Thursday Seminar
Sp 2022

Weiss Calculus & Derivative of the Identity Functor

§0. History + Motivation

Last time: [Gray 88]

$C(n) = \mathrm{hofib}(S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1})$ is a loop space.

Mahowald's program $(p=2)$

$$\begin{tikzcd}
① \quad S^1 \arrow[r, "E^2"] & \Omega^2 S^3 \arrow[r, "E^2"] & \Omega^4 S^5 \arrow[r, "E^2"] & \cdots \arrow[r] & QS^1 \\
C(1) \arrow[u] & \Omega^2 C(2) \arrow[u] & \Omega^4 C(3) \arrow[u] & &
\end{tikzcd}$$

$\rightsquigarrow E_1^{k,n} = \pi_{k+2n} C(n) \Rightarrow \pi^S(S^1)$

[Serre] Rationally ① is the constant filtration

[Mahowald 82] $p=2$, Y finite w/ self-map $v_1 : \Sigma^2 Y \to Y$.

$v_1^{-1}\pi_*(C(n); Y) \cong v_1^{-1}\pi_*^s(A_0; Y)$

↖ Mod 2 Moore space, $H^*(A_0) \cong A(0)$

So v_1^{-1}① coincides with the sseq. obtained by applying $v_1^{-1}\pi_*(-;Y)$ to the filtration

$$\Sigma^\infty \mathbb{RP}^2 \subset \Sigma^\infty \mathbb{RP}^4 \subset \cdots \subset \Sigma^\infty \mathbb{RP}^\infty.$$

[Thompson 90] $p > 2$.

[Mahowald-Thompson 96] "Iterate E^2" ⇝ v_2-periodic analogue.

$C(n) \to \Omega^2 C(n+1)$ is null. Instead:

$C^{(2)}(n) := \mathrm{hofib}\,(C(n) \to \Omega^4 C(n+1))$.

⇝ Filtration of the stable Moore space

$$\begin{tikzcd}
C(1) \arrow[r] & \Omega^4 C(2) \arrow[r] & \Omega^8 C(3) \arrow[r] & \cdots \arrow[r] & Q(A_0) & \textcircled{2} \\
C^{(2)}(1) \arrow[u] & \Omega^4 C^{(2)}(2) \arrow[u] & \Omega^8 C^{(2)}(3) \arrow[u] & & &
\end{tikzcd}$$

Observation: the first 8-cells of $C^{(2)}(n)$, $n \geq 2$ form a cplx A_1, $H^*(A_1) \cong A(1)$ as $A(1)$-modules.

Thm $v_2^{-1}\pi_*(C^{(2)}(n); M) \cong v_2^{-1}\pi_*^s(A_1; M)$, $n \geq 2$ for some type 2 M.

The filtration v_2^{-1} ② modulo $W(1)$:

$$(\Omega^4 W(2), W(1)) \to (\Omega^8 W(3), W(1)) \to \cdots \to (Q(A_0), W(1))$$

coincides with fitration of $\Sigma^\infty(\mathbb{RP}_3^\infty \cup \mathbb{CP}_5^\infty)$ by stunted projective spaces.

⚠ Hard to generalize to higher chromatic level without more tools.

[Arone - Mahowald 98]

Apply Goodwillie calculus to $id : Top_* \to Top_*$

- **Thm** $\forall$ prime p, X odd sphere,
  - $D_n(X) \simeq *$, $n \neq p^k$ for some k.
  - $H^*(\mathbb{D}_{p^k}(X) ; \mathbb{F}_p)$ is free over $A(k-1)$

    $\Rightarrow v_{k-1}^{-1} \pi_*(D_{p^j}(X)) = 0$, $j \geq k$
  - Goodwillie tower of id converges in v_{k-1}-periodic homotopy at X.

$\Rightarrow X \to P_{p^k}(X)$ is a v_j-periodic equiv for $0 \leq j \leq k$, $\forall k$.

[Arone 98] Use Weiss calculus to produce

$$\left\{\begin{array}{l} F_1(X) \to X \xrightarrow{w_1} \Omega^2 S^2 X \\ F_2(X) \to F_1(X) \xrightarrow{w_2} \Omega^4 F_1(S^2X) \\ \vdots \\ F_m(X) \to F_{m-1}(X) \xrightarrow{w_m} \Omega^{2m} F_{m-1}(S^2X) \end{array}\right.$$

and compute $D_n F_m(X)$

- **Thm** X odd sphere, p-local, $j \leq k$

  $F_m(X) \to P_{p^k} F_m(X)$ is a v_j-periodic equiv.

  When $m = p^k - 1$, $F_m(X) \simeq D_{p^k} F_m(X)$ is the infinite loop space of a type k spectrum (Mitchell spectrum).

§1. Calculus

$S^*F : \mathcal{J} \xrightarrow{V \mapsto S^V} Top_* \xrightarrow{F} Top_*$

Goodwillie	Weiss [$k=\mathbb{R}$, Weiss ; $k=\mathbb{C}$, Taggart]
$F : Top_* \to Top_*$, "nice" homotopy functor	$F : \mathcal{J}_0 \to Top_*$ e.g. $V \mapsto BU(V)$ $V \mapsto \Omega^V F(S^V X)$ $\mathcal{J}$ = Top_*-enriched cat. of finite dim. inner product subspaces of k^∞ $\mathcal{J}_0(U,V) = \mathcal{J}(U,V)_+$ $\mathcal{J}(U,V)$ = Stiefel mfld of linear isometries
F is **n-excisive** strongly coCartesian n-cubes $\downarrow F$ Cartesian n-cubes e.g. $n=0$, htpy constant. $n=1$ $\begin{tikzcd} & (1) \arrow[d] \\ (2) \arrow[r] & (1,2) \end{tikzcd}$ pushout $\to$ pullback	F is **n-polynomial** $F(V) \xrightarrow{\simeq} \tau_n F(V) := \operatorname{holim}_{0 \neq U \subseteq k^{n+1}} F(U \oplus V)$, $\forall V$ · $n=0$, htpy constant. $F(V) \simeq F(V \oplus k) \simeq \cdots \simeq F(k^\infty)$ · $n=1$, holim is indexed by the topological poset $\{0 \neq U \subseteq k^2\}$, which has $k\mathbb{P}^1$ worth of 1-dim subspaces
n-excisive approx. $P_nF = \operatorname{hocolim}_m \rho_n^m F$ $n=1$, $P_1F = \Omega^\infty F \Sigma^\infty$. $\begin{tikzcd} \cdots \arrow[r] & P_2F \arrow[r] & P_1F \arrow[r] & P_0F \\ & D_2F \arrow[u] & D_1F \arrow[u] & \end{tikzcd}$	**n-polynomial approx.** $T_nF = \operatorname{hocolim}_m \tau_n^m F$ (Universal property). $T_0F(V) = \operatorname{hocolim} \tau_0^m F(V) = \operatorname{hocolim} F(V \oplus k^m) = F(k^\infty)$ $\begin{tikzcd} \cdots \arrow[r] & T_2F \arrow[r] & T_1F \arrow[r] & T_0F \\ & D_2^W F \arrow[u] & D_1^W F \arrow[u] & \end{tikzcd}$ ← **nth layer**
n-homogeneous functors $X \mapsto \Omega^\infty(\Theta_F \otimes X^{\otimes n})_{h\Sigma_n}$	$V \mapsto \Omega^\infty(\Psi_F \otimes S^{\mathbb{R}^n \otimes V})_{hO(n)}$ $V \mapsto \Omega^\infty(\Psi_F \otimes S^{\mathbb{C}^n \otimes V})_{hU(n)}$

D_nF, $D_n^W F$ are n-homogeneous respectively.

We will construct the nth derivatives $F^{(n)}$ of F, s.t.

① Structure maps

$$\begin{tikzcd}
F^{(n)}(V) \arrow[r, "\sigma^{ad}"] \arrow[dr] & \Omega^{nU} F^{(n)}(V \oplus U) \arrow[d, "ev_0"] \\
 & F^{(n)}(V \oplus U)
\end{tikzcd}$$

② $F^{(n+1)}(V) = \mathrm{hofib}\,(F^{(n)}(V) \longrightarrow \Omega^{2n} F^{(n)}(V \oplus \mathbb{C}))$

If we take $F(V) = \Omega^V S^V X$ for fixed $X \in Top_*$.

- $F^{(1)}(\mathbb{C}^0) = \mathrm{hofib}\,(X \to \Omega^2 S^2 X) =: F_1(X)$

  $F^{(1)}(\mathbb{C}) = \mathrm{hofib}\,(\Omega^2 S^2 X \to \Omega^6 S^4 X) = \Omega^2 F_1(S^2 X)$

  $\Rightarrow \sigma^{ad} : F_1(X) \to \Omega^4 F_1(S^2 X)$

- $F^{(2)}(\mathbb{C}^0) = \mathrm{hofib}\,(F^{(1)}(\mathbb{C}^0) \to \Omega^2 F^{(1)}(\mathbb{C})) =: F_2(X)$

  $= \mathrm{hofib}\,(F_1(X) \to \Omega^4 F_1(S^2 X))$

  $F^{(2)}(\mathbb{C}) = \mathrm{hofib}\,(F^{(1)}(\mathbb{C}) \to \Omega^4 F^{(1)}(\mathbb{C} \oplus \mathbb{C}))$

  $= \mathrm{hofib}\,(\Omega^2 F_1(S^2 X) \to \Omega^6 F_1(S^4 X))$

  $= \Omega^2 \,\mathrm{hofib}\,(F_1(S^2 X) \to \Omega^4 F_1(S^4 X)) = \Omega^2 F_2(S^2 X)$

  $\Rightarrow \sigma^{ad} : F_2(X) \to \Omega^6 F_2(S^2 X)$.

- Inductively, $F_m(X) = F^{(m)}(\mathbb{C}^0)$,

  $\sigma^{ad} : F_m(X) \to \Omega^{2m+2} F_m(S^2 X)$

- Def The nth jet category J_n has

  - objects = Ob J = {fin. dim inner product subspaces of $\mathbb{C}^\infty$}

  - $J_n(U,V) = Th(\gamma_n(U,V))$, where $\gamma_n(U,V)$ is a vector bundle over $J(U,V)$ with total space
    $\gamma_n(U,V) = \{(f,x) : f \in J(U,V),\ x \in \mathbb{C}^n \otimes f(U)^\perp\}$

w/ Composition induced by

$$\begin{tikzcd}
\gamma_n(V,W) \times \gamma_n(U,V) \arrow[r] \arrow[d, "(f{,}x)"', "(g{,}y)"] & \gamma_n(U,W) \arrow[d, "(fg{,}\ x+f_*y)"] \\
J(V,W) \times J(U,V) \arrow[r] & J(U,W)
\end{tikzcd}$$

- In particular, there is a restricted composition map
  $J_n(\mathbb{C}\oplus V, W) \wedge (\mathbb{C}^n)^c \xrightarrow{\Gamma} J_n(V,W)$,
  where $(\mathbb{C}^n)^c$ is identified with the closure of the subspace of $J_n(V, \mathbb{C}\oplus V)$ consisting of $(i : V \to \mathbb{C}\oplus V, x)$.

Let $\mathcal{E}_n$ = Cat. of Top_*-enriched functors $J_m \to Top_*$

The inclusion $\mathbb{C}^m \hookrightarrow \mathbb{C}^n$ of the first m coordinates induces a functor $i^n_m : J_m \to J_n$

- Precomposition ⇝ restriction $\mathrm{res}^n_m : \mathcal{E}_n \to \mathcal{E}_m$

  $\perp$
- Right Kan extension ⇝ induction $\mathrm{ind}^n_m : \mathcal{E}_m \to \mathcal{E}_n$

$\Rightarrow \mathrm{ind}^n_m F(V) \cong \mathcal{E}_m(J_n(V,-), F)$

- **Def**. For $F \in \mathcal{E}_0$, it's nth-derivative is $F^{(n)} := \mathrm{ind}_0^n F$.

  This can be computed inductively: $\mathrm{ind}_0^n = \mathrm{ind}_{n-1}^n \, \mathrm{ind}_{n-2}^{n-1} \cdots \mathrm{ind}_0^1$

- **Prop**. $\mathrm{res}_n^{n+1} \mathrm{ind}_n^{n+1} F(V) \to F(V) \to \Omega^{2n} F(V \oplus \mathbb{C})$ is a fiber seq $\forall F \in \mathcal{E}_n$.

  Pf. There is a cofiber seq.

  $$J_n(\mathbb{C} \oplus V, -) \wedge (\mathbb{C}^n)^c \xrightarrow{\Gamma} J_n(V, -) \to J_{n+1}(V, -)$$

  (Linear alg., see Weiss §5 Prop 1.2)

Apply the corepresentable functor $\mathcal{E}_n(-, F)$, get fiber seq

$$\begin{tikzcd}
\mathcal{E}_n(J_{n+1}(V,-), F) \arrow[r] \arrow[d, equal] & \mathcal{E}_n(J_n(V,-), F) \arrow[r] \arrow[d, equal] & \mathcal{E}_n(J_n(V \oplus \mathbb{C}, -) \wedge S^{2n}, F) \arrow[d, equal] \\
\mathrm{res}_n^{n+1} \mathrm{ind}_n^{n+1} F(V) & F(V) & \Omega^{2n} F(V \oplus \mathbb{C})
\end{tikzcd} \quad \square$$

Thus we can inductively compute $F^{(n)} \in \mathcal{E}_n$.

$\rightsquigarrow F^{(n)} \to F^{(n-1)} \to \cdots \to F^{(1)} \to F$

- **Rmk** $F \in \mathcal{E}_m$ is determined by
  - It's restriction to a functor $J_0 \to Top_*$
  - A natural transformation of functors $J_0 \times J_0 \to Top_*$

    $\sigma : (\mathbb{C}^n \otimes V)^c \wedge F(W) \to F(V \oplus W)$

- **Prop**. There is a fiber seq $F^{(n+1)} \to F \to \tau_n F$, $\forall F \in \mathcal{E}_0$

  $\Rightarrow$ If F is n-poly, then $F^{(n+1)}$ is trivial.

- Example $F = BU(-)$. $BU^{(1)}(V) \simeq \Sigma S^V \to BU(V) \to BU(V \oplus \mathbb{C})$

  $F(V) = \Omega^V G(S^V X)$, $F^{(1)}(\mathbb{C}^0) \to G(X) \to \Omega^2 G(\Sigma^2 X)$

§2. Applying Weiss calculus to $D_*(id)$

[Johnson, Arone-Mahowald]

$$D_n(id) = \Omega^\infty \mathrm{Map}_*(K_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$$

- K_n: nth partition cplx modeled by $\dfrac{N.(P_n)}{N.(P_n - \hat{0}) \cup N.(P_n - \hat{1})}$

where P_n is the poset of partitions of $[n] = \{1, 2, \cdots, n\}$ ordered by refinement, with $\hat{1} = \{[n]\}$ and $\hat{0}$ the discrete partition.

- Nonequivariantly, $K_n \simeq \bigvee^{(n-1)!} S^{n-1}$, and $\Sigma_1 \times \Sigma_{n-1} \subset \Sigma_n$ acts freely on K_n.

Thm [AM] X odd sphere; $n > 1$

Then $D_n \simeq *$, $n \neq p^k$ $\qquad |d_j| = 2p^k - 2p^j$

$$H^*(D_{p^k}; \mathbb{F}_p) \cong A_{k-1} \otimes \mathbb{F}_p[d_0, \cdots, d_{k-1}]$$

$A[k-1]$ is the sub Hopf-algebra of the Steenrod algebra generated by $Sq^1, \cdots, Sq^{2^{k-1}}$; $p = 2$ $\quad \beta, P^1, \cdots, P^{p^{k-1}}$, $p > 2$.

Idea: Use Weiss calculus to further subdivide D_{p^k}.

- Prop. Θ: spectrum with $U(n)$-action. Then $F(U) = \Omega^\infty (S^{\mathbb{C}^n \otimes U} \otimes \Theta)_{hU(n)}$ is n-polynomial with

$$F^{(i)}(U) = \Omega^\infty (S^{\mathbb{C}^n \otimes U} \otimes \Theta)_{hU(n-i)}, \quad i \leq n$$

where $U(n-i) \subset U(n)$ fixes the first i coordinates.

<u>Pf Sketch</u>: • $F[i](U) = \Omega^\infty(S^{\mathbb{C}^n \otimes U} \otimes \Theta)_{hU(n-i)} \in \mathcal{E}_i$,

i.e. $\exists$ structure maps

$$\sigma : S^{iU} \wedge F[i](V) = S^{iU} \wedge \Omega^\infty(S^{nV} \otimes \Theta)_{hU(n-i)}$$

$$\hookrightarrow \Omega^\infty(S^{iU} \otimes S^{nV} \otimes \Theta)_{hU(n-i)} \hookrightarrow \Omega^\infty(S^{nU} \otimes S^{nV} \otimes \Theta)_{hU(n-i)}$$

$$\to \Omega^\infty(S^{n(U \oplus V)} \otimes \Theta)_{hU(n-i)} = F[i](U \oplus V).$$

• Straightforward to check that $F[i+1] \simeq F[i]^{(1)}$. $\square$.

Now take $\Theta = \mathrm{Ind}_{\Sigma_n}^{U(n-1)} \mathrm{Map}_*(K_n, \Sigma^\infty X^{\wedge n})$ where Σ_n is considered as a subgp of $U(n-1)$ via the reduced standard rep.

$F_m(X) := F^{(m)}(\mathbb{C}^0)$, $F(V) = \Omega^V \Sigma^V X$

• <u>Cor</u> $F_m D_n(X) \simeq$

$$\begin{cases} \Omega^\infty\left(\mathrm{Map}_*(K_n, \Sigma^\infty X^{\wedge n}) \underset{h\Sigma_n}{\otimes} U(n-1)/U(n-m-1)_+\right), & m<n \\ *, & m \geq n. \end{cases}$$

• <u>Claim</u>: $F_m D_n(X) \simeq D_n F_m(X)$.

$$\left\{\begin{array}{l} D_n F_1(X) \to D_n(X) \to D_n(\Omega^2 S^2 X) \\ F_1 D_n(X) \to D_n(X) \to \Omega^2 D_n(S^2 X) \simeq D_n(\Omega^2 S^2 X) \end{array}\right. \Rightarrow D_n F_1(X) \simeq F_1 D_n(X)$$

$$\left\{\begin{tikzcd} D_n F_2(X) \arrow[r] & D_n F_1(X) \arrow[r] \arrow[d, phantom, "\wr"] & D_n \Omega^4 F_1(S^2 X) \arrow[d, phantom, "\wr"] \\ F_2 D_n(X) \arrow[r] & F_1 D_n(X) \arrow[r] & F_1(\Omega^4 D_n(S^2 X)) \end{tikzcd}\right. \Rightarrow D_n F_2(X) \simeq F_2 D_n(X)$$

Induct. $\square$.

- Cor. $D_m w_m : D_m F_{m-1}(-) \xrightarrow{\simeq} D_m \Omega^{2m} F_{m-1}(S^2 \wedge -)$

  ⇒ mapping telescopes generalizing Mahowald-Thompson

$$\left\{\begin{array}{l} X \xrightarrow{w_1} \Omega^2 S^2 X \xrightarrow{w_1} \Omega^4 S^4 X \to \cdots \to \Omega^\infty D_1 F_0(X) \\ F_1(X) \xrightarrow{w_2} \Omega^4 F_1(S^2 X) \xrightarrow{w_2} \Omega^8 F_1(S^4 X) \to \cdots \to \Omega^\infty D_2 F_1(X) \\ \qquad\qquad \vdots \\ F_{m-1}(X) \xrightarrow{w_m} \Omega^{2m} F_{m-1}(S^2 X) \to \cdots \to \Omega^\infty D_m F_{m-1}(X) \end{array}\right.$$

(b/c connectivity of $D_n \Omega^{2km} F_{m-1}(S^{2k} \wedge -)$ increases with k when $n > m$)

- Thm (Arone). Let $n = p^k$, X^d odd sphere. Then

$$H^*(D_n F_m(X)) \cong H^*(\mathrm{Map}_*(K_n, \Sigma^\infty X^{\wedge n}) \underset{h\Sigma_n}{\wedge} U(n-1)/U(n-m-1)_+)$$

$$\cong A[k-1] \otimes E \otimes P \text{ as free } A[k-1]\text{-modules.}$$

P is rank one over $\mathbb{F}_p[d_j = c_{p^k - p^j},\ m < p^j < p^k]$, E is rank one over $\Lambda\{\bar{c}_i \mid p^k - m \leq i \leq p^k - 1$ and $i \neq p^k - p^j$ for some $j\}$.

Idea: [Arone-Dwyer]

$$T_k : \Sigma^2 \left| \text{cat of proper subspace of } (\mathbb{F}_p)^k \right| \circlearrowleft \overset{\overset{\Sigma_k}{\cup}}{\mathrm{Aff}_k(\mathbb{F}_p)} = GL_k(\mathbb{F}_p) \ltimes \overset{\overset{\text{Acts trivially}}{\downarrow}}{(\mathbb{Z}/p\mathbb{Z})^k}$$

$$\mathrm{Map}_*(K_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \xrightarrow{\simeq_p} \mathrm{Map}_*(T_k, \Sigma^\infty X^{\wedge n})_{h\mathrm{Aff}_k(\mathbb{F}_p)}$$

$$\simeq \mathrm{Map}_*(T_k, (\Sigma^\infty X^{\wedge n})_{h(\mathbb{Z}/p\mathbb{Z})^k})_{hGL_k(\mathbb{F}_p)}$$

$$\circledast \simeq \mathrm{Map}_*(T_k, \Sigma^\infty (B(\mathbb{Z}/p\mathbb{Z})^k)^{d\gamma})_{hGL_k(\mathbb{F}_p)}$$

with cohom. $\cong^{st.}$ $H^*((B(\mathbb{Z}/p\mathbb{Z})^k)^{d\gamma})$

[Mitchell-Priddy] ⇒ free over $A[k-1]$.

$\circledast$ $X^{\wedge n}_{h(\mathbb{Z}/p\mathbb{Z})^k}$ is the Thom space over $B(\mathbb{Z}/p\mathbb{Z})^k$ assoc. to $d\gamma$

γ: regular rep. of $(\mathbb{Z}/p\mathbb{Z})^k$

The generators $c_{p^k - p^j}$ of P are Chern classes of the reduced regular rep. of $(\mathbb{Z}/p\mathbb{Z})^k$, or Dickson polynomials evaluated at the poly. generators of $H^*(B(\mathbb{Z}/p\mathbb{Z})^k; \mathbb{F}_p)$.

generators of $\mathbb{F}_p[y_1, \dots, y_k]^{GL_k(\mathbb{F}_p)}$.

So $\bar{c}_i \in E$ are the Chern classes in $H^*(B(U(n-1)/U(n-m-1)))$ that are not Dickson classes, and $d_j \in P$ are Dickson classes in $H^*(BU(n-m-1))$.

Cor. When $m = n-1 = p^k - 1$, $H^*(D_n F_m(X)) \cong A[k-1] \otimes E$.

This is essentially Mitchell's construction of finite cplx with $A[k-1]$-free cohomology.

§3. v_i-periodic homotopy

X : odd sphere, $p=2$ for simplicity. $n=p^k$ $k>0$.

Thm [AM]. $X \to P_n(X)$ is a v_i-periodic equiv., $\forall\, i \le k$.

Pf. 1). $H^*(D_n(X))$ is free over $A[k-1] \Rightarrow D_{2^{k+\ell}}$ is v_k-trivial if $\ell>0$.

[Anderson-Davis] If M is an $\mathcal{A}$-module and $P^{s_0}_{t_0}$ is the P^s_t $\leftarrow$ dual to $\xi_t^{p^s} \in \mathcal{A}_*$

of lowest degree s.t. $s<t$ and $H(M, P^s_t) \ne 0$.

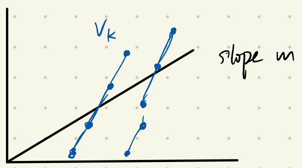
Then $\mathrm{Ext}_{\mathcal{A}}^{i,j}(M, \mathbb{F}_2)=0$ for $di > j+c$, where $d=\deg(P^{s_0}_{t_0})$

and $\frac{c}{d-1} \approx t-2$.

Since $P^s_t \in A[k-1]$ if $s+t \le k$, $\begin{cases} s_0 = \lfloor \frac{k}{2} \rfloor \\ t_0 = \lceil \frac{k}{2} \rceil + 1 \end{cases}$

$\Rightarrow |P^{s_0}_{t_0}| = 2^{s_0}(2^{t_0}-1) = 2^{k+1} - 2^{s_0} > 2^k - 1$.

⊛ Hence the Adam's sseq of $D_{2^{k+\ell}}$ has a vanishing line of slope $m < \frac{1}{2^{k+\ell}-2} = \frac{1}{|v_{k+\ell-1}|}$ and intercept $< k+\ell$. Since v_k acts on ASS as multiplization by an element of slope $\frac{1}{|v_k|}$, $D_{2^{k+\ell}}$ is v_k-trivial for $\ell>0$.



Upshot: in v_k-periodic homotopy, the Goodwillie tower has only $k+1$ nontrivial layers $D_{p^0}, \dots, D_{p^k}$.

2). Need to show that the Goodwillie tower for v_k-periodic homotopy of X converges.

Fix finite space V with v_k-self map and write $\pi_* = \pi_*(-;V)$.

<u>WTS</u>: $v_k^{-1}\pi_*(X) = v_k^{-1}\pi_*(\operatorname{holim} P_j) \cong v_k^{-1}\left(\varprojlim \pi_*(P_j)\right)$

Set $Q_j = \mathrm{fib}(P_{p^{k+j}} \to P_{p^k})$, then suffices to show that $v_k^{-1}\pi_*(\operatorname{holim} Q_j) \cong v_k^{-1}\varprojlim \pi_*(Q_j) = 0$.

$$\begin{tikzcd}
\vdots \arrow[d] & \\
Q_j \arrow[d, "f_j"'] & D_{p^{k+j}} \arrow[l] \\
Q_{j-1} \arrow[d, "f_{j-1}"'] & D_{p^{k+j-1}} \arrow[l] \\
\vdots \arrow[d] & \\
Q_1 = D_{p^k+1} &
\end{tikzcd}$$

Take any $\alpha = (\cdots, \alpha_2, \alpha_1) \in \varprojlim \pi_*(Q_j)$ w/ $|\alpha| = d$, so $\alpha_j \in \pi_d(Q_j)$. Can assume that $\alpha_1 = 0$ by applying v_1 k_1 times. Set $d_2 = |v_k^{k_1}(\alpha_1)| = d + k_1(2^{k+1}-1)$.

Now induct on j. If $\alpha_{j-1} = 0$, then α_j can be identified with an element in $\pi_{d_j}(D_{p^{k+j}})$ thought of as an E_∞-term of the ASS. with bidegree $(0, d_j)$. Let k_j be the smallest integer s.t. $v_k^{k_j}(\alpha_j) = 0$. Then $\circledast$ implies that

$$d_{j+1} = |v_k^{k_j}(\alpha_j)| = d_j + k_j(2^{k+1}-1) < 2^{k+1}(k+j+1) + \frac{3}{2}d_j.$$

Hence the sequence $\{d_j\}$ has growth rate $\le \frac{3}{2}$, whereas the connectivity of $D_{p^{k+j}}$ has growth rate 2. □.

<u>Cor</u>. $F_m(X) \to P_{p^k}F_m(X)$ is an v_i-equiv. for $i \le k$.

Let k be the smallest s.t. $m \le p^k - 1$, then

$$F_m(X) \to P_{p^k}F_m(X) \xleftarrow{\cong} D_{p^k}F_m(X)$$ is a v_k-equiv.