tau(p)}$.

The inclusion $\nu_E \hookrightarrow n\epsilon_E$ yields the map $\Sigma^n B_+ \to E^{\nu_E} \to \Sigma^n E_+$. It follows that the transfer map can be viewed as a stable map $\Sigma^\infty_+ B \to \Sigma^\infty_+ E$. The composite map $\Sigma^\infty_+ B \to \Sigma^\infty_+ E \xrightarrow{p} \Sigma^\infty_+ B$ induces multiplication by the Euler characteristic $\chi(p)$ of the fiber on ordinary cohomology (with arbitrary coefficients).

Remark 2.2. Let us briefly mention why this map is called the transfer. If $p : E \to B$ is a finite covering space (explored in more detail in Construction 4.1 below), then the transfer induces a map $p_* : \mathrm{H}^*(E) \to \mathrm{H}^*(B)$, which can be described as follows in the case of de Rham cohomology, for instance. Let $[\omega] \in \mathrm{H}^k(E)$, and let ω be a k-form representing this cohomology class. If $x \in E$ and $y \in p^{-1}(x)$, then we can identify $T^*_x E \cong T^*_y B$. We then have $(p_*\omega)(x) = \sum_{y\in p^{-1}(x)} \omega(y)$, which is precisely integration over the fiber. A similar example works for complex K-theory: if ξ is a vector bundle over E, then $p_*\xi$ is the bundle over B whose fibers are $(p_*\xi)_x = \bigoplus_{y\in p^{-1}(x)} \xi_y$.

3. The proof of Theorem 1.1

Let us now proceed to discuss the proof of Theorem 1.1. Our first order of business will be to define the map ν_n appearing in the statement of the theorem. As in Theorem 1.1, let G_n denote $U(n)$, $Sp(n)$, or $O(2n)$. Define subgroups H_n by the wreath product $G_1 \wr \Sigma_n$. We can therefore describe BH_n as $E\Sigma_n \times_{\Sigma_n} BG_1^n$. We then have the following result, a proof of which can be found in [CMT79]:

Lemma 3.1. *There is a stable splitting*

$$\Sigma^\infty_+ BH_n \simeq \bigvee_{0\leq j\leq n} \Sigma^\infty BH_j/BH_{j-1}.$$

Proof sketch. For $0 \leq k \leq n$, there is a H_n/H_k-bundle $BH_k \to BH_n$, so the transfer gives a stable map $\Sigma^\infty_+ BH_n \to \Sigma^\infty_+ BH_k$. The composite down to $\Sigma^\infty BH_k/BH_{k-1}$ gives a map $\Sigma^\infty_+ BH_n \to \bigvee_{0\leq j\leq n} \Sigma^\infty BH_j/BH_{j-1}$. The same argument as the one used below to prove the first part of Theorem 1.1 will show that this map is an equivalence. □

There is a G_n/H_n-bundle $\pi_n : BH_n \to BG_n$, so we obtain a stable transfer map $\tau_n : \Sigma^\infty_+ BG_n \to \Sigma^\infty_+ BH_n$. We shall implicitly use the following proposition below; its proof will be postponed to the end of the lecture.

Proposition 3.2. *There is a homotopy commutative diagram*

$$\begin{tikzcd} BG_n \arrow[r, "\tau_n"] \arrow[d] & QBH_n \arrow[d] \\ BG_{n+1} \arrow[r, "\tau_n"'] & QBH_{n+1}. \end{tikzcd}$$

We now turn to the proof of the first part of Theorem 1.1. There is a map

$$\nu_n : \Sigma^\infty_+ BG_n \xrightarrow{\tau_n} \Sigma^\infty_+ BH_n \xrightarrow{\sim} \bigvee_{0\leq t\leq n} \Sigma^\infty BH_t/BH_{t-1} \xrightarrow{\bigvee \pi_t/\pi_{t-1}} \bigvee_{0\leq t\leq n} \Sigma^\infty BG_t/BG_{t-1}.$$

By Whitehead's theorem, statement (1) of Theorem 1.1 will follow if we can prove that ν_n induces an isomorphism in homology. We shall prove this by induction on n. The claim is obvious when $n = 1$, so we only need to establish the inductive step. Assume that ν_{n-1} is a homotopy equivalence. Then, the composite

$$\Sigma^\infty_+ BG_n \xrightarrow{\nu_n} \bigvee_{0\leq t\leq n} \Sigma^\infty BG_t/BG_{t-1} \to \Sigma^\infty BG_n/BG_{n-1}$$

is homotopic to the composite

$$\Sigma^\infty_+ BG_n \to \Sigma^\infty BG_n/BG_{n-1} \xrightarrow{\tau_n/\tau_{n-1}} \Sigma^\infty BH_n/BH_{n-1} \xrightarrow{\pi_n/\pi_{n-1}} \Sigma^\infty BG_n/BG_{n-1},$$

where the existence of the map τ_n/τ_{n-1} is deduced from Proposition 3.2. However, the map $\pi_n \circ \tau_n$ induces the identity in homology, and the map $\mathrm{H}_*(BG_n) \to \mathrm{H}_*(BG_n/BG_{n-1})$ is surjective with kernel given by $\mathrm{H}_*(BG_{n-1})$, so we conclude by induction that ν_n induces a homology isomorphism.

The proof of statement (2) of Theorem 1.1 is a little more subtle. We begin by defining a few maps. When G_n is as in Theorem 1.1, we know that BG_∞ is an infinite loop space. It follows that the inclusion of G_1 into G_∞ induces a map $\lambda : QBG_1 \to BG_\infty$. There is also a map going the other way, as we now explain.

By Proposition 3.2, we obtain a map $\tau : BG_\infty \to QBH_\infty$. There is a map $i : BH_\infty \to QBG_1$, defined as follows. Recall that QX (for simply-connected X, at least) has an operadic filtration by subspaces C_nX. Choosing the Barratt-Eccles model for the $\mathbf{E}_\infty$-operad, we find that there is a map $i_n : E\Sigma_n \times_{\Sigma_n} X^n \to C_nX$. In particular, since $BH_n = E\Sigma_n \times_{\Sigma_n} (BG_1)^n$, we obtain maps $i_n : BH_n \to C_nBG_1$. These maps are compatible, in the sense that $i_n|_{BH_{n-1}} = i_{n-1}$. It follows that $i = \operatorname{colim} i_n$ defines a map $BH_\infty \to QBG_1$. The composite

$$BG_\infty \xrightarrow{\tau} QBH_\infty \xrightarrow{i} QQBG_1 \to QBG_1$$

is the map hinted to in the previous paragraph. We shall denote this map by τ_G.

We then have:

Proposition 3.3. *The composite $\lambda \circ \tau_G$ is a homotopy equivalence. Moreover, if R is any torsion-free ring such that $\mathrm{H}_*(BG_\infty; R)$ is torsion-free, then the composite $BH_\infty \xrightarrow{i} QBG_1 \xrightarrow{\lambda} BG_\infty$ induces the same map as $\pi_\infty : BH_\infty \to BG_\infty$ in R-homology.*

Remark 3.4. This implies, for example, that BU splits off of $Q\mathbf{C}P^\infty$. This result was originally proved by Segal in [Seg73] using Brauer induction, and is a space-level statement of the splitting principle for vector bundles. It can also be used to provide a slick proof of the complex Adams conjecture. More is true: this splitting result is true C_2-equivariantly, for the complex conjugation actions on BU and $Q\mathbf{C}P^\infty$. It is also true in the $\mathbf{C}$-motivic category, although we do not know of a reference for these latter two claims.

The equivalence in statement (2) of Theorem 1.1 comes from the following two maps:

$$f : \bigvee_{t\geq 0} \Sigma^\infty BG_t/BG_{t-1} \xrightarrow{\bigvee \tau_t/\tau_{t-1}} \bigvee_{t\geq 0} \Sigma^\infty BH_t/BH_{t-1} \xrightarrow{\sim} \Sigma^\infty_+ BH_\infty \xrightarrow{i} \Sigma^\infty_+ QBG_1 \xrightarrow{\lambda} \Sigma^\infty_+ BG_\infty$$

and

$$g : \Sigma^\infty BG_\infty \xrightarrow{\tau_\infty} \Sigma^\infty_+ BH_\infty \xrightarrow{\sim} \bigvee_{t\geq 0} \Sigma^\infty BH_t/BH_{t-1} \xrightarrow{\bigvee \pi_t/\pi_{t-1}} \bigvee_{t\geq 0} \Sigma^\infty BG_t/BG_{t-1}.$$

Here, we have used Lemma 3.1. It is easy to use Proposition 3.3 to conclude.

It remains to provide proofs of Proposition 3.2 and Proposition 3.3.

4. The proof of Proposition 3.3

We first show that $\lambda \circ \tau_G$ is a homotopy equivalence if the second statement of Proposition 3.3 is true. By Whitehead's theorem, we know that $\lambda \circ \tau_G$ is a homotopy equivalence if it induces an isomorphism in integral homology (at least when $G_n = U(n), Sp(n)$; if $G_n = O(2n)$, then we need to show it induces an isomorphism in homology with $\mathbf{Z}[1/2]$ and $\mathbf{Z}/2$-coefficients). The map $\pi_n \circ \tau_n$ induces multiplication by the Euler characteristic $\chi(G_n/H_n)$ in homology; but $\chi(G_n/H_n) = 1$, so we find that $\pi_n \circ \tau_n$ induces an isomorphism in homology. The second part of Proposition 3.3 now implies that $\lambda \circ \tau_G$ also induces the identity on integral (and $\mathbf{Z}[1/2]$) homology. The proof that $\lambda \circ \tau_G$ induces the identity on $\mathbf{Z}/2$-homology when $G = O$ is a bit more subtle, so we will not address it here.

It therefore suffices to prove the second statement of Proposition 3.3. It clearly suffices to show that the map $BH_n \xrightarrow{i_n} QBG_1 \xrightarrow{\lambda} BG_\infty$ induces the same map in homology as π_n for each finite n. Before proceeding, we need to recall the construction of the Kahn-Priddy transfer (from [KP72]). This is an elaboration of Construction 2.1 in the case of a finite covering.

Construction 4.1. Let $\pi : E \to B$ be a n-fold covering with B connected, locally path-connected, and semi-locally simply connected (these conditions are to ensure the existence of a universal cover). Suppose that $\pi_1(E) = H$ and $\pi_1(B) = H$, so that $|G/H| = n$. If X is the universal cover of B on which G acts freely on the right, then π is homeomorphic to the cover $p : X/H \to X/G$. The transitive action of G on $G/H = \{g_1H, \cdots, g_nH\}$ defines a homomorphism $G \to \Sigma_n$, and hence a map $\rho : EG \to E\Sigma_n$. This allows us to define a map

$$B = X/G = X \times_G EG \to (X/H)^n \times_{\Sigma_n} E\Sigma_n = E^n \times_{\Sigma_n} E\Sigma_n$$

by sending a pair $(x, w) \in X \times_G EG$ to $(\overline{xg_1}, \cdots, \overline{xg_n}, \rho(w))$. Here, $\overline{xg_i}$ is the image of $xg_i \in X$ under the quotient $X \to X/H$.

This map is related to the transfer from Construction 2.1 as follows. There is a map $E^n \times_{\Sigma_n} E\Sigma_n \to QE$, and the composite $B \to E^n \times_{\Sigma_n} E\Sigma_n \to QE$ is precisely the Becker-Gottlieb transfer.

In the general situation above, suppose we have a map $E \to X$, where X is an infinite loop space. Then, the composite

$$B \to E^n \times_{\Sigma_n} E\Sigma_n \to X^n \times_{\Sigma_n} E\Sigma_n \to X,$$

where the final map uses the infinite loop structure on X, is called the Kahn-Priddy transfer of the map $E \to X$ with respect to the covering π. (Note that we do not need the entire infinite loop structure on X to construct such a map: it suffices to have a H^n_∞-structure.)

Returning to the situation at hand, we learn that $\lambda \circ i_n : BH_n \to BG_\infty$ is exactly the Kahn-Priddy transfer of the composite

$$BG_1 \times BH_{n-1} \to BG_1 \to BG_\infty$$

with respect to the n-fold covering $BG_1 \times BH_{n-1} \to BH_n$. Here, $BG_1 \to BG_\infty$ classifies the virtual vector bundle $\xi - \dim\xi$, where ξ is the universal complex/quaternionic line bundle or the universal 2-plane real vector bundle over BG_1.

Let τ_ξ denote the transfer of the map $BG_1 \times BH_{n-1} \to BG_1 \xrightarrow{\xi} BG_\infty$ with respect to the covering $BG_1 \times BH_{n-1} \to BH_n$, and similarly for $\tau_{\dim\xi}$. Concretely, the above discussion implies that if $\iota : BG_\infty \to BG_\infty$ is the map representing negation in reduced K-theory, then the map induced by $\lambda \circ i_n$ on homology sends:

$$\widetilde{\mathrm{H}}_*(BH_n) \ni x \mapsto \sum (\tau_\xi)_*(x_i)(\iota \circ \tau_{\dim\xi})_*(x_j) \in \widetilde{\mathrm{H}}_*(BG_\infty)$$

where the diagonal of $\widetilde{\mathrm{H}}_*(BH_n)$ sends x to $\sum x_i \otimes x_j$. This is the same as the map induced by π_n on homology if $\tau_{\dim\xi}$ kills $\widetilde{\mathrm{H}}_*(BH_n)$ and τ_ξ is the same as π_n.

We leave it as an easy exercise to the reader to use the triviality of $\dim\xi$ over BG_1 to prove that $\tau_{\dim\xi}$ is the composite of the map $BH_n \to B\Sigma_n$ contracting G_1 to a point with the map $B\Sigma_n \to BG_n \to BG_\infty$. In particular, $(\tau_{\dim\xi})_*$ factors through $\widetilde{\mathrm{H}}_*(B\Sigma_n)$ — but this is torsion, so $(\tau_{\dim\xi})_*$ must be zero, as desired.

It remains to show that τ_ξ is the same as π_n. The map τ_ξ is given by the composite

$$BH_n \to E\Sigma_n \times_{\Sigma_n} (BG_1 \times BH_{n-1})^n \to E\Sigma_n \times_{\Sigma_n} (BG_1)^n \to E\Sigma_n \times_{\Sigma_n} (BG_\infty)^n \to BG_\infty.$$

There is an equivalence $BH_n = E\Sigma_n \times_{\Sigma_n} (BG_1)^n$, and the composite to $E\Sigma_n \times_{\Sigma_n} (BG_1)^n$ is the identity. The desired claim reduces to proving that there is a commutative diagram

$$\begin{tikzcd} BH_n \arrow[r] \arrow[d] & BG_n \arrow[d] \\ E\Sigma_n \times_{\Sigma_n} (BG_\infty)^n \arrow[r] & BG_\infty. \end{tikzcd}$$

A proof of this appears in [May77, Chapter VIII].

In order to finish the case when $G_n = O(2n)$, we need to also address the case of homology with $\mathbf{Z}/2$-coefficients. Although this is the case we care about, it requires a lot more care, so we will omit the argument.

5. The proof of Proposition 3.2

In order to prove Proposition 3.2, we need the following technical result from differential topology.

Lemma 5.1. *Let G be a compact manifold. Let F and Y be compact G-manifolds with a free action of G such that F has no boundary (i.e., F is closed), and let $F \to Y \times_G F = E \xrightarrow{\pi} Y/G = X$ be a differentiable fiber bundle. Suppose $F_1 \subseteq F$ is a G-submanifold, with equivariant tubular neighborhood N. Suppose ρ is an equivariant vector field on F such that on ∂N, it is homotopic through nowhere zero vector fields to an outward normal and satisfies $|\rho(x)| = 1$ for $x \notin N$. Let π' denote the differentiable bundle $F_1 \to Y \times_G F_1 \xrightarrow{\pi'} X$. Then, there is a homotopy commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\tau(\pi')"] \arrow[dr, "\tau(\pi)"'] & Q(Y \times_G F_1) \arrow[d] \\ & Q(E). \end{tikzcd}$$

Proof sketch. The fiber bundle $N \to Y \times_G N \xrightarrow{\pi''} X$ is fiber homotopy equivalent to $F_1 \to Y \times_G F_1 \xrightarrow{\pi'} X$, so the composite $X \xrightarrow{\tau(\pi')} Q(Y \times_G F_1) \to Q(E)$ in the diagram above is homotopic to the composite $X \xrightarrow{\tau''} Q(Y \times_G N) \to Q(E)$. It therefore suffices to show that the latter map is homotopy equivalent to $\tau(\pi)$.

The construction of the map $\tau(\pi)$ can be rephrased as follows. Let V be a finite-dimensional G-vector space for which there is a G-equivariant embedding $F \subseteq V$, and let N_1 be the normal bundle. Then, if $f : N_1 \to V$ is the choice of a tubular neighborhood for F, we have the Pontryagin-Thom map $\gamma : \mathrm{Th}(V) = S^V \to V/(V - f(N_i)) = \mathrm{Th}(N_1)$. There is an inclusion $N_1 \subseteq N_1 \oplus TF = F \times V$, so this can be further composed with the map $\mathrm{Th}(N_1) \to \mathrm{Th}(N_1 \oplus TF) = F_+ \wedge \mathrm{Th}(V)$. This produces a map $\mathrm{Th}(V) \to F_+ \wedge \mathrm{Th}(V)$; the transfer $\tau(\pi)$ is obtained by taking the product of this map with the identity of Y, quotienting out by the G-action, and then stabilizing.

We can define a family of maps $i_s : \mathrm{Th}(N_1) \to F_+ \wedge \mathrm{Th}(V)$ as follows:

$$i_s(v) = \begin{cases} \frac{1}{1 - s|\rho(x)|}(v, s\rho(x)) & \text{if } s|\rho(x)| < 1 \\ \infty & \text{else.} \end{cases}$$

Clearly i_0 is the map induced by the inclusion $N_1 \subseteq F \times V$, so i_0 composed with the Pontryagin-Thom map γ is the transfer map $\tau(\pi)$. However, the composite map $i_1 \circ \gamma$ is precisely the map used to define $\tau(\pi'')$ when ρ is an outward normal on ∂N. □

We need to specialize a little bit in order to be able to apply Lemma 5.1 in our situation. Let ρ be a G-equivariant vector field on F which is non-degenerate on its singular set F_1 (this means that the Jacobian does not have vanishing determinant at any critical point of ρ). Assume that F_1 is connected with transitive G-action. Let $\epsilon > 0$, and define $N_\epsilon = \{f \in F : |\rho(f)| \leq \epsilon\}$. Because ρ is non-degenerate, we can choose ϵ small enough so that ρ has a nonzero component on ∂N near $f_0 \in F_1$ in the normal direction to ∂N. By the transitivity of the G-action on F_1 and the equivariance of ρ, we can do this compatibly near all $f_0 \in F_1$. Then, N_ϵ is an equivariant tubular neighborhood of F_1 in F. Moreover, ρ has a nonzero (outward, if we wish) normal component at each point of ∂N. If ρ has this property, then ρ is homotopic on ∂N to an outward normal vector field by linearly shrinking the tangential component of ρ to zero. We can therefore apply Lemma 5.1 to such a situation.

Using Lemma 5.1, we have:

Proof. There is a pullback diagram:

$$\begin{tikzcd} G_n/H_n \arrow[r] \arrow[d, equal] & EG_n \times_{G_{n-1}} G_n/H_n \arrow[r, "\pi'_n"] \arrow[d, "j'"] & BG_{n-1} \arrow[d] \\ G_n/H_n \arrow[r] & EG_n \times_{G_n} G_n/H_n = BH_n \arrow[r, "\pi_n"'] & BG_n. \end{tikzcd}$$

This gives rise to a commutative diagram

$$\begin{tikzcd} BG_{n-1} \arrow[r] \arrow[d, "\tau'_n"] & BG_n \arrow[d, "\tau_n"] \\ Q\left(EG_n \times_{G_{n-1}} G_n/H_n\right) \arrow[r, "Qj'"'] & QBH_n. \end{tikzcd}$$

We therefore need to understand the map τ'_n. The discussion above shows that Lemma 5.1 can be applied to the situation where ρ is a G-equivariant vector field on F which is non-degenerate on its singular set F_1, the latter of which is connected with transitive G-action. Suppose that we can construct G_{n-1}-equivariant vector fields on G_n/H_n which have singular sets G_{n-1}/H_{n-1} on which they are non-degenerate (this will be done below). It follows from Lemma 5.1 applied to the bundle π'_n that there is a commutative diagram

$$\begin{tikzcd} BG_{n-1} \arrow[r, "\tau_{n-1}"] \arrow[dr, "\tau'_n"'] & QBH_{n-1} \arrow[d] \\ & Q\left(EG_n \times_{G_{n-1}} G_n/H_n\right). \end{tikzcd}$$

Gluing these two commutative diagrams proves the desired result. □

It remains to construct the vector fields used in the above proof. Let G be a compact Lie group as above, and let $\mathfrak{g}$ be its associated Lie algebra. For $v \in \mathfrak{g}$ we can define a vector field ϕ_v on G as follows: $\phi_v(g) = (Dr_g)(v) \in T_zG$, where r_g denotes right translation by $g \in G$ and D denotes the derivative of r_g. It is not hard to prove the following lemma.

Lemma 5.2. *Let ℓ_g denote left translation by $g \in G$. If $h \in G$ is in the centralizer of $\exp(v) \in G$, then $D(\ell_h)(\phi_v(g)) = \phi_v(hg)$. If $k \in G$ is any element, then $D(r_k)(\phi_v(g)) = \phi_v(gk)$.*

Now, suppose that $H \subseteq G$ is a closed subgroup of G, and suppose $0 \neq v \in \mathfrak{g}$. Then, we can find a vector field ρ_v on G/H such that $\rho_v(gH) = 0$ if and only if $g^{-1}\exp(v)g \in H$, and $D(\ell_k)(\rho_v(gH)) = \rho_v(kgH)$ if k is in the centralizer of $\exp(v)$. The idea is to take ρ_v to be the quotient of ϕ_v by the right H-action.

We now specialize even further to the special case when G is one of our groups G_n. We will only concentrate on the case $G_n = U(n)$ and $H_n = N_{U(n)}(T)$ for now. Let $v \in \mathfrak{u}(n)$ denote an element such that $w = \exp(v) = \begin{pmatrix} x & & & \\ & \ddots & & \\ & & x & \\ & & & y \end{pmatrix} \in T$ with $x \neq y$, so that the centralizer of w is $U(n-1) \times U(1)$. This implies that ρ_v is a left $U(n-1) \times U(1)$-equivariant vector field on $U(n)/N_{U(n)}(T)$. In order to conclude, we need to show that its singular set is equivariantly homeomorphic to $U(n-1)/N_{U(n-1)}(T)$.

We know that $\rho_v(gH) = 0$ if and only if $g^{-1}wg \in H$. If $g^{-1}wg \in H$, then there is a $\sigma \in \Sigma_n$ such that $\sigma^{-1}g^{-1}wg\sigma \in T$, so $g\sigma T\sigma^{-1}g^{-1}$ is a maximal torus which contains w. Now, it is a general fact that the identity component of the normalizer of an element g in a compact Lie group is the union of the maximal tori containing g. In particular, there is $b \in U(n-1) \times U(1)$ such that $bTb^{-1} = g\sigma T\sigma^{-1}g^{-1}$, so we conclude that $g \in (U(n-1) \times U(1))H$, so the singular set of ρ_v is contained in $U(n-1)/N_{U(n-1)}(T)$. Conversely, the same argument proves that every element of $U(n-1)/N_{U(n-1)}(T)$ is in the singular set of ρ_v.

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