WOOD'S THEOREM (NOTES)

This document should be read with care.

1. Introduction

2. Generalities

Lemma 1. *Let $X \to \mathcal{M}_{\mathrm{FG}}$ be a flat map from a* Noetherian *Deligne-Mumford stack which admits an even periodic refinement $\mathfrak{X}$. Then the $\mathbf{E}_\infty$-ring $A = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is L_n-local for some n.*

Proof. There is a descending sequence of closed substacks of $\mathcal{M}_{\mathrm{FG}}$ given by $\mathcal{M}_{\mathrm{FG}}^{\geq m}$, and each $\mathcal{M}_{\mathrm{FG}}^{\geq m}$ is obtained from $\mathcal{M}_{\mathrm{FG}}^{\geq m-1}$ by taking the substack corresponding to the vanishing locus of a regular element. As X is Noetherian, there is some n such that $X \times_{\mathcal{M}_{\mathrm{FG}}} \mathcal{M}_{\mathrm{FG}}^{\geq n}$ is empty. It suffices to show that if that $\operatorname{Spec} B \to X$ is a flat morphism, then the associated Landweber exact spectrum $\widetilde{B}$ is E_n-local. Indeed, since $A = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a homotopy limit of $\mathcal{O}_{\mathfrak{X}}(\operatorname{Spec} B \to X)$ over all étale maps $\operatorname{Spec} B \to X$, it follows from the fact that L_n-local spectra are closed under limits that $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is also L_n-local.

We now prove the fact used above, that if that $\operatorname{Spec} B \to X$ is a flat morphism with B a Noetherian ring, then the associated Landweber exact spectrum $\widetilde{B}$ is E_n-local. Let Z be an E_n-acyclic spectrum, so we obtain an associated quasicoherent sheaf $\mathcal{F}_Z$ on $\mathcal{M}_{\mathrm{FG}}$. Let $f : \operatorname{Spec} B \to X \to \mathcal{M}_{\mathrm{FG}}^{\geq n} \to \mathcal{M}_{\mathrm{FG}}$ be the associated morphism; we need to show that $f^*\mathcal{F}_Z = 0$. It suffices to show that $\mathcal{F}_Z|_{\mathcal{M}_{\mathrm{FG}}^{\geq n}} = 0$. However, the map $g : \operatorname{Spec} \pi_0 E_n \to \mathcal{M}_{\mathrm{FG}}^{\geq n}$ is a faithfully flat cover, so we need to show that $g^*\mathcal{F}_Z = 0$. However, this is immediate since $g^*\mathcal{F}_Z$ is the $\pi_0 E_n$-module associated to $E_n \wedge Z$, which vanishes since Z is E_n-acyclic. □

We note the following consequence:

Proposition 2. *Let $X \to \mathcal{M}_p(n)$ be a formally étale map from a Noetherian and separated Deligne-Mumford stack which is of finite type over* $\operatorname{Spec} \mathbf{Z}_p$*, such that the map $X \to \mathcal{M}_{\mathrm{FG}}$ is representable. Let $\mathfrak{X}$ denote the even-periodic Deligne-Mumford stack $\mathfrak{X}$ via* [BL10, Theorem 8.1.4]*. Let $A = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, so that A is L_n-local by Lemma 1. Then, there is an equivalence*

$$L_{K(n)} A \simeq \prod_{x \in X^{[n]}(\overline{\mathbf{F}_p})} E_n^{h\,\mathrm{Aut}(x) \rtimes \mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)}.$$

Proof. We begin by proving that if $X^{[n]}$ is the locus of $X \times_{\operatorname{Spec} \mathbf{Z}_p} \operatorname{Spec} \mathbf{F}_p$ where the formal group H over X determined by the map $X \to \mathcal{M}_{\mathrm{FG}}$ has height exactly n, then $X^{[n]}$ is étale over $\operatorname{Spec} \mathbf{F}_p$; in particular, $X^{[n]}$ is zero-dimensional and finite. By our assumptions on X, there is a finite étale cover $f : Y \to X$ of X by a scheme Y. It follows that the map $f' : Y^{[n]} \to X^{[n]}$ is also finite étale. Since X is of finite type over $\operatorname{Spec} \mathbf{Z}_p$ by assumption, we know that Y is of finite type over $\operatorname{Spec} \mathbf{Z}_p$, and that $X^{[n]}$ is of finite type over $\operatorname{Spec} \mathbf{F}_p$. To prove that $X^{[n]}$ is étale over $\operatorname{Spec} \mathbf{F}_p$, it therefore suffices to prove that the map $Y^{[n]} \to \operatorname{Spec} \mathbf{F}_p$ is étale. Since $Y^{[n]}$ is of finite type over $\operatorname{Spec} \mathbf{F}_p$, it suffices to prove that $Y^{[n]}$ is formally étale over $\operatorname{Spec} \mathbf{F}_p$.

Let $x : \operatorname{Spec} k \to Y$ be a point of Y lying inside the subscheme $Y^{[n]}$, where k is a finite field of characteristic p. This defines a p-divisible group H_x over k, given by the map $\operatorname{Spec} k \to Y \to \mathcal{M}_p(n)$. Since the map $Y \to \mathcal{M}_p(n)$ is formally étale, there is an isomorphism $(X \times_{\operatorname{Spec} \mathbf{Z}_p} \operatorname{Spec} \mathbf{F}_p)^\wedge_x \cong \operatorname{Spf} k[\![u_1, \cdots, u_{n-1}]\!]$. The subscheme $(Y^{[n]})^\wedge_x$ consists of those deformations of H_x

of height exactly n. This implies that $u_1, \cdots, u_{n-1}$ must vanish in $(Y^{[n]})^\wedge_x$, so $(Y^{[n]})^\wedge_x \cong \operatorname{Spec} k$. Since this is étale over $\operatorname{Spec} \mathbf{F}_p$, it follows that $Y^{[n]}$ is formally étale over $\operatorname{Spec} \mathbf{F}_p$, as desired.

Since the map $X \to \mathcal{M}_p(n) \to \mathcal{M}_{\mathrm{FG}}$ is representable, we learn that for each $x \in X^{[n]}(\overline{\mathbf{F}_p})$, the automorphism group $\operatorname{Aut}(x)$ injects into the Morava stabilizer group of automorphisms of the associated height n formal group. Since X is separated, this automorphism group is finite.

We can now prove the proposition. In the case when X is affine, this result can be proved by arguing as in [BL10, Proposition 14.4.6]. The general case is now a formal consequence of descent. By our assumptions on X, there is a finite étale cover $f_0 : Y \to X$ of X by an affine scheme $Y = \operatorname{Spec} B_0$. This refines to a finite étale cover $f : \mathcal{Y} \to \mathcal{X}$, where $\mathcal{Y} = \operatorname{Spec} B$ with B an even periodic Noetherian $\mathbf{E}_\infty$-ring. It follows that there is an equivalence $A \to \varprojlim \operatorname{Tot}^s B^{\otimes_A \bullet}$. Since $B^{\otimes_A \bullet} \simeq \Gamma(\mathcal{Y}^{\times_{\mathcal{X}} m}, \mathcal{O}_{\mathcal{Y}^{\times_{\mathcal{X}} m}})$, we learn that $B^{\otimes_A \bullet}$ is an even periodic Noetherian $\mathbf{E}_\infty$-ring. It follows that there is a $K(n)$-local equivalence $L_{K(n)} A \to \varprojlim \operatorname{Tot}^s L_{K(n)} B^{\otimes_{L_{K(n)}A} \bullet}$, and each term $L_{K(n)} B^{\otimes_{L_{K(n)}A} \bullet}$ is a wedge of copies of E_n. The desired result now follows from the fact that there is an equivalence $E_n^{hG} \to \varprojlim \operatorname{Tot}^s E_n^{\otimes_{E_n^{hG}} \bullet} \simeq \varprojlim \operatorname{Tot}^s \prod_{g \in G^\bullet} E_n$. □

The even-periodic derived stack $\mathfrak{B}\mathbf{Z}/2$ defining real K-theory has "finite cohomological dimension", while the same claim is false for the classical stack $B\mathbf{Z}/2$. Making this statement precise involves a discussion of the Adams-Novikov spectral sequence. We begin our discussion with a construction, outlined in [DFHH14, Chapter 9]. The reader should be warned that we will be particularly egregious in implicitly localizing at a prime in this section.

Construction 3. Let R be a p-local homotopy commutative ring spectrum. Define an algebraic stack $\mathcal{M}_R$ over $\operatorname{Spec} \mathbf{Z}_{(p)}$ as follows. The stack associated to the Hopf algebroid $(\mathrm{MU}_{2*}, \mathrm{MU}_{2*}\mathrm{MU})$ is $\mathcal{M}_{\mathrm{FG}}$, so that the global sections functor supplies a symmetric monoidal equivalence between $\mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}})$ and the category of evenly graded $(\mathrm{MU}_{2*}, \mathrm{MU}_{2*}\mathrm{MU})$-comodules. This is explained in [Nau07, Remark 34]. If R is a p-local homotopy commutative ring spectrum such that $\mathrm{MU}_{2*}R$ is an algebra object in evenly graded $(\mathrm{MU}_{2*}, \mathrm{MU}_{2*}\mathrm{MU})$-comodules, then $\mathrm{MU}_{2*}R$ corresponds to a quasicoherent sheaf $\mathcal{F}(R)$ of algebras on $\mathcal{M}_{\mathrm{FG}}$. Thus, we can define a stack $\mathcal{M}_R$ as the relative spec $\underline{\operatorname{Spec}}(\mathcal{F}(R))$ of $\mathcal{F}(R)$; explicitly,

$$\mathcal{M}_R = \operatorname{colim}_{\Delta^{op}} \operatorname{Spec} \mathcal{F}(R)(\mathrm{MU}^{\wedge \bullet + 1})/\mathbf{G}_m = \operatorname{colim}_{\Delta^{op}} \operatorname{Spec} \pi_*(\mathrm{MU}^{\wedge \bullet + 1} \wedge R)/\mathbf{G}_m,$$

where the $\mathbf{G}_m$-action enforces the grading.

Theorem 4. *Assume that $X \to \mathcal{M}_{\mathrm{FG}}$ is a flat and affine map from a Noetherian and locally separated Deligne-Mumford stack X such that X refines to an even periodic derived Deligne-Mumford stack $\mathcal{X}$. Let $E = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ denote its global sections. Then $\mathcal{M}_E$ can be identified with the underlying ordinary Deligne-Mumford stack X.*

Proof. There is a flat covering map $\mathcal{M}_{\mathrm{FG}}^{\mathrm{coord}} = \operatorname{Spec} L \to \mathcal{M}_{\mathrm{FG}}$. Since $X \to \mathcal{M}_{\mathrm{FG}}$ is an affine morphism, the pullback $X^{\mathrm{coord}} = \mathcal{M}_{\mathrm{FG}}^{\mathrm{coord}} \times_{\mathcal{M}_{\mathrm{FG}}} X$ is an affine scheme, and the morphism $p : X^{\mathrm{coord}} \to X$ is a flat and affine cover of X. Let us first show that for any étale cover $\operatorname{Spec} A \to X$, there is natural isomorphism

$$\pi_*(\mathcal{O}_{\mathcal{X}}(\operatorname{Spec} A) \otimes \mathrm{MUP}) \cong (p_* p^* \pi_* \mathcal{O}_{\mathcal{X}})(\operatorname{Spec} A). \tag{1}$$

There is a commutative diagram, where the square is a pullback:

$$\begin{tikzcd}
 & X^{\mathrm{coord}} \arrow[r] \arrow[d, "p"] & \mathcal{M}_{\mathrm{FG}}^{\mathrm{coord}} \arrow[d, "q"] \\
\operatorname{Spec} A \arrow[r] & X \arrow[r, "f"'] & \mathcal{M}_{\mathrm{FG}}
\end{tikzcd}$$

Since $\mathcal{X}$ is an even periodic refinement of $X \to \mathcal{M}_{\mathrm{FG}}$, we have identifications

$$\begin{aligned}\pi_*(\mathcal{O}_{\mathcal{X}}(\operatorname{Spec} A) \otimes \mathrm{MUP}) = \mathrm{MUP}_*(\mathcal{O}_{\mathcal{X}}(\operatorname{Spec} A)) = (q_*\omega^{\otimes *})(\operatorname{Spec} A) \\ \cong (\pi_*\mathcal{O}_{\mathcal{X}})(\operatorname{Spec} A \times_{\mathcal{M}_{\mathrm{FG}}} X^{\mathrm{coord}}) = (p_*p^*\pi_*\mathcal{O}_{\mathcal{X}})(\operatorname{Spec} A).\end{aligned}$$

Equation (1) implies that $\pi_*(\mathcal{O}_{\mathcal{X}} \otimes \mathrm{MUP}) \simeq p_*p^*\pi_*\mathcal{O}_{\mathcal{X}}$. There is a descent spectral sequence

$$\mathrm{H}^*(X; \pi_*(\mathcal{O}_{\mathcal{X}} \otimes \mathrm{MUP})) \Rightarrow \pi_*(E \otimes \mathrm{MUP}).$$

Since $X \to \mathcal{M}_{\mathrm{FG}}$ is affine, the main result of [MM15] implies that $\Gamma(\mathcal{O}_{\mathcal{X}} \otimes \mathrm{MUP}) = E \otimes \mathrm{MUP}$. There is therefore an isomorphism $\mathrm{H}^*(X; p_*p^*(\pi_*\mathcal{O}_{\mathcal{X}})) \cong \mathrm{H}^*(X; \pi_*(\mathcal{O}_{\mathcal{X}} \otimes \mathrm{MUP}))$. The map p is affine, so the Leray spectral sequence degenerates to give an isomorphism

$$\mathrm{H}^s(X; p_*p^*(\pi_*\mathcal{O}_{\mathcal{X}})) \simeq \mathrm{H}^s(X^{\mathrm{coord}}; p^*(\pi_*\mathcal{O}_{\mathcal{X}})) \cong \begin{cases}\Gamma(X^{\mathrm{coord}}; p^*\pi_*\mathcal{O}_{\mathcal{X}}) & \text{if } s = 0 \\ 0 & \text{else.}\end{cases}$$

We conclude that $\Gamma(X^{\mathrm{coord}}; p^*\pi_*\mathcal{O}_{\mathcal{X}}) \simeq \pi_*(E \otimes \mathrm{MUP})$. By running the same argument with X^{coord} replaced by $(\mathcal{M}_{\mathrm{FG}}^{\mathrm{coord}})^{\times_{\mathcal{M}_{\mathrm{FG}}}\bullet} \times_{\mathcal{M}_{\mathrm{FG}}} X = (X^{\mathrm{coord}})^{\times_X \bullet}$, we learn that $\Gamma((X^{\mathrm{coord}})^{\times_X\bullet}; p^*\pi_*\mathcal{O}_{\mathcal{X}}) \simeq \pi_*(E \otimes \mathrm{MUP}^\bullet)$. This implies that the Hopf algebroid

$$E_*(\mathrm{MUP}) \rightrightarrows E_*(\mathrm{MUP} \otimes \mathrm{MUP}) \Rrightarrow \cdots$$

is exactly the presentation of X via the flat cover $X^{\mathrm{coord}} \to X$; but this is precisely the definition of $\mathcal{M}_E$. □

Theorem 4 provides another proof of one the main results of [Ban14].

Let $\mathcal{X}$ be as in Theorem 4. Recall that the descent spectral sequence runs

$$E_2^{*,*} = \mathrm{H}^*(X; \pi_*\mathcal{O}_{\mathcal{X}}) \Rightarrow \pi_*\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}).$$

There is also an Adams-Novikov spectral sequence

$$'E_2^{*,*} = \mathrm{Ext}^*_{\mathrm{MUP}_*\mathrm{MUP}}(\mathrm{MUP}_*, \Sigma^*\mathrm{MUP}_*(\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))) \cong \mathrm{H}^*(\mathcal{M}_E; \pi_*\mathcal{O}_{\mathcal{X}}) \Rightarrow \pi_*\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}).$$

It follows from Theorem 4 that there is an isomorphism of spectral sequences $E_2^{*,*} \cong' E_2^{*,*}$. In other words:

Corollary 5. *Let $\mathcal{X}$ be as in Theorem 4. Then, the descent spectral sequence for $\pi_*\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is isomorphic to the Adams-Novikov spectral sequence.*

There are numerous interesting consequences of Corollary 5. For instance:

Proposition 6. *Let $\mathcal{X}$ be as in Theorem 4. Then, the descent spectral sequence for $\pi_*\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ degenerates at a finite page with a horizontal vanishing line.*

Proof. By Corollary 5, the descent spectral sequence is isomorphic to the Adams-Novikov spectral sequence. Lemma 1 implies that $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is L_n-local for some n, so the desired result follows from the smash product theorem, which is equivalent to the statement that the Adams-Novikov spectral sequence for a L_n-local spectrum degenerates at a finite page with a horizontal vanishing line. □

In particular, each element in $\pi_*\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ has finite Adams-Novikov filtration, even if $\mathcal{X}$ has infinite cohomological dimension.

3. G-equivariant Wood equivalences

3.1. A Wood equivalence for KU_G. In this section, we prove:

Proposition 7. *Let G be a compact abelian Lie group, i.e., a group isomorphic to $T \times F$ where T is a torus and F is a finitely generated abelian group. There is a G-equivariant equivalence $\mathrm{KO}_G \wedge C\eta \simeq \mathrm{KU}_G$, where $C\eta$ is given the trivial G-action.*

It is possible to give a very classical proof of Proposition 7, but we provide an algebro-geometric proof which will easily generalize to TMF.

We first recall how the spectra KO_G and KU_G are constructed in spectral algebraic geometry. Let $\mathcal{X}$ denote the derived stack associated to KO, so that the underlying stack is $B\mathbf{Z}/2$, and the map $B\mathbf{Z}/2 \to \mathcal{M}_{\mathrm{FG}}$ classifies the multiplicative formal group $\widehat{\mathbf{G}_m}$. This has a degree two finite étale cover by the scheme $\operatorname{Spec} \mathbf{Z}$, which refines to a finite étale cover $p : \mathcal{Y} = \operatorname{Spec} \mathrm{KU} \to \mathcal{X}$.

Let $\mathbf{G}$ (resp. $\mathbf{G}_m$) denote the derived p-divisible group defined over $\mathcal{X}$ (resp. $\mathcal{Y}$) via the construction of [Lur18]. Then, the spectrum KO_G can be defined as follows (see [Lur09, Section 3.4]): for a G-space X, the cohomology $\mathrm{KO}_G(X)$ is the global sections of a quasicoherent sheaf $\mathcal{F}_G(X)$ defined over the mapping stack $\operatorname{Map}(G^\vee, \mathbf{G})$. Here, $G^\vee$ is the Pontryagin dual of G (so it is a finitely generated abelian group). Similarly, we obtain a definition of $\mathrm{KU}_G(X)$.

Proof of Proposition 7. We begin by noting that the classical Wood cofiber sequence, stating that $\mathrm{KO} \wedge C\eta \simeq \mathrm{KU}$, translates to the statement that $p_* \mathcal{O}_\mathcal{Y} \simeq \mathcal{O}_\mathcal{X} \wedge \mathcal{F}(C\eta)$, where $\mathcal{F}(C\eta)$ is the sheaf over $\mathcal{X}$ associated to $C\eta$. Define $\mathcal{Y}_G = \operatorname{Map}(G^\vee, \mathbf{G}_m)$ and $\mathcal{X}_G = \operatorname{Map}(G^\vee, \mathbf{G})$, and let $p^G : \mathcal{Y}_G \to \mathcal{X}_G$ denote the obvious map of mapping stacks. It follows from the above discussion that it suffices to prove that there is an equivalence $p^G_* \mathcal{O}_{\mathcal{Y}_G} \simeq \mathcal{F}(C\eta)$. There is a Cartesian diagram

$$\begin{tikzcd} \mathcal{Y}_G \arrow[r, "f'"] \arrow[d, "p^G"'] & \mathcal{Y} \arrow[d, "p"] \\ \mathcal{X}_G \arrow[r, "f"'] & \mathcal{X}. \end{tikzcd}$$

It follows that

$$p^G_* \mathcal{O}_{\mathcal{Y}_G} \simeq p^G_* f'^* \mathcal{O}_\mathcal{Y} \simeq f^* p_* \mathcal{O}_\mathcal{Y} \simeq f^*(\mathcal{O}_\mathcal{X} \wedge \mathcal{F}(C\eta)) \simeq \mathcal{O}_{\mathcal{X}_G} \wedge f^* \mathcal{F}(C\eta).$$

In order to finish the proof, it will therefore suffice to show that $f^* \mathcal{F}(C\eta) \simeq \mathcal{F}_G(C\eta)$.

This is true more generally: if X is a finite CW-complex on which G acts trivially, then $\mathcal{F}_G(X) \simeq f^* \mathcal{F}(X)$. The proof is by induction on the number of cells. When $X = *$, we know that $\mathcal{F}_G(*) = \mathcal{O}_{\mathcal{X}_G}$, in which case the result is obvious. Every G-space with the trivial G-action is built from the trivial G-orbit $G/G = *$, so the desired result follows from the observation that $\mathcal{F}_G$ takes finite homotopy colimits of G-spaces to homotopy limits of quasicoherent sheaves by [Lur09, Theorem 3.2(2)]. □

The same method of proof holds more generally. Let $DA(1)$ and X_3 denote the spectra of [Mat16]. Then, the same argument shows:

Proposition 8. *Let G be a compact abelian Lie group. Then, there is a G-equivariant 2-local equivalence $\mathrm{TMF}_G \wedge DA(1) \simeq \mathrm{TMF}_1(3)_G$, where $DA(1)$ is given the trivial G-action. There is also a G-equivariant 3-local equivalence $\mathrm{TMF}_G \wedge X_3 \simeq \mathrm{TMF}_1(2)_G$, where X_3 is again given the trivial G-action.*

Remark 9. Let G be a compact Lie group which is possibly nonabelian. If one could construct derived stacks $\mathcal{X}_G$ and $\mathcal{Y}_G$, functors $\mathcal{F}_G$, and an analogue of [Lur09, Proposition 3.2], then the same proof would show that Proposition 7 and Proposition 8 are true for every compact Lie group. We know that Proposition 7 is indeed true for G a general compact Lie group; see, e.g., [MNN17, Theorem 9.8]. See [Lur09, Section 5.1] for the case of TMF_G.

4. Real Wood equivalences

4.1. **A Real Wood equivalence for $\mathrm{KU}_{\mathbf{R}}$.** Let $\mathrm{KU}_{\mathbf{R}}$ denote the genuine C_2-spectrum given by Atiyah's **R**eal K-theory spectrum (see [Ati66]); it is a Borel C_2-spectrum whose underlying spectrum is KU, and whose fixed points $(\mathrm{KU}_{\mathbf{R}})^{C_2} \simeq (\mathrm{KU}_{\mathbf{R}})^{hC_2}$ is KO. Let $k\mathbf{R}$ denote the connective cover of $\mathrm{KU}_{\mathbf{R}}$; this is a C_2-spectrum such that the underlying spectrum $k\mathbf{R}^e$ is the connective cover bu of KU, and such that $k\mathbf{R}^{C_2}$ is the connective cover of $\mathrm{KU}_{\mathbf{R}}^{C_2} = \mathrm{KO}$.

The equivalence $C\eta \simeq \Sigma^{-2}\mathbf{C}P^2$ suggests a natural C_2-action on $C\eta$, given by complex conjugation. In order to define this precisely, we will need to recall a few things. We will denote by σ the sign (also known as the reduced regular) representation of C_2, and ρ the regular representation. In the group $RO(C_2)$, we have $\rho = 1+\sigma$. If X is a C_2-spectrum, we will denote by $\pi_{p,q}X$ the $RO(C_2)$-graded homotopy group of X consisting of C_2-equivariant maps $S^{p+q\sigma} \to X$. The C_2-spectrum $C\widetilde{\eta}$ is then defined to be the cofiber of an element $\widetilde{\eta} \in \pi_{0,1}S^0$. This element is a map $S^\sigma \to S^0$, and is the stable representative of the following unstable map. Let $\mathbf{C}^2 - \{0\} \to \mathbf{C}P^1$ denote the C_2-equivariant map $(x,y) \mapsto [x:y]$, where both the source and target are given the complex conjugation action. Then this map can be identified with the map $S^{\sigma+\rho} = S^{1+2\sigma} \to S^\rho$, so we get a stable map $\widetilde{\eta} : S^\sigma \to S^0$. We then have (see [GHIR17, Proposition 10.12]):

Theorem 10. *There is a C_2-equivariant equivalence $\mathrm{bo}_{C_2} \wedge C\widetilde{\eta} \simeq k\mathbf{R}$.*

An easy consequence is the existence of a C_2-equivariant equivalence $\mathrm{KO}_{C_2} \wedge C\widetilde{\eta} \simeq \mathrm{KU}_{\mathbf{R}}$.

Proof of Theorem 10. We need to show that there is a cofiber sequence $\Sigma^\sigma \mathrm{bo}_{C_2} \xrightarrow{\widetilde{\eta}} \mathrm{bo}_{C_2} \to k\mathbf{R}$. On the underlying spectra, this is precisely the statement of Wood's theorem. We therefore need to show that the induced composite on C_2-fixed points is still a cofiber sequence. By construction, $(\mathrm{bo}_{C_2})^{C_2} = \mathrm{bo} \vee \mathrm{bo}$, and $k\mathbf{R}^{C_2} = \mathrm{bo}$. Therefore, it suffices to show that $(\Sigma^\sigma \mathrm{bo}_{C_2})^{C_2} \simeq \mathrm{bo}$, and that the induced sequence $\mathrm{bo} \to \mathrm{bo}\vee\mathrm{bo} \to \mathrm{bo}$ is a cofiber sequence.

We begin by showing that $(\Sigma^\sigma \mathrm{bo}_{C_2})^{C_2} = \mathrm{bo}$. There is a cofiber sequence $C_{2+} \to S^0 \hookrightarrow S^\sigma$, so we obtain a cofiber sequence $\mathrm{bo}_{C_2} \wedge C_{2+} \to \mathrm{bo}_{C_2} \to \Sigma^\sigma \mathrm{bo}_{C_2}$. Taking C_2-fixed points, we get a cofiber sequence $\mathrm{bo} \to \mathrm{bo}\vee\mathrm{bo} \to (\Sigma^\sigma \mathrm{bo}_{C_2})^{C_2}$. The first map, however, is simply the inclusion of $\mathbf{Z}$ into $RO(C_2)$ which sends 1 to ρ; this inclusion is split, so we obtain a splitting of the above cofiber sequence. It follows that $(\Sigma^\sigma \mathrm{bo}_{C_2})^{C_2} = \mathrm{bo}$, as desired.

It remains to show that the induced composite $\mathrm{bo} = (\Sigma^\sigma \mathrm{bo}_{C_2})^{C_2} \to \mathrm{bo}\vee\mathrm{bo} = (\mathrm{bo}_{C_2})^{C_2} \to \mathrm{bo} = k\mathbf{R}^{C_2}$ is a cofiber sequence. The second map $\mathrm{bo}\vee\mathrm{bo} \to \mathrm{bo}$ is simply the fold map ∇, so it suffices to show that the first map $\widetilde{\eta}^{C_2} : \mathrm{bo} \to \mathrm{bo}\vee\mathrm{bo}$ is given by $(k,-k)$ for some integer k. To show this, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{bo} \arrow[r, "\simeq"] \arrow[d] & (\Sigma^\sigma \mathrm{bo}_{C_2})^{C_2} \arrow[r] \arrow[d] & (\mathrm{bo}_{C_2})^{C_2} \arrow[r, "\simeq"] \arrow[d] & \mathrm{bo}\vee\mathrm{bo} \arrow[d, "\nabla"] \\
\Sigma\mathrm{bo} \arrow[r, "\simeq"] & (\Sigma^\sigma \mathrm{bo}_{C_2})^e \arrow[r] & (\mathrm{bo}_{C_2})^e \arrow[r, "\simeq"] & \mathrm{bo}
\end{tikzcd}$$

The map $\mathrm{bo} \to \Sigma\mathrm{bo}$ is zero for dimension reasons, so by commutativity of this diagram, the horizontal map $\mathrm{bo} \to \mathrm{bo}\vee\mathrm{bo}$ factors through the fiber of the fold map, which implies that it is given by $(k,-k)$ for some integer k. □

Remark 11. It is possible to prove this result in terms of the theory of derived stacks described in the previous section; however, we have elected not to do so since that presentation obscures the proof.

Corollary 12. *Let $\mathbf{C}P^{\infty}_{\mathbf{R}}$ denote the infinite-dimensional complex projective space with its conjugation C_2-action. Then, the Atiyah-Hirzebruch spectral sequence computing the C_2-equivariant KO_{C_2}-cohomology of $\mathbf{C}P^{\infty}_{\mathbf{R}}$ degenerates at a finite page.*

Proof. The Atiyah-Hirzebruch spectral sequence computing the C_2-equivariant $\mathrm{KU}_{\mathbf{R}}$-cohomology of $\mathbf{C}P^{\infty}_{\mathbf{R}}$ collapses at the E_2-page, because $\mathrm{KU}_{\mathbf{R}}$ is Real oriented. A thick subcategory argument now shows that the Atiyah-Hirzebruch spectral sequence computing the C_2-equivariant KO_{C_2}-cohomology of $\mathbf{C}P^{\infty}_{\mathbf{R}}$ degenerates at a finite page. $\square$

4.2. **C_p-equivariant Wood equivalences.** Being worked on.

References

[Ati66] M. Atiyah. K-theory and reality. *Quart. J. Math. Oxford Ser. (2)*, 17:367–386, 1966.

[Ban14] R. Banerjee. A modular description of $ER(2)$. *New York J. Math.*, 20:743–758, 2014.

[BL10] M. Behrens and T. Lawson. *Topological Automorphic forms*, volume 204 of *Mem. Amer. Math. Soc.* American Mathematical Society, 2010.

[DFHH14] C. Douglas, J. Francis, A. Henriques, and M. Hill. *Topological Modular Forms*, volume 201 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2014.

[GHIR17] B. Guillou, M. Hill, D. Isaksen, and D. Ravenel. The cohomology of C_2-equivariant $A(1)$ and the homotopy of ko_{C_2}. `https://arxiv.org/abs/1708.09568`, 2017.

[Lur09] J. Lurie. A survey of elliptic cohomology. In *Algebraic Topology*, volume 4 of *Abel. Symp.*, pages 219–277. Springer, 2009.

[Lur18] J. Lurie. Elliptic Cohomology II, 2018.

[Mat16] A. Mathew. The homology of tmf. *Homology Homotopy Appl.*, 18(2):1–20, 2016.

[MM15] A. Mathew and L. Meier. Affineness and chromatic homotopy theory. *J. Topol.*, 8(2):476–528, 2015.

[MNN17] A. Mathew, N. Naumann, and J. Noel. Nilpotence and descent in equivariant stable homotopy theory. *Adv. Math.*, 305:994–1084, 2017.

[Nau07] N. Naumann. The stack of formal groups in stable homotopy theory. *Adv. Math.*, 215(2):569–600, 2007.