Thursday seminar

14-4-2022

# Higher exponents of generalized Moore spaces

from $p$-exponents to $v_i$-exponents, $i \geq 0$

**CMN:** $\Omega_0^{2n+1} S^{2n+1}$ has exp. $p^n$ ($p$ odd)

**Wang:** $v_2$-periodic htpy groups

$\pi_* \Phi_2 S^3$ are $p$-torsion

and $v_1^2$ acts trivially ($p \geq 5$)

**guess:** $v_i$ act trivially on $\pi_* \Phi_n S^\ell$ for $i < n$

**however:** the spectrum $\Phi_2 S^3$ does not have

a $v_1$-exponent

$E_*(\Phi_2 S^3)$ is not $v_1$-torsion

CMN, Neisendorfer: for $p$ odd, $k \geq 2$, the space $\Omega^2 S^k/p^r$ has exponent $p^{r+1}$

Theriault: for $p=2$, $r \geq 2$, get exponent

$S^k/2$ ?

What about generalized Moore spaces?

$$S^k/(p^{r_0}, v_1^{r_1}, \ldots, v_{n-1}^{r_{n-1}})$$

(or just general finite type $n$ space $V$?)

guess: these have a $v_i$-exponent for $i < n$

obs: the suspension spectrum of $V$ certainly does

Goal: this is true after inverting $v_h$ (for arbitrary $h \geq 0$).

in particular: $\Phi_h(\Sigma V)$ has $v_i$-exponent $i < n$

---

preliminaries

(1) $V$ pointed finite space of type $i$,

$$\Sigma^d V \xrightarrow{v} V \quad v_i \text{ self-map}$$

then $X \in \mathcal{S}_*$ has a $v_i$-exponent if

$$\mathrm{Map}_*(V, X) \xrightarrow{v^*} \Omega^d \mathrm{Map}_*(V, X) \xrightarrow{v^*} \cdots$$

is nilpotent

generally: $\mathcal{C}$ pointed $\infty$-cat. with finite limits

$$X^V \xrightarrow{v^*} \Omega^d X^V \xrightarrow{v^*} \cdots$$

② "inverting $v_h$":

localize $\mathcal{S}_*$ by killing cofiber of $v_h$

fix finite space $F$ of type $h+1$,

a suspension

$d$: dim'n of bottom cell of $F$

$L_h^f$ Bousfield localization w.r.t. $F \to *$

Define $L_h^f \mathcal{S}_*\langle d \rangle$ for full subcategory of $\mathcal{S}_*$

on $d$-connected, $L_h^f$-local spaces.

<u>Thm</u> (Bousfield): A map $X \to Y$ of $d$-conn-

spaces is a $L_h^f$-equiv. if and only if

it is a iso on $v_i$-periodic htpy gps

for $0 \le i \le h$.

**Facts:** + $\mathcal{S}_*\langle d\rangle \xrightarrow{L_n^f} L_n^f \mathcal{S}_*\langle d\rangle$ preserves

colimits + finite limits

+ $L_n^f \mathcal{S}_*\langle d\rangle$ admits Goodwillie calculus,

" $P_k(\mathrm{id}_{L_n^f \mathcal{S}_*\langle d\rangle}) = L_n^f P_k \mathrm{id}$ "

**Main Thm:** If $X \in \mathcal{S}_*\langle d\rangle$ such that

$L_n^f \Sigma^\infty X$ has a $v_i$-exp., $\forall i < n$, then

so does the space $L_n^f \Sigma X$.

**Cor:** The spectrum $\Phi_n(\Sigma X)$ then also has

a $v_i$-exponent, $i < n$.

outline of pf:

part A: "H-spaces are nilpotent in $\mathcal{L}_h^f \mathcal{S}_* \langle d \rangle$"

fact: if $X$ is a rational H-space, then in fact it is an inf. loop space (product of EM's)

Def: An object $X \in \mathcal{L}_h^f \mathcal{S}_* \langle d \rangle$ is htpy nilpotent (of exp. $k$) if $X \longrightarrow P_k X$ admits a retraction.

(cf. Biedermann–Dwyer)

(from joint work with Brantner, Hahn, Yuan)

<u>Thm A</u>: Any H-space $X \in L_n^f \mathcal{S}_*\langle d\rangle$ is htpy nilpotent of exponent $p^h$.

<u>proof B</u>:

<u>Thm B</u> (Mathew): If $Sp \xrightarrow{F} Sp$ is a reduced polynomial functor (in the sense of Goodwillie) and $Y$ has a $v_i$-exp, $i<n$, then $F(Y)$ also has a $v_i$-exp, $i<n$.

---

<u>proof of Main Thm</u>

take $X \in \mathcal{S}_*\langle d\rangle$, s.t. $L_n^f \Sigma^\infty X$ has $v_i$-exp.

suffices to show $L_n^f \Omega\Sigma X$ has a $v_i$-exp.

this is htpy nilpotent of exponent $p^h$ (Thm A),

so suffices to find exponents for

$$D_k(\Omega\Sigma X) \cong \Omega^\infty\left(\partial_k \mathrm{id} \otimes \left(L_h^f \Sigma^\infty \Omega\Sigma X\right)^{\otimes k}\right)_{h\Sigma_k}$$

with $k \le p^h$

apply Thm B, observing that

$$L_h^f \Sigma^\infty \Omega\Sigma X \simeq \bigoplus_{j\ge 1} L_h^f \Sigma^\infty X^{\otimes j}$$

has <u>same</u> exponent as $L_h^f \Sigma^\infty X$. $\square$

---

Proof of Thm A:

any H-space is a retract of a loop space,

so wlog $X$ is a loop space

Step 1: $X$ loop space $\Rightarrow$ $X \xrightarrow{\eta} \Omega\Sigma X$ has retraction

$\Rightarrow$ suffices to establish factorization

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr] & \Omega\Sigma X \\ & P_{ph} X \arrow[u, dashed] \end{tikzcd}$$

will do this for any $X$ (not necessarily loop space)

Step 2: functors involved preserve sifted colimits, so suffices to do this for generators

$$X = L_h^f(\underbrace{S^{d+1} \vee \dots \vee S^{d+1}}_{k \text{ copies}})$$

Step 3: Hilton-Milnor thm gives identification of $\eta$

as

weak inf. product over words $w$ in a basis for free Lie alg. on $x_1, \dots, x_k$

$$
\begin{tikzcd}
S^{d+1} \vee \dots \vee S^{d+1} \arrow[r, "\eta"] \arrow[d, dash] & \prod'_{w \in \mathrm{Lie}(k)} \Omega\Sigma\, w(S^{d+1}, \dots, S^{d+1}) \\
S^{d+1} \times \dots \times S^{d+1} \arrow[r] & \Omega\Sigma S^{d+1} \times \dots \times \Omega\Sigma S^{d+1} \arrow[u]
\end{tikzcd}
$$

inclusion of "linear" words $x_1, \dots, x_k$

<u>Step 4:</u> <u>Arone–Mahowald:</u>

$S^l \longrightarrow P_{p^h}(S^l)$ is a $L_h^f$-equivalence if $l$ odd

(if $l$ even: $2p^h$ if $l$ even)

$$
\begin{tikzcd}
S^{d+1} \times \cdots \times S^{d+1} \arrow[r] \arrow[d] & \Omega\Sigma S^{d+1} \times \cdots \times \Omega\Sigma S^{d+1} \arrow[dd] \\
P_p^h(S^{d+1} \times \cdots \times S^{d+1}) \arrow[d] \arrow[ur, dashed] & \\
P_p^h(S^{d+1}) \times \cdots \times P_p^h(S^{d+1}) \arrow[r] & \Omega P_p^h(\Sigma S^{d+1}) \times \cdots \times \Omega P_p^h(\Sigma S^{d+1})
\end{tikzcd}
$$

<u>one</u> of the vertical composites is a $L_n^f$-equivalence by Arone–Mahowald, so get lift in square

$\square$

<u>part B</u>: polynomial functors & exponents

$\mathcal{C}$ stable $\infty$-category

(in particular, it makes sense to tensor objects of $\mathcal{C}$ with finite spectra)

$F$ a fin. spectrum

__Def:__ (1) $X \in \mathcal{C}$ is $F$-__nilpotent__ if it's in the thick subcat. generated by objects of the form $F \otimes Y$, $Y \in \mathcal{C}$.

(2) $X \in \mathcal{C}$ is $F$-__torsion__ if it's a colimit of $F$-nilpotent objects

__Lem:__ $X \in \mathcal{C}$ has a $v_i$-exp. $\forall\ i \leq n$ $\iff$ $X$ is $V$-nilpotent for any type $n+1$ cplex $V$

__Pf:__ "$\Rightarrow$": say $v_i^k \otimes X$ is null ($\Sigma^d F \xrightarrow{v_i} F$ $v_i$ self-map)

then $\Sigma^{kd} F \otimes X \underset{v_i^k}{\xrightarrow{0}} F \otimes X \longrightarrow F/v_i^k \otimes X$

$F \otimes X \Rightarrow F/_{v_1}{}^k$ - nilp.

"$\Leftarrow$": category of $X \in \mathcal{C}$ with $v_1$-maps $\Rightarrow$ thick,

so suffices to do this for $V \otimes Y$,

$Y \in \mathcal{C}$

but $V$ has such a eq. by

Periodicity Thm $\square$

<u>Thm B</u> (Mathew): Let $P: \mathcal{C} \longrightarrow \mathcal{D}$ be

a reduced polynomial functor between stable

$\infty$-cat's. The $P$ preserves $F$-nilpotent

objects

<u>Pf</u>: Embed $\mathcal{C}, \mathcal{D}$ into $\mathrm{Ind}(\mathcal{C})$ & $\mathrm{Ind}(\mathcal{D})$.

Extend $P$ to $\hat{P} : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$

preserving filtered colimits.

By induction on Goodwillie tower, suffices to treat

homogeneous $\hat{P}$.

$$\hat{P}(X) \simeq L(X, \ldots, X)_{h\Sigma_n}$$

for some multilinear $L : \mathrm{Ind}(\mathcal{C})^n \to \mathrm{Ind}(\mathcal{D})$,

so $\hat{P}$ preserves $F$-torsion objects

but this suffices!

if $X \in \mathcal{C}$ $F$-nilpotent, then

$\hat{P}(X)$ is filtered colimit of compact $F$-nilp.

objects, but also itself cpct in $\mathrm{Ind}(\mathcal{D})$

$\Rightarrow$ $\hat{P}(X)$ retract of a $F$-nilp. object. $\square$

<u>Cor</u>: Let $V$ be finite type $n$ spectrum. Then the free Lie alg.

$$L_{T(n)}\left(\bigoplus_{k\geq 1}\left(\mathbb{L}(k)\otimes V^{\otimes k}\right)_{h\Sigma_k}\right)$$

has a $(p, v_1, \ldots, v_{n-1})$-exp.

<u>by contrast</u>: $L_{T(1)}\,\mathrm{Sym}^{\geq 1}(\mathbb{S}/p)$ does <u>not</u> have a $p$-exp.