Homotopy Exponents

March 31, 2022

Contents

Dates: usually year I first saw the proof; often precedes publication date
Spaces/maps localized at prime p
Unless specified otherwise, valid for any p

Notation and Terminology: using notation from 1980

$S^m\langle m\rangle :=$ homotopy fibre of $S^m \to K(\mathbf{Z}, m)$.
$C(n) :=$ homotopy fibre of $E^2 : S^{2n-1} \to \Omega^2 S^{2n+1}$
If X an H-space, $k : X \to X$ denotes $x \mapsto x^k$. (If not homotopy-associative pick some order, say left-to-right)
$X\{k\} :=$ homotopy fibre of $k : X \to X$.
Note: If X homotopy abelian $k : X \to X$ is an H-map so $X\{k\}$ an H-space in this case.
If $k : X \to X$ is null homotopic say "k is an H-space exponent for X" (stronger than homotopy exponent)
If X co-H-space, $\underline{k} : X \to X$ denotes k-fold product (in some order) of $1_X \in [X, X]$.
$P^m(k) := S^{m-1} \cup_{\underline{k}} e^m$ Moore space
$\pi_n(X; \mathbf{Z}/p) := [P^n(p), X]$
$\mathrm{Map}_*(X, Y) :=$ space of pointed maps from X to Y.
If X, Y H-groups (homotopy associative with homotopy inverse); $f : X \to Y$
$D(f) : X \times X \to Y$ by $D(f)(a, b) := f(b)^{-1} f(a)^{-1} f(ab)$. H-deviation.

1 Preliminaries

If $p > 2$, S^{2m-1} becomes homotopy assoc/abelian H-space

Proposition 1 *X H-space.* $\mathrm{Map}_*(S^n, X\{k\}) = \mathrm{Map}_*(P^{n+1}(k), X)$

Proof. Both are the homotopy fibre of $y \mapsto y^k$ in $\mathrm{Map}_*(S^n, X) = \Omega^n X$

Proposition 2 (Univ Coeff Thm)

$$0 \to \pi_n(X) \otimes \mathbf{Z}/p \to \pi_n(X; \mathbf{Z}/p) \to \mathrm{Tor}\big(\pi_{n-1}(X), \mathbf{Z}/p\big) \to 0$$

If $p > 2$:

Sequence splits

$1_{P^n(p)}$ has order p in $[P^n(p), P^n(p)]$ thus if Y is an H-space then $\Omega^m Y\{p\}$ has space exponent p

Proof. Cofibration $S^{n-1} \to P^n(p) \to S^n$ yields

$$\pi_{n-1}(X) \xleftarrow{p} \pi_{n-1}(X) \longleftarrow \pi_n(X; \mathbf{Z}/p) \longleftarrow \pi_n(X) \xleftarrow{p} \pi_n(X)$$

For $p > 2$, special case $X = P^n(p)$: $\pi_{n-1}\big(P^n(p)\big) = \mathbf{Z}/p$; $\pi_n\big(P^n(p)\big) = 0$
implies $[P^n(p); \mathbf{Z}/p] = \mathbf{Z}/p$ □

Corollary 3 *For $p > 2$*

$$P^k(p) \wedge P^n(p) \simeq P^{n+k-1}(p) \vee P^{n+k}(p)$$

Proof. Smash cofibration $S^{n-1} \to P^n(p) \to S^n$ with $P^k(p)$ to get cofibration

$$P^{n+k-1}(p) \to P^k(p) \wedge P^n(p) \to P^{n+k}(p)$$

using above (or otherwise) check it splits. □

Fails for $p = 2$:

Why should $p = 2$ behave differently?

Difference can be traced back to the fact that the Steenrod operation Sq^1 equals the Bockstein so coproduct of Sq^2 contains $\beta \otimes \beta$; nothing corresponding to this at an odd prime.

$P^{n+k-1}(2) \wedge P^{n+k}(2)$ has a nontrivial Sq^2 (by Cartan) so cannot be a wedge.

$1_{P^n(2)}$ has order 4 in $[P^n(2), P^n(2)]$

2 Between James/Toda and Cohen/Moore/Neisendorfer

Conjecture 4 (Barratt-Mahowald, ∼1965) *(Open for $p = 2$; solved by C-M-N for $p > 2$)*
p-torsion elements of $\pi_*(S^{2n+1})$ have order at most $p^{n+\epsilon}$ where

$$\epsilon = \begin{cases} 0 & p > 2 \text{ or } p = 2 \text{ and } n \equiv 0, 3 \mod 4; \\ 1 & p = 2 \text{ and } n \equiv 1, 2 \mod 4 \end{cases}$$

First open case: S^7
Conjectural bound 8; best known bound 32

Theorem 5 (Gray, 1969) *The bounds in conjecture 4 are best possible. (I.e. there exist elements having maximum order specified)*

Theorem 6 (Moore, 1976) *(Grad Course): p^{2n} is an H-space exponent for $\Omega^{2n}(S^{2n+1}\langle 2n+1\rangle)$*

2.1 Subsequent extensions

1. **Cohen/Moore/Neisendorfer (1980)** p odd: p^n is a space exponent for $\Omega^{2n}(S^{2n+1}\langle 2n+1\rangle)$

2. **Neisendorfer/Selick (1981)** $\Omega^{2n-2}(S^{2n+1}\langle 2n+1\rangle)$ has no space exponent;

   New proof (1986):

   $K(\mathbf{Z}, 2n) \to S^{2n+1}\langle 2n+1\rangle \to S^{2n+1} \to K(\mathbf{Z}, 2n+1)$

   G locally finite group.

   Miller (Sullivan Conj, 1984) $\text{Map}_*(BG, X) \simeq *$ for any finite complex X.

   $\implies \text{Map}_*\big(BG, K(\mathbf{Z}, 2n)\big) \simeq \text{Map}_*(BG, S^{2n+1}\langle 2n+1\rangle)$

   Apply $\pi_{2n-2}(\ ) \implies [BG, K(\mathbf{Z}, 2)] \cong [BG, \Omega^{2n-2}(S^{2n+1}\langle 2n+1\rangle)]$

   If $\Omega^{2n-2}(S^{2n+1}\langle 2n+1\rangle)]$ had H-exponent M then $H^2(BG) \cong [BG, K(\mathbf{Z}, 2)]$ would have uniform (independent of G) exponent $M \Rightarrow\Leftarrow$ $\square$

3. **Selick (1982)** If 2^r is an exponent for $\pi_*(S^{4n-1})$ then 2^{r+1} is a homotopy exponent for $\pi_*(S^{4n+1})$. Thus 2-torsion elements of $\pi_*(S^{2n+1})$ have order at most $2^{3n/2+\epsilon}$ (ϵ as above).

4. **Richter (1995)** $2^{3n/2+\epsilon}$ is a space exponent for $\Omega^{2n}(S^{2n+1}\langle 2n+1\rangle)$ when $p = 2$

5. **Selick (1996)** For p odd: $\Omega^{2n-1}(S^{2n+1}\langle 2n+1\rangle)$ has no space exponent.

   Uses Anick's fibration (**Anick, 1991)**

Open question: Does $\Omega^{2n-1}(S^{2n+1}\langle 2n+1\rangle)$ have a space exponent when $p = 2$?

3 Homotopy Exponent for S^3

4 is an exponent for 2-torsion of $\pi_k(S^3)$ by James

For the remainder of this section, assume $p > 2$.

Theorem 7 (Selick; 1977) *The p torsion of S^3 has exponent p.*

Proof.

Plan: $\Omega^2 S^{2n+1}\{p\}$ has exponent p.

Construct $s : \Omega^2 S^3\langle 3\rangle \to \Omega^2 S^{2n+1}\{p\}$ and $\tau : \Omega^2 S^{2n+1}\{p\} \to \Omega^2 S^3\langle 3\rangle$ such that $\tau \circ s$ induces a homology isomorphism and thus a homotopy equivalence.

Step 1: Construct τ

$BS^3 = \mathbf{H}P^\infty \to K(Z, 4)$ loops to $S^3 \to K(\mathbf{Z}, 3)$ so its fibre is a delooping $B(S^3\langle 3\rangle)$ of $S^3\langle 3\rangle$.

Let $\alpha' : S^{2p+1} \to B(S^3\langle 3\rangle)$ represent a generator of $\pi_{2p+1}\big(B(S^3\langle 3\rangle)\big) = \mathbf{Z}/p$, the least nonvanishing homotopy group.

$$
\begin{array}{ccc}
\Omega S^{2p+1} & \xrightarrow{\Omega\alpha'} & S^3\langle 3\rangle \\
\downarrow & & \| \\
S^{2p+1}\{p\} & \overset{\bar\alpha}{\cdots\to} & S^3\langle 3\rangle \\
\downarrow{\scriptstyle i} & & \downarrow \\
S^{2p+1} & \longrightarrow & E(S^3\langle 3\rangle) \\
\downarrow{\scriptstyle p} & & \downarrow \\
S^{2p+1} & \xrightarrow{\alpha'} & B(S^3\langle 3\rangle)
\end{array}
$$

$p\alpha' = 0 \implies$ bottom square commutes.

$\bar\alpha :=$ an induced map of homotopy fibres. $\tau := \Omega^2\bar\alpha : \Omega^2 S^{2n+1}\{p\} \to \Omega^2 S^3\langle 3\rangle$

Note: Homotopy class of $\bar\alpha$ is not uniquely determined by the diagram (depends on choices made in realizing as commuting diagram of topological spaces) but any choice makes the top square commute, which is sufficient to determine its induced map on homology.

Step 2:

Naturality of James-Hopf invariant

$$\begin{array}{ccc} \Omega S^{2n+1} & \xrightarrow{H_k} & \Omega S^{2nk+1} \\ \downarrow q & & \downarrow q^k \\ \Omega S^{2n+1} & \xrightarrow{H_k} & \Omega S^{2nk+1} \end{array}$$

Set $q = k = p$. Write H for H_p.

$H \circ p = p^p \circ H = p^p H$.

Note: $H \circ p \neq pH$ since H not an H-map.

First Attempt: Loop

$p^p(\Omega H) = p(\Omega H)$

$up(\Omega H) = 0$ where $u = p^{p-1} - 1$ (unit mod p) so $p(\Omega H) = 0$.

$$
\begin{array}{ccc}
 & & \Omega^2 S^{2np+1}\{p\} \\
 & \overset{\bar{H}}{\nearrow} & \downarrow \\
\Omega^2 S^{2n+1} & \xrightarrow{\Omega H} & \Omega S^{2np+1} \\
 & & \downarrow{\scriptstyle p} \\
 & & \Omega S^{2np+1}
\end{array}
$$

Let s be $\Omega^2 S\langle 3\rangle \to \Omega^2 S^3 \xrightarrow{\bar{H}} \Omega^2 S^{2p+1}\{p\}$

Potential Difficulty: It might not be easy to calculate s_* on homology.

Examine more closely and show that s can be chosen to be an H-map. Then homology calculation requires only knowing the images of the algebra generators.

Theorem 8 *Let $f : X \to Y$, where X a homotopy associative H-space, Y an H-group. Suppose*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m^p} \\ X & \xrightarrow{f} & Y \end{array}$$

Let $j : A \to X$ where (Lusternik-Schnirelman) category of A is less than p.
Then $pfj = 0 \in [A, Y]$.

Defn: $\text{cat}(A) < m$ iff A can be covered by m subsets which are contractible in A (null homotopic inclusion maps). Renumbered from the original L-S defn so cat(contractible set) $= 0$ rather than 1.

Whitehead: $\text{cat}(A) < m$ iff $A \xrightarrow{\Delta^{m-1}} A^m$ factors through the fat wedge of A^m.

Proof. Inductively define higher H-deviations $D_m(f) : X^m \to Y$ by

$D_m(f)(x_1, \ldots, x_m) :=$
$\big(D_{m-1}(f)(x_1, \ldots, x_{m-2}, x_m)\big)^{-1} \cdot \big(D_{m-1}(f)(x_1, \ldots, x_{m-2}, x_{m-1})\big)^{-1} \cdot \big(D_{m-1}(f)(x_1, \ldots, x_{m-2}, x_{m-1}x_m)\big)^{-1}$

and define $f_m : X \to Y$ by $X \xrightarrow{\Delta^{m-1}} X^m \xrightarrow{D_m(f)} Y$.

Induction on m gives

Lemma 9

$$D_m(x_1, \dots, x_m) = \sum_{j=1}^{m} \sum_{1 \le i_1 < \dots i_j \le m} D_j(f)(x_{i_1}, \dots, x_{i_j})$$

□

Setting $x_1 = x_2 = \cdots = x_m$ gives

Corollary 10 $f \circ m = \sum_{j=1}^{m} \binom{m}{j} f_j$

Lemma 11 *Given the diagram in the theorem, $f_m = \kappa_m f$ where p divides κ_m for $m > 1$.*

Proof.

$$m^p f = f \circ m = \sum_{i=1}^{m} \binom{m}{i} f_i$$

$$f_m = (m^p - m) f - \sum_{i=2}^{m-1} \binom{m}{i} f_i$$

Apply Fermat and induction. □

Proof of Theorem

$$f_p = (p^p - p)f - \sum_{i=2}^{p-1} \binom{p}{i} f_i$$

f_i and the binomial coeffs are divisible by p but not p^2.

Therefore all terms except one are divisible by p^2. Thus $f_p = upf$ some unit $u \in \mathbf{Z}_{(p)}$.

Using Whitehead reformulation of LS-category: $\operatorname{cat}(A) < p \implies f_p \circ j = 0$ so $pfj = 0$. $\square$

3.1 Construction of H-map $\Omega^2 S^{2n+1} \to \Omega^2 S^{2np+1}\{p\}$

$\Omega S^{2n+1} = B\Omega^2 S^{2n+1}$.

$B_1(\Omega^2 S^{2n+1}) \subset B_2(\Omega^2 S^{2n+1}) \subset \ldots \subset B_k(\Omega^2 S^{2n+1}) \subset \cdots \subset B_\infty(\Omega^2 S^{2n+1}) = \Omega S^{2n+1}$ Milnor filtration

Milnor gives charts showing LS-category $B_k(G) \leq k$ for any G.

Thus Thm. $\implies B_2(\Omega^2 S^{2n+1}) \xrightarrow{r} \Omega S^{2n+1} \xrightarrow{H} \Omega S^{2np+1} \xrightarrow{p} \Omega S^{2np+1}$ is null

Note use of $p > 2$.

$\exists$ lift $\lambda : B_2(\Omega^2 S^{2n+1}) \to \Omega S^{2np+1}\{p\}$

Lemma 12 *$G \xrightarrow{j} \Omega B_1(G) \to \Omega B_k(G)$ is an H-map for $k \geq 2$.*

Given Lemma: Apply with $G = \Omega^2 S^{2n+1}$.

Let S be

$$\Omega^2 S^{2n+1} \xrightarrow{j} \Omega B_2(\Omega^2 S^{2n+1}) \xrightarrow{\Omega\lambda} \Omega^2 S^{2np+1}\{p\}$$

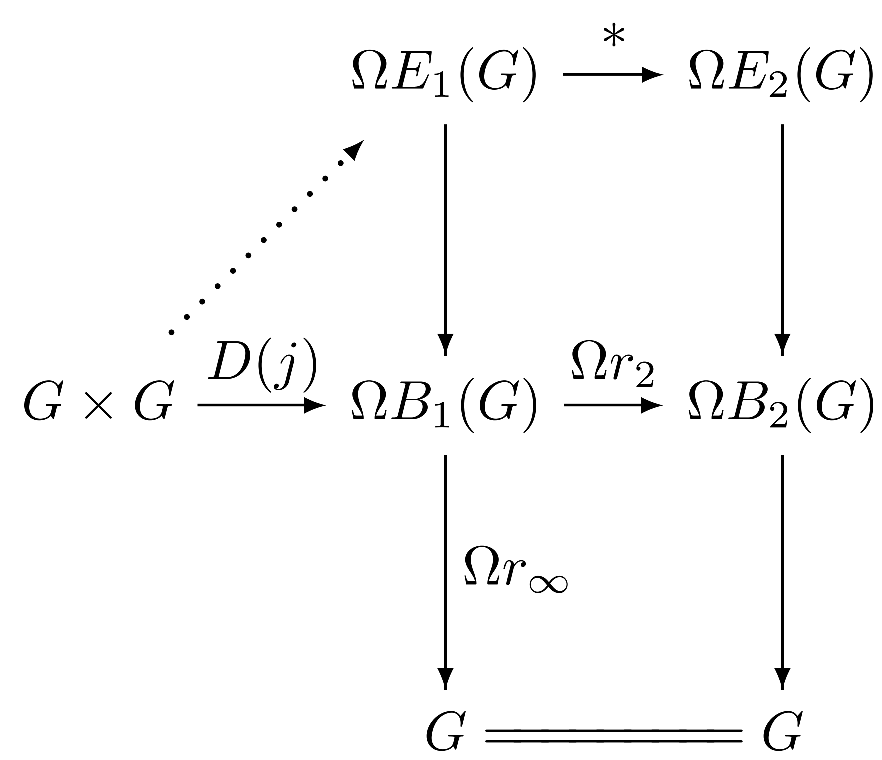
and let s be

$$\Omega^2(S^3\langle 3\rangle) \to \Omega^2 S^3 \xrightarrow{S} \Omega^2 S^{2p+1}\{p\}$$

(composite of H-maps)

Proof of Lemma: Suffices to consider $k = 2$.

$j : G \to \Omega\Sigma G = \Omega B_1(G)$



Since Ωr_∞ and 1_G are H-maps: $\Omega r_\infty \circ D(j) = D(\Omega r_\infty \circ j) = D(1_G) = 0$

Thus $D(\Omega r_\infty \circ j) = \Omega r_\infty \circ D(j)$ factors through $\Omega E_1(G) \to \Omega E_2(G)$, loop of null map $G * G \to G * G * G$

Step 3: Homology calculation

$r : B_2(\Omega S^3\langle 3\rangle) \to B_\infty(\Omega S^3\langle 3\rangle) = \Omega S^3\langle 3\rangle$
Set $\bar{j} : \Omega^2(S^3\langle 3\rangle) \to \Omega^2 S^3 \xrightarrow{j_2} \Omega B_2(\Omega^2 S^3)$ so $\Omega r \circ \bar{j} = 1_{\Omega^2 S^3}$
$\tau := \Omega^2\bar{\alpha} \qquad s := \Omega\lambda \circ \bar{j}$
$\tau \circ s = \Omega^2\bar{\alpha} \circ \Omega\lambda \circ \bar{j}$

$i : S^{2p+1}\{p\} \to S^{2p+1}$
Let H' be $\Omega S^3\langle 3\rangle \to \Omega S^3 \xrightarrow{H} \Omega S^{2p+1}$.
Recall Toda fibration

$$J_{p-1}S^2 \to \Omega S^3 \xrightarrow{H} \Omega S^{2p+1}$$

proved using

Lemma 13 (Toda, 1956) *$H_* : H_{2mp}(\Omega S^3) \to H_{2mp}(\Omega S^{2p+1})$ is multiplication by the unit $\frac{(mp)!}{m!(p!)^j}$.*

It follows that $(H' \circ \bar{\alpha})_* = (\Omega i)_* : H_*(\Omega S^{2p+1}\{p\}) \to H_*(\Omega S^{2p+1})$ up to mult by units — an automorphism of $H_*(\Omega S^{2p+1})$

As Hopf algs

$$H_*(\Omega S^3\langle 3\rangle) = (\mathbf{Z}/p)[x_{2p}] \otimes \Lambda[\beta x_{2p}]$$

$$H_*(\Omega^2 S^3\langle 3\rangle) = \otimes_{k=1}^{\infty}\big(\Lambda[a_{2p^k-1}] \otimes (\mathbf{Z}/p)[\beta a_{2p^k-1}]\big)$$

with generators primitive.

H-maps so suffices to check iso on indecomposables. Observing action of Bockstein checking odd degree incomposables will do.

Let $\sigma_* : \Sigma H_q(\Omega X) \to H_{q+1}(X)$ be homology suspension
Up to an automorphism (mult by the Toda units)

$$H'_*\sigma_*\tau_*s_* = H'_*\sigma_*(\Omega^2\bar{\alpha})_*(\Omega\lambda)_*\bar{j}_* = \sigma_*(\Omega^2 i)_*(\Omega\lambda)_*\bar{j}_* = \sigma_*(\Omega H)_*(\Omega r)_*\bar{j}_* = \sigma_*(\Omega H')_* = H'_*\sigma_*$$

But $H'\sigma_*$ is a monomorphism on $Q_{\mathrm{odd}}\big(H_*(\Omega^2 S^3\langle 3\rangle)\big)$ so $\tau_* s_*$ is iso in these dims. □

4 Later developments

Theorem 14 (Cohen/Mahowald, 1981) *Let* $f : \Omega^2 S^{2n+1} \to \Omega^2 S^{2n+1}$ *induce an isomorphism on* $H_{2n-1}(\ )$*. Suppose* $n > 1$*. Then* f *is a homotopy equivalence.*

Theorem 15 (Campbell/Peterson/Selick, 1985) *Let* $f : \Omega^k S^{m+1} \to \Omega^k S^{m+1}$ *induce an isomorphism on* $H_{m+1-k}(\ )$*. Suppose* $m > k$*. If* $p = 2$ *assume* $m \neq 1, 3, 7$*; if* $p > 2$ *assume* m *even. Then* f *is a homotopy equivalence.*

Idea of Proof: Show that commutativity with the coalgebra structure and Steenrod operations forces a homology isomorphism.

Definition 16 (Cohen/Moore/Neisendorfer) *A space whose least non-vanshing group has dimesion* 1 *and having the property that any self-map inducing an isomorphism on that group is a homotopy equivalence is called* atomic.

Using these ideas:

Theorem 17 $\Omega^2 S^3\langle 3\rangle$ *is atomic. That is, let* $f : \Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3\langle 3\rangle$ *induce an isomorphism on* $H_{2p-2}(\ )$*. Then* f *is a homotopy equivalence.*

Corollary: Return to "First "Attempt" of Step 2 of last section, skip the rest and write QED.

Assume p odd. C-M-N diagram

$$\begin{array}{ccccc}
D(n) & \longrightarrow & \Omega^2 S^{2n+1}\{p\} & \longrightarrow & C(n) \\
\| & & \downarrow & & \downarrow \\
D(n) & \longrightarrow & \Omega^2 S^{2n+1} & \xrightarrow{\pi} & S^{2n-1} \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle E^2} \\
 & & \Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1}
\end{array}$$

Corollary 18 *$D(p) \to \Omega^2 S^{2p+1}\{p\} \xrightarrow{\tau} \Omega^2 S^2\langle 3\rangle$ is a homology isomorphism.*

Proof. They have the same homology including coalgebra structure and Steenrod operations. Check iso on $H_{2p-2}(\ )$.

Corollary 19 *The fibration $D(p) \to \Omega^2 S^{2p+1}\{p\} \to C(p)$ splits. In particular, the other factor in the decomposition of $\Omega^2 S^{2p+1}\{p\}$ is $C(p)$.*

NonConjecture 20 *The fibration $D(n) \to \Omega^2 S^{2n+1}\{p\} \to C(n)$ splits*

Won't conjecture because

Theorem 21 (Selick, 1980) *$\Omega^2 S^{2np+1}\{p\}$ is indecomposable except (possibly) in cases where there exists an element of Arf invariant $1 \mod p$ in the stable homotopy group $\pi^s_{2n(p-1)-2}(S^0)$.*

Theorem 22 (Ravenel, 1978) *For $p > 3$, there exists an element of Arf invariant $1 \mod p$ in $\pi^s_{2n(p-1)-2}(S^0)$ only when $n = 1$.*

Thus for $p > 3$, Theorem 19 provides the only example where NonConjecture 20 holds.

$p = 3$: Arf invariant problem is still open. Steve Amelotte (2017) has provided an example where the fibration splits.

Theorem 23 (Amelotte, 2017) *Localized at 3, $\Omega S^{55}\{3\} \simeq B^2C(9) \times BC(27)$*

For $p = 2$

Theorem 24 (Cohen, 1983) $(\Omega^2 S^5)\{2\} \simeq \Omega^2 S^3\langle 3\rangle \times C(2)$

5 Cohen's $p = 2$ version of Selick's Thm

$p = 2$

$2 = \Omega(\underline{2}) : \Omega S^{2n+1} \to \Omega S^{2n+1}$ iff S^{2n+1} H-space.

We noted problem at $p = 2$ showing existence of H-lift $\Omega^2 S^{2n+1} \to (\Omega^2 S^{2n+1})\{2\}$

Using atomicity, don't need an H-lift: can use any lift to prove Cohen's Thm. 24, but not obvious any lift exists.

James still gives $\Omega(\underline{4}) \circ H = H \circ \underline{2}$ but even after looping this does not give $4\Omega H = 2\Omega H$.

Thm. 24 will follow from atomicity if we can show $2\Omega H = 0$ in some other way.

Theorem 25 (Cohen, 1982; Barratt was here?) $2\Omega H = 0 : \Omega^2 S^{2n+1} \to \Omega^2 S^{4n+1}$

Proof.

$\Sigma[\iota_m, \iota_m] = 0$

$2[\iota_{2n+1}, \iota_{2n+1}] = 0$

$[[\iota_{2n+1}, \iota_{2n+1}], \iota_{2n+1}] = 0$

Set $w := [\iota_{2n+1}, \iota_{2n+1}] : S^{4n} \to S^{2n+1}$

$S^{2n} \to \Omega S^{2n+1} \xrightarrow{H} \Omega S^{4n+1}$ (James)

$2H_\#(w) = 0 \in \pi_{4n}(\Omega S^{4n+1}) = \mathbf{Z}$ so $H_\#(w) = 0$

Therefore, $w = \Sigma f$, some $f \in \pi_{4n}(S^{2n})$

$f \wedge f = f \circ \Sigma^{2n} f = f \circ \Sigma^{2n-1} w = 0 \in \pi_{8n}(S^{4n})$

$$\begin{array}{ccc} J(S^{4n}) & \xrightarrow{J(f) = \Omega w} & J(S^{2n}) \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} \\ J(S^{8n}) & \xrightarrow{J(f \wedge f) = 0} & J(S^{4n}) \end{array}$$

Thus $H \circ \Omega w = 0$
Also $H \circ \Omega(\underline{-1}) = H$ James

Hilton-Milnor: $h :=$ Hilton-Hopf invariant

$$\begin{array}{ccc} \Omega S^{2n+1} & \xrightarrow{\Omega(\underline{j+k})} & \Omega S^{2n+1} \\ \downarrow{\scriptstyle \Delta^2} & & \uparrow{\scriptstyle m} \\ \Omega(S^{2n+1})^3 & \xrightarrow{\Omega(\underline{j}) \times \Omega(\underline{k}) \times (\Omega w \circ \Omega(\underline{jk}) \circ h)} & \Omega(S^{2n+1})^3 \end{array}$$

(Would have been more terms, but $[[\iota_{2n+1}, \iota_{2n+1}], \iota_{2n+1}] = 0$)
$\Omega^2(\underline{j+k}) = \Omega^2(\underline{j}) + \Omega^2(\underline{k}) + w \circ \Omega^2(\underline{jk}) \circ h \in [\Omega^2 S^{2n}, \Omega^2 S^{2n}]$
$0 = 1 + \Omega^2(\underline{-1}) + w \circ \Omega^2(\underline{-1}) \circ h.$ Compose with ΩH:
$0 = \Omega H + \Omega H \circ \Omega^2(\underline{-1}) + \Omega H \circ \Omega w \circ \Omega^2(\underline{-1}) \circ \Omega h = \Omega H + \Omega H + 0$
$2\Omega H = 0$ □

Theorem 26 (Selick, 1982) *If 2^r is an exponent for $\pi_*(S^{4n-1})$ then 2^{r+1} is a homotopy exponent for $\pi_*(S^{4n+1})$.*

Proof. Let Q be the pullback of $H : \Omega S^{2n+1} \to \Omega S^{4n+1}$ and $\Omega S^{4n+1}\{2\} \to \Omega S^{4n+1}$.

$$
\begin{array}{ccccccccc}
 & & & & \Omega S^{2n} & = & \Omega S^{2n} & & \\
 & & & & \downarrow a & & \downarrow & & \\
 & & \Omega^3 S^{4n+1} & \xrightarrow{j} & \Omega Q & \longrightarrow & \Omega^2 S^{2n+1} & & \\
 & & \| & & \downarrow b & & \downarrow \Omega H & & \\
\Omega^3 S^{4n+1} & \xrightarrow{2} & \Omega^3 S^{4n+1} & \xrightarrow{k} & \Omega^2 S^{4n+1}\{2\} & \longrightarrow & \Omega^2 S^{4n+1} & \xrightarrow{2} & \Omega^2 S^{2n+1}
\end{array}
$$

By Cohen, $2\Omega H = 0$ so principal fibration $\Omega^3 S^{4n+1} \to \Omega Q \to \Omega^2 S^{2n+1}$ splits. Therefore $\exists\ s : \Omega Q \to \Omega^3 S^{4n+1}$ such that $s \circ j = 1_{\Omega^3 S^{4n+1}}$

Let $x \in \pi_*(\Omega^3 S^{4n+1})$.

$b_\# j_\# (2x) = 2k_\#(x) = 0$

Therefore $j_\#(2x) = a_\#(y)$, some $y \in \pi_*(\Omega S^{2n})$

$S^{2n-1} \xrightarrow{E} \Omega S^{2n} \xrightarrow{H'} \Omega S^{4n-1}$
$H'_{\#}(2^r y) = 2^r H'_{\#}(y) = 0$ by exponent of $\pi_*(\Omega S^{4n-1})$ so $2^r y = E(z)$, some $z \in \pi_*(S^{2n-1})$.
$saE = 0 : S^{2n-1} \to \Omega^3 S^{4n+1}$
Therefore $0 = s_{\#} a_{\#}(Ez) = 2^r s_{\#} a_{\#}(y) = 2^{r+1} s_{\#} j_{\#}(x) = 2^{r+1}$. □

To proceed further we need to show $C(n)$ is a loop space (Gray, 1985).

6 Delooping the homotopy fibre of the double suspension

Conjecture 27 (Barratt-Mahowald∼1965) *$C(n)$ is a loop space.*
For $p > 2$ (and sometimes for $p = 2$) it is a double loop space.

$J_k S^{2n} = J_{k-1} S^{2n} \cup_{w_k} e^{2nk}$
Note: $w_2 = [\iota_{2n}, \iota_{2n}] : S^{4n-1} \to S^{2n}$ (we do not use this)

$$J_{p-1} S^{2n} \to \Omega S^{2n+1} \xrightarrow{H} \Omega S^{2np+1} \qquad \text{(James, } p = 2\text{; Toda, } p > 2\text{)}$$

$$\Longrightarrow \qquad \Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \xrightarrow{\partial} J_{p-1} S^{2n}$$

For $k \le p-1$, let $F_k \subset \Omega^2 S^{2np+1}$ be the restriction (pullback) to $J_k S^{2n} \subset J_{p-1} S^{2n}$

Ganea-style argument: Take inverse images under ∂.
Restriction to $e^{2n(p-1)}$ and its subsets are trivial.
$\Omega^2 S^{2np+1} = F_{p-2} \cup_{(S^{2n(p-1)-1} \times \Omega^2 S^{2n+1})} (e^{2n(p-1)} \times \Omega^2 S^{2n+1})$

Equivalently pushout/cofibration diagram

$$\begin{array}{ccc}
S^{2n(p-1)-1} \times \Omega^2 S^{2n+1} & \longrightarrow & e^{2n(p-1)} \times \Omega^2 S^{2n+1} \\
\downarrow & & \downarrow {\scriptstyle * \times \Omega H} \\
F_{p-2} & \longrightarrow & \Omega^2 S^{2np+1} \\
\downarrow & & \downarrow \\
F_{p-2}/(S^{2n(p-1)-1} \times \Omega^2 S^{2n+1}) & = & \Omega^2 S^{2np+1}/\Omega^2 S^{2n+1} \\
\downarrow & & \downarrow \\
\Sigma(S^{2n(p-1)-1} \times \Omega^2 S^{2n+1}) & \longrightarrow & \Sigma(e^{2n(p-1)} \times \Omega^2 S^{2n+1})
\end{array}$$

$$\begin{aligned}
\Sigma(S^{2n(p-1)-1} \times \Omega^2 S^{2n+1}) &= \Sigma S^{2n(p-1)-1} \vee \Sigma\Omega^2 S^{2n+1} \vee \Sigma(S^{2n(p-1)-1} \wedge \Omega^2 S^{2n+1}) \\
&\to \Sigma^{2n(p-1)}\Omega^2 S^{2n+1} \to S^{2n(p-1)+2n-1} = S^{2np-1}
\end{aligned}$$

Let G denote the composite

$$\Omega^2 S^{2np+1} \to \Omega^2 S^{2np+1}/\Omega^2 S^{2n+1} = F_{p-2}/(S^{2n(p-1)-1} \times \Omega^2 S^{2n+1}) \to \Sigma(S^{2n(p-1)-1} \times \Omega^2 S^{2n+1}) \to S^{2np-1}$$

Let $B(n)$ denote the homotopy fibre of G.
Note: $B(1) \simeq \Omega^2 S^3\langle 3\rangle$

Theorem 28 (Gray, 1985) $\Omega B(n) \simeq C(n)$
(Works at any prime; does not require localization.)

Proof. $\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \xrightarrow{G} S^{2np-1}$ is null by construction

$$\begin{array}{ccccc} & & \Omega^2 S^{2n+1} & & \\ & \overset{\nu}{\swarrow} & \downarrow \Omega H & & \\ B(n) & \longrightarrow & \Omega^2 S^{2np+1} & \xrightarrow{G} & S^{2np-1} \end{array}$$

Serre SS (or otherwise) shows $H_*(\Omega^2 S^{2n+1}) \cong H_*(S^{2n-1}) \otimes H_*\big(B(n)\big)$ and combining the diagram with Toda's calculation of $(\Omega H)_*$ shows that the homotopy fibre of ν is S^{2n-1}.

Therefore $\Omega B(n) \simeq C(n)$. $\square$

Major open conjecture:

Conjecture 29 *For $p > 2$, choices in Gray/C-M-N can be made so that $G = \pi : \Omega^2 S^{2np+1} \to S^{2np-1}$ up to some homotopy equivalences of domain and range. In particular $D(np) \simeq B(n)$.*

In particular, a key step in the proof above is that $G \circ \Omega H$ is null; it is conjectured but not known whether $\pi \circ \Omega H$ is null.

Remark 30 *By now there are multiple constructions of maps $\Omega^2 S^{2np+1} \to S^{2np-1}$ whose homotopy fibre loops to $C(n)$. (I think I know at least 8.) None are known to be "equivalent" to π.*

Further evidence for conjecture $B(n) = D(np)$:
Arbitrary p

Theorem 31 (Richter, 2014; Harper, 1989 for $p > 2$ after looping))
$\Omega^2 S^{2np+1} \xrightarrow{G} S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1}$ *is multiplication by p.*

Corollary 32 *Fibration $B(n) \to \Omega^2 S^{2np+1}\{p\} \to C(np)$.*

$p = 2$ analogues of $\Omega^2 S^{2p+1}\{p\}$ splitting

Must distinguish between $(\Omega^k S^{2m+1})\{2\}$ and $\Omega^k(S^{2m+1}\{\underline{2}\})$ (same in Hopf Inv 1 cases)

Theorem 33

1. $(\Omega^2 S^5)\{2\} \simeq B(1) \times C(2)$ *(Cohen, 1983)*
2. $(\Omega^2 S^9)\{2\} \simeq B(2) \times C(4)$ *(Cohen/Selick, 1986)*
3. $(\Omega^3 S^{17})\{2\} \simeq \Omega\big(B(4) \times C(8)\big)$ *(Amelotte, 2017)*

Theorem 34 (Campbell, Cohen, Peterson, Selick, 1983) $(\Omega^2 S^{2^t+1})\{2\}$ *is indecomposable unless there exists Arf Invariant one element $\theta \in \pi^S_{2^t-2}$ s.t. $\eta\theta$ is divisible by 2. This condition implies $t \leq 4$.*

Theorem 35 ($p > 3$: **Gray, 1985;** $p = 3$: **Theriault, 2006)**

Proof. Fibration $Y_k \to J_k S^{2n} \to \Omega S^{2n+1}$ defines Y_k (Thus $Y_{p-1} = \Omega^2 S^{2np+1}$)
Ganea argument yields $G_k : Y_k \to S^{2n(k+1)-1}$ for $1 \le k \le p-1$
Fibration $B_k \to Y_k \to S^{2nk+2n-1}$ defines B_k. (Thus $B_{p-1} = B(n)$)

$$\begin{array}{ccccc} & & Y_{k-1} & \longrightarrow & S^{2n(k-1)+2n-1} = S^{2nk-1} \\ & \swarrow & \downarrow & & \downarrow * \\ B_k & \longrightarrow & Y_k & \longrightarrow & S^{2nk+2n-1} \end{array}$$

$$B_1 \to Y_1 \to B_2 \to \cdots \to B_{k-1} \to Y_{k-1} \to B_k \to \cdots \to B_{p-1} = B(n) \to Y_{p-1} = \Omega^2 S^{2np+1}$$

Homology shows $B_1 \to \cdots \to B_{k-1} \to B_k \to \cdots \to B_{p-1} = B(n)$ homotopy equivalences.
(Thus if $k < p-1$ the fibration defining B_k splits to give $Y_k \simeq B(n) \times S^{2n(k+1)-1}$.)

$$\begin{array}{ccccc} Y_k \times Y_m & \longrightarrow & J_k S^{2n} \times J_m S^{2n} & \longrightarrow & \Omega S^{2n+1} \times \Omega S^{2n+1} \\ \downarrow & & \downarrow & & \downarrow \\ Y_{k+m} & \longrightarrow & J_{k+m} S^{2n} & \longrightarrow & \Omega S^{2n+1} \end{array}$$

$$B(n) \times B(n) \simeq B_1 \times B_1 \longrightarrow Y_1 \times Y_1 \longrightarrow Y_2 \longrightarrow B_2 \simeq B(n).$$

Restriction to each summand of $B(n) \vee B(n)$ is the homotopy equivalence $B(n) \simeq B_1 \simeq B_2 \simeq B(n)$ so this defines a H-space structure on $B(n)$.

Letting ν_k denote $\Omega^2 S^{2n+1} \xrightarrow{\nu} B(n) \simeq B_k$. For $p > 3$

$$\begin{array}{ccccc}
\Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1} & & \longrightarrow & & \Omega^2 S^{2n+1} \\
\downarrow{\scriptstyle \nu_1 \times \nu_1} & & & & \downarrow{\scriptstyle \nu_3} \\
B_1 \times B_1 & \longrightarrow & Y_2 & \longrightarrow & B_3
\end{array}$$

ν_k corresponds to ν under $B_k \simeq B(n)$ so ν is an H-map.
$p = 3$: Existence of H-space structure works; H-map proof fails since $B_k \simeq B(n)$ requires $k \leq p - 1$.

Theriault (2006) for $p = 3$ is complicated. (Skipped) □

Lemma 36 *Let* $F \xrightarrow{j} E \xrightarrow{q} B$ *be a fibration with a retraction* $r : \Sigma^k E \to \Sigma^k F$ *of* $\Sigma^k j$*. Then* $\Sigma^k E \simeq \Sigma^k (F \times B)$.

Proof.

$$\Sigma^k E \xrightarrow{\Sigma^k \Delta} \Sigma^k (E \times E) \simeq \Sigma^k E \vee \Sigma^k E \vee \Sigma^k (E \wedge E) \xrightarrow{r \vee \Sigma^k q \vee (r \wedge q)} \Sigma^k F \vee \Sigma^k B \vee \Sigma^k (F \wedge B) \simeq \Sigma^k (F \times B)$$

induces a homology isomorphism.

Corollary 37 $\Sigma^2 \Omega^2 S^{2n+1} \simeq \Sigma^2 \big(S^{2n-1} \times B(n) \big)$

Thus $\Sigma^2\nu : \Sigma^2\Omega^2 S^{2n+1} \to \Sigma^2 B(n)$ has a right homotopy inverse.

Lemma 38 (Theriault, 2006) *Let $f : X \to Y$ s.t. $\Sigma^2 f$ has a right homotopy inverse. Let Z be an H-space and let $g, h : Y \to Z$ such that $g \circ f \simeq h \circ f$. Then $g \simeq h$.*

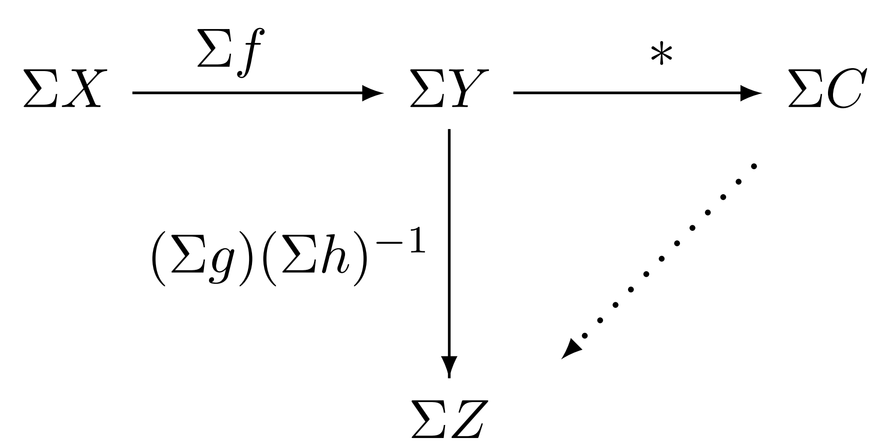
Proof. Since Z is an H-space there is a retraction of $Z \to \Omega\Sigma Z$ so it suffices to show $\Sigma g \simeq \Sigma h$.

Let C be the homotopy cofibre of f.

$$X \to Y \to C \to \Sigma X \to \Sigma Y \xrightarrow{*} \Sigma C \to \Sigma^2 X \to \Sigma^2 Y \to \dots$$

Therefore $\Sigma^2 X \simeq \Sigma^2 Y \vee \Sigma C$ and $\Sigma Y \to \Sigma C$ is null homotopic.

$\big((\Sigma g)(\Sigma h)^{-1}\big) \circ \Sigma f \simeq *$



Therefore $\Sigma g \simeq \Sigma h$. $\square$

Corollary 39 (Theriault, 2006)
$p \geq 3$

1. *$B(n)$ has a unique H-space structure such that $\nu : \Omega^2 S^{2n+1} \to B(n)$ and $B(n) \xrightarrow{\mu} \Omega^2 S^{2np+1}$ are H-maps. This H-structure is homotopy associative.*
2. *p is an H-space exponent for $B(n)$.*

Proof.

1. $\Sigma^2(\Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1}) \xrightarrow{\nu \times \nu} B(n) \times B(n)$ has a right homotopy inverse.

   If $m, m' : B(n) \times B(n) \to B(n)$ are H-maps then $m \circ (\nu \times \nu) = \nu \circ m_{\Omega^2 S^{2n+1}} = m' \circ (\nu \times \nu)$ so $m \simeq m'$.

   By Lemma, to show μ is an H-map it suffices to check that $\nu \circ \mu$ is an H-map. However $\nu \circ \mu = \Omega H : \Omega^2 S^{2n+1} \to \Omega^2 S^{2np+1}$.

   $\Sigma^2(\Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1}) \xrightarrow{\nu \times \nu \times \nu} B(n) \times B(n) \times B(n)$ has a right homotopy inverse.

   By Lemma, to show $m \circ (m \times 1_{B(n)}) = m \circ (1_{B(n)} \times m) : B(n) \times B(n) \times B(n) \to B(n)$ it suffices to show their compositions with $\nu \times \nu \times \nu$ are equal, which follows from the facts that $\Omega^2 S^{2n+1}$ is homotopy associative and ν is an H-map. Thus $B(n)$ is homotopy associative.

2. Since ν is an H-map, $p \circ \nu = \nu \circ p : \Omega^2 S^{2n+1} \to B(n)$. By C-M-N, the latter is

$$\Omega^2 S^{2n+1} \xrightarrow{\pi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} B(n)$$

which is null (last two maps are a fibration sequence).

Since $p \circ \nu = *$ and $\Sigma^2 \nu$ has a right homotopy inverse, Lemma implies $p \simeq *$.

□

7 Anick's fibration

$p \geq 3$.

Conjecture 40 (∼1980)

$\exists$ a fibration of H-space/maps $S^{2n-1} \to X(n) \to \Omega S^{2n+1}$ with the following properties

1. The boundary map $\partial(n) : \Omega^2 S^{2n+1} \to S^{2n-1}$ is M-N-C's π
2. The composite $\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{2np+1} \xrightarrow{\partial(np)} S^{2np-1}$ is null homotopic
3. $\Omega^2 X(np) \simeq C(n)$ (satisfying Barratt-Mahowald)

The last part is a consequence of the first two as follows:

$$\begin{array}{ccccc} & & \Omega^2 S^{2n+1} & & \\ & \overset{\lambda}{\swarrow} & \downarrow{\scriptstyle \Omega H} & & \\ \Omega X(np) & \longrightarrow & \Omega^2 S^{2np+1} & \xrightarrow{\partial} & S^{2np-1} \end{array}$$

By Serre (or otherwise), the homotopy fibre of λ is S^{2n-1} so $\Omega^2 X(np) = C(n)$.

Theorem 41 (Anick, 1991) *For $p \geq 5$, $\exists$ fibration $S^{2n-1} \to A(n) \to \Omega S^{2n+1}$ satisfying Part (1) of Conjecture 40.*

Subsequent work (**Anick-Gray, 1993**; **Theriault, 2001**) showed that $A(n)$ is a homotopy associative, homotopy commutative H-space, that the maps in the fibration are H-maps, and that $A(n)$ has the universal property: any map $P^{2n}(p) \to Z$ to a homotopy associative, homotopy commutative H-space Z extends uniquely to an H-map $A(n) \to Z$.

7.1 Theriault's Construction (2007)

Notation: Let $X^{(m)}$ denote m-fold smash product of X and let $X_{(m)}$ denote the m-skeleton of X.

$$\Omega^2 S^{2n+1} \to \Omega S^{2n+1}\{p\} \xrightarrow{\Omega j} \Omega S^{2n+1} = J(S^{2n})$$

Wish to get

$$\begin{array}{ccccc} C(n) & \longrightarrow & A(n) & \longrightarrow & \Omega S^{2n+1}\{p\} \\ \downarrow & & \| & & \downarrow \Omega j \\ S^{2n-1} & \longrightarrow & A(n) & \longrightarrow & \Omega S^{2n+1} \end{array}$$

There is no BS^{2n-1} (in general) but we do have a delooping of $C(n)$.
Plan:

Construct appropriate $\phi : \Omega S^{2n+1}\{p\} \to B(n)$ and take homotopy fibre to get $A(n)$.

Not using C-M-N but will use their notation.

$$\begin{array}{ccccc}
\Omega E^{2n+1}(p) & \longrightarrow & \Omega P^{2n+1}(p) & \xrightarrow{\Omega q'} & \Omega S^{2n+1}\{p\} \\
\downarrow & & \| & & \downarrow \Omega j \\
\Omega F^{2n+1}(p) & \longrightarrow & \Omega P^{2n+1}(p) & \xrightarrow{\Omega q} & \Omega S^{2n+1}
\end{array}$$

Compatible filtrations:

$\mathcal{F}_k(\Omega S^{2n+1}\{p\}) := (\Omega j)^{-1} J_k(S^{2n})$

$\mathcal{F}_k\big(\Omega P^{2n+1}(p)\big) := (\Omega q')^{-1}\big(\mathcal{F}_k(\Omega S^{2n+1}\{p\})\big) = (\Omega q)^{-1} J_k(S^{2n})$

$\mathcal{F}_0(\Omega S^{2n+1}\{p\}) = \Omega^2 S^{2n+1}$

Set $\phi_0 := \nu : \mathcal{F}_0(\Omega S^{2n+1}\{p\}) \to B(n)$.

Suppose by induction that $\phi_{k-1} : \mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\}) \to B(n)$ has been defined

$$J_k S^{2n} = J_{k-1} S^{2n} \cup_{f_k} e^{2nk}$$

Ganea:

$$\mathcal{F}_k(\Omega S^{2n+1}\{p\}) = \mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\}) \cup_{(S^{2nk-1}\times \Omega^2 S^{2n+1})} (e^{2nk} \times \Omega^2 S^{2n+1})$$

I.e. Homotopy Pushout

$$\begin{array}{ccc} S^{2nk-1} \times \Omega^2 S^{2n+1} & \xrightarrow{\pi_2} & \Omega^2 S^{2n+1} \\ \downarrow \mu & & \downarrow \\ \mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\}) & \longrightarrow & \mathcal{F}_k(\Omega S^{2n+1}\{p\}) \end{array}$$

The restriction of the "action map" μ to $\Omega^2 S^{2n+1}$ is the connecting map ∂ in the defining fibration $\mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\}) \to J_{k-1}(S^{2n}) \to \Omega S^{2n+1}$.

To verify this:

$$
\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} \\
\| & & \downarrow & & \downarrow{\scriptstyle \partial} & & \downarrow \\
\Omega^2 S^{2n+1} & \longrightarrow & S^{2nk-1} \times \Omega^2 S^{2n+1} & \xrightarrow{\mu} & \mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\}) & \longrightarrow & \Omega S^{2n+1}\{p\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & S^{2nk-1} & \xrightarrow{f_k} & J_{k-1}(S^{2n}) & \longrightarrow & \Omega S^{2n+1}
\end{array}
$$

By naturality of action maps, the middle column gives

$$
\begin{array}{ccc}
S^{2nk-1} \times \Omega^2 S^{2n+1} & \xrightarrow{\mu} & \mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\}) \\
\downarrow{\scriptstyle g_k \times 1_{\Omega^2 S^{2n+1}}} & & \| \\
\mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\}) \times \Omega^2 S^{2n+1} & \xrightarrow{\mu'} & \mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\})
\end{array}
$$

where $g_k : S^{2nk-1} \to \mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\})$ is the restriction of μ to S^{2nk-1}.

$B(n)$ homotopy associative, so any map $f : X \to B(n)$ has a unique multiplicative extension $\tilde{f} : \Omega\Sigma X \to B(n)$.

To induce an extension $\phi_k : \mathcal{F}_k(\Omega S^{2n+1}\{p\}) \to B(n)$ of ϕ_{k-1} we need to show

$S^{2nk-1} \times \Omega^2 S^{2n+1} \xrightarrow{\pi_2} \Omega^2 S^{2n+1} \xrightarrow{\nu} B(n)$

equals

$S^{2nk-1} \times \Omega^2 S^{2n+1} \xrightarrow{\mu} \mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\}) \xrightarrow{\phi_{k-1}} B(n)$ or, equivalently, that their multiplicative extensions are equal. Show:

$$\begin{array}{ccccc} S^{2nk-1} \times \Omega^2 S^{2n+1} & \longrightarrow & \Omega\Sigma(S^{2nk-1} \times \Omega^2 S^{2n+1}) & \xrightarrow{\widetilde{\phi_{k-1} \circ \mu}} & B(n) \\ & & \downarrow{\scriptstyle \Omega\Sigma\pi_2} & & \Vert \\ & & \Omega\Sigma\Omega^2 S^{2n+1} & \xrightarrow{\tilde{\nu}} & B(n) \end{array}$$

$\Omega\Sigma(S^{2nk-1} \times \Omega^2 S^{2n+1}) = \Omega\big(\Sigma S^{2nk-1} \vee \Sigma\Omega^2 S^{2n+1} \vee \Sigma(S^{2nk-1} \wedge \Omega^2 S^{2n+1})\big)$

The restrictions to $\Omega^2 S^{2n+1}$ are equal (both are ν)

Need to show triviality of

$S^{2nk-1} \to \Omega\Sigma S^{2nk-1} \to \Omega\Sigma(S^{2nk-1} \times \Omega^2 S^{2n+1}) \xrightarrow{\widetilde{\phi_{k-1} \circ \mu}} B(n)$

and

$S^{2nk-1} \wedge \Omega^2 S^{2n+1} \to \Omega\Sigma(S^{2nk-1} \wedge \Omega^2 S^{2n+1}) \to \Omega\Sigma(S^{2nk-1} \times \Omega^2 S^{2n+1}) \xrightarrow{\widetilde{\phi_{k-1} \circ \mu}} B(n)$

Since $B(n)$ is an H-space suffices to show their compositions with $B(n) \to \Omega\Sigma B(n)$ are equal.
Adjoint:

First is

$$\Sigma S^{2nk-1} \xrightarrow{\Sigma g_k} \Sigma\mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\}) \xrightarrow{\Sigma\phi_{k-1}} \Sigma B(n)$$

Second is

$$\Sigma S^{2nk-1} \wedge \Omega^2 S^{2n+1} \xrightarrow{\Sigma g_k \wedge 1_{\Omega^2 S^{2n+1}}} \Sigma\mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\}) \wedge \Omega^2 S^{2n+1} \to$$
$$\Sigma\big(\mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\}) \times \Omega^2 S^{2n+1}\big) \xrightarrow{\Sigma\mu'} \Sigma\mathcal{F}_{k-1}(\Omega S^{2n+1}\{p\}) \xrightarrow{\Sigma\phi_{k-1}} \Sigma B(n)$$

$B(n)$ is homotopy associative so the retraction $\Omega\Sigma B(n) \to B(n)$ is an H-map.

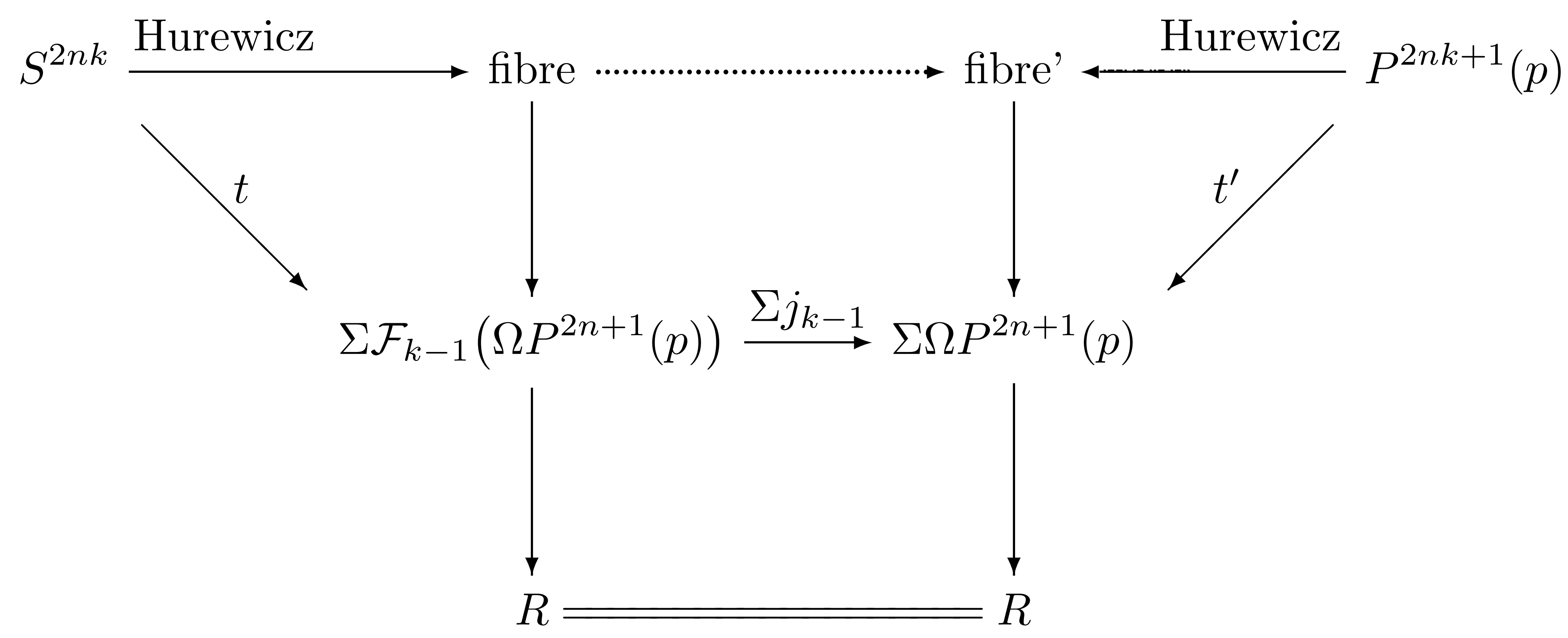
Thus since $B(n)$ has H-exponent p, to show these are trivial it suffices to show that Σg_k is divisible by p.

The similar Ganea pushout diagram for $\Omega P^{2n+1}(p)$ gives a lift $\bar{g}_k : S^{2nk-1} \to \mathcal{F}_{k-1}\big(\Omega P^{2n+1}(p)\big)$
Show Σg_k is divisible by p by showing that its lift $\Sigma\bar{g}_k$ is divisible by p.

$P^a(p) \wedge P^b(p) = P^{a+b} \vee P^{a+b-1}$ yields by induction $\big(P^{2n}(p)\big)^{(k)} = P^{2nk}(p) \vee (\ )$
James: $\Sigma\Omega P^{2n+1}(p) = \vee_{j=1}^{\infty}\Sigma\big(P^{2n}(p)\big)^{(j)} = P^{2nk+1}(p) \vee R$ for some wedge R of Moore spaces.

The homotopy fibre of the composite $\Sigma\mathcal{F}_{k-1}\big(\Omega P^{2n+1}(p)\big) \to \Sigma\Omega P^{2n+1}(p) \to R$ is $(2nk-1)$ connected with $H_{2nk}(\ ) = \mathbf{Z}$.

Let t be the composite $S^{2nk} \to \text{homotopy fibre} \to \Sigma\mathcal{F}_{k-1}\big(\Omega P^{2n+1}(p)\big)$.



Show $pt = \Sigma\bar{g}_k$.

Proof. Through the $(2nk+1)$ skeleton, the split cofibration

$$P^{2nk+1}(p) \to \Sigma\Omega P^{2n+1}(p) \to R$$

is also a fibration. In particular

$$\pi_{2nk}\big(\Sigma\Omega P^{2n+1}(p)\big) = \mathbf{Z}/p \oplus \pi_{2nk}(R).$$

The composite $S^{2nk} \xrightarrow{t} \Sigma\mathcal{F}_{k-1}\big(\Omega P^{2n+1}(p)\big) \to R$ is null, so $\Sigma j_{k-1} \circ t$ lifts to $P^{2nk+1}(p)$ and is thus determined by its image on $H_{2nk}(\ )$; checking homology shows it is the standard inclusion $i : S^{2nk} \to P^{2nk+1}(p)$.

Thus $\Sigma j_{k-1} \circ t = t' \circ i$ which is the composition
$S^{2nk} \xrightarrow{i} P^{2nk+1}(p) \to \Sigma P^{2n}(p)^{(k)} \to \Sigma\Omega P^{2n+1}(p)$.

Define X, Y, Z as the homotopy fibres in the fibration diagram

$$\begin{array}{ccccc}
X & \longrightarrow & Y & \longrightarrow & Z \\
\| & & \downarrow & & \downarrow \\
X & \longrightarrow & \Sigma\mathcal{F}_{k-1}\big(\Omega P^{2n+1}(p)\big) & \longrightarrow & \Sigma\Omega P^{2n+1}(p) \\
 & & \downarrow & & \downarrow \\
 & & R & = & R
\end{array}$$

Upon restricting to $(2nk+1)$ skeletons the diagram becomes

$$
\begin{array}{ccccc}
\Sigma S^{2nk-1} & \xrightarrow{p} & \Sigma S^{2nk-1} & \longrightarrow & \Sigma P^{2nk}(p) \\
\| & & \downarrow t & & \downarrow \\
\Sigma S^{2nk-1} & \xrightarrow{\Sigma \bar{g}_k} & \Sigma \mathcal{F}_{k-1}(\Omega P^{2n+1})_{(2nk+1)} & \longrightarrow & (\Sigma \Omega P^{2n+1})_{(2nk+1)} \\
 & & \downarrow & & \downarrow \\
 & & R_{(2nk+1)} & = & R_{(2nk+1)}
\end{array}
$$

where the top row is the Hurewicz degree so the maps are as shown since they are determined by homology.

Let $A(n)$ be the homotopy fibre of $\phi : \Omega^2 S^{2n+1}\{p\} \to B(n)$.

$$\begin{array}{ccccc}
S^{2n-1} & \longrightarrow & A(n) & \longrightarrow & \Omega S^{2n+1} \\
\downarrow & & \downarrow & & \| \\
\Omega^2 S^{2n+1} & \longrightarrow & \Omega S^{2n+1}\{p\} & \longrightarrow & \Omega S^{2n+1} \\
\downarrow \nu & & \downarrow \phi & & \\
B(n) & = & B(n) & &
\end{array}$$

7.2 Gray's Improvement (2008)

The Theriault construction gives a fibration satisfying Property 1 of the 1980 Conjecture, but it was not so obvious that it has all its properties of Anick which had been shown by Anick-Gray and Theriault.

M-N-C

$$\begin{array}{ccccccc} \Omega S^{2n+1}\{p\} & \longrightarrow & E^{2n+1}\{p\} & \longrightarrow & P^{2n+1}(p) & \longrightarrow & S^{2n+1}\{p\} \\ \downarrow & & \downarrow & & \| & & \downarrow \\ \Omega S^{2n+1} & \longrightarrow & F^{2n+1}\{p\} & \longrightarrow & P^{2n+1}(p) & \longrightarrow & S^{2n+1} \end{array}$$

As in M-N-C, $H_m(F^{2n+1}\{p\}) = \begin{cases} \mathbf{Z} & m = 2nq; \\ 0 & \text{otherwise.} \end{cases}$

$\Omega S^{2n+1} \longrightarrow F^{2n+1}\{p\}$ induces multiplication by p on homology.

Filter $F^{2n+1}\{p\}$ by skeletons $\mathcal{F}_k(F^{2n+1}\{p\}) := F^{2n+1}\{p\}_{(2nk)}$

Filtering the other spaces by inverse images produces filtrations compatible with the Theriault filtrations.

Apply the argument to $E^{2n+1}\{p\}$ rather than $\Omega S^{2n+1}\{p\}$.

By showing appropriate homotopy classes divisible by p, inductively construct $E^{2n+1}\{p\} \to B(n)$ thereby factoring Theriault's $\Omega S^{2n+1}\{p\} \to B(n)$ through $E^{2n+1}\{p\}$.

More precisely, G-T shows that the attaching maps in $F^{2n+1}\{p\}$ are divisible by p and use that to show that any map from $\mathcal{F}_0(E^{2n+1}\{p\}) = \Omega^2 S^{2n+1}$ to a connected H-space with H-exponent p extends to $E^{2n+1}\{p\}$, thus producing the map $E^{2n+1}\{p\} \to B(n)$.

This allows a proof that the total space $A(n)$ of the fibration constructed above is homotopy equivalent to that of Anick and, in particular, has additional properties (i.e. homotopy associative commutative H-space, universal property) that were shown for Anick's version.

References

[1] Amelotte, S., A homotopy decomposition of the fibre of the squaring map on $\Omega^3 S^{17}$, Homology, Homotopy and Applications **20**, (2018), 141–154.

[2] Amelotte, S., The fibre of the degree 3 map, Anick spaces and the double suspension, Proceedings of the Edinburgh Mathematical Society, **63**, No. 3, (2020), 830–843.

[3] Anick, D. Differential Algebras in Topology, Research Notes in Mathematics, A.K. Peters publisher, **3** (1993)

[4] Anick, D., Gray, B., Small H-spaces related to Moore spaces, Topology **34**, (1995) 859–881.

[5] Campbell, E., Peterson, F., Selick, P., Self-maps of Loop Space I, Trans. Amer. Math, Soc. **293** (1986), 1–39

[6] Campbell, E., Peterson, F., Selick, P., The space of maps of Moore spaces into spheres, Proc. of John Moore Conf. on Topology and Algebraic K-theory, W. Browder ed., Annals of Math Studies **113** (1987).

[7] Cohen, F., Mahowald, M, A remark on the self-maps of $\Omega^2 S^{2n+1}$, Indiana Univ Math J **30** (1981), 583–588

[8] Cohen, F., Moore, J., Neisendorfer, J., Torsion in homotopy groups, Annals of Math. **109** (1979), 121–168

[9] Cohen, F., Moore, J., Neisendorfer, J., The double suspension and exponents of the homotopy groups of spheres, Annals of Math. **110** (1979), 549–565

[10] Cohen, F., Moore, J., Neisendorfer, J., Exponents in homotopy theory, Proc. of John Moore Conf. on Topology and Algebraic K-theory, W. Browder ed., Annals of Math Studies **113** (1987).

[11] The unstable decomposition of $\Omega^2\Sigma^2X$ and its applications, Math. Zeithschrift **182** (1983), 553–568

[12] Cohen, F., Selick, P., Splittings of two function spaces, Quart. J. Math., **41** (1990), 145–153 into spheres, Proc. of John Moore Conf. on Topology and Algebraic K-theory, (1985).

[13] Gray, B. On the sphere of origin of infinite families in the homotopy grops of spheres, Topology **8**, (1969)219–232.

[14] Gray, B. On the iterated suspension, Topology **27** (1988), 301–310.

[15] Gray, B., Theriault, S., An elementary constuction of Anick's fibration, Geometry & Topology **14**, No. 1 (2010), 243–275.

[16] James, I. Reduced product spaces, Ann. of Math. **62** (1955) 170–197.

[17] Harper, J. A proof of Gray's conjecture, Contemp. Math. **96** (1989), 189–195

[18] Neisendorfer, J., 3-primary exponents, Math. Proc. Cambridge Phil. Soc., **90** (1981), 63–83

[19] Richter, W. The H-space squaring map on $\Omega^3 S^{4n+1}$ factors thruogh the double suspension, Proc. Amer. Math. Soc. **123** (1995), 3889–3900

[20] Richter, W., A conjecture of Gray and the p^{th} power map on $\Omega^2 S^{2np+1}$, Proc. Amer. Math. Soc. **142** (2014)

[21] Selick, P. Odd primary torsion in$\pi_k(S^3)$, Topology **17**, (1978), 407–412.

[22] Selick, P., A decomposition of $\pi_*(S^{2p+1}; \mathbf{Z}/(p\mathbf{Z})$, Topology **20** (1981), 175–177.

[23] Selick, P., A reformulation of the Arf invariant one mod-p problem and applications to atomic space, Pacific J. Math, **94** (1983), 53–60.

[24] Selick, P., 2-primary exponents for the homotopy groups of spheres, Topology **23** (1984), 97–99.

[25] Selick, P., Theriault S., New Perspectives on the Classifying Space of the Fibre of the Double Suspension, Proc. of the Amer. Math. Soc., **147** (2019), 1325–1333.

[26] Theriault S., The 3-primary classifying space of the fiber of the double suspension. Proc. Amer. Math. Soc, **136**, (2008) 1489–1499

[27] Toda, H., On the double suspension E^2, J. Institute Polytech. Osaka City Univ., Ser. A **7**,(1956), 103–145.