The CMN Theorem I

Throughout this talk and the next $p>3$.
We work locally.

[Part II will be Andy after Spring break]

Thm $n \geq 1$, then $p^n \cdot \pi_* S^{2n+1} = 0$.

[Emphasize the paper is quite well written and still readable today.]

Idea: Induct on n after we construct a map π

$$\begin{tikzcd} & \Omega^2 S^{2n+1} \arrow[dr, "\pi"] & \\ S^{2n-1} \arrow[ur, "E^2"] \arrow[rr, "p"'] & & S^{2n-1} \end{tikzcd}$$

How do we construct π?
We'll totally overshoot our target, then grab π as a byproduct.

$$F^m\{p^r\} \longrightarrow P^m(p^r) \xrightarrow{q_m} S^m$$

fiber of pinch

Thm $n>1$,

$$S^{2n-1} \times \prod_{k=1}^{\infty} S^{2p^k n-1}\{p^{r+1}\} \times \Omega\left(\bigvee P^{\bullet}(p^r)\right) \xrightarrow{\simeq} \Omega F^{2n+1}\{p^r\}.$$

[Now $\Omega^2 S^{2n+1} \longrightarrow \Omega F^{2n+1}\{p^r\} \longrightarrow S^{2n-1}$ (with $r=1$)
is the map π we want.]

① Produce a map ← (1a) Produce maps in from each component separately and assemble using Ω.

(1b) The components will be constructed in Andy's talk using Samelson/rel. Samelson etc.

② Prove it's an iso on homology. ← (2a) Compute homology

(2b) Understand homology + structure.

[Point out we're going → not ↓ & how dividing this proof is like cutting a disk in half.]

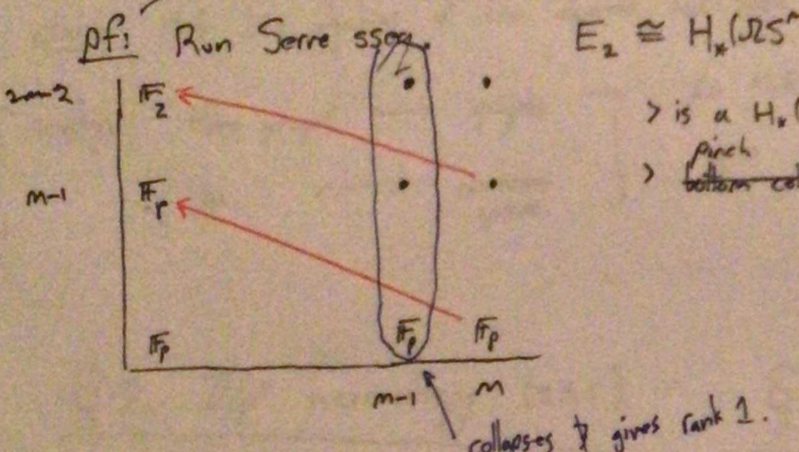
§1 $\mathbb{F}_p$ homology

order

$$\underset{③}{\Omega S^m} \longrightarrow \underset{④}{F^m\{p^r\}} \longrightarrow \underset{②}{P^m(p^r)} \longrightarrow \underset{①}{S^m}$$

$H_*(\Omega S^m;\mathbb{F}_p)$	$H_*(F^m\{p^r\};\mathbb{F}_p)$	$\tilde{H}_*(P^m(p^r);\mathbb{F}_p)$	$\tilde{H}_*(S^m;\mathbb{F}_p)$
$\mathbb{F}_p[L]$ free algebra, L is prim	has a generator lifting u, rank 1 free $\mathbb{F}_p[L]$-module, $L \longrightarrow p^r u$	$\mathbb{F}_p\langle u, v\rangle$ (u: $m-1$, v: m), both prim	$\mathbb{F}_p\langle v\rangle$, v is primitive

<u>pf:</u> Run Serre sseq.



$E_2 \cong H_*(\Omega S^m;\mathbb{F}_p) \otimes_{\mathbb{F}_p} H_*(P^m(p^r);\mathbb{F}_p)$

> is a $H_*(\Omega S^m;\mathbb{F}_p)$-module

> ~~bottom cell~~ pinch gives a diff'l.

//

From here things get more serious.

$$\Omega F^m\{p^r\} \longrightarrow \Omega P^m(p^r) \longrightarrow \Omega S^m$$

$$\begin{array}{ccccc} H_*(\Omega F^m\{p^r\};\mathbb{F}_p) \ ⑥ & \longrightarrow & H_*(\Omega P^m(p^r);\mathbb{F}_p) \ ⑤ & & \\ \cong & & \cong & & \\ \mathbb{F}_p\{x_k\}_{k\geq 1} & \longrightarrow & \mathbb{F}_p\{u, v\} & \longrightarrow & \mathbb{F}_p\{L\} \end{array}$$

$x_k \longmapsto \mathrm{ad}^{k-1}(v)(u)$

free, u: $m-2$, v: $m-1$

$v \longrightarrow L$, $u \longrightarrow 0$

<u>pf:</u> EM sseq degen since $P^m(p^r)$ is a suspension.

pf: ~~In terms of~~
Run EM sseq.

$$\mathbb{F}_p\{u, v\} \longrightarrow \mathbb{F}_p\{L\}$$

~~target~~ source is injective as an $\mathbb{F}_p\{L\}$-comodule, hence all ~~ans~~ are linz.

$$H_*(\Omega F^m\{p^r\}; \mathbb{F}_p) \cong \mathbb{F}_p\{u, v\} \,\square_{\mathbb{F}_p\{L\}}\, \mathbb{F}_p$$

we can rewrite as

UL° via $\quad L^\circ \longrightarrow L(u, v) \longrightarrow L(L)$

claim: sub Lie algebras of Lie algebras are free.

[analogy: Free groups $\longleftrightarrow$ graphs; sub $\longleftrightarrow$ covering space.]

In this case L° is free on $\mathrm{ad}^{k-1}(v)(u)$. //

§2 $\mathbb{Z}/p^s$ homology $(s \leq r)$

1. ibid.
2. ibid.
3. ibid
4. Sparsity means no Bockstein diff.
5. ibid.
6. ibid.

§3 Integral homology

1. $\tilde{H}_*(S^m) \cong \mathbb{Z}\langle L\rangle$
2. $\tilde{H}_*(P^m(p^r)) \cong \mathbb{Z}/p^r\langle u\rangle$ and $\beta^{(r)}(v) = u$.
3. $\tilde{H}(\Omega S^m; \mathbb{Z}) \cong \mathbb{Z}\{L\}$
4. Things get more serious again...

$H_*(F^m\{p^r\};\mathbb{Z})$ is a free $\mathbb{Z}[\iota]$-module on a class u.
rank 2

>no room for Bockstein.

What about as a ~~Hopf algebra~~, coalgebra, at cochain level?

<u>Lem</u>: $C_*(\Omega S^m)$ is formal as an E_1-E_1-Hopf algebra.

<u>pf</u>: $C_*(\Omega S^m)$ & $H_*(\Omega S^m)$ are both free Hopf alg on the n.u. coalgebra S^{m-1} which is prim.

<u>Lem</u>: As a $C_*(\Omega S^m)$-module coalgebra $\tilde{C}_*(F^m\{p^r\})$ is free on a prim class u in degree $m-1$, and in particular formal as well.

<u>pf</u>: $S^{m-1} \longrightarrow F^m\{p^r\}$ is the bottom cell & gives map in. extend to modules.

Let's give a second description:

$$H_*(\Omega S^m) \hookrightarrow H_*(F^m\{p^r\}) \hookrightarrow H_*(\Omega S^m)\otimes\mathbb{Q}$$

we can identify with $p^{-r}H_*(\Omega S^m)$

[There's a dual operation $p^r\cdot -$ on n.u. algebras.]

$$H_*(\Omega P^m(p^r);\mathbb{Z}) \cong H_*(\Omega\Sigma P^{m-1}(p^r);\mathbb{Z}) \cong \mathrm{Free}_{\mathbb{Z}}(\mathbb{Z}/p^r)$$

① This is entirely copies of $\mathbb{Z}/p^r$.

② The Bockstein $\beta^{(r)}$ is determined by

$$\mathbb{F}_p\{u,v\} \cong H_*(\Omega P^m(p^r);\mathbb{F}_p) \cong E_s^r(\Omega P^m\{p^r\})$$

being a dg algebra with $\beta^{(r)}(v) = u$.

One more digression, then I'll get to what we're here for.

$H^*(\Omega^2 S^m; \mathbb{F}_p)$ is

if $m=2n+1$, $\mathbb{F}_p\langle \tau_k \cdots \rangle_{k\geq 0}[\sigma_\ell \cdots]_{\ell \geq 1}$

$|\tau_k| = 2p^k n - 1 \qquad |\sigma_\ell| = 2p^\ell n - 2$

σ homology suspension (comes from EM)

$\sigma(\tau_k) = \iota^{p^k}$

$\beta\tau_0 = 0$

$\beta\tau_k \doteq \sigma_k$.

if $m=2n$, $\mathbb{F}_p[u]\langle \tau_k \rangle_{k\geq 0}[\sigma_\ell]_{\ell\geq 1}$

$|u| = 2n-2$

$|\tau_k| = 2p^k(2n+1) - 1$

$|\sigma_\ell| = 2p^\ell(2n+1) - 2$

$\sigma(u) = \iota$

$\sigma(\tau_k) = [\iota,\iota]^{p^k}$

$\beta(u) = 0$

$\beta(\tau_k) \doteq \sigma_k$.

Note: β is exact, i.e. p is a homology exponent for $\Omega^2 S^{2n+1}$!

§4 The homology of $\Omega F^m\{p^r\}$

Recall: $H_*(\Omega F^m\{p^r\}; \mathbb{F}_p) \cong UL(x_k)_{k\geq 1}$.

We run the Bockstein.

[use check marks]

① $E^1_H(\Omega F^m\{p^r\}) \cong E^r_H(\Omega F^m\{p^r\})$ ← We already proved this.

② Determine $\beta^{(r)}$.

③ $E^{r+1}_H(\Omega F^m\{p^r\}) \cong E^1_H(\Omega^2 S^m)$

④ Determine $\beta^{(r+1)}$

⑤ The spectral sseq collapses at E^{r+2}_H.

Step 2:
$$\begin{array}{ccc} E^r_H(\Omega F^m\{p^r\}) & \hookrightarrow & E^r_H(\Omega P^m(p^r)) \\ & & \cong \\ UL(x_k)_{k\geq 1} & \longrightarrow & UL(u,v) \end{array}$$

$x_k \longmapsto ad^{k-1}(v)(u). \qquad \beta^{(r)}(v) = u$

Step 5:

Using formality of $C_*(\Omega F^m\{p^r\}; \mathbb{Z})$, and the divided model.

$$H_*(\Omega F^m\{p^r\};\mathbb{Z}) \cong H_*\Big(\mathrm{cobar}\big(p^{-r} H_*(\Omega S^m;\mathbb{Z})\big)\Big)$$

this look like

$$\mathbb{Z} \longrightarrow p^{-r} H_*(\Omega S^m) \longrightarrow p^{-2r} H_*(\Omega S^m) \longrightarrow \cdots$$

multiplying by p^r always puts us in a range where it looks like cobar $H_*\Omega S^m$. $\rightsquigarrow$ If a class is p-torsion, it's simple p-torsion.

Step 3:

We need to compute $E_H^{r+1}(\Omega F^m\{p^r\})$.

We already have a SES of Hopf algebras

$$\left[\; E_H^r(\Omega F^m\{p^r\}) \longrightarrow \underset{d(v)=u}{\mathbb{F}_p\{u,v\}} \longrightarrow \mathbb{F}_p\{\iota\} \;\right]$$

compare with

$$\left[\;\begin{array}{ccccc} \boxed{} & \longrightarrow & \underset{d(v)=u}{\mathbb{F}_p\{u,v\}} & \longrightarrow & \mathbb{F}_p\{\iota\} \\ \Downarrow & & & & \\ \Omega^2 S^m & \longrightarrow & * & \longrightarrow & \Omega S^m \end{array}\;\right]$$

They are the same!

Step 4:

The primitives of $H_*(\Omega^2 S^m)$ are exactly $\sigma_\ell^{p^k}$ & τ_k.

Using collapse, we must have $\beta^{(r+1)}(\tau_k) \doteq \sigma_k$.

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