Tools of Unstable Homotopy Theory

We can study X connected by studying the group ΩX. If X is nilpotent, ΩX is nilpotent, and nilpotent discrete gps are effectively studied via their Lie algebras

The Lie alg is realized on the level of π_* via Samelson/Whitehead products.

Def the Samelson product is the map

$$\Omega X \wedge \Omega X \xrightarrow{c \,\leftarrow\, a,b \mapsto aba^{-1}b^{-1}} \Omega X$$ on homotopy

giving $\pi_i \Omega X \otimes \pi_j \Omega X \xrightarrow{\langle\, , \rangle} \pi_{i+j} \Omega X$,

In the universal case,

$$\Omega\Sigma(S^i \wedge S^j) \xrightarrow{c} \Omega\Sigma(S^i \vee S^j)$$

corepresents the Samelson product, is adjoint to

$$\Sigma S^i \wedge S^j \longrightarrow \Sigma(S^i \vee S^j).$$

The Whitehead product is the operation

$\pi_{i+1} X \otimes \pi_{j+1} X \xrightarrow[{[\, ,]}]{} \pi_{i+j+1} X$ given by

$$S^i \wedge S^1 \wedge S^j \underset{\sim}{\xrightarrow{\text{swap}}} S^1 \wedge S^i \wedge S^j \longrightarrow \Sigma(S^i \vee S^j)$$

Note $\mathrm{cof}\,(\Sigma S^i \wedge S^j \to \Sigma(S^i \vee S^j)) \simeq S^i \times S^j$

(More gen, $(\Sigma X \wedge Y \xrightarrow{c} \Sigma X \vee \Sigma Y) \simeq \Sigma X \times \Sigma Y$)

ie $(-1)^{|x|}\langle x,y\rangle = [x,y]$.

Samelson prod satisfies

1 • bilinear

2 • $\langle x,y\rangle = \langle y,x\rangle(-1)^{xy+1}$

3 • $\langle\langle x,y\rangle,z\rangle + \langle\langle y,z\rangle,x\rangle(-1)^{x(y+z)} + \langle\langle z,x\rangle,y\rangle(-1)^{z(x+y)} = 0$

& Whitehead prod satisfies

- bilinear
- $[x,y] = [y,x](-1)^{xy}$
- $[[x,y],z](-1)^{xz} + [[y,z],x](-1)^{yx} + [[z,x],y](-1)^{zy} = 0$

Pf 1&2 are easy

for 3, $\{[[x,y],z][[y,z],x][[z,x],y]\}^{=1}$ holds in any group up to comm. of length 4.

Those are null since they factor through

$$S^{n+1}\wedge S^{k+1}\wedge S^{l+1} \xrightarrow{\Delta} S^{n+1}\wedge S^{m+1}\wedge S^{k+1}\wedge S^{l+1}$$

We'll see that Iterated Samelson products & homotopy gps of spheres account for all primary homotopy operations:

Thm (Hilton - Milnor)

$$\Omega\Sigma\left(\bigvee_i^n X_i\right) \simeq \prod \Omega\Sigma X_{|w|}^{\wedge |w|} \qquad X_i \text{ connected}$$

$w \in$ basis of free Lie alg on $\{X_i\}$

The map $\longleftarrow$ is given by product of iterated Samelson prod

The maps $\longrightarrow$ are 'Hilton Hopf invariants',

Two approaches to above thm.

1. Mather's 2nd cube thm (See [DH])
2. Understanding free gps.

We'll do approach 2.

The point is: Given a Group G, LCS

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \cdots \qquad G_{i+1} = [G, G_i]$$

G_i/G_{i-1} is a Lie alg. By choosing lifts for G nilpotent, get

$\prod G_i/G_{i-1} \simeq G$. For G a free gp (ie $\Omega\Sigma S$ S a set)

S gives natural choices of lifts $[S,S]$ ...

but free gp on S is unfortunately not nilpotent.

But if you do this in families X, it still works bc

$\Omega\Sigma X$ is nilpotent for X connected.

Lem $\exists$ split ses of $\mathbb{E}_1$-algs

$$1 \to \Omega\Sigma(A \vee A \wedge \Omega\Sigma B) \longrightarrow \Omega\Sigma A \vee B \leftrightarrows \Omega\Sigma B \to 1$$

Pf Everyting preserves sifted colimits, reducing to case A, B discrete (finite).

$$\ker F(A \vee B) \longrightarrow F(B)$$

is freely gen by A & $[A, B]$.

As groups

Lem $\Omega\Sigma(A \wedge \Omega\Sigma B) \simeq \Omega\Sigma(A \wedge B \vee A \wedge B \wedge \Omega\Sigma B)$

Pf again reduce to A, B discrete and note

$F([A, FB])$ is gen freely by

$[A, B]$ $[[A, B], FB]$.

iterating above Lemma for B connected

$$\Longrightarrow \Omega\Sigma(A \wedge \Omega\Sigma B) \simeq \Omega\Sigma\Big(\bigvee_1^\infty A \wedge B^{\wedge i}\Big) \text{ as } \mathbb{E}_1\text{-algs}$$

delooping $\Longrightarrow$

Thm (James)

$$\Sigma\Omega\Sigma B \simeq \bigvee_1^\infty \Sigma B^{\wedge i}.$$

also

$$1 \to \Omega\Sigma\Big(\bigvee_0^\infty A \wedge B^{\wedge i}\Big) \to \Omega\Sigma(A \vee B) \leftrightarrows \Omega\Sigma B \to 1$$

which inductively $\Longrightarrow$ Hilton-Milner

EHP Sequence

The James splitting gives

$$\Sigma\Omega\Sigma X \simeq \bigvee_{1}^{\infty} \Sigma X^{\wedge i}$$

projections adjoint to maps

$$\Omega\Sigma X \xrightarrow{H_n} \Omega\Sigma X^{\wedge n}.$$

James used this to construct the EHP seq:

at $p=2$; it's a fibre seq

$$S^n \xrightarrow{E} \Omega\Sigma S^n \xrightarrow{H} \Omega\Sigma S^{2n} \qquad H = H_2$$

Toda at $p>2$

$$J_{p-1}S^{2n} \longrightarrow \Omega\Sigma S^{2n} \xrightarrow{H_p} \Omega\Sigma S^{2pn}$$

$$S^{2n-1} \longrightarrow \Omega J_{p-1}S^{2n} \longrightarrow \Omega\Sigma S^{2pn-2}$$

Claim: H is surj on H_{2n}.

Pf:
$$\begin{tikzcd}
\Sigma\Omega\Sigma S^n \arrow[r, "\text{proj}"'] \arrow[rr, bend left, "\Sigma H"] & \Sigma S^{2n} \arrow[r, "\text{iso on } H_{2n}"'] & \Sigma\Omega\Sigma S^{2n}
\end{tikzcd}$$

each is iso on H_{2n+1}

n odd.

$$\begin{tikzcd}
\Omega\Sigma S^{2n+1} \arrow[r] & \Omega\Sigma S^{4n+2} \\
x_{2n+1} & y_{2n+1} \arrow[l]
\end{tikzcd}$$

+SSS

Rmk works integrally for n odd, but $\Omega[\iota_{n+1},\iota_{n+1}]$ splits it at $p>2$. giving $\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}$. Also it splits at $p=2 \Leftrightarrow \exists$ Hopf inv 1.

n even $$S^{2n} \to \Omega\Sigma S^{2n} \to \Omega\Sigma S^{4n}$$

$$x_{2n}^{(2)} \leftarrow y_{4n}$$

$$x_{2n}^{(2k)} \boxed{\frac{(2k)!}{k!\,2^k}} \leftarrow y_{4n}^{(k)}$$

1 mod 2.

Rmk $P: \Omega^2 S^{2n+1} \to S^n$ is $[i_n, i_n]$ on bottom cell.

James' Torsion Bound

Thm $\exists$ factorization

$$\begin{tikzcd}
\Omega^3 S^{2n+1} \arrow[rr, "4"] \arrow[dr, dashed] & & \Omega^3 S^{2n+1} \\
& \Omega S^{2n-1} \arrow[ur, "E^2"'] &
\end{tikzcd}$$

Cor.

$$\begin{tikzcd}
\Omega^{2n+1} S^{2n+1} \arrow[rr, "2^{2n}"] \arrow[dr] & & \Omega^{2n+1} S^{2n+1} \\
& \Omega S^1 \simeq \mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

so 4^n kills $\pi_* S^{2n+1}$.

Pf We'll show $\dashrightarrow$ exists using EHP

$$\begin{tikzcd}
 & & \Omega^3 S^{2n+1} \arrow[dl, dashed] \arrow[d, "2"] \\
 & \Omega^2 S^{2n} \arrow[r, "E"] \arrow[d, "1-\Omega-1"] \arrow[dl, dashed] & \Omega^3 S^{2n+1} \arrow[d, "2"] \\
\Omega S^{2n-1} \arrow[r] & \Omega^2 S^{2n} \arrow[r, "E"] & \Omega^3 S^{2n+1}
\end{tikzcd}$$

ie suffices to show

$$\Omega S^{2n} \xrightarrow{1-\Omega-1} \Omega S^{2n} \xrightarrow{H} \Omega S^{4n-2}$$

and

$$\Omega^2 S^{2n+1} \xrightarrow{2} \Omega^2 S^{2n+1} \xrightarrow{\Omega H_2} \Omega^2 S^{4n+1}$$

are null.

for the first,

$$\begin{tikzcd}
\Omega\Sigma S^n \arrow[r, "H"] \arrow[d, "\Omega\Sigma-1"] & \Omega\Sigma S^{2n} \arrow[d, "\Omega\Sigma(-1\overset{\overset{1}{\shortparallel}}{\wedge}-1)"] \\
\Omega\Sigma S^n \arrow[r, "H"] & \Omega\Sigma S^{2n}
\end{tikzcd}$$

$\Rightarrow H \circ \Omega\Sigma-1 = H$

ΩH is a group hom.

the 2nd is trickier

$$\begin{tikzcd}
\Omega\Sigma S^{2n} \arrow[r, "\text{pinch\&flip}"] \arrow[dr] & \Omega\Sigma(S^{2n} \vee S^{2n}) \arrow[d, phantom, sloped, "\cong"] \arrow[r, "\text{fold}"] & \Omega\Sigma S^{2n} \\
 & \prod \Omega\Sigma S^{\wedge kn} \arrow[ur, "\text{prod of Samelson products}"'] &
\end{tikzcd}$$

iterated $[\ ]$ of length ≥ 3 are null for ι_{2n+1}.

$$\Rightarrow \quad 1 + \Omega - 1 + \Omega[\iota_{2n+1}, \iota_{2n+1}] \circ h = 0$$

$$\Omega H \circ 2 + \Omega H \circ \Omega^2 [\iota_{2n+1}, \iota_{2n+1}] \circ h$$

Claim: $H \circ \Omega[\iota_{2n+1}, \iota_{2n+1}] = 0$.

Pf: By EHP $[\iota_{2n+1}, \iota_{2n+1}] = \Sigma\nu$.

$$\begin{tikzcd}
\Omega\Sigma S^{4n} \arrow[r, "H"] \arrow[d, "\Omega\Sigma\nu"] & \Omega\Sigma S^{8n} \arrow[d, "\Omega\Sigma\nu\wedge\nu"] \\
\Omega\Sigma S^{2n} \arrow[r, "H"] & \Omega\Sigma S^{4n}
\end{tikzcd}$$

$\Sigma\nu\wedge\nu$ is null since it factors through $\Sigma\nu \wedge id$

$= \Sigma^{2n}\nu$

but $\Sigma^2\nu = \Sigma[\iota_{2n+1}, \iota_{2n+1}]$

$= 0$.