WHITEHEAD PRODUCTS

1. Introduction

The goal of this brief document is to study Whitehead products and provide some applications. Our first general result is the following theorem.

Theorem 1.1. *Let $\mathcal{X}$ be an ∞-category with universal pushouts. For any pointed $X \in \mathcal{X}$, there is a cofiber sequence*

$$\Sigma^n X^{\wedge n+1} \to J_n(\Sigma X) \to J_{n+1}(\Sigma X)$$

which rotates to the cofiber sequence of [DH19, Proposition 4.26].

As an application of Theorem 1.1, we pave a path to a motivic construction of the α-family, and hence to answering [AF17, Question 6] positively. We begin by introducing some terminology. If S is a scheme, let $\mathrm{H}(S)$ be the ∞-category of motivic spaces over S. We let $S^{n,w}$ denote the motivic space $S^{n-w} \wedge \mathbb{G}_m^{\wedge w}$. We show:

Theorem 1.2 ([AF17, Question 6])**.** *Let $p > 2$. Then there is an element $\widetilde{\alpha}_1 \in \pi_{2p,p+1}(\mathrm{SL}_2)(\mathbf{C})$ whose Betti realization is $\alpha_1 \in \pi_{2p}(S^3)$.*

2. Whitehead products

Recall from [DH19] that an ∞-category is said to have universal pushouts if it has finite limits and pushouts, and the base-change of a pushout is still a pushout. The key tool is the following construction.

Construction 2.1. Let $\mathcal{X}$ be an ∞-category with finite limits and pushouts, and let $f : X \to A$ and $g : Y \to B$ be two maps of pointed objects of $\mathcal{X}$. Define $\mathrm{cofib}(f, g) \in \mathcal{X}_*$ via the pushout

$$\begin{CD} A \times B @>>> \mathrm{cofib}(f) \times B \\ @VVV @VVV \\ A \times \mathrm{cofib}(g) @>>> \mathrm{cofib}(f, g). \end{CD}$$

Theorem 2.2. *Let $\mathcal{X}$ be an ∞-category with universal pushouts, and let $f : X \to A$ and $g : Y \to B$ be two maps of pointed objects of $\mathcal{X}$. There is a cofiber sequence*

$$\Sigma(X \wedge Y) \to \mathrm{cofib}(f, g) \to \mathrm{cofib}(f) \times \mathrm{cofib}(g).$$

Proof. We claim that it suffices to prove that if $f : X \to A$ and $g : Y \to B$ are two maps of pointed objects of $\mathcal{X}$, then there is a pushout square

$$\begin{CD} \mathrm{cofib}(\mathrm{id}_X, g) @>>> \mathrm{cofib}(f, g) \\ @VVV @VVV \\ \mathrm{cofib}(g) @>>> \mathrm{cofib}(f) \times \mathrm{cofib}(g). \end{CD} \tag{2.1}$$

Indeed, first note that $\mathrm{cofib}(\mathrm{id}_X, \mathrm{id}_Y)$ sits in a pushout square

$$\begin{CD} X \times Y @>>> X \\ @VVV @VVV \\ Y @>>> \mathrm{cofib}(\mathrm{id}_X, \mathrm{id}_Y), \end{CD}$$

so [DH19, Proposition 2.18] implies that $\mathrm{cofib}(\mathrm{id}_X, \mathrm{id}_Y) \simeq \Sigma(X \wedge Y)$. Therefore, setting $g = \mathrm{id}_Y$ in (2.1) produces a pushout square

$$\begin{array}{ccc} \Sigma(X \wedge Y) & \longrightarrow & \mathrm{cofib}(f, \mathrm{id}_Y) \\ \downarrow & & \downarrow \\ * \simeq \mathrm{cofib}(\mathrm{id}_Y) & \longrightarrow & \mathrm{cofib}(f) \times \mathrm{cofib}(\mathrm{id}_Y) \simeq \mathrm{cofib}(f). \end{array} \tag{2.2}$$

Attaching (2.2) to (2.1) gives a diagram

$$\begin{array}{ccccc} \Sigma(X \wedge Y) & \longrightarrow & \mathrm{cofib}(f, \mathrm{id}_Y) & \longrightarrow & \mathrm{cofib}(f, g) \\ \downarrow & & \downarrow & & \downarrow \\ * \simeq \mathrm{cofib}(\mathrm{id}_Y) & \longrightarrow & \mathrm{cofib}(f) \times \mathrm{cofib}(\mathrm{id}_Y) \simeq \mathrm{cofib}(f) & \longrightarrow & \mathrm{cofib}(f) \times \mathrm{cofib}(g). \end{array}$$

Both the left and right squares are pushouts, so the outer square is also a pushout; this is the desired result.

We now prove that (2.1) is a pushout. Consider the diagram:

$$\begin{array}{ccccc} X \times \mathrm{cofib}(g) & \longrightarrow & \mathrm{cofib}(\mathrm{id}_X, g) & \longrightarrow & \mathrm{cofib}(g) \\ \downarrow & & \downarrow & & \downarrow \\ A \times \mathrm{cofib}(g) & \longrightarrow & \mathrm{cofib}(f, g) & \longrightarrow & \mathrm{cofib}(f) \times \mathrm{cofib}(g), \end{array} \tag{2.3}$$

where the rightmost square is (2.1). Since $\mathcal{X}$ has universal pushouts, the outermost square is a pushout. To show that (2.1) is a pushout, it therefore suffices to show that the leftmost square of (2.3) is a pushout. The leftmost square of (2.3) fits into the following diagram:

$$\begin{array}{ccccc} X \times B & \longrightarrow & X \times \mathrm{cofib}(g) & \longrightarrow & A \times \mathrm{cofib}(g) \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & \mathrm{cofib}(\mathrm{id}_X, g) & \longrightarrow & \mathrm{cofib}(f, g). \end{array} \tag{2.4}$$

The leftmost square is a pushout, by the definition of $\mathrm{cofib}(\mathrm{id}_X, g)$. To prove that the rightmost square (which is the leftmost square of (2.3)) is a pushout, it therefore suffices to show that the outermost square of (2.4) is a pushout. To prove this, consider the following diagram, where the outermost square is the outermost square of (2.4):

$$\begin{array}{ccccc} X \times B & \xrightarrow{f} & A \times B & \longrightarrow & A \times \mathrm{cofib}(g) \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & \mathrm{cofib}(f) \times B & \longrightarrow & \mathrm{cofib}(f, g). \end{array}$$

Since $\mathcal{X}$ has universal pushouts, the leftmost square is a pushout. Moreover, the rightmost square is a pushout by the definition of $\mathrm{cofib}(f, g)$; therefore, the outer square is a pushout, as desired. ☐

Proof of Theorem 1.1. We construct the map $\Sigma^n X^{\wedge n+1} \to J_n(\Sigma X)$ by induction on n. When $n = 0$, this is just the projection map $X \to *$. For $n \geq 1$, recall that there is a pushout square

$$\begin{array}{ccc} \Sigma X \times J_{n-1}(\Sigma X) \sqcup_{J_{n-1}(\Sigma X)} J_n(\Sigma X) & \longrightarrow & J_n(\Sigma X) \\ \downarrow & & \downarrow \\ \Sigma X \times J_n(\Sigma X) & \longrightarrow & J_{n+1}(\Sigma X). \end{array}$$

It therefore suffices to show that there is a cofiber sequence

$$\Sigma^n X^{\wedge n+1} \to \Sigma X \times J_{n-1}(\Sigma X) \sqcup_{J_{n-1}(\Sigma X)} J_n(\Sigma X) \to \Sigma X \times J_n(\Sigma X).$$

We now apply Theorem 2.2 to the maps $f : X \to *$ and $g : \Sigma^{n-1}X^{\wedge n} \to J_{n-1}(\Sigma X)$ (where the latter map comes from the inductive hypothesis). It is clear that $\mathrm{cofib}(f) \simeq \Sigma X$, and the inductive hypothesis gives $\mathrm{cofib}(g) \simeq \Sigma X \times J_n(\Sigma X)$. To conclude, it therefore suffices to note that Construction 2.1 immediately yields an equivalence

$$\mathrm{cofib}(f, g) \simeq \Sigma X \times J_{n-1}(\Sigma X) \sqcup_{J_{n-1}(\Sigma X)} J_n(\Sigma X).$$

□

Remark 2.3. Let $\mathcal{X}$ be an ∞-category with universal pushouts, and let $X, Y \in \mathcal{X}$ be a pair of pointed objects. By the Hilton-Milnor theorem of [DH19, Theorem 3.1], there is an equivalence

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \wedge \Omega Y).$$

In particular, there is a map

$$X \wedge Y \to \Omega\Sigma X \wedge \Omega\Sigma Y \to \Omega\Sigma(\Omega X \wedge \Omega Y) \to \Omega\Sigma(X \vee Y).$$

The adjoint to this map is the Whitehead product $w : \Sigma(X \wedge Y) \to \Sigma(X \vee Y)$. It may equivalently be viewed as the map $\Sigma(X \wedge Y) \to \mathrm{cofib}(f, g)$ from Theorem 2.2 applied to the maps $f : X \to *$ and $g : Y \to *$.

3. Motivic α_1

In this section, we prove Theorem 1.2. Recall that by [DH19, Example 2.3] (see also [Hoy17, Proposition 3.15]), all small colimits are universal in $\mathrm{H}(S)$.

Corollary 3.1. *Suppose n and w are integers such that $n - w \geq 1$. Then for any scheme S, there is a map $S^{n(k+1)-1,w(k+1)} \to J_k(S^{n,w})$ in $\mathrm{H}(S)$ whose cofiber is equivalent to $J_{k+1}(S^{n,w})$.*

Proof. Since $n - w \geq 1$, the space $S^{n,w}$ is a suspension. The claim is now an immediate consequence of Corollary 1.1. □

Remark 3.2. The map $S^{n(k+1)-1,w(k+1)} \to J_k(S^{n,w})$ of Corollary 3.1 is called the *generalized Whitehead product.*

Definition 3.3. Let p be a prime, and let S be a scheme over $\mathbf{Z}_{(p)}$. Say that a motivic space X is said to be p-good if:

- X is $\mathbf{P}^1$-cellular and $(2n, n)$-connective;
- the p-localized homotopy group $\pi_{2m-1,m}(X) = 0$ for $2m < 2p + n - 3$.

Corollary 3.4. *Let $X \in \mathrm{H}(S)$ be a $(2n, n)$-connective p-good space. Let $d = \lfloor \frac{2p+n-3}{2n} \rfloor$. Then any map $S^{2n,n} \to X$ canonically extends to a map $J_d(S^{2n,n}) \to X$.*

Proof. Corollary 3.1 implies that the obstruction to extending $S^{2n,n} \to X$ to $J_k(S^{2n,n})$ lives in $\pi_{2nk-1,nk}(X)$. Since X is p-good, $\pi_{2m-1,m}(X) = 0$ for $2m < 2p + n - 3$, so all these obstructions vanish. □

From now on, our base scheme will be $\mathbf{C}$. I learned the following lemma and proposition in an email from Aravind Asok (but errors introduced below are mine):

Lemma 3.5. *Let $\Phi : \prod_{i=1}^{n-1} Q_{2i+1} \to \mathrm{SL}_n$ denote the morphism of* [AFH19, Section 5.2]. *If p is an odd prime, and $2 < i < n - 1$, then $\pi_{2i,i}$ of the morphism $s_i : Q_{2i-1} \to \mathrm{SL}_n$ induces an isomorphism after p-localization. If $i = n - 1$, and n is odd, then $\pi_{2i,i}$ of s_i is an isomorphism after p-localization on $\mathbf{C}$-points.*

Proof. Recall that we work over $\mathbf{C}$. First, note that since Q_{2i-1} is $\mathbf{A}^1$-equivalent to $\mathbf{A}^{i+1} - \{0\}$, we know that $\pi_{2i,i}(Q_{2i-1}) \cong \pi_{2i,i}(\mathbf{A}^{i+1} - \{0\}) = \mathbf{K}_1^{MW}$. Since $(n-1)!$ is invertible, the morphism Φ is an equivalence. Therefore, $\pi_{2i,i}Q_{2i-1}$ is a summand of $\pi_{2i,i}\mathrm{SL}_n$.

Suppose first that $i + 1 < n$. Then, we know that $\pi_{2i,i}\mathrm{SL}_n$ is in the stable range, i.e., is isomorphic to the Nisnevich sheafification $\mathbf{K}_1^Q$ of the Quillen K-theory presheaf. It follows that $\pi_{2i,i}$ of the morphism s_i is $\mathbf{K}_1^{MW} \to \mathbf{K}_1^Q$; one can think of this as the map sending $[a] \in K_i^{MW}(k)$ to $(a) \in K_i^Q(k)$. Thus, we need to show that the morphism $\mathbf{K}_1^{MW} \to \mathbf{K}_1^Q$ is an isomorphism upon p-localization. This is a consequence of [AF14a, Lemma 3.8 and Corollary 3.9].

If $i+1=n$ and n is odd, then we can use the same argument. Namely, by [AF14a, Theorem 1], there is an exact sequence

$$0 \to \mathbf{S}_2 \to \pi_{2n-2,n-1}\mathrm{SL}_n \to \mathbf{K}_1^Q \to 0,$$

where $\mathbf{S}_2$ admits an epimorphism $\mathbf{K}_2^M/n! \to \mathbf{S}_2$. Because $K_2^M(\mathbf{C})$ is divisible, we see that $S_2(\mathbf{C})=0$, and thus $\pi_{2n-2,n-1}\mathrm{SL}_n \cong K_1^Q(\mathbf{C})$. Thus, the morphism $\pi_{2n-2,n-1}Q_{2n-1} \to \pi_{2n-2,n-1}\mathrm{SL}_n$ is the map $K_1^{MW}(\mathbf{C}) \to K_1^Q(\mathbf{C})$, which is an isomorphism. □

Proposition 3.6. *Let p be an odd prime. The motivic space BSL_2 over $\mathbf{C}$ is p-good.*

Proof. We must show that $\pi_{2m-1,m}(\mathrm{BSL}_2)(\mathbf{C}) = \pi_{2m,m}(\mathrm{SL}_2)(\mathbf{C}) = 0$ for $m < p$. If $p=3$, then we must show that $\pi_{4,2}(\mathrm{SL}_2)(\mathbf{C}) = 0$. By [AF14b, Theorem 3], we know that (using the notation from the cited paper, albeit with different indexing on homotopy) $\pi_{4,2}(\mathrm{SL}_2)(\mathbf{C}) \cong \pi_2^{\mathbf{A}^1}(\mathrm{SL}_2)_{-2}$ is isomorphic to an extension of $(\mathbf{T}_4')_{-2}$ by $(\mathbf{K}_3^{Sp})_{-2} \cong \mathbf{GW}^0_{-1}$. The sheaf $\mathbf{GW}^0_{-1}$ is 2-torsion, so vanishes upon 3-localization. Thus it remains to show that $(\mathbf{T}_4')_{-2}$ is 3-locally zero. But upon 3-localization, $(\mathbf{T}_4')_{-2}$ is the 3-torsion subsheaf of $\mathbf{K}_2^M/12$, and $\mathbf{K}_2^M(\mathbf{C})$ is divisible.

We now consider the case $p>3$. Suppose first that $m < p-1$. By Lemma 3.5 with $n=p$, we see that the morphism $\pi_{2m,m}Q_{2m-1} \to \pi_{2m,m}\mathrm{SL}_p$ is an isomorphism after p-localization. This implies that $\pi_{2m,m}$ of the other summands in $\mathrm{SL}_p \simeq \prod_{i=1}^{p-1} Q_{2i+1}$ is zero, i.e., that $\pi_{2m,m}Q_{2i+1} = 0$ for $i \neq m-1$ and $1 < i < p-1$. In particular, $\pi_{2m,m}Q_3 = \pi_{2m,m}\mathrm{SL}_3 = 0$ for $m < p-1$. The same argument works with $m = p-1$ using that p is odd. □

Proof of Theorem 1.2. There is an equivalence $S^{3,2} \simeq \mathrm{SL}_2$ of motivic spaces. Indeed, by [ADF17, Theorem 2], it suffices to note that $\mathrm{SL}_2 \simeq Q_3$. There is therefore a map $\iota : S^{4,2} \to \mathrm{B}S^{3,2}$. We claim that ι lifts to a map $\widetilde{\iota} : J_{(p-1)/2}(S^{4,2}) \to \mathrm{B}S^{3,2}$. This follows from Corollary 3.4 (with $n=2$) and our assumption that BSL_2 is p-good (i.e., that $\pi_{2k+2,k+1}(S^{3,2})(\mathbf{C}) = 0$ if $k < p-1$). Finally, setting $n=4$, $w=2$, and $k=(p-1)/2$ in Corollary 3.1 produces a map $S^{2p+1,p+1} \to J_{(p-1)/2}(S^{4,2})$. The composite

$$S^{2p+1,p+1} \to J_{(p-1)/2}(S^{4,2}) \to \mathrm{B}S^{3,2}$$

defines an element of $\pi_{2p,p+1}(S^{3,2})(\mathbf{C})$. This recovers $\alpha_1 \in \pi_{2p}(S^3)$ upon realization, because it realizes to the p-fold Whitehead product $[\iota, \cdots, \iota] \in \pi_{2p+1}(\mathbf{H}P^\infty)$ (which is given by α_1). □

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